# EVEN GALOIS REPRESENTATIONS AND THE FONTAINE–MAZUR CONJECTURE II

FRANK CALEGARI

ABSTRACT. We prove, under mild hypotheses, that there are no irreducible two-dimensional potentially semi-stable *even* $p$-adic Galois representations of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ with distinct Hodge–Tate weights. This removes the ordinary hypotheses required in our previous work [9]. We construct examples of irreducible two-dimensional residual representations that have no characteristic zero geometric deformations.

## 1. INTRODUCTION

Let $G_{\mathbf{Q}}$ denote the absolute Galois group of $\mathbf{Q}$, and let

$$\rho : G_{\mathbf{Q}} \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$$

be a continuous irreducible representation unramified away from finitely many primes. In [21], Fontaine and Mazur conjecture that if $\rho$ is semi-stable at $p$, then either $\rho$ is the Tate twist of an even representation with finite image or $\rho$ is modular. In [32], Kisin establishes this conjecture in almost all cases under the additional hypotheses that $\rho|D_p$ has distinct Hodge–Tate weights and $\rho$ is *odd* (see also [19]). The oddness condition in Kisin's work is required in order to invoke the work of Khare and Wintenberger [29, 30] on Serre's conjecture. If $\rho$ is even and $p > 2$, however, then $\overline{\rho}$ will never be modular. Indeed, when $\rho$ is even and $\rho|D_p$ has distinct Hodge–Tate weights, the conjecture of Fontaine and Mazur predicts that $\rho$ does not exist. In [9], some progress was made towards proving this claim under the additional assumption that $\rho$ was *ordinary* at $p$. The main result of this paper is to remove this condition. Up to conjugation, the image of $\rho$ lands in $\mathrm{GL}_2(\mathcal{O})$ where $\mathcal{O}$ is the ring of integers of some finite extension $L/\mathbf{Q}_p$ (see Lemme 2.2.1.1 of [8]). Let $\mathbf{F}$ denote the residue field, and let $\overline{\rho} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{F})$ denote the corresponding residual representation. We prove:

1.1. **Theorem.** *Let $\rho : G_{\mathbf{Q}} \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ be a continuous Galois representation which is unramified except at a finite number of primes. Suppose that $p > 7$, and, furthermore, that*

(1) *$\rho|D_p$ is potentially semi-stable, with distinct Hodge–Tate weights.*
(2) *The residual representation $\overline{\rho}$ is absolutely irreducible and not of dihedral type.*
(3) *$\overline{\rho}|D_p$ is not a twist of the representation $\begin{pmatrix} \omega & * \\ 0 & 1 \end{pmatrix}$ where $\omega$ is the mod-$p$ cyclotomic character.*

*Then $\rho$ is modular.*

Taking into account the work of Colmez [15] and Emerton [19], this follows directly from the main result of Kisin [32] when $\rho$ is odd. Thus, it suffices to assume that $\rho$ is even and derive a contradiction. As in [9], the main idea is to use *potential automorphy* to construct from $\rho$ a RAESDC automorphic representation $\pi$ for $\mathrm{GL}(n)$ over some totally real field $F$ whose existence is

Supported in part by NSF Career Grant DMS-0846285 and the Sloan Foundation. MSC2010 classification: 11R39, 11F80.

incompatible with the evenness of $\rho$. It was noted in [9] that improved automorphy lifting theorems would lead to an improvement in the main results of that paper. Using the recent work of Barnet–Lamb, Gee, Geraghty, and Taylor [2], it is a simple matter to deduce the main theorem of this paper if $\rho$ is a twist of a crystalline representation sufficiently deep in the Fontaine–Laffaille range (explicitly, if *twice* the difference of the Hodge–Tate weights is at most $p-2$). However, if one wants to apply the main automorphy lifting theorem (Theorem 4.2.1) of [2] more generally, then (at the very least) one has to assume that $\rho|D_p$ is potentially crystalline. Even under this assumption, one runs into the difficulty of showing that $\rho|D_p$ is *potentially diagonalizable* (in the notation of that paper) which seems out of reach at present. Instead, we use an idea we learnt from Gee (which is also crucially used in [2, 3, 4]) of tensoring together certain "shadow" representations in order to manoeuvre ourselves into a situation in which we can show a certain representation (which we would like to prove is automorphic) lies on the same component (of a particular local deformation ring) as an automorphic representation. In [2, 3], it is important that one restricts, following the idea of M. Harris, to tensoring with representations induced from characters, since then one is still able to prove the modularity of the original representation. In contrast, we shall need to tensor together representations with large image. Ultimately, we construct (from $\rho$) a regular algebraic self dual automorphic representation for $\mathrm{GL}(9)$ over a totally real field $E^+$ with a corresponding $p$-adic Galois representation $\varrho : G_{E^+} \to \mathrm{GL}_9(\overline{\mathbf{Q}}_p)$. If $\rho$ is even, then (by construction) it will the case that $\mathrm{Trace}(\varrho(c)) = +3$ for any complex conjugation $c$. This contradicts the main theorem of [41], and thus $\rho$ must be odd. In order to understand the local deformation rings that arise, and in order to construct an appropriate shadow representation, we shall have to use the full strength of the results of Kisin [32] for totally real fields in which $p$ splits completely. This is the reason why condition 3 of Theorem 1.1 is required, even when $\rho$ is even.

It will be convenient to prove the following, which, in light of the main theorem of [32], implies Theorem 1.1. Recall that $\omega$ denotes the mod-$p$ cyclotomic character.

1.2. **Theorem.** *Let $F^+$ be a totally real field in which $p$ splits completely. Let $\rho : G_{F^+} \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ be a continuous Galois representation unramified except at a finite number of primes. Suppose that $p > 7$, and, furthermore, that*

1. *$\rho|D_v$ is potentially semi-stable, with distinct Hodge–Tate weights, for all $v|p$.*
2. *The representation $\mathrm{Sym}^2\overline{\rho}|_{G_{F^+(\zeta_p)}}$ is irreducible.*
3. *If $v|p$, then $\overline{\rho}|D_v$ is independent of $v|p$ and is not a twist of the representation $\begin{pmatrix}\omega & * \\ 0 & 1\end{pmatrix}$.*

*Then, for every real place of $F^+$, $\rho$ is odd.*

1.3. **Remark.** Under the conditions of Theorem 1.2, it follows that $\rho$ is potentially modular over an extension in which $p$ splits completely (see Remark 3.8).

In section 5, we give some applications of our theorem to universal deformation rings. In particular, we construct (unrestricted) universal deformation rings of large dimension such that none of the corresponding Galois representations are geometric.

1.4. **Remark.** *A word on notation.* There are only finitely many letters that can plausibly be used to denote a global field, and thus, throughout the text, we have resorted to using subscripts. In order to prepare the reader, we note now the existence in the text of a sequence of inclusions of totally real fields:

$$F^+ \subseteq F_1^+ \subseteq F_2^+ \subseteq F_3^+ \subseteq F_4^+ \subseteq F_5^+ \subseteq F_6^+,$$

and corresponding degree two CM extensions $F_3 \subseteq \ldots \subseteq F_6$. The subscript implicitly records (except for one instance) the number of times a theorem of Moret-Bailly (Theorem 3.1) is applied. (This is not literally true, since many of the references we invoke also appeal to variations of this theorem.)

As usual, the abbreviations RAESDC and RACSDC for an automorphic representation $\pi$ for $\mathrm{GL}(n)$ stand for regular, algebraic, essentially-self-dual, and cuspidal; and regular, algebraic, conjugate-self-dual, and cuspidal, respectively.

**Acknowledgements.** I would like to thank Toby Gee, Matthew Emerton, and Richard Taylor for useful conversations, Jordan Ellenberg for a discussion about the inverse Galois problem, Toby Gee for explaining some details of the proof of Theorem 2.2.1 of [2] as well as keeping me informed of changes between the first and subsequent versions of [2], Mark Kisin for explaining how his results in [32] could be used to deduce that every component of a certain local deformation ring contained a global point, Brian Conrad for help in proving Lemma 7.3 and discussions regarding the material of Section 7, and Florian Herzig for conversations about adequateness and the cohomology of Chevalley groups.

## 2. Local Deformation Rings

Let $E$ be a finite extension of $\mathbf{Q}_p$ (the coefficient field), and let $V$ be a $d$-dimensional vector space over $E$ with a continuous action of $G_K$, where $K/\mathbf{Q}_p$ is a finite extension. Let us suppose that $V$ is potentially semi-stable [22]. Let

$$\tau : I_K \to \mathrm{GL}_d(\overline{\mathbf{Q}}_p).$$

be a continuous representation of the inertia subgroup of $K$. Fix an embedding $K \hookrightarrow \overline{\mathbf{Q}}_p$. Attached to $V$ is a $d$-dimensional representation of the Weil–Deligne group of $K$. If the restriction of this representation to the inertia subgroup is equivalent to $\tau$, we say that $V$ is of *type* $\tau$. Also associated to $V$ is a $p$-adic Hodge type $\mathbf{v}$, which records the breaks in the Hodge filtration associated to $V$ considered as a de Rham representation (cf. [31], §2.6). Let $\mathbf{F}$ be a finite field of characteristic $p$, and let us now fix a representation

$$\overline{\rho} : G_K \to \mathrm{GL}_n(\mathbf{F}).$$

Let $R^{\square}_{\overline{\rho}}$ be the universal framed deformation ring of $\overline{\rho}$. The following theorem is a result of Kisin (see [31], Theorem 2.7.6).

2.1. **Theorem** (Kisin). *There exists a quotient $R^{\square,\tau,\mathbf{v}}_{\overline{\rho}}$ of $R^{\square}_{\overline{\rho}}$ such that the $\overline{\mathbf{Q}}_p$-points of the scheme $\mathrm{Spec}(R^{\square,\tau,\mathbf{v}}_{\overline{\rho}}[1/p])$ are exactly the $\overline{\mathbf{Q}}_p$ points of $\mathrm{Spec}(R^{\square}_{\overline{\rho}})$ that are potentially semi-stable of type $\tau$ and Hodge type $\mathbf{v}$. It is unique if it is assumed to be reduced and $p$-torsion free.*

Note that restricting $\overline{\rho}$ to some finite index subgroup $G_L$ induces a functorial map of corresponding local deformation rings:

$$\mathrm{Spec}(R^{\square,\tau,\mathbf{v}}_{\overline{\rho}}[1/p]) \to \mathrm{Spec}(R^{\square,\tau,\mathbf{v}}_{\overline{\rho}|G_L}[1/p]),$$

where, by abuse of notation, $\tau$ in the second ring denotes the restriction of $\tau$ to $I_L$ (and correspondingly with $\mathbf{v}$). We use $\mathbb{1}$ to denote the trivial type.

