ESSENTIAL DIMENSION OF SIMPLE ALGEBRAS IN POSITIVE CHARACTERISTIC

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Abstract. Let p be a prime integer, $1 \leq s \leq r$ integers, F a field of characteristic p. Let Dec_{p^r} denote the class of the tensor product of r p-symbols and Alg_{p^r,p^s} denote the class of central simple algebras of degree p^r and exponent dividing p^s. For any integers $s < r$, we find a lower bound for the essential p-dimension of Alg_{p^r,p^s}. Furthermore, we compute upper bounds for Dec_{p^r} and $\mathit{Alg}_{8,2}$ over $\mathrm{char}(F) = p$ and $\mathrm{char}(F) = 2$, respectively. As a result, we show $\mathrm{ed}_2(\mathit{Alg}_{4,2}) = \mathrm{ed}(\mathit{Alg}_{4,2}) = \mathrm{ed}_2(\mathbf{GL}_4/\boldsymbol{\mu}_2) = \mathrm{ed}(\mathbf{GL}_4/\boldsymbol{\mu}_2) = 3$ and $3 \leq \mathrm{ed}(\mathit{Alg}_{8,2}) = \mathrm{ed}(\mathbf{GL}_8/\boldsymbol{\mu}_2) \leq 10$ over a field of characteristic 2.

1. Introduction

A numerical invariant, essential dimension of algebraic groups was introduced by Reichstein and was generalized to algebraic structures by Merkurjev. We refer to [9] for the definition of essential dimension and denote by ed and ed_p the essential dimension and essential p-dimension, respectively.

Let F be a field, Fields/F the category of field extensions over F, and Sets the category of sets. For every integer $n \geq 1$, a divisor m of n and any field extension E/F, let

$$\mathit{Alg}_{n,m} : \mathit{Fields}/F \to \mathit{Sets}$$

be the functor taking a field extension K/F to the set of isomorphism classes of central simple K-algebras of degree n and exponent dividing m. Then, there is a natural bijection between $H^1(K, \mathbf{GL}_n/\boldsymbol{\mu}_m)$ and $\mathit{Alg}_{n,m}(K)$ (see [1, Example 1.1]), thus $\mathrm{ed}(\mathit{Alg}_{n,m}) = \mathrm{ed}(\mathbf{GL}_n/\boldsymbol{\mu}_m)$ and $\mathrm{ed}_p(\mathit{Alg}_{n,m}) = \mathrm{ed}_p(\mathbf{GL}_n/\boldsymbol{\mu}_m)$.

Let F be a field of characteristic p. For $a \in F$ and $b \in F^{\times}$, we write $\big[a, b\big)_p$ for the central simple algebra over F generated by u and v satisfying $u^p - u = a$, $v^p = b$ and $vu = uv + v$ (it is called a symbol p-algebra). For a field extension E/F, let

$$\mathit{Dec}_{p^r} : \mathit{Fields}/F \to \mathit{Sets}$$

be the functor taking a field extension K/F to the set of isomorphism classes of the tensor product of r p-symbols over E.

Some computations of the essential dimension and essential p-dimension of $\mathit{Alg}_{m,n}$ have been done. But most of them have the restriction $\mathrm{char}(F) \neq p$ on the base field F. In this paper, for any integers $r > s$, we find a new

The work has been supported by NSERC Discovery 385795-2010, Accelerator Supplement 396100-2010 grants, and Erhard Neher's NSERC Discovery grant.

lower bound for $\mathrm{ed}_p(\mathsf{Alg}_{p^r,p^s})$ over $\mathrm{char}(F)=p$. Moreover, we compute upper bounds for Dec_{p^r} and $\mathsf{Alg}_{8,2}$ over $\mathrm{char}(F)=p$ and $\mathrm{char}(F)=2$, respectively. As a result, we get:

Theorem 1.1. *Let F be a field of characteristic 2. Then*

$$\mathrm{ed}_2(\mathsf{Alg}_{4,2})=\mathrm{ed}(\mathsf{Alg}_{4,2})=\mathrm{ed}_2(\mathbf{GL}_4/\boldsymbol{\mu}_2)=\mathrm{ed}(\mathbf{GL}_4/\boldsymbol{\mu}_2)=3.$$

Proof. The lower bound $3\le \mathrm{ed}_2(\mathsf{Alg}_{4,2})$ follows from Corollary 2.4. By a Theorem of Albert, we have $\mathsf{Dec}_4=\mathsf{Alg}_{4,2}$ for $p=2$, thus we get $\mathrm{ed}(\mathsf{Alg}_{4,2})\le 3$ by Proposition 3.2. As $\mathrm{ed}_2(\mathsf{Alg}_{4,2})\le \mathrm{ed}(\mathsf{Alg}_{4,2})$, the result follows. □

Corollary 2.4 and Corollary 3.5 give the following:

Theorem 1.2. *Let F be a field of characteristic 2. Then*

$$3\le \mathrm{ed}(\mathsf{Alg}_{8,2})=\mathrm{ed}(\mathbf{GL}_8/\boldsymbol{\mu}_2)\le 10.$$

2. Lower bounds

Theorem 2.1. (*Tsen*) *Let K be a field of transcendental degree 1 over an algebraically closed field F. Then, for any central division algebra A over K, $\mathrm{ind}(A)=\exp(A)=1$, i.e., $A=K$.*

As an application of Theorem 2.1, Reichstein obtained the following result:

Corollary 2.2. [10, Lemma 9.4(a)] *Let F be an arbitrary field and A be a division algebra of degree $n\ge 2$. Then $\mathrm{ed}(A)\ge 2$. In particular, for any integers r, s and any prime p, $\mathrm{ed}_p(\mathsf{Alg}_{p^r,p^s})\ge 2$.*

Initially, the following theorem is proved under the additional condition that $\mathrm{char}(F)$ does not divide $\exp(A)$ in [5]. In subsequent papers [6, Theorem 1.0.2] and [8, Theorem 4.2.2.3], this condition is removed:

Theorem 2.3. (*de Jong*) *Let K be a field of transcendental degree 2 over an algebraically closed field F. Then, for any central simple algebra A over K, $\mathrm{ind}(A)=\exp(A)$.*

As an application of Theorem 2.3, we have the following result:

Corollary 2.4. *Let F be an arbitrary field and p be a prime. For any integers r and s with $s<r$, $\mathrm{ed}_p(\mathsf{Alg}_{p^r,p^s})\ge 3$.*

Proof. By [9, Proposition 1.5], we may replace the base field F by an algebraically closure of F. Let K be a field extension of F and A be a central simple algebra over K of $\mathrm{ind}(A)=p^r$ and $\exp(A)|p^s$. Let E be a field extension of K of degree prime to p. As $\mathrm{ind}(A)$ is relatively prime to $[E:K]$, we have $\mathrm{ind}(A_E)=\mathrm{ind}(A)=p^r$. Suppose that $A_E\simeq B\otimes E$ for some $B\in \mathsf{Alg}_{p^r,p^s}(L)$ and $\mathrm{tr.deg}_F(L)=2$. Then, by Theorem 2.3, we have $\mathrm{ind}(B)=\exp(B)$. As $p^r=\mathrm{ind}(A_E)|\,\mathrm{ind}(B)=\exp(B)$, we get $p^r|\exp(B)$. But it contradicts to $\exp(B)|p^s$. By Corollary 2.2, the result follows. □

Remark 2.5. As we see in [2, Theorem], the above lower bound 3 is much less than the best known lower bounds, but these lower bounds are valid only for $\mathrm{char}(F) \neq p$. Hence, our main application of Corollary 2.4 is for the case of $\mathrm{char}(F) = p$.

