Fourier multipliers for non-symmetric Lévy processes

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Abstract

We study Fourier multipliers resulting from martingale transforms of general Lévy processes.

1 Introduction

For each bounded function $M : \mathbb{R}^d \to \mathbb{C}$ there is a unique bounded linear operator $\mathcal{M}$ on $L^2(\mathbb{R}^d)$ defined in terms of the Fourier transform as follows,

$$\widehat{\mathcal{M}f} = M\hat{f}\,. \tag{1.1}$$

The operator norm of $\mathcal{M}$ on $L^2(\mathbb{R}^d)$ is $\|\mathcal{M}\| = \|M\|_\infty$. It has long been of interest to study *symbols* M for which the *Fourier multiplier* $\mathcal{M}$ extends to a bounded linear operator on $L^p(\mathbb{R}^d)$ for $p \in (1, \infty)$. Fourier multipliers resulting from transforming jumps of symmetric Lévy process have been recently obtained in [2]. By using Burkholder's inequalities for differentially subordinate continuous

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time martingales with jumps [7] in the general form of Wang [25], we proved that their operator norms on $L^p(\mathbb{R}^d)$ do not exceed

$$p^* - 1 = \max\{p-1, \frac{1}{p-1}\}\,. \tag{1.2}$$

For a broad discussion of Burkholder's method and its many extensions and applications, we refer the reader to [1]. In this note we adapt the methods of [2] to non-symmetric Lévy processes. The resulting multipliers are given in Theorem 1.1 below. We remark that for $\mu = 0$ and symmetric V the result was proved in [2, Theorem 1]. The present Theorem 1.1 is a generalization, but the symbols (1.4) are very similar to those given in [2].

Given a Borel measure $V \geq 0$ on $\mathbb{R}^d$ such that $V(\{0\}) = 0$ and

$$\int_{\mathbb{R}^d} \min(|z|^2, 1) V(dz) < \infty \tag{1.3}$$

(that is, a Lévy measure), a finite Borel measure $\mu \geq 0$ on the unit sphere $\mathbb{S}$ in $\mathbb{R}^d$, and Borel measurable complex-valued functions ϕ on $\mathbb{R}^d$ and φ on $\mathbb{S}$ such that $\|\varphi\|_\infty \leq 1$ and $\|\phi\|_\infty \leq 1$, we define

$$M(\xi) = \frac{\frac{1}{2}\int_{\mathbb{S}} (\xi,\theta)^2 \varphi(\theta)\,\mu(d\theta) + \int_{\mathbb{R}^d} [1-\cos(\xi,z)]\,\phi(z)\,V(dz)}{\frac{1}{2}\int_{\mathbb{S}} (\xi,\theta)^2 \mu(d\theta) + \int_{\mathbb{R}^d} [1-\cos(\xi,z)]\,V(dz)}\,, \tag{1.4}$$

where we let $M(\xi) = 0$ if the denominator equals zero. Clearly, $\|M\|_\infty \leq 1$. Here and for the rest of this paper, the pairing between vectors,

$$(\xi, \eta) = \sum_{n=1}^d \xi_n \eta_n\,, \qquad \text{if } \xi, \eta \in \mathbb{R}^d \text{ or } \mathbb{C}^d\,, \tag{1.5}$$

is without complex conjugation. We also denote $|\xi|^2 = \sum_{n=1}^d |\xi_n|^2 = (\xi, \overline{\xi})$. If M vanishes on a set of positive Lebesgue measure, then $V = 0$, $\mu = 0$ and hence $M \equiv 0$. This was proved in [2].

Theorem 1.1. *If $1 < p < \infty$ and $\mathcal{M}$ is defined by (1.1) and (1.4) then*

$$\|\mathcal{M}f\|_p \leq (p^*-1)\|f\|_p\,, \quad f \in L^p(\mathbb{R}^d)\,. \tag{1.6}$$

In particular, letting $V = 0$ in (1.4) yields the symbol

$$M(\xi) = \frac{\int_{\mathbb{S}} (\xi,\theta)^2 \varphi(\theta)\,\mu(d\theta)}{\int_{\mathbb{S}} (\xi,\theta)^2 \mu(d\theta)}\,, \tag{1.7}$$

or

$$M(\xi) = \frac{(\mathbb{A}\xi, \xi)}{(\mathbb{B}\xi, \xi)}\,, \tag{1.8}$$

where $\mathbb{A} = [\mathbb{A}_{k,l}]_{k,l=1,\dots,d}$ and $\mathbb{B} = [\mathbb{B}_{k,l}]_{k,l=1,\dots,d}$ are given by

$$\mathbb{A}_{k,l} = \int_{\mathbb{S}} \theta_k\theta_l \,\varphi\,(\theta)\,\mu(d\theta)\,, \quad \mathbb{B}_{k,l} = \int_{\mathbb{S}} \theta_k\theta_l \,\mu(d\theta)\,. \tag{1.9}$$

These matrices are symmetric, and $\mathbb{B}$ is nonnegative definite. We have

$$\mathbb{A}\xi = \int_{\mathbb{S}} \theta(\xi,\theta)\phi(\theta)\mu(\theta)\,, \quad \xi \in \mathbb{R}^d\,,$$

and $(\mathbb{B}\xi,\xi) = \int_{\mathbb{S}}(\xi,\theta)^2\mu(d\theta)$, hence

$$|(\mathbb{A}\xi,\xi)| \le (\mathbb{B}\xi,\xi)\,, \quad \xi \in \mathbb{R}^d\,. \tag{1.10}$$

For instance, the approach yields the bound p^*-1 for the multiplier with the symbol $-2\xi_1\xi_2/|\xi|^2$ via

$$\mathbb{A} = \begin{bmatrix} 0 & -1 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

and $\mathbb{B} = \mathbb{I}$, the identity matrix. We thus obtain $2R_1R_2$, the second order Riesz transform multiplied by two. It is known that the norm of this operator indeed equals p^*-1 [12, Corollary 3.2], so the constant in (1.6) cannot be improved in general. On the other hand, our method will only give the upper bound $2(p^*-1)$ for the norm of the operator resulting from

$$\mathbb{A} = \begin{bmatrix} 1 & -i \\ -i & -1 \end{bmatrix} \tag{1.11}$$

and $\mathbb{B} = \mathbb{I}$. Namely, we will show in Lemma 4.2 below that in this case the representation (1.9) may only hold with $\|\phi\|_\infty \ge 2$. We should remark that $|\mathbb{A}\xi| = |\xi|$ for $\xi \in \mathbb{R}^2$ ($|\mathbb{A}\xi| \le 2|\xi|$ for $\xi \in \mathbb{C}^2$), and it is known that the estimate $2(p^*-1)$ is not optimal; see [1] and the discussion in Section 4.

The paper is organized as follows. Section 2 has a didactic purpose. We namely consider $\mathbb{B} = \mathbb{I}$ in (1.9). This case can be resolved by means of the standard Itô calculus for the Brownian motion. A similar argument was first given in [4], and has since appeared in many different places and settings, but we believe it is worth repeating here with notation emphasizing analogies with Section 3. In this way we hope to make the rest of the paper more readable for those less familiar with the stochastic calculus of Lévy processes. In Section 3 we give the proof and a discussion of Theorem 1.1. First of all, by using a simple algebra we reduce the symbols (1.4) to those of [2, Theorem 1]. This gives a proof but not much insight, since [2] only concerns symmetric Lévy processes. Therefore in the remainder of Section 3 we present the stochastic calculus leading to the symbols (1.4). Our main purpose is to explain why the non-symmetry of the process is not reflected in the symbol. For instance we will see in (3.28) that the drift of the Lévy process does not contribute to M. Examples and further discussion are given in Section 4.