2.2. **Definition.** *A point of $\mathrm{Spec}(R^{\square,\tau,\mathbf{v}}_{\overline{\rho}}[1/p])$ is* very smooth *if it defines a smooth point on $\mathrm{Spec}(R^{\square,\tau,\mathbf{v}}_{\overline{\rho}|G_L}[1/p])$ for every finite extension $L/K$.*

In sections §1.3 and §1.4 of [2], various notions of equivalence are defined between representations. We would like to define a mild (obvious) extension of these definitions when $v|p$. Suppose that $\rho_1$ and $\rho_2$ are two continuous $d$-dimensional representations of $G_K$ with coefficients in some finite extension $E$ over $\mathbf{Q}_p$. Let $\mathcal{O}$ denote the ring of integers of $E$. Let us assume that $\rho_1$ and $\rho_2$ come with a specific integral structure, i.e., a given $G_K$-invariant $\mathcal{O}$-lattice. Equivalently, we may suppose that $\rho_1$ and $\rho_2$ are representations $G_K \to \mathrm{GL}_d(\mathcal{O})$. In particular, the mod-$p$ reductions $\overline{\rho}_1$ and $\overline{\rho}_2$ are well defined. Such extra structure arises, for example, if the representations $\rho_i$ are the local representations attached to global representations whose mod-$p$ reductions are absolutely irreducible.

2.3. **Definition.** *Suppose $\rho_1$ and $\rho_2$ are two continuous $G_K$-representations with given integral structure. If $\rho_1$ and $\rho_2$ are potentially semi-stable, we say that $\rho_1 \lightning \rho_2$ (respectively, $\rho_1 \rightsquigarrow \rho_2$) if $\overline{\rho}_1 \simeq \overline{\rho}_2$, the representations $\rho_1$ and $\rho_2$ have the same type $\tau$, the same Hodge type $\mathbf{v}$, and lie on the same irreducible component of $\mathrm{Spec}(R^{\square,\tau,\mathbf{v}}_{\overline{\rho}}[1/p])$, and, furthermore, that $\rho_1$ corresponds to a very smooth point of $\mathrm{Spec}(R^{\square,\tau,\mathbf{v}}_{\overline{\rho}}[1/p])$ (respectively, smooth point of $\mathrm{Spec}(R^{\square,\tau,\mathbf{v}}_{\overline{\rho}}[1/p])$).*

2.4. **Remark.** If $\rho_1 \lightning \rho_2$, we say (following [2], §1.3, §1.4) that $\rho_1$ *very strongly connects* to $\rho_2$ (or $\rho_1$ "zap" $\rho_2$). (If $\rho_1 \rightsquigarrow \rho_2$, then $\rho_1$ *strongly connects* to $\rho_2$, or $\rho_1$ "squig" $\rho_2$.) If $\rho_1 \lightning \rho_2$, then clearly $\rho_1 \rightsquigarrow \rho_2$, and moreover $\rho_1|_{G_L} \lightning \rho_2|_{G_L}$ (and hence $\rho_1|_{G_L} \rightsquigarrow \rho_2|_{G_L}$) for any finite extension $L/K$.

2.5. **Remark.** If $\rho_1$ and $\rho_2$ are both potentially crystalline representations, then one may also consider the ring $R^{\square,\tau,\mathbf{v},cr}_{\overline{\rho}}$ parametrizing representations which are potentially crystalline (cf. [31]). One may subsequently define the notions $\lightning$ and $\rightsquigarrow$ relative to this ring. Since $\mathrm{Spec}(R^{\square,\tau,\mathbf{v},cr}_{\overline{\rho}}[1/p])$ is smooth (*ibid.*), the relationships $\rho_1 \lightning \rho_2$ and $\rho_1 \rightsquigarrow \rho_2$ are symmetric, and one may simply write $\rho_1 \sim \rho_2$ ("$\rho_1$ connects to $\rho_2$", cf. [2]). The scheme $\mathrm{Spec}(R^{\square,\tau,\mathbf{v}}_{\overline{\rho}}[1/p])$ is not in general smooth, so we must impose strong connectedness in the potentially semi-stable case in order for the arguments of [2] to apply. In this paper, whenever we have $p$-adic representations $\rho_1$ and $\rho_2$, we shall write $\rho_1 \sim \rho_2$ only when $\rho_1$ and $\rho_2$ are potentially crystalline, and by writing this we mean that they are connected relative to $\mathrm{Spec}(R^{\square,\tau,\mathbf{v},cr}_{\overline{\rho}}[1/p])$.

In order to deduce in any particular circumstance that $\rho_1 \lightning \rho_2$, it will be useful to have some sort of criteria to determine when $\rho_1$ corresponds to a very smooth point. If $V$ is a potentially semi-stable representation of $G_K$, let $D = D_{\mathrm{pst}}(V)$ denote the corresponding weakly admissible $(\varphi, N, \mathrm{Gal}(\overline{K}/K))$-module. Let

$$D(k) \subset \overline{\mathbf{Q}}_p \otimes_{\mathbf{Q}_p^{nr}} D$$

denote the subspace generated by the (generalized) eigenvectors of Frobenius of slope $k$. Since $N\varphi = p\varphi N$, there is a natural map $N : D(k+1) \to D(k)$.

2.6. **Lemma.** *Let $V$ be a potentially semi-stable representation of $G_K$ of type $\tau$ and Hodge type $\mathbf{v}$. Suppose that $N : D(k+1) \to D(k)$ is an isomorphism whenever the target and source are non-zero. Then $V$ is a very smooth point on $\mathrm{Spec}(R^{\square,\tau,\mathbf{v}}_{\overline{\rho}}[1/p])$.*

*Proof.* The explicit condition follows from the proof of Lemma 3.1.5 of [31]. □

The condition of Lemma 2.6 is a (somewhat brutal) way of insisting that the monodromy operator $N$ is as nontrivial as possible, given the action of Frobenius. By Theorem 3.3.4 of [31], $\mathrm{Spec}(R^{\square,\tau,\mathbf{v}}_{\overline{\rho}}[1/p])$ admits a formally smooth dense open subscheme.

2.7. **Example.** *If $V$ is a 2-dimensional representation that is potentially semi-stable but not potentially crystalline, then $\mathrm{Sym}^{n-1}(V)$ is very smooth for all $n$.*

2.8. **Remark.** One expects that a local $p$-adic representation associated to an RACDSC automorphic representation $\pi$ is very smooth on the corresponding local deformation ring. In fact, this would follow by the proof of Lemma 1.3.2 of [2] if one had local–global compatibility at all primes (cf. Conjecture 1.1 and Theorem 1.2 of [44]). Since local–global compatibility is still unknown, however, we must take more care in ensuring that the local representations associated to automorphic representations we construct are (very) smooth.

## 3. Realizing local representations

It will be useful in the sequel to quote the following extension of a theorem of Moret–Bailly.

3.1. **Theorem.** *Let $E$ be a number field and let $S$ be a finite set of places of $E$. Let $F/E$ be an auxiliary finite extension of number fields. Suppose that $X/E$ is a smooth geometrically connected variety. Suppose that: For $v \in S$, $\Omega_v \subset X(E_v)$ is a non-empty open (for the $v$-topology) subset. Then there is a finite Galois extension $H/E$ and a point $P \in X(H)$ such that*

1. *$H/E$ is linearly disjoint from $F/E$.*
2. *Every place $v$ of $S$ splits completely in $H$, and if $w$ is a prime of $H$ above $v$, there is an inclusion $P \in \Omega_v \subset X(H_w)$.*
3. *Suppose that for any place $u$ of $\mathbf{Q}$, $S$ contains either every place $v|u$ of $E$ or no such places. Then one can choose $H$ to be a compositum $EM$ where:*
   - (a) *$M/\mathbf{Q}$ is a totally real Galois extension.*
   - (b) *If there exists a $v \in S$ and a prime $p$ such that $v|p$, then $p$ splits completely in $M$.*

*Proof.* Omitting part (3), this is (a special case of) Proposition 2.1 of [27]. To prove the additional statement, it suffices to apply Proposition 2.1 of [27] to the restriction of scalars $Y = \mathrm{Res}_{E/\mathbf{Q}}(X)$. □

As a first application of this theorem, we prove the following result, which shows that the inverse Galois problem can be solved "potentially", even with the imposition of local conditions at a finite number of primes.

3.2. **Proposition.** *Let $G$ be a finite group, let $E/\mathbf{Q}$ be a finite extension, and $S$ a finite set of places of $E$. Let $F/E$ be an auxiliary finite extension of number fields. For each finite place $v \in S$, let $D_v \subset G$ be a subgroup that occurs as the automorphism group of some finite Galois extension of $E_v$. For each real infinite place $v \in S$, let $c_v \in G$ be an element of order dividing 2. There exists a number field $K/E$ and a finite Galois extension of number fields $L/K$ with the following properties:*

1. *There is an isomorphism $\mathrm{Gal}(L/K) = G$.*
2. *$L/E$ is linearly disjoint from $F/E$.*
3. *All places in $S$ split completely in $K$.*
4. *For all finite places $w$ of $K$ above $v \in S$, the decomposition group $D_w \subset G$ is conjugate to the group $D_v$.*
5. *For all real places $w|\infty$ of $K$ above $v \in S$, complex conjugation $c_w \in G$ is conjugate to $c_v$.*

*Proof.* Suppose that $G$ acts faithfully on $n$ letters, and let $G \hookrightarrow \Sigma$ denote the corresponding map from $G$ to the symmetric group. (Any group admits such a faithful action, e.g., the regular representation.) There is an induced action of $G$ on $\mathbf{Q}[x_1, x_2, \ldots, x_n]$, and we may let $X_G = \mathrm{Spec}(\mathbf{Q}[x_1, x_2, \ldots, x_n]^G)$. There are corresponding morphisms

$$\mathbf{A}^n \to X_G \to X_\Sigma.$$

The scheme $X_G$ is affine, irreducible, geometrically connected, and contains a Zariski dense smooth open subscheme. The variety $X_\Sigma$ is canonically isomorphic to affine space $\mathbf{A}^n$ over $\mathrm{Spec}(\mathbf{Q})$ via the symmetric polynomials. Under the projection to $X_\Sigma$, A $K$-point of $X_G$ (for any perfect field $K$) gives a polynomial over $K$ such that the Galois group of its splitting field $L$ is a (not necessarily transitive) subgroup of $G$. Without loss of generality, we may enlarge $S$ in the following way: For each conjugacy class $\langle g \rangle \in G$, we add to $S$ an auxiliary finite place $v$ and impose a local condition that the decomposition group at $v$ is unramified and is the subgroup generated by $g$. For all $v \in S$, let $\Omega_v \subset X_G(E_v)$ denote the smooth points of $X_G$ such that the corresponding extension $L_v/E_v$ has Galois group $D_v$ (if $v$ is finite) or $\langle c_v \rangle$ (if $v$ is a real infinite place). By assumption, these sets are non-zero, and by Krasner's Lemma they are open. We deduce by Theorem 3.1 (applied to the smooth open subscheme of $X_G$) that there exists a Galois extension $L/K$ with Galois group $H \subset G$ with the required local decomposition groups at each place $w$ above $v$. By construction, for every $g \in G$ there exists a finite unramified place $w$ in $K$ such that the conjugacy class of Frobenius at $w$ in $\mathrm{Gal}(L/K)$ is the conjugacy class of $g$. It follows that the intersection of $H$ with every conjugacy class of $G$ is nontrivial, and hence $H = G$ by a well known theorem of Jordan (see Theorem 4′ of [38]). Thus the theorem is established. □

3.3. **Remark.** A weaker version of Proposition 3.2, namely, that every finite group $G$ occurs at the Galois group $\mathrm{Gal}(L/K)$ for some extension of number fields, is a trivial consequence of the fact that $\Sigma = S_n$ occurs as the Galois group of some extension of $\mathbf{Q}$, since we may take $L$ to be any such extension and $K = L^G$. If we insist that some place $v$ splits completely in $K$, however, this will typically force $L$ to also split completely at $v$.

