

Leibniz's Principles and Topological Extensions

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Abstract

Three philosophical principles are often quoted in connection with Leibniz: “objects sharing the same properties are the same object”, “everything can possibly exist, unless it yields contradiction”, “the ideal elements correctly determine the real things”.

Here we give a precise formulation of these principles within the framework of the Topological Extensions of [8], structures that generalize at once compactifications, completions, and nonstandard extensions. In this topological context, the above Leibniz's principles appear as a property of separation, a property of compactness, and a property of analyticity, respectively.

Abiding by this interpretation, we obtain the somehow surprising conclusion that these Leibniz's principles *can be fulfilled in pairs, but not all three together*.

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Introduction

Three philosophical principles are often quoted in connection with Leibniz:

Identity of indiscernibles

“objects sharing the same properties are the same object”

There are never in nature two beings which are perfectly identical to each other, and in which it is impossible to find any internal difference ... (*Monadology*)

Two indiscernible individuals cannot exist. [...] To put two indiscernible things is to put the same thing under two names.

(*Fourth letter to Clarke*)

[...] dans les choses sensibles on n'en trouve jamais deux indiscernables [...]¹

(*Fifth letter to Clarke*, [?], p. 132)

Possibility as consistency

“everything can possibly exist, unless it yields contradiction”

- *Impossible* is what yields an absurdity.
- *Possible* is not impossible.
- *Necessary* is that, whose opposite is impossible.
- *Contingent* is what is not necessary.

(*unpublished*, 1680 ca.)

[...] nothing is absolutely necessary, when the contrary is possible. [...]

Absolutely necessary is [...] that whose opposite yields a contradiction.

(*Dialogue between Theofile and Polydore*)

Transfer principle

“the ideal elements correctly determine the real things”

Perhaps *the infinite and infinitely small* [numbers] that we conceive *are imaginary, nevertheless* [they are] *suitable to determine the real things*, as usually do the imaginary roots. They are situated in the ideal regions, from where things are ruled by laws, even though they do not lie in the part of matter.

(*Letter to Johann Bernoulli*, 1698)

¹ among sensible things one never finds two [that are] indiscernible.

In this paper, we try and give a precise mathematical formulation of these principles in the context of the *Topological Extensions* of [8], structures which generalize at once compactifications, completions, and nonstandard models (see also [4]).

Given a set M , a *topological extension* of M is a T_1 space $*M$, where M is a dense subspace and every function $f : M \rightarrow M$ has a distinguished continuous extension $*f : *M \rightarrow *M$ that preserves compositions and local identities. The operator $*$ can be appropriately defined so as to provide also all properties P and relations R with distinguished extensions $*P, *R$ to $*M$.

Following the basic idea that the elements of the [“standard”] set M are the “real objects” of the “actual world”, whereas the extension $*M$ contains also the “ideal elements” of all “possible worlds”, an appropriate interpretation of the Leibniz’s principles in the context of topological extensions might be

Ind *different elements of $*M$ are separated by the extension $*P$ of some property P of M ;*

Poss *if the extensions of a family \mathcal{F} of properties of M are not simultaneously satisfied in $*M$, then there are finitely many properties of \mathcal{F} that are not simultaneously satisfied in M ;*

Tran *a statement involving elements, properties and relations of M is true if and only if the corresponding statement about their extensions is true in $*M$.²*

We shall see that the extended properties correspond exactly to the clopen subsets of $*M$, and so the above principles turn out to be respectively a property of *separation*, of *compactness*, and of *analyticity* of the topology of $*M$. Grounding on results of [8, 4], we obtain the somehow surprising consequence that the Leibniz’s principles *can be fulfilled in pairs, but not all three together*.

The paper is organized as follows. In Section 1, we give the precise definition of *topological extension* and we recall the main properties stated in [8]. In particular, in Subsection 1.1, we introduce the *S-topology* and we

² Clearly one has to admit only “first order” statements, so as to avoid trivial inconsistencies.

determine its connection with the principles **Ind** and **Poss**. In Subsection 1.2 we study the *canonical map* from a topological extension of X into the Stone-Čech compactification βX of the discrete space X : we obtain *inter alia* that the Stone-Čech compactification itself is essentially the *unique* topological extension that satisfies *both principles* **Ind** and **Poss**.