3. Upper bounds

3.1. An upper bound for $\mathrm{ed}(\mathsf{Dec}_{p^r})$.

Lemma 3.1. [3, Example 2.3 and page 298] *Let F be a field of characteristic p and $r \geq 1$ be an integer. If $|F| \geq p^r$, then $\mathrm{ed}((\mathbb{Z}/p\mathbb{Z})^r) = 1$.*

Proof. From the exact exact sequence

$$0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{G}_a \xrightarrow{\wp} \mathbb{G}_a \to 0,$$

we have $H^1(E, \mathbb{Z}/p\mathbb{Z}) = E/\wp(E)$ for any field extension E/F where $\wp(x) = x^p - x$ for $x \in E$, hence $\mathrm{ed}(\mathbb{Z}/p\mathbb{Z}) = 1$. By [3, Proposition 4.11], we have $\mathrm{ed}((\mathbb{Z}/p\mathbb{Z})^r) \geq 1$.

As $|F| \geq p^r$, we have an exact sequence

$$0 \to (\mathbb{Z}/p\mathbb{Z})^r \to \mathbb{G}_a \xrightarrow{\wp} \mathbb{G}_a \to 0.$$

It follows from $H^1(E, \mathbb{G}_a) = 0$ that $\mathbb{G}_a(E) \to H^1(E, (\mathbb{Z}/p\mathbb{Z})^r)$ is surjective. By [9, Proposition 1.3], we have $\mathrm{ed}((\mathbb{Z}/p\mathbb{Z})^r) \leq \dim(\mathbb{G}_a) = 1$. □

Proposition 3.2. *Let p be a prime integer, F be a field of characteristic p. If $|F| \geq p^r$, then $\mathrm{ed}(\mathsf{Dec}_{p^r}) \leq r + 1$.*

Proof. Let

$$A = \otimes_{i=1}^r \big[a_i, b_i\big)_p \in \mathsf{Dec}_{p^r}(E)$$

for a field extension E/F. As $\mathrm{ed}((\mathbb{Z}/p\mathbb{Z})^r) = 1$ by Lemma 3.1, there exists a sub-extension E_0/F of E/F and $c_i \in E_0$ for all $1 \leq i \leq r$ such that $c_i \equiv a_i \mod \wp(E)$ and $\mathrm{tr.deg}_F(E_0) \leq 1$. Therefore, A is defined over $L = E_0(b_1, \cdots, b_r)$ and $\mathrm{tr.deg}_F(L) \leq r + 1$. Hence, $\mathrm{ed}(A) \leq r + 1$ and $\mathrm{ed}(\mathsf{Dec}_{p^r}) \leq r + 1$.

□

3.2. An upper bound for $\mathrm{ed}(\mathsf{Alg}_{8,2})$.

In this subsection we assume that the base field F is of characteristic 2. The upper bound 8 for $\mathrm{ed}(\mathsf{Alg}_{8,2})$ over the base field F of characteristic different from 2 was determined in [1, Theorem 2.12]. We use a similar method to find an upper bound for $\mathrm{ed}(\mathsf{Alg}_{8,2})$ over the base field F of characteristic 2.

For a commutative F-algebra R, $a \in R$ and $b \in R^\times$ we write $\big[a, b\big)_R$ for the quaternion algebra $R \oplus Ru \oplus Rv \oplus Rw$ with the multiplication table $u^2 + u = a, v^2 = b, uv = w = vu + v$. The class of $\big[a, b\big)_R$ in the Brauer group $\mathrm{Br}(R)$ will be denoted by $\big\{a, b\big\} = \big\{a, b\big\}_R$.

Let $a \in R$ and $S = R[\alpha] := R[t]/(t^2 + t + a)$ with $\alpha^2 = \alpha + a$ the quadratic extension of R. We write $N_R(a)$ for the subgroup of $R^\times$ of all element of the form $x^2 + xy + ay^2$ with $x, y \in R$, i.e., $N_R(a)$ is the image of the norm homomorphism $N_{S/R} : S^\times \to R^\times$. If $b \in N_R(a)$, then the quaternion algebra $\big[a, b\big)_R$ is isomorphic to the matrix algebra $M_2(R)$.