Let $\rho : G_{F^+} \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ be as in Theorem 1.2. After increasing $F^+$ (if necessary), we may assume that $\rho$ is semi-stable at all primes of residue characteristic different from $p$. Attached to $\rho$ is a residual representation $\overline{\rho} : G_{F^+} \to \mathrm{GL}_2(\mathbf{F})$ for some finite field $\mathbf{F}$ of characteristic $p$.

3.4. **Proposition.** *There exists a totally real field $F_1^+/F^+$ and a residual Galois representation $\overline{r}_{\mathrm{res}} : G_{F_1^+} \to \mathrm{GL}_2(\mathbf{F})$ with the following properties:*

1. *All primes above $p$ split completely in $F_1^+$.*
2. *The residual representation $\overline{r}_{\mathrm{res}} : G_{F_1^+} \to \mathrm{GL}_2(\mathbf{F})$ has image containing $\mathrm{SL}_2(\mathbf{F}_p)$.*
3. *For each $v|p$ in $F^+$, and for each $w$ above $v$ in $F_1^+$, there is an isomorphism $\overline{r}_{\mathrm{res}}|D_w \simeq \overline{\rho}|D_v$.*
4. *If $v \nmid p$, then $\overline{r}_{\mathrm{res}}|D_v$ is unramified.*
5. *$\overline{r}_{\mathrm{res}}$ is totally odd at every real place of $F_1^+$.*
6. *$F_1^+ \cap \mathbf{Q}(\zeta_p) = \mathbf{Q}$.*

*Proof.* Proposition 3.2 immediately guarantees a residual representation satisfying all the conditions with the possible exception of (4), which can be achieved by a further base extension. □

3.5. **Lemma.** *After possibly increasing $\mathcal{O}$, there exists a global lift $r_{\mathrm{res}} : G_{F_1^+} \to \mathrm{GL}_2(\mathcal{O})$ of $\overline{r}_{\mathrm{res}}$ such that:*

1. *If $\overline{r}_{\mathrm{res}}|D_v$ is reducible for all $v|p$, then $r_{\mathrm{res}}$ is ordinary and crystalline with distinct Hodge–Tate weights.*
2. *If $\overline{r}_{\mathrm{res}}|D_v$ is irreducible for all $v|p$, then $r_{\mathrm{res}}$ is crystalline in the Fontaine–Laffaille range with distinct Hodge–Tate weights.*
3. *If $v \nmid p$, then $r_{\mathrm{res}}|D_v$ is unramified.*

**Remark**. For any $\overline{r}_{\mathrm{res}}$, then either (1) or (2) holds, since we are assuming that $\overline{r}_{\mathrm{res}}|D_v$ for $v|p$ does not depend on $v$.

*Proof.* Theorems of this kind (minimal lifting theorems) were first proved by Khare–Wintenberger, see in particular Corollary 4.7 of [30]. We avoid appealing directly to [30] only because the results of *ibid.* are only formulated for $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ representations. Instead, we may appeal to Proposition 3.2.1 of [2] (in the ordinary case) or Proposition 4.3.1 of [2] (for both cases), since these theorems are conveniently formulated for totally real fields. To apply these theorems, we remark that our running assumption $p > 7$ implies that $p > 6$, that we may take the CM field $F_1$ to be any CM field in which all primes in $p$ split completely, and that any local residual Galois representation:

$$\overline{r}_{\mathrm{local}} : \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p) \to \mathrm{GL}_2(\mathbf{F})$$

clearly admits a crystalline ordinary lift if $\overline{r}_{\mathrm{local}}$ is reducible and a Fontaine–Laffaille crystalline lift if it is irreducible. □

3.6. **Proposition.** *There exists a totally real field $F_2^+/F_1^+$ and a Hilbert modular form $f$ for $F_2^+$ with a corresponding residual representation $\overline{\rho}_f : G_{F_2^+} \to \mathrm{GL}_2(\mathbf{F})$ with the following properties:*

1. *There is an isomorphism $\overline{\rho}_f \simeq \overline{r}_{\mathrm{res}}|G_{F_2^+}$.*
2. *All primes above $p$ split completely in $F_2^+$.*
3. *$F_2^+ \cap \mathbf{Q}(\zeta_p) = \mathbf{Q}$.*

3.7. **Remark.** *This remark may be omitted on first reading.* A general strategy of proving such results was developed by Taylor in a sequence of two papers [42, 43]. Theorem 1.6 of [42] implies Proposition 3.6 under the additional assumption that $\overline{\rho}|_{D_v}$ is reducible for $v|p$ and $\det(\overline{\rho}) = \omega$. The idea, loosely speaking, is as follows. For some totally real field $E$, consider the moduli space $X$ of polarized Hilbert–Blumenethal abelian varieties $A$ with an action of $\mathcal{O}_E$ such that:

1. For a prime $\mathfrak{p} \subset \mathcal{O}_E$, there is an isomorphism $A[\mathfrak{p}] \simeq \overline{r}_{\mathrm{res}}$.
2. For a prime $\lambda \subset \mathcal{O}_E$ with residue characteristic some auxiliary prime $l \neq p$, there is an isomorphism $A[\lambda] \simeq \mathrm{Ind}^{\mathbf{Q}}_M \chi \mod \lambda$ with some irreducible induced modular representation.

Using Theorem 3.1, one deduces the existence of a suitable totally real field $F_2^+$. In [43], Taylor considers the case when $\overline{r}_{\mathrm{res}}|D_p$ is irreducible, and in Proposition 4.1 of [43] proves the potential modularity of $\overline{\rho}$ *without* any restriction on the determinant except that it is totally odd. The main innovation is to consider a *twisted* moduli space $X_\mu$, where $\mu$ is a finite character lifting $\det(\overline{r}_{\mathrm{res}})\omega^{-1}$. The author expects that a "fibre product" of this argument can be constructed to deduce Proposition 3.6, even under the weaker condition that $\overline{r}_{\mathrm{res}}|D_v$ for $v|p$ need not be independent of $v$. Instead, we present an alternative argument using minimal lifting theorems (of Khare–Wintenberger type) as well as recent modularity lifting theorems, which allows us to avoid generalizing the arguments of [42, 43].

*Proof of Proposition 3.6.* We divide the proof into two cases, depending on whether $\overline{r}_{\mathrm{res}}|D_v$ is reducible or not (for $v|p$).

Suppose that $\overline{r}_{\mathrm{res}}|D_v$ is reducible. We first construct a cyclic Galois extension $A/F_1^+$ such that:

1. $\det(\overline{r}_{\mathrm{res}})|G_{AF_1^+} = \omega \cdot \psi^2|G_{AF_1^+}$ for some character $\psi$ of $G_{\mathbf{Q}}$.
2. $A \cap \mathbf{Q}(\zeta_p) = \mathbf{Q}$.
3. $A$ is totally real.

Suppose that $\det(\overline{r}_{\mathrm{res}})\omega^{-1} = \mu$. The character $\mu$ is totally even. Choose an auxiliary prime $l \equiv 1 \mod 2 \cdot \deg(\mu)$ which is unramified in $F_1^+(\ker(\mu))$. Then there exists a character $\psi$ of $G_{\mathbf{Q}}$ of degree $2\det(\mu)$ which is totally ramified at $l$. Let $A = F_1^+(\mu\psi^{-2})$. Since $\mu$ and $\psi^2$ are totally even, $A$ is totally real. Clearly $A$ is totally ramified at $l$, and thus $A \cap \mathbf{Q}(\zeta_p) = F_1^+ \cap \mathbf{Q}(\zeta_p) = \mathbf{Q}$. Yet, by construction, $\det(\overline{r}_{\mathrm{res}})|G_{AF_1^+} = \omega\mu = \omega\psi^2$. We now apply Theorem 1.6 of [42] to $\psi^{-1}\overline{r}_{\mathrm{res}}|G_{AF_1^+}$ (modified using the restriction of scalars trick as in part (3) of Theorem 3.1) to deduce that $\overline{r}_{\mathrm{res}}$ is modular over an extension of the form $AF_2^+$, where all primes above $p$ split completely in $F_2^+/F_1^+$.

By Theorem 6.1.7 of [4], we may assume, moreover, that the representation is *ordinarily* modular. Consider the representation:

$$r_{\mathrm{res}}|G_{AF_2^+} \to \mathrm{GL}_2(\mathcal{O}).$$

By assumption, this representation is ordinary, and residually ordinarily modular. Hence it is modular by Theorem B of [2]. Since $A$ is solvable, we deduce by a standard base change argument (Theorem 4.2 (p.202) of [1]) that $r_{\mathrm{res}}|G_{F_2^+}$ is modular, and we are done.