Throughout the paper we only consider Borel functions, measures and sets in $\mathbb{R}^d$. For $1 \leq p < \infty$ we let $L^p = L^p(\mathbb{R}^d, dx)$ be the class of complex-valued functions f on $\mathbb{R}^d$ with finite $\|f\|_p = \left[\int_{\mathbb{R}^d} |f(x)|^p dx\right]^{1/p}$. $L^\infty = L^\infty(\mathbb{R}^d)$ are those functions f for which $\|f\|_\infty = \sup_{x \in \mathbb{R}^d} |f(x)| < \infty$, $C_b^1 = \{f \in L^\infty : \|\nabla f\|_\infty < \infty\}$, and C_c^1 consists of those functions in C_b^1 which have compact support. Similarly, L_c^∞ are compactly supported functions in L^∞. We recall that C_c^1 is dense in L^p for each $p \in [1, \infty)$. Our convention for the Fourier transform will be

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{i(\xi,z)} f(z) dz\,, \quad \xi \in \mathbb{R}^d\,.$$

If ρ is a probability measure on $\mathbb{R}^d$ and $k \in L^1$, then Fubini's theorem yields

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} k(x+y)\rho(dy)dx = \int k(x)dx\,. \tag{1.12}$$

2 Brownian martingales and Itô calculus

In this section we present a simple approach to Fourier multipliers with symbols of the form (1.8). We will use the familiar Itô calculus for the Brownian motion, for which we refer the reader to [17], [18] or [19]. The main ideas will be similar to those in Section 3 below, but the calculations are shorter and simpler. As already mentioned, we hope that this will be easier to read for those familiar with the basics of the Itô calculus but perhaps not as familiar with the stochastic calculus of jump processes used in Section 3.

We let $\mathbf{P}$ and $\mathbf{E}$ be the probability and expectation for a family of *Brownian increments* $B_{s,t}$. Namely, let $B_t^{(1)}$, $B_t^{(2)}$, $t \geq 0$, be independent Brownian motions in $\mathbb{R}^d$ starting at the origin, let $B_u = -B_{-u}^{(1)}$ if $u < 0$ and $B_u = B_u^{(2)}$ if $u \geq 0$. For $-\infty < s < t < \infty$ we let $B_{s,t} = B_t - B_s$. These increments are independent for disjoint time intervals. They are also Gaussian and centered, with variance $t - s$. We will consider the filtration

$$\mathcal{F}_t = \sigma\{B_{s,t}\,;\ s \leq t\}\,, \quad t \in \mathbb{R}\,,$$

and the Gaussian convolution semigroup

$$p_t(x) = (2\pi t)^{-d/2} \exp(-|x|^2/2t)\,, \quad t > 0,\ x \in \mathbb{R}^d\,. \tag{2.1}$$

It is well-known that

$$\widehat{p_t}(\xi) = e^{-t|\xi|^2/2}\,, \quad \xi \in \mathbb{R}^d\,, \tag{2.2}$$

and that the heat equation holds for $p_t(x)$,

$$\frac{\partial}{\partial_t} p_t(x) = \frac{1}{2}\Delta_x p_t(x)\,. \tag{2.3}$$

We also have that $p_{t-s}(x)dx$ is the distribution of $B_{s,t}$, for $s < t$. Let $g \in C_b^1$. For $x \in \mathbb{R}^d$ and finite $t < u$, we define

$$P_{t,u}g(x) = \mathbf{E}g(x + B_{t,u}) = p_{u-t} * g(x)\,, \tag{2.4}$$

and $P_{u,u}g(x) = g(x)$. For $s \leq t \leq u$ we define the following Brownian parabolic martingale

$$G_t = G_t(x; s, u; g) = P_{t,u}g(x + B_{s,t})\,. \tag{2.5}$$

Indeed, $t \mapsto G_t$ is an $\{\mathcal{F}_t\}$-martingale on $[s, u]$. This follows from the Markov property of the Brownian motion; see [2, Lemma 2]. Note that regardless of t, the entire time interval $[s, u]$ is involved in G_t as the "evolution" from s to t proceeds via the Brownian motion, while that from t to u goes by its expectations. In fact, the martingale equals an Itô integral plus a constant. This follows from (2.4) and (2.3) by simply applying the Itô formula to the process $t \mapsto (u - t, B_t - B_s)$,

$$\begin{aligned} G_t - G_s &= \int_s^t \left(\frac{\partial}{\partial_v} P_{v,u}g\right)(x + B_{s,v})dv + \int_s^t \nabla_x P_{v,u}g(x + B_{s,v})dB_v \\ &+ \int_s^t \frac{1}{2}\Delta_x P_{v,u}g(x + B_{s,v})dv = \int_s^t \nabla_x P_{v,u}g(x + B_{s,v})dB_v\,. \end{aligned} \tag{2.6}$$

G is bounded, hence square integrable. The quadratic variation of G is

$$[G,G]_t = |G_s|^2 + \int_s^t |\nabla_x P_{v,u} g(x+B_{s,v})|^2\, dv\,. \tag{2.7}$$

Let $\mathbb{A}$ be a real or complex $d \times d$ matrix such that

$$|\mathbb{A}z| \le |z|\,, \quad z \in \mathbb{C}^d\,. \tag{2.8}$$

Let $f \in C_c^1$. For $s \le t \le u$ we consider the martingale

$$F_t = F_t(x; s, u; f, \mathbb{A}) = \int_s^t \mathbb{A}\nabla_x P_{v,u} f(x+B_{s,v}) dB_v. \tag{2.9}$$

The quadratic variation of F is

$$[F,F]_t = \int_s^t |\mathbb{A}\nabla_x P_{v,u} f(x+B_{s,v})|^2\, dv\,. \tag{2.10}$$

By (2.8), $F = F(x; s, u; f, \mathbb{A})$ is *differentially subordinate* to $G = G(x; s, u; f)$, in the following sense introduced in [5]:

$$0 \le [G,G]_t - [F,F]_t \quad \text{is non-decreasing for } \ t \in [s,u]\,. \tag{2.11}$$

Let $p \in (1,\infty)$. By [5, Theorem 2],

$$\mathbf{E}|F_t(x; s, u; f, \mathbb{A})|^p \le (p^*-1)^p\, \mathbf{E}|G_t(x; s, u; f)|^p\,, \quad s \le t \le u\,, \tag{2.12}$$

provided $p \in (1,\infty)$. By (2.12), (1.12) and (2.1),

$$\begin{aligned} \int_{\mathbb{R}^d} \mathbf{E}|F_u(x; s, u; f, \mathbb{A})|^p dx \ &\le \ (p^*-1)^p \int_{\mathbb{R}^d} \mathbf{E}|G_u(x; s, u; f)|^p dx \\ &= \ (p^*-1)^p \|f\|_p^p\,. \end{aligned} \tag{2.13}$$

Let $q = p/(p-1)$. By Hölder's inequality for $\mathbf{P} \otimes dx$, (2.13) and (1.12),

$$\int_{\mathbb{R}^d} \mathbf{E}|F_u(x; s, u; f, \mathbb{A}) g(x+B_{s,u})| dx \le (p^*-1)\|f\|_p \|g\|_q\,. \tag{2.14}$$

Therefore there is a unique function $h \in L^p$ such that

$$\Lambda(g) := \int_{\mathbb{R}^d} \mathbf{E} F_u(x; s, u; f, \mathbb{A}) g(x+B_{s,u}) dx = \int_{\mathbb{R}^d} h(x) g(x) dx\ , \tag{2.15}$$

if $g \in L^q$, and we have

$$\|h\|_p \le (p^* - 1)\|f\|_p \, . \tag{2.16}$$

For a deeper understanding of Λ we will use the classical Burkholder-Gundy inequalities (see for example, [10, p. 155] or the original paper [8]). Namely, for each $r \in (0, \infty)$ there is constant $0 < c_r < \infty$ depending only on r such that

$$c_r^{-1}\mathbf{E}(F_u^*)^r \le \mathbf{E}[F, F]_u^{r/2} \le c_r\mathbf{E}(F_u^*)^r \tag{2.17}$$

where $F_u^* = \sup_{s\le v\le u} |F_v|$ is the maximal function of the martingale F_t. Applying (2.17) for $r = 1$ yields the following integrability of F_u,