In Section 2, we present two simple properties that characterize all those topological extensions (*hyperextensions*) that satisfy the transfer principle **Tran**. A complete characterization of the hyperextensions satisfying also **Ind** is derived at once, and with it the *impossibility of satisfying simultaneously the three Leibniz's principles*. In Subsection 2.1 we show how to topologize arbitrary nonstandard models, so as to obtain also topological extensions where the principles **Poss** and **Tran** hold together.

A few concluding remarks and open questions, in particular the set-theoretic problems originated by the combination of **Ind** with **Tran**, can be found in the final Section 3.

In general, we refer to [10] for all the topological notions and facts used in this paper, and to [6] for definitions and facts concerning ultrapowers, ultrafilters, and nonstandard models. General references for nonstandard Analysis could be [13, 1]; specific for our “elementary” approach is [4].

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1. Topological extensions and the Identity of Indiscernibles

In this section we review the main features of the *topological extensions* introduced in the paper [8]; these structures naturally accommodate, within a general unified framework, both *Stone-Čech compactifications of discrete spaces* and *nonstandard models* (see also [4]). The most important characteristic shared by compactifications and completions in topology, and by nonstandard models of analysis is the existence of a distinguished extension $*f : *X \rightarrow *X$ for each function $f : X \rightarrow X$. Given an arbitrary set X , we consider here a topological extension of X as a sort of “topological completion” $*X$, where the “*” operator provides a *distinguished continuous extension* of each function $f : X \rightarrow X$.

Definition 1.1. The T_1 topological space $*X$ is a *topological extension* of X if X is a *discrete dense* subspace of $*X$, and a *distinguished continuous*

extension $*f : *X \rightarrow *X$ is associated to each function $f : X \rightarrow X$, so as to satisfy the following conditions:

- (c) $*g \circ *f = *(g \circ f)$ for all $f, g : X \rightarrow X$, and
- (i) if $f(x) = x$ for all $x \in A \subseteq X$, then $*f(\xi) = \xi$ for all $\xi \in \overline{A}$.

Since a finite set cannot have *nontrivial* topological extensions, we are interested only in infinite sets, and for convenience we stipulate that $\mathbb{N} \subseteq X$.

It is easily seen that the operator $*$ preserves also constant and characteristic functions (see Lemma 1.2 of [8]). So, by using the characteristic functions, the operator $*$ provides also an *extension* $*A$ for every *subset* $A \subseteq X$, which turns out to be a *clopen* superset of A , and actually the *closure* \overline{A} of A in $*X$.

Notice that, if the topological extension $*X$ of X is Hausdorff, then $*f$ is the *unique* continuous extension of f , because X is dense. Therefore properties (c) and (i) are automatically satisfied (see [3], where Hausdorff topological extensions have been introduced and studied). However considering only Hausdorff spaces would have turned out too restrictive: we shall see below that the Hausdorff topological extensions of X are particular subspaces of the Stone-Ćech compactification βX of the discrete space X that, in general, are not nonstandard extensions. In fact, the existence of Hausdorff nonstandard extensions, although consistent, has not yet been proved in ZFC alone (see [8, 9]) and Section 3 below. These are the reasons why we only require that topological extensions be T_1 spaces.

1.1. The S -topology

In order to study our versions of the Leibniz's principles for topological extensions, it is useful to consider on $*X$ the so called *S -topology*,³ *i.e.* the topology generated by the (clopen) sets $*A = \overline{A}$ for $A \subseteq X$. The S -topology is obviously *coarser than* or *equal to* the original topology of $*X$, and we can characterize the respective *separation properties* as in Theorem 1.4 of [8]:

Theorem 1.2. *Let $*X$ be a topological extension of X . Then*

1. *The S -topology of $*X$ is either 0-dimensional or not T_0 .*

³ The S -topology (for *Standard* topology) is a classical notion of nonstandard Analysis, already considered since [15].

2. $*X$ is Hausdorff if and only if the S -topology is T_1 , hence 0-dimensional.
3. $*X$ is regular if and only if the S -topology is the topology of $*X$ (and so $*X$ is 0-dimensional).

Proof.