Suppose that $\overline{r}_{\mathrm{res}}|D_v$ is irreducible. (This is the easier case, because Taylor's construction in [43] has no restiction on the determinant other than being totally odd.) We modify the proof of Proposition 4.1 of [43] as follows. Although the formulation of Taylor's result is for Galois representations of $G_{\mathbf{Q}}$, the argument remains unchanged for Galois representations of totally real fields in which $p$ splits completely and for which $\overline{r}_{\mathrm{res}}|D_v$ is irreducible and independent of $v|p$. Apply Theorem 3.1 to Taylor's twisted moduli space $X_\mu$ over $F_1^+$. We deduce as in [43] that $\overline{r}_{\mathrm{res}}$ is modular over an extension $AF_2^+$, where all primes above $p$ split completely in $F_2^+/F_1^+$. Moreover, as follows from the arguments of §5 of *ibid.* (Corollary 5.2 and its application in Lemma 5.6, see also [23]), that $\overline{r}_{\mathrm{res}}$ arises as the mod-$p$ representation associated to a Hilbert modular form $\pi$ of parallel weight $k$ and level co-prime to $p$ for some $2 \le k \le p-1$. Consider the representation:

$$r_{\mathrm{res}}|G_{F_2^+} \to \mathrm{GL}_2(\mathcal{O}).$$

From the discussion above, we know that this representation is residually modular from an automorphic representation $\pi$ whose associated local representations are crystalline in the Fontaine–Laffaille range. Hence it is modular by Theorem 4.2.1 of [2], and we are done. □

3.8. **Remark.** If $\rho$ is *odd* for all infinite places of $F_1^+$, then we may take $\overline{r}_{\mathrm{res}}$ to be $\overline{\rho}$, and Proposion 3.6 implies that $\overline{\rho}|G_{F_2^+}$ is modular. By Theorem 2.2.18 of [33], it follows in this case that $\rho$ is modular over $F_2^+$. (This is not literally correct, because $\overline{\rho}$ may not have image containing $\mathrm{SL}_2(\mathbf{F}_p)$ and so not virtually satisfy Condition 2 of Proposition 3.4. However, one may check that the only fact used about the image of $\overline{r}$ so far is that it is irreducible.)

Having realized the representations $\overline{\rho}|D_v$ for $v|p$ inside the mod-$p$ reduction of some Hilbert modular form $f$, we now realize the representations $\rho|D_v$ in characteristic zero as coming from Hilbert modular forms (to the extent that it is possible).

3.9. **Proposition.** *Let $p > 2$ be prime. There exists a Hilbert modular form $g$ over $F_2^+$ with the following properties:*

1. *The residual representation $\overline{\rho}_g : G_{F_2^+} \to \mathrm{GL}_2(\mathbf{F})$ is equal to $\overline{\rho}_f$.*
2. *For each place $v|p$ of $F_2^+$, $\rho_g|D_v \sim \rho|D_w$ if $\rho|D_w$ is potentially crystalline, and $\rho_g|D_v \leadsto \rho|D_w$ otherwise.*
3. *For each finite place $v$ of $F_2^+$ away from $p$, $\rho_g|D_v$ is unramified.*

*Proof.* Consider the modular representation $\overline{\rho}_f = \overline{r}_{\mathrm{res}}|G_{F_2^+}$ constructed in Proposition 3.6. By construction, it is modular of minimal level and is unramified outside $p$. It follows from Theorem 2.2.18 and Corollary 2.2.17 of [32] that (in the notation of *ibid.*) $M_\infty$ is faithful as an $\bar{R}_\infty$-module. Recall that $\bar{R}_\infty = \bar{R}^{\square,\psi}_{\Sigma_p}[\![x_1,\ldots,x_g]\!]$ is a power series ring over a tensor product of local deformation rings. Here $\Sigma_p$ denotes the set of places dividing $p$. Consider a component $Z$ of $\mathrm{Spec}(\bar{R}_\infty)$ such that the characteristic zero points of $Z$ lie on the same local component as $\rho|D_v$ for $v|p$ (if $\rho|D_v$ lies on multiple components, choose any component). Since all the (equivalent) conditions of Lemma 2.2.11 of *ibid.* hold, we know (as in the proof of and notation of that lemma) that $\bar{R}_\infty$ is a finite torsion free

$\mathcal{O}[\![\Delta_\infty]\!]$-module. In particular, $Z$ surjects onto $\mathrm{Spec}(\mathcal{O}[\![\Delta_\infty]\!])$. In particular, there is a non-trivial fibre at 0. Since $M_0 = M_\infty \otimes_{\mathcal{O}[\![\Delta_\infty]\!]} \mathcal{O}$ is a space of *classical* modular forms (of minimal level), we deduce the existence of $g$. Note that if $\rho|D_w$ is semi-stable but not crystalline, then $\rho_g$ is very smooth by Example 2.7, and hence $\rho_g|D_v \leftrightsquigarrow \rho|D_w$ for $v|p$. □

3.10. **Remark.** The faithfulness of $M_\infty$ as a $\bar{R}_\infty$-module is not a clear consequence of the Fontaine–Mazur conjecture. That is, *a priori*, the collection of all global representations may surreptitiously conspire to avoid a given local component. Thus, without any further ideas, the new cases of the Fontaine–Mazur conjecture proved by Emerton [19] do not allow us to realize all local representations globally when

$$\overline{\rho}|D_p \sim \begin{pmatrix} \omega & * \\ 0 & 1 \end{pmatrix}.$$

We have now constructed a Hilbert modular form $g$ whose $p$-adic representation is a "shadow" of $\rho$, that is, lies on the same component as $\rho$ of every local deformation space at a place dividing $p$. However, the global mod-$p$ representations $\overline{\rho}$ and $\overline{\rho}_g$ are unrelated. In order to prove a modularity statement, we will need to construct a second pair of shadow representations with the same residual representation as $\overline{\rho}$ and $\overline{\rho}_g$. It will never be the case, however, that $\overline{\rho}$ will be automorphic over a totally real field unless $\overline{\rho}$ is totally odd. The main idea of [9] was to consider the representation $\mathrm{Sym}^2(\overline{\rho})$ as a conjugate self-dual representation over some CM field. In the sequel, we shall construct a pair of RACSDC ordinary crystalline shadow representations which realize the mod-$p$ representations $\mathrm{Sym}^2(\overline{\rho})$ and $\mathrm{Sym}^2(\overline{\rho}_g)$. By abuse of notation, let $\mathbf{v}$ denote the Hodge type of $\rho$ for any prime dividing $p$, so $\mathbf{v}$ is literally a collection of Hodge types for each $v|p$ in $F^+$ (adding subscripts would add nothing to the readability of the following argument).

3.11. **Proposition.** *Let $p > 7$ be prime. There exists a totally real field $F_4^+/F_2^+$, a CM extension $F_4/F_2^+$, and a RACDSC automorphic representation $\pi$ over $F_4$ with a corresponding Galois representation $\rho_\pi : G_{F_4} \to \mathrm{GL}_3(\overline{\mathbf{Q}}_p)$ such that:*

1. *$\rho_\pi$ is unramified at all places not dividing $p \cdot \infty$.*
2. *For every $v|p$, $\rho_\pi|G_v$ is ordinary and crystalline with Hodge type $\mathrm{Sym}^2\mathbf{w}$, where $\mathbf{w}$ is a Hodge type of some 2-dimensional de Rham representation.*
3. *For all $v|p$ and for all $i$, there is an inequality $\dim \mathrm{gr}^i(\mathrm{Sym}^2\mathbf{v} \otimes \mathrm{Sym}^2\mathbf{w}) \leq 1$.*
4. *The image of the restriction of $\overline{\rho}$ to $G_{F_4^+}$ is the image of $\overline{\rho}$ on $G_{\mathbf{Q}}$.*
5. *The restriction of $\overline{\rho}_g$ to $G_{F_4^+}$ has image containing $\mathrm{SL}_2(\mathbf{F}_p)$.*
6. *The residual Galois representation $\overline{\rho}_\pi : G_{F_4} \to \mathrm{GL}_3(\mathbf{F})$ is isomorphic to the restriction of $\mathrm{Sym}^2(\overline{\rho})$ to $G_{F_4}$.*
7. *The Hilbert modular form $g$ remains modular over $F_4^+$.*
8. *The compatible family of Galois representations associated to $\pi$ is irreducible after restriction to any finite index subgroup of $G_{F_4}$.*
9. *If $\rho_g|D_p$ is not potentially crystalline, the representation $\rho_\pi \otimes \mathrm{Sym}^2(\rho_g)|D_p$ is a very smooth point of*
$$\mathrm{Spec}(R^{\square,\mathrm{Sym}^2(\tau)\otimes\mathrm{Sym}^2(\mathbb{1}),\mathrm{Sym}^2\mathbf{v}\otimes\mathrm{Sym}^2\mathbf{w}}[1/p]).$$
10. *$F_4 \cap \mathbf{Q}(\zeta_p) = \mathbf{Q}$.*

*Proof.* Let $F_3^+/F_2^+$ be a totally real field for which $\mathrm{Sym}^2(\overline{\rho})$ becomes ordinary at all $v|p$. (In general, the field $F_3^+$ will be highly ramified at $p$). Increasing $F_3^+$ if necessary, assume that the restriction of $\mathrm{Sym}^2(\overline{\rho})$ is unramified outside $v|p$, and that there exists a CM extension $F_3/F_3^+$ which is totally unramified at all finite places. By Proposition 3.3.1 of [2], $\mathrm{Sym}^2(\overline{\rho})$ admits minimal ordinary automorphic lifts over some CM extension $F_4 = F_4^+.F_3$. (We use the fact that $\overline{\rho}$ is not of dihedral type, so $\mathrm{Sym}^2(\overline{\rho})$ is irreducible, and that $p \geq 2(3+1)$.) The resulting automorphic representation $\pi$ satisfies condition (1). Note that $F_4$ may be chosen to be disjoint from any auxiliary field. This implies that we may construct $F_4$ so that conditions (1), (4), (5), (6), and (10) hold, The modularity of $g$ (by construction) arises from the fact that at some auxiliary prime $\lambda$, the mod-$\lambda$ representation associated to $g$ is induced from a character. Choosing $F_5$ to be disjoint from the fixed field of the kernel of this representation ensures that condition (7) holds. Note that the freedom to choose the Hodge type follows from the freedom to choose $\mu$ in Proposition 3.2.1 of [2], and thus we may chose $\pi$ to satisfy conditions (2) and (3). To verify that the compatible system associated to $\pi$ is irreducible over any finite index subgroup of $G_{F_4}$ (and so verifies condition (8)), we invoke Theorem 2.2.1 of [6]. It suffices to note that $\overline{\rho}_\pi$ has non-solvable image, and thus $\pi$ is not induced from an algebraic Hecke character over a solvable extension. Finally, we must show that $\pi$ can be chosen to satisfy (9). The slopes of Frobenius of a crystalline representation are given by the breaks in the Hodge filtration. In particular, we may choose a $\mathbf{w}$ so that the integers $i$ such that $\mathrm{gr}^i\mathrm{Sym}^2(\mathbf{w}) \neq 0$ each differ by $\geq 4$. If $\mathrm{Sym}^2(\rho_g)$ is potentially semistable but not potentially crystalline, it follows from Lemma 2.6 that for such a $\mathbf{w}$ that *any* such tensor product $\rho_\pi \otimes \mathrm{Sym}^2(\rho_g)$ will be very smooth. Thus, choosing $\mathbf{w}$ appropriately, very smoothness is automatically satisfied. □

We may now construct a Hilbert modular form $h$ as follows.