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mathbf{E}|F_u(x; s, u; f, \mathbb{A})|dx \le c_1 \int_{\mathbb{R}^d} \mathbf{E}\left(\int_s^u |\nabla_x P_{v,u} f(x + B_{s,v})|^2 dv\right)^{1/2} dx \\
&\le c_1\|\nabla f\|_\infty^{1/2} \int_{\mathbb{R}^d} \mathbf{E}\left(\int_s^u |\nabla_x P_{v,u} f(x + B_{s,v})| dv\right)^{1/2} dx \\
&\le c_1\|\nabla f\|_\infty^{1/2} \int_{\mathbb{R}^d} \left(\mathbf{E}\int_s^u |\nabla_x P_{v,u} f(x + B_{s,v})| dv\right)^{1/2} dx \\
&\le c_1\|\nabla f\|_\infty^{1/2}(u - s)^{1/2} \int_{\mathbb{R}^d} (P_{s,u}|\nabla f|(x))^{1/2}\, dx < \infty \, ,
\end{aligned} \tag{2.18}$$

because $P_{s,u}|\nabla f|(x)$ decays exponentially as $|x| \to \infty$. It follows that $|\Lambda(g)| \le c\|g\|_\infty$. By consider g approximating $\overline{h}/|h|$ we see that $h \in L^1$. We will now give an alternative representation of Λ. We have

$$a\bar{b} = (|a + b|^2 - |a - b|^2 + i|a + ib|^2 - i|a - ib|^2)/4 \, , \quad a, b \in \mathbb{C} \, . \tag{2.19}$$

By this, (2.7) and (2.10),

$$\begin{aligned}
\mathbf{E}F_u G_u &= \mathbf{E}F_u(G_u - G_s) \\
&= \mathbf{E}\int_s^u \big(\mathbb{A}\nabla_x P_{v,u} f(x + B_{s,v}), \nabla_x P_{v,u} g(x + B_{s,v})\big) dv.
\end{aligned} \tag{2.20}$$

We claim that

$$\Lambda(g) = \int_{\mathbb{R}^d}\int_s^u\int_{\mathbb{R}^d} \big(\mathbb{A}\nabla_x P_{v,u} f(x+y), \nabla_x P_{v,u} g(x+y)\big)\, p_{v-s}(dy)dvdx \, . \tag{2.21}$$

Indeed, (2.21) follows from Fubini's theorem, since

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_s^u\int_{\mathbb{R}^d} |P_{v,u}\nabla f(x + y)|\, |P_{v,u}\nabla g(x + y)|\, p_{v-s}(dy)dvdx \\
&\le \|\nabla g\|_\infty\|\nabla f\|_1(u - s) < \infty \, .
\end{aligned}$$

Arguing as in (2.18) the reader may also verify that

$$\int_{\mathbb{R}^d}\int_s^u\int_{\mathbb{R}^d}|\nabla_x P_{v,u}f(x+y)|\,|P_{v,u}g(x+y)|\,p_{v-s}(dy)dvdx \leq c_pc_q\|f\|_p\|g\|_q\,,$$

but this will not be used in the sequel, and (2.12) gives a better constant.

Let $\xi\in\mathbb{R}^d$, $e_\xi(x)=e^{i(\xi,x)}\in C_b^1$, $\mathcal{E}_t(x;s,u;\xi)=G_t(x;s,u;e_\xi)$. We have

$$P_{v,u}e_\xi(x)=\int_{\mathbb{R}^d}e^{i(\xi,x+y)}p_{u-v}(dy)=e^{-(u-v)|\xi|^2/2}e_\xi(x)\,,$$

and

$$\nabla_x P_{v,u}e_\xi(x)=e^{-(u-v)|\xi|^2/2}e_\xi(x)i\xi\,.$$

By (2.15), (2.21) and (1.12) we obtain

$$\begin{aligned}
\hat{h}(\xi) &= \int_{\mathbb{R}^d}h(x)e_\xi(x)dx=\Lambda(e_\xi) && (2.22)\\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_s^u\big(\mathbb{A}\nabla_x P_{v,u}f(x+y),i\xi\big)e^{-(u-v)|\xi|^2/2}e_\xi(x+y)dvp_{v-s}(y)dydx\\
&= \int_s^u\int_{\mathbb{R}^d}\big(\mathbb{A}\nabla_x P_{v,u}f(x),i\xi\big)e^{-(u-v)|\xi|^2/2}e_\xi(x)dxdv\\
&= \int_s^u\big(-i\mathbb{A}\xi e^{-(u-v)|\xi|^2/2}\hat{f}(\xi),i\xi\big)e^{-(u-v)|\xi|^2/2}dv\\
&= \hat{f}(\xi)\frac{(\mathbb{A}\xi,\xi)}{|\xi|^2}[1-e^{-(u-s)|\xi|^2}]\quad \text{if } \xi\neq 0,
\end{aligned}$$

and $\hat{h}(0)=0$. Thus the map $f\mapsto h$ is a Fourier multiplier with the symbol

$$(\mathbb{A}\xi,\xi)|\xi|^{-2}[1-e^{-(u-s)|\xi|^2}].$$

The reader puzzled by the fact that $h\in L^1$ may find comfort in noticing that this symbol is continuous at the origin. We let $u=0$ and $s\to-\infty$. The symbol converges to $(\mathbb{A}\xi,\xi)/|\xi|^2$ and the function h converges in L^2 by Plancherel's theorem. A subsequence converges almost everywhere, and by Fatou's lemma and (2.16) the limit has L^p norm bounded by $(p^*-1)\|f\|_p$. Since C_c^1 is dense in L^p, the Fourier multiplier with the symbol $(\mathbb{A}\xi,\xi)/|\xi|^2$ extends to L^p, with the norm not exceeding p^*-1.

If $\mathbb{A}$ is a general square real or complex $d\times d$ matrix, then $\mathbb{A}/\|\mathbb{A}\|$ satisfies (2.8), hence the Fourier multiplier with the symbol $(\mathbb{A}\xi,\xi)/|\xi|^2$ has the norm at most $\|\mathbb{A}\|(p^*-1)$ on L^p. Here $\|\mathbb{A}\|$ is the (spectral) operator norm of $\mathbb{A}$, induced by the Euclidean norm on $\mathbb{C}^d$. On occasions, if $P_{v,u}f(x)$ has a restricted range of values, (2.9) need only to hold in this range. In particular, the multiplier given by (1.8) and (1.11) has the norm at most $2(p^*-1)$ when acting on complex-valued functions, and at most $\sqrt{2}(p^*-1)$ when restricted to real-valued functions.

As already mentioned in the Introduction, the symbols (1.8) and their L^p estimates are not new. We refer the reader to [1] for a detailed discussion of further symbols that can be obtained by transformations of more general Itô integrals, and for their applications. We also like to note that our calculations of the symbol may be considered a probabilistic counterpart of the identity

$$\frac{(\mathbb{A}\xi,\xi)}{|\xi|^2} = \frac{1}{2}(\mathbb{A}\xi,\xi)\int_0^\infty e^{-t|\xi|^2/2}dt\,. \tag{2.23}$$

A semigroup counterpart of (2.23) is mentioned in [2].