1. The S -topology has a clopen basis by definition. In this topology the closure of a point ξ is $M_\xi = \bigcap_{\xi \in \bar{A}} \bar{A}$. If $M_\xi = \{\xi\}$ for all $\xi \in *X$, then the S -topology is T_1 , hence 0-dimensional. Otherwise let $\eta \neq \xi$ be in M_ξ . Then η belongs to the same clopen sets as ξ , and the S -topology is not T_0 . In fact, given $A \subseteq X$, $\xi \in \bar{A}$ implies $\eta \in \bar{A}$, by the choice of η . Similarly $\xi \notin \bar{A}$ implies $\eta \in X \setminus \bar{A}$, hence $\eta \in X \setminus A$ and $\eta \notin \bar{A}$.

2. By point 1, the S -topology is Hausdorff (in fact 0-dimensional) whenever it is T_1 . Therefore also the topology of $*X$ is Hausdorff, being finer than the S -topology. For the converse, let U, V be disjoint neighborhoods of the points $\xi, \eta \in *X$, and put $A = U \cap X$, $B = V \cap X$. Then $\xi \in \bar{A}$, $\eta \in \bar{B}$, and $\bar{B} \cap \bar{A} = \emptyset$. Therefore $\eta \notin M_\xi$, and the S -topology is T_1 .

3. The closure of an open subset $U \subseteq *X$ is the clopen set $\overline{U \cap X}$. Therefore any closed neighborhood of $\xi \in *X$ includes a clopen one. Since the clopen sets are a basis of the S -topology, $*X$ can be regular if and only if its original topology is the S -topology (and so the latter is T_1 , hence 0-dimensional). □

Now the principle **Ind** simply means that the S -topology of $*X$ is Hausdorff. On the other hand, the principle **Poss** states that every proper filter of clopen sets has nonempty intersection, *i.e.* that the S -topology of $*X$ is quasi-compact.⁴ So we have

Corollary 1.3. *Let $*X$ be a topological extension of X . Then*

1. *the principle **Ind** holds if and only if $*X$ is Hausdorff;*
2. *the principle **Poss** holds if and only if the S -topology of $*X$ is quasi-compact;*
3. *both principles **Ind** and **Poss** hold in $*X$ if and only if the S -topology of $*X$ is compact. So either $*X$ is compact, or it is not regular, but becomes compact by suitably weakening its topology, still maintaining all functions $*f$ continuous.*

⁴ Following [10], we call compact only Hausdorff spaces.

Proof. We only need to prove the last assertion of Point 3. If $*X$ is regular, then its topology agrees with the S -topology. If not, then all functions $*f$ are continuous also with respect to the coarser S -topology, because the inverse images of clopen sets are clopen. □

1.2. The canonical map and the principle **Ind**

Any topological extension of X is canonically mappable into the Stone-Čech compactification βX of the discrete space X .⁵

If X is a discrete space, identify βX with the set of all ultrafilters over X , endowed with the topology having as basis $\{\mathcal{O}_A \mid A \subseteq X\}$, where \mathcal{O}_A is the set of all ultrafilters containing A . So the embedding $e : X \rightarrow \beta X$ is given by the principal ultrafilter

$$e(x) = \{A \subseteq X \mid x \in A\},$$

and the unique continuous extension $\bar{f} : \beta X \rightarrow \beta X$ of $f : X \rightarrow X$ can be defined by putting

$$\bar{f}(\mathcal{U}) = \{A \subseteq X \mid f^{-1}(A) \in \mathcal{U}\}.$$

Given a topological extension $*X$ of X and a point $\xi \in *X$, put

$$\mathcal{U}_\xi = \{A \subseteq X \mid \xi \in *A\},$$

which is an ultrafilter over X , and define the *canonical map*

$$v : *X \rightarrow \beta X \text{ by } v(\xi) = \mathcal{U}_\xi.$$

Then we can reformulate Theorem 2.1 of [8] in terms of the Leibniz's principles **Ind** and **Poss**. Namely

Theorem 1.4. *Let $*X$ be a topological extension of X , and let βX be the Stone-Čech compactification of X . Then*

1. *The canonical map $v : *X \rightarrow \beta X$ is the unique continuous extension to $*X$ of the embedding $e : X \rightarrow \beta X$, and*

$$v \circ *f = \bar{f} \circ v \text{ for all } f : X \rightarrow X.$$

⁵ For various definitions and properties of the Stone-Čech compactification see [10].

2. The map v is injective if and only if **Ind** holds in $*X$.
3. The map v is surjective if and only if **Poss** holds in $*X$.