3.12. **Proposition.** *Let $p > 5$. There exists a totally real field $F_5^+/F_4^+$ and a Hilbert modular form $h$ over $F_5^+$ with a corresponding Galois representation $\rho_h : G_{F_5^+} \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ such that:*

1. *For every $v|p$, $\rho_h|D_v$ is ordinary with Hodge type $\mathbf{w}$.*
2. *The residual representation $\overline{\rho}_h : G_{F_5^+} \to \mathrm{GL}_2(\mathbf{F})$ is isomorphic to the restriction of $\overline{\rho}_g$.*
3. *The images of $\overline{\rho}$ and $\overline{\rho}_g$ remain unchanged upon restriction to $\mathrm{SL}_2(\mathbf{F}_p)$.*
4. *The Hilbert modular form $g$ remains modular over $F_5^+$, and the RACDSC representation $\pi$ remains modular over $F_5 = F_5^+.F_4$.*
5. *For all places $v$ not dividing $p$, $\rho_h|D_v \rightsquigarrow \rho|D_v$.*
6. *$F_5 \cap \mathbf{Q}(\zeta_p) = \mathbf{Q}$.*

*Proof.* We may prove this by modifying the proof of Proposition 3.11 as follows. Modify the field $F_3^+$ so that $\overline{\rho}_g$ is ordinary at all $v|p$ and is unramified at all other finite places (but still has large image), and such that $\overline{\rho}_g|D_v$ admits a ramified semi-stable lift for those primes $v \nmid p$ which ramify in $\rho$. Then let $F_5^+$ denote a field for which we can simultaneously establish the modularity of the GL(6) forms arising in the proof of Lemma 3.11 and the GL(2) form associated to ordinary deformations of $\overline{\rho}_g$. (It may have been more consistent to have combined Propositions 3.11 and 3.12 into a single Lemma, but it would have been more unwieldy.) □

## 4. The proof of Theorem 1.2

Let $L/\mathbf{Q}$ denote a field which contains the coefficient field of $g$ and $\pi$. Let $(L, \rho_{g,\lambda})$ denote the compatible family of Galois representations associated to $g$. There exists a prime of $\mathcal{O}_F$ dividing $p$ such that the corresponding mod-$p$ representation is $\overline{\rho}_g$, which, by construction, has non-solvable image. It follows that the form $g$ does not have complex multiplication, and hence the images of $\rho_{g,\lambda}$ for all $\lambda$ contains an open subgroup of $\mathrm{SL}_2(\mathbf{Z}_l)$ where $\lambda|l$.

4.1. **Proposition.** *There exists a totally real field $F_6^+/F_5^+$, a CM extension $F_6/F_6^+$ and a RACDSC automorphic representation $\Pi$ such that:*

(1) $\Pi$ *corresponds to* $\mathrm{Sym}^2(g) \otimes \pi$.
(2) $\overline{\rho}$ *and* $\overline{\rho}_h$ *have image containing* $\mathrm{SL}_2(\mathbf{F}_p)$ *after restriction to* $G_{F_6^+}$.

*Proof.* This is equivalent to showing that the compatible family

$$(L, \mathrm{Sym}^2(\rho_{g,\lambda}) \otimes \rho_{\pi,\lambda})$$

is potentially automorphic. This compatible system is essentially self-dual, orthogonal (automatically since $n = 9$ is odd), and has distinct Hodge–Tate weights (by assumption 3 of Lemma 3.11). Let us verify that it is irreducible. Since the image of $\rho_{g,\lambda}$ restricted to any finite index subgroup contains an open subgroup of $\mathrm{SL}_2(\mathbf{Z}_l)$, and since the image of $\rho_{\pi,\lambda}$ restricted to any open subgroup is irreducible (by part 8 of Proposition 3.11), their tensor product will be irreducible unless $\rho_{\pi,\lambda}$ is (on some open subgroup) a twist of $\mathrm{Sym}^2(\rho_{g,\lambda})$. This implies that $\rho_{\pi,\lambda}$ is already equal to a twist of $\mathrm{Sym}^2(\rho_{g,\lambda})$, since the latter representation has no inner twists. By multiplicity one [28] for $\mathrm{GL}(3)$, we deduce that $\mathrm{Sym}^2(g)$ is a twist of $\pi$. This contradicts the fact that the mod-$p$ residual representations representations $\mathrm{Sym}^2(\overline{\rho}_g)$ and $\overline{\rho}_\pi$ are not twists of each other, since one extends to a totally odd representation of $G_{F_5^+}$ and the other to a representation of $G_{F_5^+}$ which is even at some infinite place. The potential automorphy follows from Theorem A of [2]. □

Let us now write $E^+ = F_6^+$ and $E = F_6$, and consider the representations $\rho$, $\rho_g$, $\rho_\pi$, and $\rho_h$ as representations of $G_E$. Without loss of generality, we may assume that $\rho$ is even for at least one real place of $F^+$ (and hence also of $E^+$). Let us consider the representation

$$\varrho : \mathrm{Sym}^2(\rho) \otimes \mathrm{Sym}^2(\rho_h) : G_E \to \mathrm{GL}_9(\overline{\mathbf{Q}}_p).$$

By construction, we observe that $\overline{\varrho} = \mathrm{Sym}^2(\overline{\rho}) \otimes \mathrm{Sym}^2(\overline{\rho}_h) = \overline{\rho}_\pi \otimes \mathrm{Sym}^2(\overline{\rho}_g) = \overline{\rho}(\Pi)$ is residually modular. Moreover, we find that

$$\rho(\Pi)|D_v = \rho_\pi \otimes \mathrm{Sym}^2(\rho_g)|D_v \rightsquigarrow \mathrm{Sym}^2(\rho) \otimes \mathrm{Sym}^2(\rho_h)|D_v = \varrho|D_v$$

for all $v$, with the possible exception of $v|p$. By Lemma 3.4.3 of Geraghty [24], the ordinary deformation rings are smooth and connected, and hence, for $v|p$,

$$\rho_\pi|D_v \sim \mathrm{Sym}^2(\rho_h)|D_v.$$

On the other hand, by construction (Proposition 3.9 (2)), we also have (for $v|p$) that

$$\mathrm{Sym}^2(\rho_g)|D_v \leftrightsquigarrow \mathrm{Sym}^2(\rho)|D_v.$$

If $\rho|D_v$ is potentially crystalline, then all four representations are potentially crystalline at $v$, and we deduce that

$$\rho_\pi \otimes \mathrm{Sym}^2(\rho_g)|D_v \sim \mathrm{Sym}^2(\rho) \otimes \mathrm{Sym}^2(\rho_h)|D_v.$$

On the other hand, if $\rho|D_v$ is not potentially crystalline, then neither is $\rho_g|D_v$, and we deduce from condition (9) of Lemma 2.6 that the left hand side corresponds to a very smooth point of the corresponding local deformation ring $\mathrm{Spec}(R^{\square,\mathrm{Sym}^2(\tau)\otimes\mathrm{Sym}^2(\mathbb{1}),\mathrm{Sym}^2\mathbf{v}\otimes\mathrm{Sym}^2\mathbf{w}}[1/p])$. Hence

$$\rho(\Pi)|D_v = \rho_\pi \otimes \mathrm{Sym}^2(\rho_g)|D_v \leftrightsquigarrow \mathrm{Sym}^2(\rho) \otimes \mathrm{Sym}^2(\rho_h)|D_v = \varrho|D_v,$$

and thus $\rho(\Pi)|D_v \rightsquigarrow \varrho|D_v$. Since the fixed fields corresponding to $\mathrm{Sym}^2(\overline{\rho})$ and $\mathrm{Sym}^2(\overline{\rho}_h)$ are disjoint by construction, and since both representations have adequate image (in the sense of [45]), the representation $\overline{\varrho}$ is also adequate, by Lemma 2(ii) of [25]. It follows from Theorem 7.1 below (c.f. Theorem 2.2.1 of [2]) that $\varrho$ is modular over $E$. (Since $n = 9$ is odd, the weakly regular condition is vacuous.) Since the Galois representations $\rho$ and $\rho_h$ extend to the totally real subfield $E^+$, so does the representation $\varrho$, and hence (by [1]) $\varrho$ comes from a RAESDC representation for

$\mathrm{GL}(9)/E^+$. By assumption, however, there exists a real place of $E^+$ and a correponding complex conjugation $c_v \in G_{E^+}$ such that $\rho(c_v)$ is a scalar. It follows that the image $\varrho(c_v)$ of $c_v$ is conjugate to

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix},$$

which has trace $+3$. Since the representation $\varrho$ is irreducible, automorphic, and 9 is odd, this contradicts the main theorem (Proposition A) of [41], which says that such representations must be "odd" in the sense that the trace of complex conjugation must be $\pm 1$. This completes the proof of Theorem 1.2.

4.2. **Remark.** In [9], we proved (Theorem 1.3) that ordinary representations $\rho : G_K \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ (satisfying certain supplementary hypotheses) had parallel weight. One may ask whether the methods of this paper can be used to generalize that result (presuming that $p$ splits in $K$). Starting with the tensor representation $\psi = \rho \otimes \rho^c$, one is lead to a 16 dimensional representation $\varrho = \psi \otimes \rho_f \otimes \rho_g$ for non-CM Hilbert modular forms $f$ and $g$ which one may show is modular for $\mathrm{GL}(16)$ over some totally real field $E^+$. It is not apparent, however, how this might lead to a contradiction, since $\varrho(c)$ has trace 0 for any complex conjugation $c$, and known cases of functoriality do not yet allow one to deduce the modularity of $\psi$ from the modularity of $\varrho$. Another approach is to consider the representation $\psi = \mathrm{Sym}^2 \rho \otimes \mathrm{Sym}^2 \rho^c$, and a corresponding 81 dimensional representation $\varrho = \psi \otimes \mathrm{Sym}^2 \rho_f \otimes \mathrm{Sym}^2 \rho_g$. In this case, one should be able to deduce that $\varrho$ is modular for $\mathrm{GL}(81)/E^+$ for some totally real field $E^+$, and that $\varrho(c)$ is conjugate to

$$\begin{pmatrix} 1 & 0 & 0 & & & & & & \\ 0 & 1 & 0 & & & & & & \\ 0 & 0 & 1 & & & & & & \\ & & & 0 & 1 & & & & \\ & & & 1 & 0 & & & & \\ & & & & & 0 & 1 & & \\ & & & & & 1 & 0 & & \\ & & & & & & & 0 & 1 \\ & & & & & & & 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix},$$

which has trace 3, contradicting the main theorem of [41]. However, this approach only works when $\psi$ has distinct Hodge–Tate weights, and correspondingly one may only deduce (if $\rho$ has Hodge–Tate weights $[0, m]$ and $[0, n]$ with $n \geq m$ positive) that either $n = m$ (which is the expected conclusion) *or* $n = 2m$.