3 Lévy-Itô calculus and Fourier multipliers

Proof of Theorem 1.1. We will first consider $\mu = 0$ in (1.4), i.e. we will prove the theorem for symbols of the form

$$\frac{\int\left[1-\cos(\xi,z)\right]\phi\left(z\right)V\left(dz\right)}{\int\left[1-\cos(\xi,z)\right]V\left(dz\right)}. \tag{3.1}$$

For $A \subset \mathbb{R}^d$ we let $\breve{V}(A) = [V(A)+V(-A)]/2$ (the symmetrization of V), $\tilde{V}(A) = [V(A)-V(-A)]/2$ (the antisymmetric part of V). We also define $\breve{\phi}(z) = [\phi(z)+\phi(-z)]/2$, $\tilde{\phi}(z) = [\phi(z)-\phi(-z)]/2$ for $z\in\mathbb{R}^d$. The function $z\mapsto \cos(\xi,z)$ is symmetric, hence $\int_{\mathbb{R}^d}[1-\cos(\xi,z)]V(dz) = \int_{\mathbb{R}^d}[1-\cos(\xi,z)]\breve{V}(dz)$. We note that

$$\phi V = (\breve{\phi}+\tilde{\phi})(\breve{V}+\tilde{V}) = \left(\breve{\phi}\breve{V}+\tilde{\phi}\tilde{V}\right)+\left(\breve{\phi}\tilde{V}+\tilde{\phi}\breve{V}\right)$$

as measures, and so for every $\xi\in\mathbb{R}^d$ we have

$$\frac{\int_{\mathbb{R}^d}[1-\cos(\xi,z)]\phi(z)V(dz)}{\int_{\mathbb{R}^d}[1-\cos(\xi,z)]V(dz)} = \frac{\int_{\mathbb{R}^d}[1-\cos(\xi,z)]\left(\breve{\phi}\breve{V}+\tilde{\phi}\tilde{V}\right)(dz)}{\int_{\mathbb{R}^d}[1-\cos(\xi,z)]\breve{V}(dz)}\,. \tag{3.2}$$

Since $\breve{V}+\tilde{V} = V \geq 0$, we have that $\tilde{V} = k\breve{V}$, with an antisymmetric real function k such that $|k|\leq 1$. Thus, in the numerator of (3.2) we eventually integrate against

$\phi^*\breve{V}$, where $\phi^* = \breve{\phi} + k\tilde{\phi} = \frac{1+k}{2}(\breve{\phi} + \tilde{\phi}) + \frac{1-k}{2}(\breve{\phi} - \tilde{\phi})$, a convex combination. If $|\phi| \leq 1$ on $\mathbb{R}^d$ then $|\breve{\phi} \pm \tilde{\phi}| \leq 1$ on $\mathbb{R}^d$. By convexity we see that $|\phi^*| \leq 1$. Application of [2, Theorem 1] to $\breve{V}$ and ϕ^* gives the L^p estimate (1.6) for the Fourier multiplier with the symbol (3.1).

We will now prove the general result. Consider M given by (1.4) and let $\varepsilon > 0$. In polar coordinates $(r, \theta) \in (0, \infty) \times \mathbb{S}$ we define Lévy measure

$$\nu_\varepsilon(drd\theta) = \varepsilon^{-2}\delta_\varepsilon(dr)\mu(d\theta)\,.$$

Here δ_ε is the probability measure concentrated on $\{\varepsilon\}$. We consider multiplier $\mathcal{M}_\varepsilon$ on $L^2(\mathbb{R}^d)$ with symbol M_ε defined by (3.1) where the Lévy measure is replaced by $\mathbf{1}_{\{|z|>\varepsilon\}}V + \nu_\varepsilon$ and the jump modulator is replaced by $\mathbf{1}_{\{|z|>\varepsilon\}}\phi(z) + \mathbf{1}_{\{|z|=\varepsilon\}}\varphi(z/|z|)$. We let $\varepsilon \to 0$ and note that

$$\begin{aligned} \int_{\mathbb{R}^d} [1 - \cos(\xi, z)]\varphi(z/|z|)\nu_\varepsilon\,(dz) &= \int_{\mathbb{S}} (\xi, \theta)^2 \varphi(\theta) \frac{[1 - \cos(\xi, \varepsilon\theta)]}{(\xi, \varepsilon\theta)^2} \mu(d\theta) \\ &\to \frac{1}{2} \int_{\mathbb{S}} (\xi, \theta)^2 \varphi(\theta)\mu(d\theta)\,. \end{aligned} \tag{3.3}$$

If $f \in L^2(\mathbb{R}^d)$, then $\mathcal{M}_\varepsilon f \to \mathcal{M}f$ in $L^2(\mathbb{R}^d)$ by Plancherel's theorem and bounded pointwise convergence of the symbols. A sequence, $\mathcal{M}_{\varepsilon_n} f$, converges to $\mathcal{M}f$ almost everywhere, as $\varepsilon_n \to 0$. If $f \in L^2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ then by Fatou's lemma and the first part of the proof applied to $\mathcal{M}_{\varepsilon_n}$ we have that $\|\mathcal{M}f\|_p \leq (p^* - 1)\|f\|_p$. This proves the general case because $\mathcal{M}$ extends uniquely to the whole of $L^p(\mathbb{R}^d)$ without increasing the norm. □

In the remainder of this section we will show how the symbol M in (3.1) is obtained from transforming parabolic martingales related to non-symmetric Lévy processes. Our main purpose is to elucidate as clearly as possible at which point the drift and asymmetry of the Lévy measure disappear from the picture, so that only symmetric symbols remain. The phenomenon was quite a surprise to the authors and it may be important in extending the methods of this paper. We will closely follow the development of [2]. The reader may also consult [17] or [19] for general information about the stochastic calculus of jump processes.

For a measure μ, set A, function f, and point a, we define the quantities $\breve{\mu}(A) = \mu(-A)$, $\mu(f) = \int f(x)\mu(dx)$, $(f\mu)(A) = \int_A f(x)\mu(dx)$, $f^a(x) = f(x+a)$, and $(\mu)^a(f) = \int f(x+a)\mu(dx) = \mu(f^a)$.

Let $\nu \geq 0$ be an arbitrary *finite* nonzero measure on $\mathbb{R}^d$ not charging the origin. Let $|\nu| = \nu(\mathbb{R}^d)$ and $\widetilde{\nu} = \nu/|\nu|$. Let $\mathbf{P}$ and $\mathbf{E}$ be the probability and expectation for a family of independent random variables T_i and Z_i, $i = \pm 1, \pm 2, \ldots$, where each T_i is exponentially distributed with $\mathbf{E}T_i = 1/|\nu|$, and each Z_i has $\widetilde{\nu}$ as its distribution. We let $S_i = T_1 + \ldots + T_i$, for $i = 1, 2, \ldots$, and $S_i = -(T_{-1} + \ldots + T_i)$, for $i = -1, -2, \ldots$. For $-\infty < s < t < \infty$ we let $X_{s,t} = \sum_{s<S_i\le t} Z_i$, $X_{s,t-} = \sum_{s<S_i<t} Z_i$ and $\Delta X_{s,t} = X_{s,t} - X_{s,t-}$. We note that $\mathcal{N}(B) = \#\{i : (S_i, Z_i) \in B\}$ is a Poisson random measure on $\mathbb{R} \times \mathbb{R}^d$ with intensity measure $dv\, \nu(dx)$, and $X_{s,t} = \int_{s<v\le t} x\mathcal{N}(dvdx)$ is the Lévy-Itô decomposition of X; see [21]. Alternatively, we may also consider $\mathcal{N}$ as the initial datum, and then (S_i, Z_i) may be defined as the atoms of $\mathcal{N}$. The number of *signals* S_i such that $s < S_i \le t$ equals $N(s,t) = \mathcal{N}((s,t] \times \mathbb{R}^d)$. We consider the generic compound Poisson process with the drift,

$$X^b_{s,t} = X_{s,t} + (t-s)b\,. \tag{3.4}$$

Here $b \in \mathbb{R}^d$. It is well-known that every Lévy process on $\mathbb{R}^d$ can be obtained as a limit of such processes. Again, we refer the reader to [21]. As we will see, the study of $\{X^b_{s,t}\}$ easily reduces to that of $\{X_{s,t}\}$, or to the case of $b = 0$. For instance, our notation gives

$$\mathbf{E}f(X^b_{s,t}) = \mathbf{E}f^{(t-s)b}(X_{s,t})\,. \tag{3.5}$$

Lemma 3.1. *For bounded $F : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$, and finite $s \le t$,*

$$\mathbf{E}\sum_{s<S_i\le t} F(S_i, X^b_{s,S_i-}, X^b_{s,S_i}) = \mathbf{E}\int_s^t \int_{\mathbb{R}^d} F(v, X^b_{s,v-}, X^b_{s,v-} + z)\nu(dz)dv\,. \tag{3.6}$$

Proof. By considering $F^*(v,x,y) = F(v, x + (v-s)b, y + (v-s)b)$ we can assume that $b = 0$ in (3.6). In this case the proof of [2, Lemma 1] applies (the symmetry of ν was not used in that proof). For clarity we note that $N(s,t)$ is exponentially integrable, and so is the sum in (3.6). $\square$