3.2.1. *Rowen's construction.* Rowen extended the Tignol's theorem [13] to a field of characteristic 2. We recall Rowen's argument in [11]. Let A be a central simple F-algebra in $\mathsf{Alg}_{8,2}(F)$. By [11], there is a triquadratic splitting extension $F(\alpha, \beta, \gamma)/F$ of A such that $\alpha^2 + \alpha = a, \beta^2 + \beta = b$, and $\gamma^2 + \gamma = c$ for some $a, b, c \in F$. Let $L = F(\alpha)$. By [11, Corollary 5], we have

$$\big\{A\big\}_L = \big\{b, s\big\} + \big\{c, t\big\} \tag{1}$$

in $\mathrm{Br}(L)$ for some $s, t \in L^\times$.

Taking the corestriction for the extension L/F in (1), we get

$$0 = 2\big\{A\big\} = \big\{b, N_{L/F}(s)\big\} + \big\{c, N_{L/F}(t)\big\}$$

in $\mathrm{Br}(F)$, hence $\big\{b, N_{L/F}(s)\big\} = \big\{c, N_{L/F}(t)\big\}$. By the chain lemma [11, Lemma 3], we have

$$\big\{b, N_{L/F}(s)\big\} = \big\{d, N_{L/F}(s)\big\} = \big\{d, N_{L/F}(t)\big\} = \big\{c, N_{L/F}(t)\big\}$$

in $\mathrm{Br}(F)$ for some $d \in F$. Therefore, we get $\big\{b + d, N_{L/F}(s)\big\} = \big\{c + d, N_{L/F}(t)\big\} = \big\{d, N_{L/F}(st)\big\} = 0$. By the proof of [4, Lemma 2.3],

$$\begin{aligned}\big\{b + d, s\big\} &= \big\{b + d, k\big\},\\ \big\{c + d, t\big\} &= \big\{c + d, l\big\},\\ \big\{d, st\big\} &= \big\{d, m\big\}.\end{aligned}$$

in $\mathrm{Br}(L)$ for some $k, l, m \in F^\times$. It follows from (1) that

$$\big\{A\big\}_L = \big\{b + d, k\big\}_L + \big\{c + d, l\big\}_L + \big\{d, m\big\}_L$$

in $\mathrm{Br}(L)$. Hence

$$\big\{A\big\} = \big\{a, e\big\} + \big\{b, k\big\} + \big\{c, l\big\} + \big\{d, klm\big\}$$

in $\mathrm{Br}(F)$ for some $e \in F^\times$.

We shall need the following result:

Lemma 3.3. *Let R be a commutative F-algebra, $a, b \in R$, $T = R[\alpha] := R[t]/(t^2 + t + a)$ and $x + y\alpha \in T^\times$ such that $x^2 + xy + ay^2 = u^2 + uv + bv^2$ for some $u, v \in R$. If $v + y \in R^\times$, then $(v + y)(x + y\alpha) \in N_T(b)$. In particular,*

$$\big\{b, x + y\alpha\big\}_T = \big\{b, v + y\big\}_T.$$

Proof. The result comes from the following equality

$$(x + y\alpha + u)^2 + (x + y\alpha + u)v + bv^2 = (x + y\alpha)(v + y).$$

□

3.2.2. *Classifying Azumaya algebra for* $\mathsf{Alg}_{8,2}$. Consider the affine space $\mathbb{A}^{13}_F$ with coordinates $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{m}, \mathbf{n}$ and define the rational functions:

$$\begin{aligned}
\mathbf{f} &= \mathbf{xz} + \mathbf{wz} + \mathbf{xy},\\
\mathbf{g} &= \mathbf{wy} + \mathbf{xza},\\
\mathbf{r} &= (\mathbf{g}^2 + \mathbf{gf} + \mathbf{f}^2\mathbf{a} + \mathbf{m}^2 + \mathbf{mn}),\\
\mathbf{h} &= (\mathbf{w}^2 + \mathbf{wx} + \mathbf{x}^2\mathbf{a} + 1 + \mathbf{u} + \mathbf{u}^2\mathbf{d}),\\
\mathbf{l} &= (\mathbf{y}^2 + \mathbf{yz} + \mathbf{z}^2\mathbf{a} + 1 + \mathbf{v} + \mathbf{v}^2\mathbf{d}),\\
\mathbf{p} &= (\mathbf{u} + \mathbf{x})(\mathbf{v} + \mathbf{z})(\mathbf{n} + \mathbf{f}).
\end{aligned}$$

Let X be the quasi-affine variety defined by

$$\mathbf{q} := \mathbf{abcdep}(\mathbf{w}^2 + \mathbf{wx} + \mathbf{x}^2\mathbf{a})(\mathbf{y}^2 + \mathbf{yz} + \mathbf{z}^2\mathbf{a})(\mathbf{g}^2 + \mathbf{gf} + \mathbf{f}^2\mathbf{a}) \neq 0,$$

$$\mathbf{bu}^2 = \mathbf{h}, \mathbf{cv}^2 = \mathbf{l}, \mathbf{dn}^2 = \mathbf{r}$$

i.e., $X = \mathrm{Spec}(R)$ with

$$R = F[\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{m}, \mathbf{n}, \mathbf{q}^{-1}]/ < \mathbf{bu}^2 + \mathbf{h}, \mathbf{cv}^2 + \mathbf{l}, \mathbf{dn}^2 + \mathbf{r} > .$$

Let $T = R[\alpha]$ and $S = R[\alpha, \beta, \gamma]$ with $\alpha^2 = \alpha + \mathbf{a}, \beta^2 = \beta + \mathbf{b}, \gamma^2 = \gamma + \mathbf{c}$. Consider the Azumaya R-algebra

$$\mathcal{B}' = \big[\mathbf{a}, \mathbf{e}\big)_R \otimes \big[\mathbf{b}, \mathbf{x} + \mathbf{u}\big)_R \otimes \big[\mathbf{c}, \mathbf{z} + \mathbf{v}\big)_R \otimes \big[\mathbf{d}, \mathbf{p}\big)_R. \tag{2}$$

By Lemma 3.3, we get the followings :

$$\begin{aligned}
(\mathbf{x} + \mathbf{u})(\mathbf{w} + \mathbf{x}\alpha) &\in N_T(\mathbf{b} + \mathbf{d}) \subset N_S(\mathbf{d}),\\
(\mathbf{z} + \mathbf{v})(\mathbf{y} + \mathbf{z}\alpha) &\in N_T(\mathbf{c} + \mathbf{d}) \subset N_S(\mathbf{d}),\\
(\mathbf{n} + \mathbf{f})(\mathbf{w} + \mathbf{x}\alpha)(\mathbf{y} + \mathbf{z}\alpha) &\in N_T(\mathbf{d}) \subset N_S(\mathbf{d}).
\end{aligned}$$

It follows from (2) that

$$\big\{\mathcal{B}'\big\}_T = \big\{\mathbf{b}, \mathbf{w} + \mathbf{x}\alpha\big\} + \big\{\mathbf{c}, \mathbf{y} + \mathbf{z}\alpha\big\}$$

in $\mathrm{Br}(T)$.

Since $\mathbf{p} \in N_S(\mathbf{d})$, $\big[\mathbf{d}, \mathbf{p}\big)_S$ is isomorphic to the matrix algebra $M_2(S)$. In particular,

$$M_2(R) \subset M_2(S) \simeq \big[\mathbf{d}, \mathbf{p}\big)_S \subset \mathcal{B}'$$

and hence $\mathcal{B}' \simeq M_2(\mathcal{B})$ for the centralizer $\mathcal{B}$ of $M_2(R)$ in $\mathcal{B}'$ by the proof of [7, Theorem 4.4.2]. Then $\mathcal{B}$ is an Azumaya R-algebra of degree 8 that is Brauer equivalent to $\mathcal{B}'$ by [12, Theorem 3.10].