## 5. Applications to Universal Deformation Rings

If $F/\mathbf{Q}$ is a number field, and $\overline{\rho} : G_F \to \mathrm{GL}_n(\mathbf{F})$ is an irreducible continuous Galois representation, let $\mathfrak{X} = \mathfrak{X}_S(\overline{\rho})$ denote the rigid analytic space corresponding to the universal deformation ring of $\overline{\rho}$ unramified outside a finite set of primes $S$. The space $\mathfrak{X}$ contains a (presumably countable) set of points which are de Rham (= potentially semi-stable) at all places above $p$. We call such deformations *geometric*, and denote the corresponding set of points by $\mathfrak{X}^{\mathrm{geom}}$. Gouvêa and Mazur, using a beautiful construction they called the infinite fern, showed (under the assumption that $\overline{\rho}$ was unobstructed) that when $F = \mathbf{Q}$, $n = 2$, and $\overline{\rho}$ is odd that $\mathfrak{X}^{\mathrm{geom}} \subset \mathfrak{X}$ is Zariski dense (see [36]). This raises the general question of when $\mathfrak{X}^{\mathrm{geom}}$ is Zariski dense in $\mathfrak{X}$. The work of Gouvêa and Mazur has been extended (in the same setting) by others, in particular Böckle [7] (cf. Theorem 1.2.3 of [19]). Chenevier [11] has recently generalized the infinite fern argument to apply to certain conjugate self dual representations for $n \geq 3$ over CM fields, and has shown that the Zariski closure of $\mathfrak{X}^{\mathrm{geom}}$ is (in some precise sense) quite large. As a consequence of our main result, however, we prove the following theorem.

5.1. **Theorem.** *Let $p > 7$, let $F^+/\mathbf{Q}$ be a totally real field in which $p$ splits completely, and let $\overline{\rho} : G_{F^+} \to \mathrm{GL}_2(\mathbf{F})$ be a continuous irreducible representation whose image contains $\mathrm{SL}_2(\mathbf{F}_p)$. Suppose that $\overline{\rho}$ is even for at least one real place of $F^+$. Suppose that for all $v|p$,*

$$\overline{\rho}|I_v \sim \begin{pmatrix} \psi_v & * \\ 0 & 1 \end{pmatrix}$$

*where $\sim$ denotes up to twist, and $\psi_v \neq \omega$ is assumed to have order $> 2$, and $* \neq 0$. Assume moreover that $\overline{\rho}|_{D_v}$ is independent of $v$ for $v|p$. Let $S$ be any finite set of places of $F^+$. Then $\mathfrak{X}^{\mathrm{geom}}$ is empty, that is, $\overline{\rho}$ has no geometric deformations.*

*Proof.* Assume otherwise. Let $\rho$ be a point of $\mathfrak{X}^{\mathrm{geom}}$. By Theorem 1.2, there exists at least one $v|p$ such that the Hodge–Tate weights of $\rho$ at $v$ are equal. To be potentially semi-stable (or even Hodge–Tate) of parallel weight zero is equivalent to being unramified over a finite extension. Thus, up to twist, $\rho|I_v$ has finite image, and, in particular, the projective image of $\rho|I_v$ is finite. The only finite subgroups of $\mathrm{PGL}_2(\overline{\mathbf{Q}}_p) \simeq \mathrm{PGL}_2(\mathbf{C})$ are either cyclic, dihedral, $A_4$, $S_4$, or $A_5$. Thus, the projective image of $\overline{\rho}|I_v$ must be one of these groups. By assumption, the projective image of $\overline{\rho}|I_v$ is a non-dihedral group of order divisible by $p > 7$, hence $\rho|I_v$ can not be finite up to twist either, and $\rho$ does not exist. □

We have the following corollary:

5.2. **Corollary.** *There exist absolutely irreducible representations $\overline{\rho}$ such that the subset $\mathfrak{X}^{\mathrm{geom}} \subset \mathfrak{X}$ is* not *Zariski dense. In fact, there exist representations such that $\mathfrak{X}^{\mathrm{geom}}$ is empty, but $\mathfrak{X}$ has arbitrary large dimension.*

*Proof.* Let $F^+/\mathbf{Q}$ be a totally real field, and let $\overline{\rho} : G_{F^+} \to \mathrm{GL}_2(\mathbf{F})$ be a continuous irreducible representation satisfying the conditions of Theorem 5.1. Suppose, furthermore, that $\psi_v \neq \omega^{-1}$ for any $v|p$. The existence of such representations is guaranteed by Proposition 3.2. By Theorem 5.1, $\mathfrak{X}^{\mathrm{geom}} = \emptyset$ for any finite set of auxiliary primes $S$. However, arguing exactly as in Theorem 1(a) of [37], one deduces the existence of sets $S$ for which $\mathfrak{X}$ contains a smooth subvariety of dimension $(1+\delta)+2r$, where $\delta$ is the Leopoldt defect and $r$ is the number of infinite places of $F^+$ at which $\overline{\rho}$ is odd. In order to see that Theorem 5.1 allows us to make $r$ arbitrarily large, first find an auxiliary totally real extension $E^+/\mathbf{Q}$ in which $p$ splits completely, and then apply Theorem 5.1 with $S$ containing all infinite places such that, for $v|\infty$, $c_v$ has order two at all but one infinite place. □

Similarly, we note the following cases of the Fontaine–Mazur which do not require any assumption on the Hodge–Tate weights or the parity of $\overline{\rho}$:

5.3. **Corollary.** *$\rho : G_{\mathbf{Q}} \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ be a continuous Galois representation which is unramified except at a finite number of primes. Suppose that $p > 7$, and, furthermore, that*

(1) *$\rho|D_p$ is potentially semi-stable.*
(2) *The residual representation $\overline{\rho}$ is absolutely irreducible and is not of dihedral type.*
(3) *$\overline{\rho}|D_p$ is of the form: $\begin{pmatrix} \psi_1 & * \\ 0 & \psi_2 \end{pmatrix}$ where:*
    (a) *$*$ is ramified.*
    (b) *$\psi_1/\psi_2 \neq \omega$, and $\psi_1/\psi_2|I_p$ has order $> 2$.*

*Then $\rho$ is modular.*

Although proposition 3.2 guarantees the existence of infinitely many even Galois representations over totally real fields with image containing $\mathrm{SL}_2(\mathbf{F}_p)$, it may also be of interest to construct at least one example over $\mathbf{Q}$ (with $p \geq 11$). We shall do this now.

5.4. **Lemma.** *Let $K/\mathbf{Q}$ be a degree 11 extension with splitting field $L/\mathbf{Q}$ such that:*

(1) *11 is totally ramified in $K$.*
(2) *$G = \mathrm{Gal}(L/\mathbf{Q}) = \mathrm{PSL}_2(\mathbf{F}_{11})$.*
(3) *$\mathrm{ord}_{11}(\Delta_{K/\mathbf{Q}}) \not\equiv 0 \mod 10$.*

*Let $\mathfrak{p}$ denote a prime above 11 in $L$, and let $I \subseteq D \subset G$ denote the corresponding inertia and decomposition groups. Then $I = D$ has order 55 and is the full Borel subgroup of $G$.*

*Proof.* Since 11 is totally ramified in $K/\mathbf{Q}$, the inertia group $I$ has order divisible by $[K : \mathbf{Q}] = 11$. Since $I \subset D$ is solvable, it follows that $D$ is contained inside a Borel subgroup of $G$. Let $F$ and $E$ denote the images of $L$ and $K$ under their embedding into $\overline{\mathbf{Q}}_p$ corresponding to $\mathfrak{p}$. We note that $D = \mathrm{Gal}(F/\mathbf{Q}_p)$, and we have the following diagram:

$$\begin{array}{ccccc} L & \text{———} & K & \text{———} & \mathbf{Q} \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ F & \underset{ef = 1,5}{\text{———}} & E & \underset{e = 11}{\text{———}} & \mathbf{Q}_p \end{array}$$

It suffices to assume that $|I| = 11$ and deduce a contradiction. Suppose that $|D| = 11$, so $F = E$ is abelian over $\mathbf{Q}_p$. By local class field theory, $F/\mathbf{Q}_p$ is (up to an unramified twist) given by the degree $p = 11$ subfield of the $p^2$-roots of unity. Thus

$$\Delta_{E/\mathbf{Q}_p} = \Delta_{F/\mathbf{Q}_p} = 11^{20},$$

and thus $\mathrm{ord}_{11}(\Delta_{E/\mathbf{Q}_p}) \equiv 0 \mod 10$, a contradiction.

Suppose that $|D| = 55$ and $|I| = 11$. Let $I_n \subseteq I$ denote the lower ramification groups. The $p$-adic valuation of the discriminant of $F/\mathbf{Q}_p$ is given by the following formula:

$$\mathrm{ord}_p(\Delta_{F/\mathbf{Q}_p}) = \frac{|D|}{|I|} \cdot \sum_{n=0}^{\infty} (|I_n| - 1)\,.$$

By assumption, $|D|/|I| = 5$ and $|I_n| = 11$ or 1 for all $n$. We deduce that $\mathrm{ord}_p(\Delta_{F/\mathbf{Q}_p}) \equiv 0 \mod 50$. On the other hand,

$$\Delta_{F/\mathbf{Q}_p} = N_{F/E}(\Delta_{F/E}) \cdot (\Delta_{F/E})^5.$$

Since $F/E$ is unramified, we deduce that

$$\mathrm{ord}_{11}(\Delta_{E/\mathbf{Q}_p}) = \frac{1}{5}\mathrm{ord}_{11}(\Delta_{F/\mathbf{Q}_p}) \equiv 0 \mod 10.$$

Since $\mathrm{ord}_{11}(\Delta_{K/\mathbf{Q}}) = \mathrm{ord}_{11}(\Delta_{E/\mathbf{Q}_p})$, the lemma follows. □

5.5. **Corollary.** *There exists a surjective even representation $\overline{\rho} : G_{\mathbf{Q}} \rightarrow \mathrm{SL}_2(\mathbf{F}_{11})$ with no geometric deformations.*

*Proof.* Let $K$ be the field given by a root of the irreducible polynomial

$$\begin{aligned} &x^{11} + 154 \cdot x^{10} + 8591 \cdot x^9 + 207724 \cdot x^8 + 1846031 \cdot x^7 - 2270598 \cdot x^6 - 63850600 \cdot x^5 \\ &\quad + 73646034 \cdot x^4 + 582246423 \cdot x^3 - 1610954576 \cdot x^2 + 1500989952 \cdot x - 481890304 = 0. \end{aligned}$$