Proof.

1. For all $x \in X$, \mathcal{U}_x is the principal ultrafilter generated by x , hence v induces the canonical embedding of X into βX . If \mathcal{O}_A is a basic open set of βX , then $v^{-1}(\mathcal{O}_A) = \overline{A}$, hence v is continuous w.r.t. the S -topology, and *a fortiori* w.r.t. the (not coarser) topology of $*X$. On the other hand, let a continuous map $\varphi : *X \rightarrow \beta X$ be given. Since \mathcal{O}_A is clopen, also $\varphi^{-1}(\mathcal{O}_A)$ is clopen and so it is the closure \overline{B} of some $B \subseteq X$. If φ is the identity on X , then $\overline{B} \cap X = A$, hence $B = A$ (see Lemma 1.2 of [8]). Therefore all points of $M_\xi = \{\eta \in *X \mid \forall A \subseteq X (\xi \in *A \implies \eta \in *A)\}$ are mapped by φ onto $v(\xi)$, and so $v = \varphi$.

Moreover, for all $\xi \in *X$, one has $\xi \in \overline{A} \Leftrightarrow *f(\xi) \in \overline{f(A)}$, or equivalently $A \in \mathcal{U}_\xi \Leftrightarrow f(A) \in \mathcal{U}_{*f(\xi)}$ (see Lemma 1.3 of [8]); hence $\overline{f} \circ v = v \circ *f$, and Poiny 1. is completely proved.

2. The map v is injective if and only if the S -topology is T_1 , and this fact is equivalent to $*X$ being Hausdorff, by Theorem 1.2, or to **Ind**, by Corollary 1.3. Moreover in this case v is a homeomorphism w.r.t. the S -topology, which is the same as the topology of $*X$ if and only if the latter is regular (hence 0-dimensional).

3. The map v is surjective if and only if every maximal filter in the field of all clopen sets of $*X$ has nonempty intersection. This is equivalent to every proper filter having nonempty intersection, which in turn is equivalent to the S -topology of $*X$ being quasi-compact, i.e. to the principle **Poss**, by Corollary 1.3. □

Notice that the map v induces a bijection between the basic open sets \mathcal{O}_A of βX and the clopen subsets $*A$ of $*X$. Therefore v is open if and only if $*X$ has the S -topology.

Call *invariant* a subspace Y of $*X$ (respectively of βX) if

$$*f(\xi) \in Y \text{ (resp. } \overline{f}(\xi) \in Y) \text{ for all } f : X \rightarrow X \text{ and all } \xi \in Y.$$

It is easily seen that any invariant subspace Y of $*X$ is itself a topological extension of X , and it is mapped by v onto an invariant subspace of βX . If $*X$ is homeomorphic to a subspace of βX , then it is 0-dimensional, hence it has the S -topology, by Theorem 1.2. Conversely, if $*X$ has the S -topology,

then v is injective. Moreover, for all $A \subseteq X$, $v(A) = \mathcal{O}_A \cap v(*X)$, hence v is a homeomorphism between $*X$ and its image. On the other hand, if $*X$ is Hausdorff but not regular, then v is injective and continuous, but not open.

Whenever $*X$ verifies the principle **Ind**, the map v can always be turned into a homeomorphism, either by endowing $v(*X)$ with a suitably finer topology, or by taking on $*X$ the (coarser) S -topology. So any such extension makes use of *the same “function-extending mechanism”* as the Stone-Čech compactification. Moreover, if also **Poss** holds, then $*X$ can be taken to be βX itself, possibly endowed with a suitably finer topology.

More precisely, the above discussion provides the same characterization of all topological extensions satisfying the principle **Ind** that has been given in Corollary 2.2 of [8], namely:

Corollary 1.5. *A topological extension $*X$ of X satisfies **Ind** if and only if the canonical map v provides a continuous bijection (a homeomorphism when $*X$ is regular) onto an invariant subspace of βX .*

*Moreover $*X$ satisfies also **Poss** if and only if v is onto βX . □*

2. Topological hyperextensions and the Transfer Principle

The Transfer Principle **Tran** is the very ground of the usefulness of the nonstandard methods in mathematics. It allows for obtaining correct results about, say, the real numbers by using ideal elements like actual infinitesimal or infinite numbers.