Proposition 3.4. *The Azumaya algebra* $\mathcal{B}$ *is classifying for* $\mathsf{Alg}_{8,2}$*, i.e, the corresponding* $\mathbf{GL}_8/\boldsymbol{\mu}_2$*-torsor over* X *is classifying.*

Proof. Let $A \in \mathit{Alg}_{8,2}(K)$, where K is a field extension of F. We shall find a point $p \in X(K)$ such that $A \simeq \mathcal{B}(p)$, where $\mathcal{B}(p) := \mathcal{B} \otimes_R K$ with the F-algebra homomorphism $R \to K$ given by the point p.

Following Rowen's construction, there is a triquadratic splitting extension $K(\alpha, \beta, \gamma)/K$ of A such that $\alpha^2 + \alpha = a, \beta^2 + \beta = b$, and $\gamma^2 + \gamma = c$ for some $a, b, c \in K$. Let $L = K(\alpha)$, so

$$\{A\}_L = \{b, s\} + \{c, t\}$$

in $\mathrm{Br}(L)$ for some $s = w + x\alpha$, and $t = y + z\alpha \in L^\times$. We have

$$\{b, w^2+wx+x^2a\} = \{d, w^2+wx+x^2a\} = \{d, y^2+yz+z^2a\} = \{c, y^2+yz+z^2a\}$$

in $\mathrm{Br}(K)$ for some $d \in K$, so $\{b+d, w^2+wx+x^2a\} = \{c+d, y^2+yz+z^2a\} = \{d, (w^2+wx+x^2a)(y^2+yz+z^2a)\} = 0$. Hence

$$\begin{aligned} w^2 + wx + x^2a &= {u'}^2 + u'u + u^2(b+d), \\ y^2 + yz + z^2a &= {v'}^2 + v'u + v^2(c+d), \\ (w^2 + wx + x^2a)(y^2 + yz + z^2a) &= m^2 + mn + n^2d \end{aligned}$$

for some u, u', v, v', m, n in K. Moreover, we may assume that $u' \neq 0$. Replacing w, x and u by wu', xu' and $u'u$ respectively, we may assume that $u' = 1$. Similarly, we may assume that $v' = 1$.

We also may assume that $u \neq x$ by replacing u by $u/(b+d)$. Similarly, we may assume that $v \neq z$ and $n + xz + wz + xy \neq 0$. It follows from Lemma 3.3 that

$$\begin{aligned} \{b+d, w+x\alpha\} &= \{b+d, u+x\}_L, \\ \{c+d, y+z\alpha\} &= \{c+d, z+v\}_L, \\ \{d, (w+x\alpha)(y+z\alpha)\} &= \{d, n+xz+wz+xy\}_L \end{aligned}$$

in $\mathrm{Br}(L)$. Hence

$$\{A\} = \{a, e\} + \{b, u+x\} + \{c, z+v\} + \{d, (u+x)(z+v)(n+xz+wz+xy)\}$$

in $\mathrm{Br}(K)$ for some $e \in K^\times$.

Let p be the point $(a, b, c, d, e, u, v, w, x, y, z, m, n)$ in $X(K)$. We have $\{\mathcal{B}(p)\} = \{A\}$ and hence $\mathcal{B}(p) \simeq A$ as $\mathcal{B}(p)$ and A have the same dimension. □

Corollary 3.5. $\mathrm{ed}(\mathit{Alg}_{8,2}) \leq 10$.

Proof. There is surjective morphism $X \to \mathit{Alg}_{8,2}$ by Proposition 3.4. By [9, Proposition 1.3], $\mathrm{ed}(\mathit{Alg}_{8,2}) \leq \dim(X) = 10$. □

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