This polynomial was obtained by specializing the parameters $a$ and $t$ of a polynomial found by Malle (Theorem 9.1 of [35]) to $a = 14$ and $t = -419$ respectively. One may verify that 11 is totally ramified in $K/\mathbf{Q}$, that the splitting field $L/\mathbf{Q}$ is totally real with Galois group $G = \mathrm{PSL}_2(\mathbf{F}_{11})$, and that the discriminant has a prime factorization as follows:

$$\Delta_{K/\mathbf{Q}} = 11^{12} \cdot 1334620886698412181914^4.$$

Since $\mathrm{ord}_{11}(\Delta_{K/\mathbf{Q}}) = 12$, it follows from Lemma 5.4 that the inertia group $D$ at 11 is the full Borel subgroup of $G$. We note also the factorization

$$1334620886698412181914 \cdot \mathcal{O}_K = \mathfrak{p}_1 \mathfrak{p}_2^2 \mathfrak{p}_3^2 \mathfrak{q}_1^2 \mathfrak{q}_2,$$

where $\mathfrak{p}_i$ and $\mathfrak{q}_i$ have residue degree 1 and 2 respectively. It follows that the residue degree and the ramification index of every prime above 1334620886698412181914 in $L$ is even (in fact, 2). Since $1334620886698412181914 \equiv 3 \mod 4$, it follows from Theorem 1.1 of [34] that $L$ embeds in a $\mathrm{SL}_2(\mathbf{F}_{11})$-extension $N/\mathbf{Q}$ which is totally real. Let $\overline{\rho} : G_{\mathbf{Q}} \to \mathrm{Gal}(N/\mathbf{Q}) = \mathrm{SL}_2(\mathbf{F}_{11})$ denote the corresponding representation. We now show that $\overline{\rho}|D_{11}$ satisfies the conditions of Theorem 5.1. Since the decomposition group at 11 maps surjectively onto the Borel of $\mathrm{PSL}_2(\mathbf{F}_{11})$, it is contained in the Borel of $\mathrm{SL}_2(\mathbf{F}_{11})$. Any such representation may be twisted (in $\mathrm{GL}_2(\mathbf{F}_{11})$) to be of the form

$$\begin{pmatrix} \psi & * \\ 0 & 1 \end{pmatrix}$$

for some character $\psi$. If $\psi$ has order two, then the image of $D_{11}$ will not surject onto the Borel of $\mathrm{PSL}_2(\mathbf{F}_{11})$ (twisting does not affect this projection). If $\psi = \omega$, however, then $\det(\overline{\rho}) = \omega \cdot \chi^2$ for some character $\chi$. Yet $\overline{\rho}$ has image in $\mathrm{SL}_2(\mathbf{F}_{11})$ and thus has trivial determinant, whilst $\omega$ is not the square of any character. Thus $\psi \neq \omega$, and Theorem 5.1 applies. □

## 6. Some remarks on the condition $p > 7$

One may wonder if the condition that $p > 7$ is used in an essential way in this argument. At the very least, one will require that the representation

$$\mathrm{Sym}^2\overline{\rho} : G_{\mathbf{Q}} \to \mathrm{GL}_3(\mathbf{F}_p)$$

be adequate. This fails to have adequate image if the image of $\overline{\rho}$ is $\mathrm{SL}_2(\mathbf{F}_p)$ and $p \leq 7$. The author expects that for $p = 7$ it should be sufficient to assume that the projective image of $\overline{\rho}$ is either $A_4$, $S_4$, $A_5$ or contains $\mathrm{PSL}_2(\mathbf{F}_{49})$, that for $p = 5$ that projective image is either $A_4$, $S_4$, or contains $\mathrm{PSL}_2(\mathbf{F}_{25})$, and that for $p = 3$ the image contains $\mathrm{PSL}_2(\mathbf{F}_{27})$. The main technical issue to address is exactly what form of adequateness is required in Proposition 3.2.1 of [2], although another issue is that many of the references we cite include assumptions on $p$ which would also need to be modified (using [45]). The methods (in principle) also apply with $p = 2$, although many more technical ingredients would need to be generalized in this case, in particular, the work of [32].

## 7. A remark on potential modularity theorems

In recent modularity lifting results [5, 2, 3, 4, 24] for $l$-adic representations, a weak form of local-global compatibility at primes $v|l$ is invoked (in this section only, we work with $l$-adic representations rather than $p$-adic representations in order to be most compatible with [5]), namely, that automorphic forms of level co-prime to $l$ give rise to crystalline Galois representations (of the correct weight). In general, local–global compatibility for RACDSC cuspidal forms for $\mathrm{GL}(n)$ is only known in the crystalline case (as follows from [44]), although partial results are known in the semi-stable case. In this section, we show how to prove modularity results similar to Theorem 2.2.1

of [5] without local–global compatibility, allowing for a modularity lifting theorem in the potentially semi-stable case. We claim no great originality, as the proof is essentially the same as the proof of Theorem 2.2.1 of [2] (or Theorem 7.1 of [45]) with the addition of one simple ingredient (Lemma 7.3 below). (The authors of [2] inform me that they have a different method for dealing with the potentially semi-stable case which was not included in [2] for space reasons.) (One should also compare the statement of this theorem to Theorem 7.1 of [45].)

7.1. **Theorem.** *Let $F$ be an imaginary CM field with maximal totally real subfield $F^+$. Suppose $l$ is odd and let $n$ be a positive integer. Let*

$$r : G_F \to \mathrm{GL}_n(\overline{\mathbf{Q}}_l)$$

*be a continuous representation and let $\overline{r}$ denote the corresponding residual representation. Also, let*

$$\mu : G_{F^+} \to \overline{\mathbf{Q}}^{\times}$$

*be a continuous homomorphism. Suppose that $(r, \mu)$ enjoys the following properties:*

1. *$r^\vee \simeq r^c \epsilon_l^{n-1} \mu|_{G_F}$.*
2. *$\mu(c_v)$ is independent of $v|\infty$.*
3. *the reduction $\overline{r}$ is absolutely irreducible and $\overline{r}(G_{F(\zeta_l)}) \subset \mathrm{GL}_n(\overline{\mathbf{F}}_l)$ is adequate.*
4. *There is a RAECDSC automorphic representation $(\pi, \chi)$ of $\mathrm{GL}_n(\mathbf{A}_F)$ with the following properties.*
   - (a) *$(\overline{r}, \overline{\mu}) \simeq (\overline{r}_{l,\iota}(\pi), \overline{r}_{l,\iota}(\chi))$.*
   - (b) *For all places $v \nmid l$ of $F$ at which $\pi$ or $r$ is ramified, we have*
   $$r_{l,\iota}(\pi)|_{G_{F_v}} \sim r|_{G_{F_v}}.$$
   - (c) *For all places $v|l$ of $F$, $r|_{G_{F_v}}$ is potentially semi-stable and we have*
   $$r_{l,\iota}(\pi)|_{G_{F_v}} \rightsquigarrow r|_{G_{F_v}}.$$
   - (d) *If $n$ is even, $\pi$ has slightly regular weight.*

*Then $(r, \mu)$ is automorphic.*

7.2. **Remark.** The only difference between this theorem and Theorem 2.2.1 of [2] is that:

1. We do *not* assume that $\pi$ is potentially unramified above $l$.
2. We require for $v|l$ that $r_{l,\iota}(\pi)|_{G_{F_v}} \rightsquigarrow r|_{G_{F_v}}$ rather than $r_{l,\iota}(\pi)|_{G_{F_v}} \sim r|_{G_{F_v}}$.
3. We impose that $\pi$ has slightly regular weight (this is only a condition when $n$ is even). This is because we require that the Galois representation associated to $\pi$ can be realized geometrically. Perhaps using the methods of [12] this assumption can be eliminated. Alternatively, one could try to work with the representation $r_{l,\iota}(\pi)^{\otimes 2}$ which can be realized geometrically (see [10]).

*Proof.* We make the following minor adjustment to the proof of Theorem 7.1 of [45] (cf. Theorem 2.2.1 of [2]). The character $\chi$ may be untwisted after some solvable ramified extension. We now modify the deformation problem consided in the proof of Theorem 3.6.1 of [3] as follows. At $v|l$, we consider deformations (with a fixed finite collection of Hodge types $\mathbf{v}$) that become potentially semi-stable over some fixed extension $L/K$, where $L$ will be determined below. The argument proceeds in the same manner *providing* there exists a map from $R \to \mathbf{T}$, and, (in light of the fact that the deformation rings at $l$ might have non-smooth points in characteristic zero) the hypothesis that $r_{l,\iota}(\pi)|_{G_{F_v}}$ corresponds to a smooth point in the local deformation ring. For any fixed level structure, the Galois representations arising from quotients of $\mathbf{T}_{Q_n}$ are potentially semi-stable over some extension $L/K$ by a theorem of Tsuji [46]. We are required, however, to show that we may find a *fixed* $L/K$ such that the Galois representations obtained by adding any set of auxiliary Taylor–Wiles primes are semi-stable over the same field $L$.

7.3. **Lemma.** *Let $K/\mathbf{Q}_l$ be a finite extension, and let $X$ be a proper flat scheme over* $\mathrm{Spec}(\mathcal{O}_K)$ *with smooth generic fibre. Then there exists a finite extension $L/K$ with the following property: For every finite étale map $\pi : Y \to X$, the étale cohomology groups $H^i(Y_{\overline{K}}, \overline{\mathbf{Q}}_l)$ become semi-stable as representations of $G_L$.*

*Proof.* After making a finite extension $L/K$, there exists (via the theory of alterations [17]) a proper hypercovering $X_\bullet$ of $X$ such that for all $n \le 2\dim(X)$:

(1) $X_n$ is proper and flat over $\mathrm{Spec}(\mathcal{O}_L)$
(2) $X_n$ has smooth generic fibre and semi-stable special fibre.

By cohomological descent, there is a spectral sequence

$$H^m(X_{n,\overline{L}}, \overline{\mathbf{Q}}_l) \Rightarrow H^{m+n}(X_{\overline{L}}, \overline{\mathbf{Q}}_l).$$

The cohomology groups on the left are semi-stable by Tsuji's proof of $C_{\mathrm{st}}$. Since the property of being semi-stable is preserved by taking sub-quotients, it follows that the $G_L$-representation $H^i(X_{\overline{L}}, \overline{\mathbf{Q}}_l)$ (for $i \le 2\dim(X)$) has an exhaustive filtration by semi-stable $G_L$-modules. Hence the semi-simplification of $H^i(X_{\overline{L}}, \overline{\mathbf{Q}}_l)$ is semi-stable. Since $H^i(X_{\overline{L}}, \overline{\mathbf{Q}}_l)$ is also de Rham [20], it follows that $H^i(X_{\overline{K}}, \overline{\mathbf{Q}}_l) = H^i(X_{\overline{L}}, \overline{\mathbf{Q}}_l)$ is itself semi-stable as a $G_L$-representation for $i \le 2\dim(X)$. The cohomology of $X_{\overline{K}}$ vanishes outside this range, so the claim follows for all $i$. This recovers Tsuji's Theorem ($C_{\mathrm{pst}}$). Let us now consider a finite étale morphism $Y \to X$. We may form a hypercovering $Y_\bullet = X_\bullet \times_X Y$ of $Y$. The properties (1) and (2) of the hypercovering $X_\bullet$ are preserved under base change by a finite étale map, and thus the cohomology of $Y$ is also semi-stable over $L$. □

7.4. **Remark.** For an expositional account of the theory of hypercoverings and cohomological descent in the étale topology, see [16].