In particular, for finite $s \le t$ and bounded F we have

$$\begin{aligned}
&\mathbf{E}\sum_{s<S_i\le t} \left[F(S_i, X^b_{s,S_i-}, X^b_{s,S_i}) - F(S_i, X^b_{s,S_i-}, X^b_{s,S_i-})\right] \\
&= \mathbf{E}\int_s^t \int_{\mathbb{R}^d} \left[F(v, X^b_{s,v-}, X^b_{s,v-} + z) - F(v, X^b_{s,v-}, X^b_{s,v-})\right] \nu(dz)dv\,.
\end{aligned} \tag{3.7}$$

In what follows we will consider the filtration

$$\mathcal{F}_t = \sigma\{X_{s,t}\,;\ s \le t\} = \sigma\{X^b_{s,t}\,;\ s \le t\}\,, \quad t \in \mathbb{R}\,.$$

For $t \in \mathbb{R}$ we define

$$p_t = e^{*t(\nu - |\nu|\delta_0)} = \sum_{n=0}^{\infty} \frac{t^n}{n!}(\nu - |\nu|\delta_0)^{*n} = e^{-t|\nu|} \sum_{n=0}^{\infty} \frac{t^n}{n!}\nu^{*n} . \tag{3.8}$$

The series converges in the norm of absolute variation of measures. Clearly,

$$\frac{\partial}{\partial t} p_t = (\nu - |\nu|\delta_0) * p_t , \quad t \in \mathbb{R} , \tag{3.9}$$

and $p_{t_1} * p_{t_2} = p_{t_1+t_2}$ for $t_1, t_2 \in \mathbb{R}$. By (3.8) we have $p_t \geq 0$ for $t \geq 0$. In fact, p_{v-s} is the distribution of $X_{s,v}$, as well as of $X_{s,v-}$, whenever $s \leq v$. In particular, if $b = 0$ then the sides of (3.6) equal

$$\int_s^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} F(v, y, y+z)\nu(dz)p_{v-s}(dy)dv , \tag{3.10}$$

and the extension to $b \neq 0$ is straightforward, see, e.g., (3.25). Let

$$\Psi(\xi) = \int_{\mathbb{R}^d} \left[e^{i(\xi,z)} - 1\right] \nu(dz) , \quad \xi \in \mathbb{R}^d . \tag{3.11}$$

We directly verify that Ψ is bounded and continuous on $\mathbb{R}^d$, $\Psi(-\xi) = \overline{\Psi(\xi)}$, $\Re\psi(\xi) = \int_{\mathbb{R}^d}[\cos(\xi, z) - 1]\nu(dz)$ (compare the denominator in (3.1)), and

$$\widehat{p_t}(\xi) = \int_{\mathbb{R}^d} e^{i(\xi,x)} p_t(dx) = e^{t\Psi(\xi)} , \quad \xi \in \mathbb{R}^d . \tag{3.12}$$

Ψ is the Lévy-Khinchine exponent and (3.12) is the Lévy-Khinchin formula. We also consider the convolution semigroup with the drift of speed b,

$$p_t^b = (p_t)^{tb} , \quad t \geq 0 ,$$

that is $p_t^b(f) = p_t(f^{tb})$. We have

$$\widehat{p_t^b}(\xi) = \int_{\mathbb{R}^d} e^{i(\xi,x+tb)} p_t(dx) = e^{it(\xi,b)+t\Psi(\xi)} , \quad \xi \in \mathbb{R}^d . \tag{3.13}$$

In what follows $f \in L_c^\infty$ and $g \in L^\infty$. For $x \in \mathbb{R}^d$ and finite $t \leq u$ we define

$$P_{t,u}^b g(x) = \mathbf{E} g(x + X_{t,u}^b) = \int_{\mathbb{R}^d} g(x+y) p_{u-t}^b(dy) . \tag{3.14}$$

This is the convolution with the reflection of p^b_{u-t}, and we have

$$\widehat{P^b_{t,u}f}(\xi) = \hat{f}(\xi)\widehat{p^b_{u-t}}(-\xi) = \hat{f}(\xi)e^{-i(u-t)(\xi,b)+(u-t)\Psi(-\xi)}\,, \quad \xi \in \mathbb{R}^d\,. \tag{3.15}$$

We denote $P_{t,u} = P^0_{t,u}$. By (3.14) we get $P^b_{t,u}g = P_{t,u}(g^{(u-t)b})$.

For $s \le t \le u$ we define the following *parabolic martingale*

$$G^b_t = G^b_t(x;s,u;g) = P^b_{t,u}g(x+X^b_{s,t}) = P_{t,u}g^{(u-s)b}(x+X_{s,t})\,. \tag{3.16}$$

We will also write $G_t = G^0_t$. By [2, Lemma 2] and (3.16), $t \mapsto G^b_t$ is indeed a (bounded) $\{\mathcal{F}_t\}$-martingale on $[s,u]$ (see also the discussion below).

Let ϕ be complex-valued and let $|\phi(z)| \le 1$ for $z \in \mathbb{R}^d$.
For $x \in \mathbb{R}^d$ and $s \le t \le u$, we define $F^b_t = F^b_t(x;s,u;g,\phi)$ as

$$\sum_{s<S_i\le t}\left[P^b_{S_i,u}g(x+X^b_{s,S_i}) - P^b_{S_i,u}g(x+X^b_{s,S_i-})\right]\phi(X^b_{s,S_i}-X^b_{s,S_i-})$$
$$-\int_s^t\int_{\mathbb{R}^d}\left[P^b_{v,u}g(x+X^b_{s,v-}+z) - P^b_{v,u}g(x+X^b_{s,v-})\right]\phi(z)\nu(dz)dv\,. \tag{3.17}$$

We let $F_t = F^0_t$, and note that $F^b_t(x;s,u;g,\phi) = F_t(x;s,u;f^{(u-s)b},\phi)$. It now follows from [2, Lemma 3 and Lemma 4] that F^b_t is an $\{\mathcal{F}_t\}$-martingale in $t \in [s,u]$, and $\mathbf{E}|F_t|^p < \infty$ for every $p > 0$. We have

$$\begin{aligned} G^b_t(x;s,u;g) &= G_t(x;s,u;g^{(u-s)b}) = F_t(x;s,u;g^{(u-s)b},1) + P_{s,u}(g^{(u-s)b})(x) \\ &= F^b_t(x;s,u;g,1) + G^b_s(x;s,u;g)\,, \end{aligned} \tag{3.18}$$

where we have used [2, Lemma 5]. The equality (3.18) may also be considered a consequence of Itô formula for the space-time process $t \mapsto (u-t, X_{s,t})$. In fact, this is very simple because $X_{s,t}$ is piecewise constant. We have

$$\begin{aligned} G_t(x;s,u;g) \;-\; & G_s(x;s,u;g) = \int_s^t \left(\frac{\partial}{\partial_v}P_{v,u}\right) g(x+X_{s,v-})dv \\ &+ \sum_{s<v\le t}\left[P_{v,u}g(x+X_{s,v}) - P_{v,u}g(x+X_{s,v-})\right], \end{aligned} \tag{3.19}$$

where the sum is taken over v such that $X_{s,v} \ne X_{s,v-}$ (or see [17, Theorem II.31], [11, p. 140]). Using (3.9) and Lemma 3.1 we now obtain that the expression has zero expectation and, moreover, it is a martingale. Let $t^k_n = s + k(t-s)/n$, where

$k = 0, \ldots, n$, and $n \to \infty$. Since F_t^b is square integrable (in fact, exponentially integrable), by orthogonality of increments we have

$$\begin{aligned}
\mathbf{E}\left|F_t^b\right|^2 &= \mathbf{E}\sum_{k=1}^{n}\left|F^b_{s,t_k^n} - F^b_{s,t_{k-1}^n}\right|^2 \\
&\to \mathbf{E}\sum_{s<S_i\le t}\left|P^b_{S_i,u}g(x+X^b_{s,S_i}) - P^b_{S_i,u}g(x+X^b_{s,S_i-})\right|^2\left|\phi(\Delta X^b_{s,S_i})\right|^2 ,
\end{aligned}$$