In fact, both properties (c) and (i) of Definition 1.1 are instances of the transfer principle, for they correspond to the statements

$$\forall x \in X . f(g(x)) = (f \circ g)(x) \quad \text{and} \quad \forall x \in A . f(x) = x,$$

respectively. So all topological extensions already satisfy several important cases of the transfer principle. *E.g.*, if f is constant, or injective, or surjective, then so is $*f$. More important, we have already used the fact that the extension of the characteristic function of any subset $A \subseteq X$ is the characteristic function of the closure \overline{A} of A in $*X$, thus we can put $*A = \overline{A}$ and obtain a Boolean isomorphism between the field $\mathcal{P}(X)$ of all subsets of X and the field $\mathcal{C}\ell(*X)$ of all clopen subsets of $*X$ (see [8], Lemmata 1.2 and 1.3).

On the other hand, many basic cases of the transfer principle may fail, because topological extensions comprehend, besides nonstandard models, also

all invariant subspaces of the Stone-Ćech compactifications of discrete spaces. In order to obtain the full principle **Tran**, we postulated in [8] two additional properties, namely

Definition 2.1. *The topological extension $*X$ of X is a hyperextension⁶ if*

- (a) *for all $f, g : X \rightarrow X$*
 $f(x) \neq g(x)$ for all $x \in X \iff *f(\xi) \neq *g(\xi)$ for all $\xi \in *X$;
- (p) *there exist $p, q : X \rightarrow X$ such that*
*for all $\xi, \eta \in *X$ there exists $\zeta \in *X$ such that $\xi = *p(\zeta)$ and $\eta = *q(\zeta)$.*

The property (a), called *analyticity* in [8], isolates a fundamental feature that marks the difference between *nonstandard* and *ordinary continuous* extensions of functions: “*disjoint functions have disjoint extensions*”. It is obtained by **Tran** from the statement $\forall x \in X . f(x) \neq g(x)$, and it can be viewed as the *empty set case* of a general “*principle of preservation of equalizers*”:

- (e) $\{\xi \in *X \mid *f(\xi) = *g(\xi)\} = *\{x \in X \mid f(x) = g(x)\}$.

The property (p), called *coherence* in [8], provides a sort of “internal coding of pairs”, useful for extending multivariate functions “parametrically”: this possibility is essential in order to get the full principle **Tran**, which involves relations of any arities.⁷ Notice that the property (p) could seem *prima facie* an illegal instance of the *Transfer Principle*, for it is given in a second order formulation. On the contrary, a *strong uniform version* of that property can be obtained by fixing p, q as the compositions of a given bijection $\delta : X \rightarrow X \times X$ with the ordinary projections $\pi_1, \pi_2 : X \times X \rightarrow X$, and then applying **Tran** to the statement

$$\forall x, y \in X . \exists z \in X . p(z) = x, q(z) = y.$$

⁶ Topological hyperextensions are in fact *hyper-extensions* in the sense of [4] (*i.e.* non-standard models), by Theorem 2.2 below.

⁷ The *ratio* of considering only unary functions lies in the following facts that hold in every topological hyperextension $*X$ of X (see Section 5 of [8]):

- For all $\xi_1, \dots, \xi_n \in *X$ there exist $p_1, \dots, p_n : X \rightarrow X$ and $\zeta \in *X$ such that $*p_i(\zeta) = \xi_i$.
- If $p_1, \dots, p_n, q_1, \dots, q_n : X \rightarrow X$ and $\xi, \eta \in *X$ satisfy $*p_i(\xi) = *q_i(\eta)$, then

$$*(F \circ (p_1, \dots, p_n))(\xi) = *(F \circ (q_1, \dots, q_n))(\eta) \text{ for all } F : X^n \rightarrow X.$$

It follows that there is a unique way of assigning an extension $*F$ to every function $F : X^n \rightarrow X$ in such a way that all compositions are preserved. By using the characteristic functions in n variables one can assign an extension $*R$ also to all n -ary relations R on X .

We shall see below that there are invariant subspaces of the Stone-Čech compactification βX where (a) holds whereas (p) fails and *vice versa*, as well as invariant subspaces where both fail or hold. So the properties (a) and (p) are independent, also when **Ind** holds.