Consider the (compact) Shimura variety Sh (over $\mathrm{Spec}(\mathcal{O}_K)$) associated to the unitary similitude group $G$ as in [14, 26, 39], where $\mathscr{L} = \mathscr{L}_\xi$ is an automorphic vector bundle for an irreducible algebraic representation $\xi$ of $G$. Let $\mathcal{A}^m$ denote the $n$th self-product of the universal abelian variety over Sh, and let $\pi : \mathcal{A}^m \to \mathrm{Sh}$ denote the (smooth, proper) projection. For a suitable $m$, one can write $\mathscr{L} = eR^m\pi_*\overline{\mathbf{Q}}_l(r)$ for some $m = m_\xi$ and $r = r_\xi$, and $e$ is some idempotent (cf. [26], p.98). Finally, let $\mathrm{Sh}(N)$ denote the finite étale cover of Sh corresponding to the addition of auxiliary level $N$-structure for some $N$ co-prime to $p$. Let $\mathcal{A}^m(N)$ denote the base change of $\mathcal{A}^m$ to $\mathrm{Sh}(N)$; it is finite étale over $\mathcal{A}^m$. The Leray spectral sequence gives a map

$$H^p(\mathrm{Sh}(N)_{\overline{K}}, R^q\pi_*\overline{\mathbf{Q}}_l(r)) \Rightarrow H^{p+q}(\mathcal{A}^m(N)_{\overline{K}}, \overline{\mathbf{Q}}_l(r)).$$

Multiplication by $n$ on $\mathcal{A}$ induces the map $n^j$ on $R^j\pi_*\overline{\mathbf{Q}}_l$. The formation of the spectral sequence is compatible with this map, and hence it commutes with the differentials in the spectral sequence, which correspondingly degenerates (cf. the argument of Deligne, p.169 of [18]). Thus

$$H^n(\mathrm{Sh}(N)_{\overline{K}}, \mathscr{L}) = eH^n(\mathrm{Sh}(N)_{\overline{K}}, R^m\pi_*\overline{\mathbf{Q}}_l(r))$$

occurs as a subquotient of $H^i(\mathcal{A}(N)_{\overline{K}}, \overline{\mathbf{Q}}_l(r))$ for some $i$. Let $X = \mathcal{A}^m$ and $Y = \mathcal{A}^m(N)$. By Lemma 7.3, we deduce that

$$H^n(\mathrm{Sh}(N)_{\overline{K}}, \mathscr{L})$$

is semi-stable over a fixed extension $L/K$ for all $N$ depending only on $n$ and $\xi$. If $\pi$ has weakly regular weight, the the Galois representation associated to $\pi$ in [39] can be realized geometrically in the étale cohomology of an automorphic sheaf on Sh as considered above. Moreover, the Galois representations corresponding to automorphic forms arising in the Taylor–Wiles constructions at auxiliary primes arise in the étale cohomology of the same sheaf on $\mathrm{Sh}(N)$ for some auxiliary level $N$. It follows that the local Galois representations associated to the Hecke rings $\mathbf{T}_{Q_n}$ are all quotients of a local deformation ring involving a fixed finite set of types, which is the necessary local input for the modularity lifing theorem (Theorem 7.1) of [45]. □

## References


[1] J. Arthur, L. Clozel, *Simple algebras, base change, and the advanced theory of the trace formula*, Annals of Mathematics Studies, 120. Princeton University Press, Princeton, NJ, 1989. xiv+230 pp.

[2] T. Barnet–Lamb, T. Gee, D. Geraghty, R. Taylor, *Potential Automorphy and Change of Weight*, preprint.

[3] T. Barnet–Lamb, T. Gee, D. Geraghty, *The Sato–Tate conjecture for Hilbert modular forms*, to appear in J.A.M.S.,

[4] T. Barnet–Lamb, T. Gee, D. Geraghty, *Congruences betwen Hilbert modular forms: constructing ordinary lifts*, preprint.

[5] T. Barnet–Lamb, D. Geraghty, M. Harris, R. Taylor, *A family of Calabi–Yau varieties and potential automorphy II*, to appear in P.R.I.M.S.

[6] D. Blasius, D. Rogawski, *Tate classes and arithmetic quotients of the two-ball*, in *The zeta functions of Picard modular surfaces*, 421–444, CRM Publications, Montréal, 1992.

[7] G. Böckle, *On the density of modular points in universal deformation spaces*, Amer. J. Math. 123 (2001), 985–1007.

[8] C. Breuil, A. Mezard, *Multiplicités modulaires et représentations de* $\mathrm{GL}_2(\mathbf{Z}_p)$ *et de* $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ *en* $l = p$, With an appendix by Guy Henniart. Duke Math. J. 115 (2002), no. 2, 205–310.

[9] F. Calegari, *Even Galois representations and the Fontaine–Mazur conjecture*, preprint.

[10] A. Caraiani, *Local-global compatibility and the action of monodromy on nearby cycles*, preprint.

[11] G. Chenevier, *On the infinite fern of Galois representations of unitary type*, preprint.

[12] G. Chenevier, *Une application des variétés de Hecke des groupes unitaires*, to appear, (séminaire G.R.F.A. Paris 7)

[13] L. Clozel, M. Harris, R. Taylor, Automorphy for some l-adic lifts of automorphic mod l representations. Pub. Math. IHES 108 (2008), 1-181.

[14] L. Clozel, M. Harris, J.-P. Labesse, B.-C. Ngo, *Stabilisation de la formule des traces, variétés de Shimura, et applications arithmétiques*, to appear, (séminaire G.R.F.A. Paris 7)

[15] P. Colmez, *Représentations de* $\mathrm{GL}_2(\mathbf{Q}_p)$ *et* $(\varphi, \Gamma)$*-modules*, Astérisque 330 (2010), 281-509.

[16] B. Conrad, *Cohomological descent*, preprint.

[17] A. de Jong, *Smoothness, semi-stability and alterations*, Inst. Hautes tudes Sci. Publ. Math. No. 83 (1996), 51–93.

[18] P. Deligne, *Formes modulaires et représentations $l$-adiques*, Séminaire Bourbaki, 11 (1968-1969), Exp. No. 355, 34 p.

[19] M. Emerton, *Local–Global compatibility in the $p$-adic Langlands Programme*, preprint.

[20] G. Faltings, *$p$-adic Hodge Theory*, J. Amer. Math. Soc. 1 (1988), no. 1, 255299.

[21] J–M. Fontaine, B. Mazur, *Geometric Galois representations. Elliptic curves, modular forms, & Fermat's last theorem*, (Hong Kong, 1993), 41–78, Ser. Number Theory, I, Int. Press, Cambridge, MA, 1995.

[22] J–M. Fontaine, *Représentations p-adiques semi-stables*, Périodes $p$-adiques, Astérisque **223**, Société Mathématique de France, 113–184, 1994.

[23] T. Gee, *On the weights of mod-p Hilbert modular forms*, to appear in Invent. Math.

[24] D. Geraghty, *Modularity Lifting Theorems for Ordinary Representations*, preprint.

[25] R. Guralnick, F. Herzig, R. Taylor, J. Thorne, *Adequate subgroups*, appendix to 'On the automorphy of l-adic Galois representations with small residual image' by J.Thorne, preprint.

[26] M. Harris, R. Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Mathematics Studies, 151. Princeton University Press, Princeton, NJ, 2001. viii+276 pp.

[27] M. Harris, N. Shepherd–Barron, R. Taylor, *A family of Calabi–Yau varieties and potential automorphy*, Annals of Math. 171 (2010), 779-813.
[28] H. Jacquet, J. A. Shalika, *On Euler products and the classification of automorphic forms. II*, Amer. J. Math. 103 (1981), no. 4, 777–815.
[29] C. Khare, J–P. Wintenberger, *Serre's Modularity Conjecture I,* Invent. Math. 178 (2009), no. 3, 485504.
[30] C. Khare, J–P. Wintenberger, *Serre's Modularity Conjecture II,* Invent. Math. 178 (2009), no. 3, 505586
[31] M. Kisin, *Potentially semi-stable deformation rings*, J.A.M.S. 21(2) (2008), 513–546.
[32] M. Kisin, *The Fontaine–Mazur conjecture for* $\mathrm{GL}_2$, J.A.M.S. 22(3) 2009, 641–690.
[33] M. Kisin, *Modularity of* 2*-adic Barsotti–Tate representations*, Invent. Math. 178(3) (2009), 587-634.
[34] J. Klüners, *A polynomial with Galois group* $\mathrm{SL}_2(11)$, Algorithmic methods in Galois theory. J. Symbolic Comput. 30 (2000), no. 6, 733–737.
[35] G. Malle, *Multi-parameter Polyomials with Given Galois Group* J. Symbolic Computation (2000) 21, 1–15.
[36] B. Mazur, *An "infinite fern" in the universal deformation space of Galois representations*, Journées Arithmétiques (Barcelona, 1995). Collect. Math. 48 (1997), no. 1-2, 155-193.
[37] R. Ramakrishna, *Deforming Galois representations and the conjectures of Serre and Fontaine–Mazur*, Annals of Math, **156** (2002) 115–154.
[38] J–P. Serre, *On a theorem of Jordan*, Bull. Amer. Math. Soc. 40 (2003), 429-440.
[39] S. W. Shin, *Galois representations arising from some compact Shimura varieties*, to appear in Ann. of math.
[40] R. Taylor, *Automorphy for some l-adic lifts of automorphic mod l representations. II*, Pub. Math. IHES 108 (2008), 183-239.
[41] R. Taylor, *The image of complex conjugation on* $l$*-adic representations associated to automorphic forms*, preprint.
[42] R. Taylor, *Remarks on a conjecture of Fontaine and Mazur*, Journal of the Institute of Mathematics of Jussieu 1 (2002), 1–19.
[43] R. Taylor, *On the meromorphic continuation of degree two* $L$*-functions*, Documenta Mathematica, Extra Volume: John Coates' Sixtieth Birthday (2006), 729–779.
[44] R. Taylor, T. Yoshida, *Compatibility of local and global Langlands correspondences*, Journal: J. Amer. Math. Soc. 20 (2007), 467–493.
[45] J. Thorne, *On the automorphy of* $l$*-adic Galois representations with small residual image*, preprint.
[46] T. Tsuji, *p-adic Hodge theory in the semi-stable reduction case*, Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998). Doc. Math. 1998, Extra Vol. II, 207–216