The convergence follows from the fact that the integral in (3.17) is Lipschitz continuous in t. Hence the quadratic variation ([17], [9]) of F^b is

$$[F^b, F^b]_t = \sum_{s<S_i\le t}\left|P^b_{S_i,u}g(x+X^b_{s,S_i}) - P^b_{S_i,u}g(x+X^b_{s,S_i-})\right|^2\left|\phi(\Delta X_{s,S_i})\right|^2 . \tag{3.20}$$

By (3.18), the quadratic variation of G^b is

$$[G^b, G^b]_t = |P^b_{s,u}g(x)|^2 + \sum_{s<S_i\le t}\left|P^b_{S_i,u}g(x+X^b_{s,S_i}) - P^b_{S_i,u}g(x+X^b_{s,S_i-})\right|^2 .$$

Thus, $F^b(x; s, u; g, \phi)$ is differentially subordinate to $G^b(x; s, u; g)$, see (2.11). This time we appeal to the result of Wang [25, Theorem 1] for general martingales with jumps, to conclude that for $p \in (1, \infty)$,

$$\mathbf{E}|F^b_t(x; s, u; f, \phi)|^p \le (p^* - 1)^p\, \mathbf{E}|G^b_t(x; s, u; f)|^p , \quad s \le t \le u . \tag{3.21}$$

We have $G^b_u(x; s, u; f) = f(x + X^b_{s,u})$, and using (3.21) and (1.12) we obtain

$$\int_{\mathbb{R}^d}\mathbf{E}|F^b_u(x; s, u; f, \phi)|^p dx \le (p^* - 1)^p\int_{\mathbb{R}^d}\mathbf{E}|f(x + X^b_{s,u})|^p dx = (p^* - 1)^p\|f\|_p^p .$$

By Hölder's inequality and (1.12) we get

$$\int_{\mathbb{R}^d}\mathbf{E}|F^b_u(x; s, u; f, \phi)g(x + X^b_{s,u})|dx \le (p^* - 1)\|f\|_p\|g\|_q . \tag{3.22}$$

Therefore there is a unique function $h \in L^p$ such that

$$\int_{\mathbb{R}^d}\mathbf{E}F^b_u(x; s, u; f, \phi)g(x + X^b_{s,u})dx = \int_{\mathbb{R}^d}h(x)g(x)dx, \tag{3.23}$$

if $g \in L^q$, and we have

$$\|h\|_p \le (p^* - 1)\|f\|_p . \tag{3.24}$$

We will identify h. By (3.20), (2.19) and Lemma 3.1,

$$
\begin{aligned}
\mathbf{E} F_u^b G_u^b &= \mathbf{E} F_u^b(x;s,u;f,\phi)[G_u^b(x;s,u;g) - P_{s,u}^b g(x)] \\
&= \mathbf{E} \sum_{s<S_i\le u} \left[P_{S_i,u}^b f(x+X_{s,S_i}^b) - P_{S_i,u}^b f(x+X_{s,S_i-}^b)\right] \\
&\qquad\qquad [P_{S_i,u}^b g(x+X_{s,S_i}^b) - P_{S_i,u}^b g(x+X_{s,S_i-}^b)]\phi(\Delta X_{s,S_i}) \\
&= \mathbf{E} \int_s^u \int_{\mathbb{R}^d} \left[P_{v,u}^b f(x+X_{s,v-}^b+z) - P_{v,u}^b f(x+X_{s,v-}^b)\right] \\
&\qquad\qquad [P_{v,u}^b g(x+X_{s,v-}^b+z) - P_{v,u}^b g(x+X_{s,v-}^b)]\phi(z)\nu(dz)dv \\
&= \int_s^u \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left[P_{v,u}^b f(x+y+z) - P_{v,u}^b f(x+y)\right] \\
&\qquad\qquad [P_{v,u}^b g(x+y+z) - P_{v,u}^b g(x+y)]\phi(z)\nu(dz)p_{v-s}^b(dy)dv\,.
\end{aligned} \tag{3.25}
$$

To justify applications of Fubini's theorem in what follows, we note that (1.12) and the finiteness of ν imply

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \int_s^u \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left|P_{v,u}^b f(x+y+z) - P_{v,u}^b f(x+y)\right| \\
&\qquad\qquad \left|P_{v,u}^b g(x+y+z) - P_{v,u}^b g(x+y)\right| \phi(z)\nu(dz)p_{v-s}^b(dy)dvdx \\
&\le 4\|g\|_\infty |\nu| \int_s^u \|P_{v,u}^b f\|_1 dv \le 4|\nu|(u-s)\|g\|_\infty \|f\|_1 < \infty\,.
\end{aligned} \tag{3.26}
$$

In particular, $h \in L^1$, see (3.23). Consider $\xi \in \mathbb{R}^d$, $e_\xi(x) = e^{i(\xi,x)}$, and $\mathcal{E}_t^b(x;s,u;\xi) = G_t^b(x;s,u;e_\xi)$, $s \le t \le u$. We have

$$
P_{v,u}^b e_\xi(x) = \int_{\mathbb{R}^d} e^{i(\xi,x+y)} p_{u-v}^b(dy) = e_\xi(x) e^{(u-v)[i(\xi,b)+\Psi(\xi)]}\,, \quad v \le u\,,
$$

thus

$$
\begin{aligned}
\mathbf{E} F_u^b \mathcal{E}_u^b &= \int_s^u \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left[P_{v,u}^b f(x+y+z) - P_{v,u}^b f(x+y)\right] \\
&\qquad e^{i(\xi,x+y)}[e^{i(\xi,z)} - 1] e^{(u-v)[i(\xi,b)+\Psi(\xi)]} \phi(z)\nu(dz)p_{v-s}^b(dy)dv\,.
\end{aligned} \tag{3.27}
$$

We recall (3.23), (3.25), (3.26) and (3.27), and conclude that

$$
\begin{aligned}
\hat{h}(\xi) &= \int_{\mathbb{R}^d} \int_s^u \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left[P_{v,u}^b f(x+y+z) - P_{v,u}^b f(x+y)\right] \\
&\qquad e^{i(\xi,x+y)}[e^{i(\xi,z)} - 1] e^{(u-v)[i(\xi,b)+\Psi(\xi)]} \phi(z)\nu(dz)p_{v-s}^b(dy)dvdx\,.
\end{aligned}
$$

Using (1.12) and (3.15) we obtain

$$
\begin{aligned}
\hat{h}(\xi) &= \int_s^u \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left[P^b_{v,u} f(x+z) - P^b_{v,u} f(x)\right] \\
&\qquad\qquad e^{i(\xi,x)} dx\, [e^{i(\xi,z)} - 1] e^{(u-v)[i(\xi,b)+\Psi(\xi)]} \phi(z)\nu(dz)dv \\
&= \int_s^u \int_{\mathbb{R}^d} [e^{-i(\xi,z)} - 1] \hat{f}(\xi) e^{(u-v)[-i(\xi,b)+\Psi(-\xi)]} \\
&\qquad\qquad [e^{i(\xi,z)} - 1] e^{(u-v)[i(\xi,b)+\Psi(\xi)]} \phi(z)\nu(dz)dv \\
&= \hat{f}(\xi) \int_s^u e^{2(u-v)\Re\Psi(\xi)} dv \int_{\mathbb{R}^d} |e^{i(\xi,z)} - 1|^2 \phi(z)\nu(dz) \qquad (3.28) \\
&= \hat{f}(\xi) \left[1 - e^{2(u-s)\Re\Psi(\xi)}\right] \int_{\mathbb{R}^d} [\cos(\xi,z) - 1]\phi(z)\nu(dz)/\Re\Psi(\xi)\,,
\end{aligned}
$$

if $\Re\Psi(\xi) = \int[\cos(\xi,z)-1]\nu(dz) < 0$, and $\hat{h}(\xi) = 0$ if $\Re\Psi(\xi) = 0$. This identifies the function h in (3.23). We conclude that $f \mapsto h$ is a Fourier multiplier with the symbol

$$
\frac{\int_{\mathbb{R}^d}[\cos(\xi,z) - 1]\phi(z)\nu(dz)}{\int_{\mathbb{R}^d}[\cos(\xi,z) - 1]\nu(dz)} \left[1 - e^{2(u-s)\Re\Psi(\xi)}\right] . \qquad (3.29)
$$

By (3.24) and the density of C^1_b in L^p, the operator norm of the multiplier does not exceed $p^* - 1$ on L^p. The above readily yields the symbols (3.1) with the same L^p bound $p^* - 1$ for the corresponding operators. Indeed, if V is an arbitrary (i.e. not necessarily finite or symmetric) Lévy measure, then we consider $\varepsilon > 0$ and define ν as the restriction of V to $\{z : |z| > \varepsilon\}$. We then let $u = 0$, $\varepsilon \downarrow 0$, $s \to -\infty$ in (3.29), and employ a limiting argument similar to the one following (3.3).