We consider very remarkable the fact that the combination of *four natural, simple instances* of the transfer principle, like (c), (i), (a), and (p), gives to topological hyperextensions the strongest *Transfer Principle Tran*. In reason of its importance, we have already given three different proofs of this fact in preceding papers of ours: a “logical” and a “logico-algebraic” proof in [8], and a “purely algebraic” proof in [11] (see also the survey in [4]). So we state here without proof the following theorem:

Theorem 2.2. *A topological extension $*X$ of X satisfies the principle **Tran** if and only if it is a hyperextension.* \square

We are now able to characterize all topological extensions satisfying both principles **Ind** and **Tran**. These extensions are spaces of ultrafilters, according to Corollary 1.5. So we use the reformulation in terms of ultrafilters given in [8] for the condition (e) above.

Call an ultrafilter \mathcal{U} on X *Hausdorff*⁸ if, for all $f, g : X \rightarrow X$,

$$(H) \quad \overline{f}(\mathcal{U}) = \overline{g}(\mathcal{U}) \iff \{x \in X \mid f(x) = g(x)\} \in \mathcal{U}.$$

Call *directed* a subspace Y of βX where the property (p) holds, *i.e.* there exist $p, q : X \rightarrow X$ such that for all $\mathcal{U}, \mathcal{V} \in Y$ there exists $\mathcal{W} \in Y$ such that $\mathcal{U} = \overline{p}(\mathcal{W})$ and $\mathcal{V} = \overline{q}(\mathcal{W})$.

By combining Theorem 2.2 with Corollary 1.5 we obtain

Theorem 2.3. *A topological extension $*X$ of X satisfies both principles **Ind** and **Tran** if and only if the canonical map v is a continuous bijection between $*X$ and a directed invariant subspace of βX that contains only Hausdorff ultrafilters.*

*Moreover v is a homeomorphism if and only if $*X$ has the S -topology, or equivalently is regular.* \square

Now it is easy to show that the properties (a) and (p) are independent.

⁸ The property (H) has been introduced in [7] under the name (C). Hausdorff ultrafilters are studied in [9] and [2].

For $\mathcal{U} \in \beta X$ let $Y_{\mathcal{U}} = \{\overline{f(\mathcal{U})} \mid f : X \rightarrow X\}$ be the invariant subspace generated by \mathcal{U} . Clearly $Y_{\mathcal{U}}$ is directed, so **(p)** holds for all ultrafilters \mathcal{U} , whereas **(a)** holds if and only if \mathcal{U} is Hausdorff. On the other hand, let \mathcal{U} and \mathcal{V} be Hausdorff ultrafilters such that neither of them belongs to the invariant subspace generated by the other one: then $Y_{\mathcal{U}} \cup Y_{\mathcal{V}}$ is an invariant subspace where **(a)** holds, but it is not directed, hence **(p)** fails.

We shall deal in the final section with the set theoretic strength of the combination of **Ind** with **Tran**. By now we simply recall that there are plenty of non-Hausdorff ultrafilters (e.g. all diagonal tensor products $\mathcal{U} \otimes \mathcal{U}$). Thus we can easily conclude

Corollary 2.4. *No topological extension satisfies at once the three Leibniz's principles **Ind**, **Poss**, and **Tran**.* □

2.1. The star topology

We are left with the task of combining **Poss** with **Tran**. To this aim we recall that a nonstandard model whose S -topology is quasi-compact is commonly called *enlargement*. It is well known that every structure has arbitrarily saturated elementary extensions (see e.g. [6]), and obviously a $2^{|X|^+}$ -saturated extension of X is an enlargement (see e.g. [1] or [4]). So, if we can topologize every nonstandard extension of X in such a way that all functions $*f$ become continuous, then we get a lot of topological hyperextensions satisfying **Poss**.

This task has been already accomplished in [8], where the *coarsest* such topology is defined.

Every topological extension $*X$ should be a T_1 space, so all sets of the form $E(f, \eta) = \{\xi \in *X \mid *f(\xi) = \eta\}$, for $f : X \rightarrow X$ and $\eta \in *X$, should be closed in $*X$. The (arbitrary) intersections of finite unions of such sets are the *closed* sets of a topology, called the *Star topology*, which is by construction the *coarsest* T_1 topology on $*X$ that makes all functions $*f$ continuous.