Here we should note that the asymmetry of the Lévy measure and the drift given by b have disappeared from our formulas in (3.28).

4 Further discussion and examples

We will comment on the relation between (1.7) and (1.8).

Lemma 4.1. *If $\mathbb{A}$ is a complex symmetric $d \times d$ matrix, and $|\mathbb{A}\xi| \le |\xi|$ for $\xi \in \mathbb{R}^d$, then $\mu \ge 0$ and φ exist such that $\|\varphi\|_\infty \le 2$, and*

$$
(\mathbb{A}\xi, \xi) = \int_{\mathbb{S}} (\xi,\theta)^2 \varphi(\theta)\, \mu(d\theta) \quad \textit{and} \quad \int_{\mathbb{S}} (\xi,\theta)^2 \mu(d\theta) = (\xi,\xi)\,, \quad \xi \in \mathbb{R}^d .
$$

If $\Re\mathbb{A}$ and $\Im\mathbb{A}$ commute, then we may select $\|\varphi\|_\infty \le 1$.

Proof. Recall that $\mathbb{A}$ is symmetric but not necessarily Hermitian. Assume first that $\mathbb{A}$ is normal, that is $\Re\mathbb{A}$ and $\Im\mathbb{A}$ commute. Then they have common eigenvectors $a_k \in \mathbb{R}^d$, and $\mathbb{A}a_k = \lambda_k a_k$, where $\lambda_k \in \mathbb{C}$, and $|\lambda_k| \le 1$ for $k = 1, \ldots, d$. For $\xi \in \mathbb{R}^d$,

$$\sum_{k=1}^{d} (\xi, a_k)^2 = |\xi|^2 \,,$$

and

$$(\mathbb{A}\xi, \xi) = \left(\sum_{k=1}^{d} (\xi, a_k)\mathbb{A}a_k, \sum_{k=1}^{d} (\xi, a_k)a_k \right) = \sum_{k=1}^{d} \lambda_k (\xi, a_k)^2 \,.$$

Let $\mu = \sum_{k=1}^{d} \delta_{a_k}$ and $\varphi(a_k) = \lambda_k$, so that $\|\varphi\|_\infty \le 1$. Here δ_a is the Dirac measure at a.

If $\Re\mathbb{A}$ and $i\Im\mathbb{A}$ do not commute then we consider each of them separately as in the first part of the proof, and we add the respective measures μ, and the measures $\varphi\mu$. We see that the resulting φ is bounded by 1 but we only obtain a representation of $(\mathbb{A}\xi, \xi)/[2(\xi, \xi)]$.

$\square$

We consider the Beurling-Ahlfors operator. It is the singular integral on the complex plane $\mathbb{C}$ (identified with $\mathbb{R}^2$), defined for smooth compactly supported functions f as follows

$$Bf(z) = -\frac{1}{\pi} \, p.v. \int_{\mathbb{C}} \frac{f(w)}{(z-w)^2} dm(w) \,, \quad z \in \mathbb{C} \,. \tag{4.1}$$

Here m is the planar Lebesgue measure. It is well known that B is a Fourier multiplier with the symbol

$$M(\xi) = \frac{\overline{\xi}^2}{|\xi|^2} = e^{-2i \arg \xi} \,, \tag{4.2}$$

where $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ is identified with $\xi_1 + i\xi_2 \in \mathbb{C}$. For a detailed discussion of B, its numerous connections and applications in analysis, partial differential equations and quasiconformal mappings, we refer to [1] and the many references given there.

The above symbol M is precisely the one given by (1.8) and (1.11). In particular, Lemma 4.1 and Theorem 1.1 apply, and the operator norm of B on L^p does not exceed $2(p^* - 1)$. In fact, μ uniform on $\{1, i, e^{i\pi/4}, e^{-i\pi/4}\}$, and $\phi(1) = 2$,

$\phi(i) = -2$, $\phi(e^{i\pi/4}) = -2i$, $\phi(e^{-i\pi/4}) = 2i$, give a representation (1.7) of (4.2). We note that the bound $2(p^* - 1)$ was first obtained in [24] using certain Bellman function constructed from Burkholder's discrete martingale inequalities. The Itô calculus approach was presented in [4] to get this bound, as in our Section 2. The best bound to date for the operator norm of B on L^p is given in [3]. We refer the reader to [1] for a thorough discussion of T. Iwaniec's conjecture that $\|B\| = p^* - 1$, and further references.

As it stands, our approach cannot improve the bound $2(p^* - 1)$ for (4.2) because of the following fact, which should be compared with (1.7).

Lemma 4.2. *If φ and nonzero $\mu \geq 0$ on $\mathbb{S} \subset \mathbb{R}^2$ are such that*

$$\int_{\mathbb{S}} (\xi, \theta)^2 \, \varphi(\theta)\mu(d\theta) = e^{-2i \arg \xi} \int_{\mathbb{S}} (\xi, \theta)^2 \, \mu(d\theta) \,, \quad \xi \in \mathbb{R}^2 \,, \tag{4.3}$$

then $\|\varphi\|_\infty \geq 2$.

Proof. We can assume that φ is bounded. We denote $t = \arg \xi$, $s = \arg \theta$, and identify $\varphi(\theta)$ and $\mu(d\theta)$ with $\varphi(s)$ and $\mu(ds)$, correspondingly. We have

$$(\xi, \theta)^2 = \cos^2(t - s) = \frac{1}{2}\big(\cos[2(t - s)] + 1\big) = \frac{1}{2} + \frac{1}{4}e^{2it}e^{-2is} + \frac{1}{4}e^{-2it}e^{2is} \,,$$

and hence the left-hand side of (4.3) is

$$\frac{1}{2}\int_{\mathbb{S}} \varphi(s)\mu(ds) + \frac{1}{4}e^{2it}\int_{\mathbb{S}} e^{-2is}\varphi(s)\mu(ds) + \frac{1}{4}e^{-2it}\int_{\mathbb{S}} e^{2is}\varphi(s)\mu(ds).$$

However, the right-hand side equals

$$\frac{1}{2}e^{-2it}\int_{\mathbb{S}} \mu(ds) + \frac{1}{4}\int_{\mathbb{S}} e^{-2is}\mu(ds) + \frac{1}{4}e^{-4it}\int_{\mathbb{S}} e^{2is}\mu(ds) \,.$$

In particular,

$$\frac{1}{4}\int_{\mathbb{S}} e^{2is}\varphi(s)\mu(ds) = \frac{1}{2}\int_{\mathbb{S}} \mu(ds) \,,$$

which is impossible if $\|\varphi\|_\infty < 2$. $\square$

Let $\mu(ds) = ds$. In view of the above proof, $\varphi(s) = e^{iks}$ with integer $k \neq -2, 0, 2$, yields the zero symbol. If $\varphi(s) = e^{\pm 2is}$ then we arrive at $e^{\pm 2i \arg \xi}/2$, in particular we obtain an elegant representation of (4.2).

Let V be the Lévy measure of a non-zero symmetric α-stable Lévy process in $\mathbb{R}^d$, with $\alpha \in (0,2)$. In polar coordinates we have (see, e.g., [21], [6])

$$V(drd\theta) = r^{-1-\alpha} dr \sigma(d\theta)\,, \quad r>0\,,\ \theta \in \mathbb{S}\,, \tag{4.4}$$

where the so-called spectral measure σ is finite and non-zero on $\mathbb{S}$. Let ϕ be complex-valued on $\mathbb{R}^d$ and such that $|\phi(z)| \le 1$ and $\phi(z) = \phi\left(z/|z|\right)$ for $z \neq 0$. Let $c_\alpha = \int_0^\infty [1-\cos s] s^{-1-\alpha} ds$. By a change of variable,

$$\begin{aligned}\int_{\mathbb{R}^d} [1-\cos(\xi,z)]\phi(z)V(dz) &= \int_{\mathbb{S}} \int_0^\infty [1-\cos(\xi, r\theta)]\phi(r\theta) r^{-1-\alpha} dr\, \sigma(d\theta)\\ &= c_\alpha \int_{\mathbb{S}} |(\xi,\theta)|^\alpha\, \phi(\theta)\sigma(d\theta)\,. \end{aligned} \tag{4.5}$$

Theorem 1.1 yields a multiplier bounded in L^p by p^*-1, with the symbol

$$M(\xi) = \frac{\int_{\mathbb{S}} |(\xi,\theta)|^\alpha\, \phi(\theta)\sigma(d\theta)}{\int_{\mathbb{S}} |(\xi,\theta)|^\alpha\, \sigma(d\theta)}\,. \tag{4.6}$$

In particular, for $j = 1, \ldots, d$, we obtain

$$M(\xi) = \frac{|\xi_j|^\alpha}{|\xi_1|^\alpha + \cdots + |\xi_d|^\alpha}\,, \quad \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d\,. \tag{4.7}$$

These are Marcinkiewicz-type multipliers, as in [22, p. 110].

In the next example we will specialize to $\mathbb{R}^2$. Let σ be a Lebesgue measure on the circle, and $\phi(\theta) = e^{-2i\arg\theta}$, as in the comment following Lemma 4.2. Let $\xi \in \mathbb{R}^2$ and $t = \arg \xi$. In view of (4.5), the numerator of the symbol is

$$\begin{aligned} c_\alpha |\xi|^\alpha \int_{\mathbb{S}} |\cos(t-s)|^\alpha\, e^{-2is} ds &= c_\alpha |\xi|^\alpha\, e^{-2it} \int_{\mathbb{S}} |\cos(v)|^\alpha\, e^{2iv} dv\\ &= c_\alpha |\xi|^\alpha\, e^{-2it} \int_{\mathbb{S}} |\cos(v)|^\alpha \cos(2v) dv\,. \end{aligned}$$

For $a, b > -1$ we have

$$\int_0^{\frac{\pi}{2}} \sin^a v \cos^b v\, dv = \frac{1}{2}\mathcal{B}\left(\frac{a+1}{2}, \frac{b+1}{2}\right) = \frac{1}{2}\frac{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}{\Gamma\left(\frac{a+b+2}{2}\right)}\,,$$

see, e.g., [15, Chapter I]. Therefore

$$\begin{aligned} &\int_0^{2\pi} |\cos(v)|^\alpha \cos(2v) dv = \int_0^{2\pi} |\cos(v)|^\alpha\, (2\cos^2(v) - 1) dv\\ &= 4\mathcal{B}\left(\frac{\alpha+3}{2}, \frac{1}{2}\right) - 2\mathcal{B}\left(\frac{\alpha+1}{2}, \frac{1}{2}\right) = \frac{2\alpha}{\alpha+2}\mathcal{B}\left(\frac{\alpha+1}{2}, \frac{1}{2}\right)\,, \end{aligned}$$

where we used $\Gamma(x+1) = x\Gamma(x)$. Since

$$\int_0^{2\pi} |\cos(v)|^\alpha \, dv = 2\mathcal{B}\left(\frac{\alpha+1}{2}, \frac{1}{2}\right),$$

we obtain the symbol

$$M(\xi) = \frac{\alpha}{\alpha+2} e^{-2i \arg \xi}.$$

For $\alpha \to 2$ we recover the bound $2(p^*-1)$ for the Beurling-Ahlfors transform.

We will consider more general Lévy measures in $\mathbb{R}^d$ of product form in polar coordinates,

$$V(drd\theta) = \rho(dr)\sigma(d\theta), \quad r > 0, \; \theta \in \mathbb{S}. \tag{4.8}$$

Here σ is finite on $\mathbb{S}$ and $\int_0^\infty r^2 \wedge 1 \; \rho(dr) < \infty$. An interesting class of such measures are the so-called tempered stable Lévy processes ([20], [23]). The following example is on the borderline of the tempered stable processes. Let

$$\rho(dr) = e^{-r} \frac{dr}{r}.$$

In view of the calculations following (4.8) we like to note that

$$\int_0^\infty [1 - \cos(\xi, r\eta)]\rho(dr) = \int_0^\infty [1 - \cos x] e^{-x/|(\xi,\theta)|} \frac{dx}{x}.$$

The Laplace transform of $(1 - \cos x)/x$ equals $0.5 \ln(1 + s^{-2})$. Theorem 1.1 yields a multiplier bounded in L^p by $p^* - 1$, with the symbol

$$M(\xi) = \frac{\int_{\mathbb{S}} \ln\left[1 + (\xi,\theta)^{-2}\right] \phi(\theta)\sigma(d\theta)}{\int_{\mathbb{S}} \ln\left[1 + (\xi,\theta)^{-2}\right] \sigma(d\theta)}. \tag{4.9}$$

For instance, for $j = 1, \ldots, d$, we obtain

$$M(\xi) = \frac{\ln\left(1 + \xi_j^{-2}\right)}{\ln\left(1 + \xi_1^{-2}\right) + \cdots + \ln\left(1 + \xi_d^{-2}\right)}, \quad \xi \in \mathbb{R}^d. \tag{4.10}$$

We conclude with a few general remarks. It is well known that the stochastic calculus of the Brownian motion can be used to obtain non-symmetric Fourier symbols via harmonic (rather than parabolic) martingales. This goes back to the pioneering paper of Gundy and Varopoulos [13] for Riesz transform, and we again refer the reader to the survey paper [1] for further discussion. Surprisingly, non-symmetric Lévy processes do not bring about non-symmetric symbols. We owe to Mateusz Kwaśnicki yet another explanation of this phenomenon, using time reversal of Lévy processes (private communication). Our present discussion leaves wide open the problem of modifying the jumps of Lévy processes in such a way as to obtain non-symmetric multipliers.

We also note that McConnell studied in [16] extensions of the Hörmander multiplier theorem. He used the Cauchy process composed with harmonic functions on the upper half-space in $\mathbb{R}^{d+1}$. This may be considered a special case of our parabolic martingales; see [16, Lemma 2.1]. However, the Cauchy process is obtained by optional stopping of the $(d+1)$-dimensional Brownian motion on the half-space, and so [16] is more related to the work of Gundy and Varopoulos [13] than to the parabolic martingales of Bañuelos and Méndez-Hernandéz [4].

The relationship between (1.10), (2.8) and the condition in Lemma 4.1 calls for further study. We also wonder if the bound $\|\varphi\|_\infty \le 2$ in the proof of Lemma 4.1 may be improved for non-normal $\mathbb{A}$.

If Lévy measures satisfy $\nu_1 \le \nu_2$, then

$$M(\xi) = \frac{\int_{\mathbb{R}^d} [1-\cos(\xi,z)]\, \nu_1(dz)}{\int_{\mathbb{R}^d} [1-\cos(\xi,z)]\, \nu_2(dz)}\,, \tag{4.11}$$

defines an L^p multiplier with the norm not exceeding p^*-1, which follows from Theorem 1.1 with $V=\nu_2$, $\phi = 1\wedge d\nu_1/d\nu_2$ and $\mu=0$. The observation allows to study inclusions between anisotropic Sobolev spaces ([14]).

An interesting problem, indirectly touched upon by Lemma 4.1, is the following: Can we handle a class of Fourier multipliers bounded on L^p by specifying the denominator and some boundedness and differentiability properties of the ratio (1.4), so to recover bounded φ and ϕ from these?

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