When $*X$ is a nonstandard extension of X , the four defining properties **(c)**, **(i)**, **(a)**, and **(p)** of topological hyperextensions are fulfilled by hypothesis. So one has only to prove that X is dense in $*X$ in order to obtain

Theorem 2.5 (Theorem 3.2 of [8]). *Any nonstandard extension $*X$ of X , when equipped with the Star topology, becomes a topological hyperextension of X . Conversely, any topological hyperextension $*X$ of X is a nonstandard extension, possibly endowed with a topology finer than the Star topology.*

Proof. We have only to prove that X is dense.

If $X \subseteq \bigcup_{1 \leq i \leq n} E(f_i, \eta_i)$, then we may consider w.l.o.g. only those components with $\eta_i \in X$, because each $*f_i$ maps any point $x \in X$ to the point $f_i(x) \in X$. Hence the *Transfer Principle* of the nonstandard extensions may be applied to the statement

$$\forall x \in X (f_1(x) = \eta_1 \vee \dots \vee f_n(x) = \eta_n),$$

thus producing

$$\forall \xi \in *X (*f_1(\xi) = \eta_1 \vee \dots \vee *f_n(\xi) = \eta_n).$$

So the whole space $*X$ is included in $\bigcup_{1 \leq i \leq n} E(f_i, \eta_i)$, and X is dense. □

So, in order to get topological extensions satisfying both principles **Poss** and **Tran**, we have only to put the *star topology* on any *nonstandard enlargement* of X .

3. Final remarks and open questions

We have seen that (at least) one of the three principles that we have investigated has to be left out. The most reasonable choice seems to be that of dropping **Ind**. In fact, even if one neglects the set theoretic problems that will be outlined below, one should pay attention to Leibniz himself.

[...] cette supposition de deux indiscernables [...] paroist possible en termes abstraits, mais elle n'est point compatible avec l'ordre des choses [...]

Quand je nie qu'il y ait [...] deux corps indiscernables, je ne dis point qu'il soit impossible absolument d'en poser, mais que c'est une chose contraire la sagesse divine [...]

Les parties du temps ou du lieu [...] sont des choses ideales, ainsi elles se rassemblent parfaitement comme deux unités abstraites. Mais il n'est pas de même de deux Uns concrets [...] c'est à dire veritablement actuels.

Je ne dis pas que deux points de l’Espace sont un meme point,
ny que deux instans du temps sont un meme instant comme il
semble qu’on m’impute [...] ⁹

(*Fifth letter to Clarke*, [14], pp. 131-135)

It appears that Leibniz considered the identity of indiscernibles as a “physical” rather than a “logical” principle: it is actually true, but its negation is non-contradictory in principle, so *it could fail in some possible world*. Moreover only “properties of the real world” M are considered in all these principle: so it seems natural, and not absurd, to assume that objects indiscernible by these “real” properties may be separated by some abstract, “ideal” property of $*M$.

On this ground we finally decide to call *Leibnizian* a topological extension that satisfies both **Poss** and **Tran**, and so necessarily not **Ind**. Thus the existence of plenty of *Leibnizian extensions* is granted by the final results of Section 2, without any need of supplementary set theoretic hypotheses.

3.1. Existence of Hausdorff extensions

As shown by Theorem 2.3, combining **Ind** with **Tran** requires special ultrafilters, named *Hausdorff* in Section 2. Despite the apparent weakness of their defining property (H), which is actually true whenever any of the involved functions is injective (or constant), not much is known about Hausdorff ultrafilters.

On *countable* sets, the property (H) is satisfied by *selective* ultrafilters as well as by *products of pairwise nonisomorphic selective* ultrafilters (see [9]), but their existence in pure ZFC is still unproved. However any hypothesis providing infinitely many nonisomorphic selective ultrafilters over \mathbb{N} , like the Continuum Hypothesis CH or Martin Axiom MA, provides infinitely many non-isomorphic hyperextensions of \mathbb{N} that satisfy **Ind**.

⁹... this supposition of two indiscernibles ... seems abstractly possible, but it is incompatible with the order of things...

When I deny that there are ... two indiscernible bodies, I do not say that [this existence] is absolutely impossible to assume, but that it is a thing contrary to Divine Wisdom ...

The parts of time or place ... are ideal things, so they perfectly resemble like two abstract unities. But it is not so with two concrete Ones,... that is truly actual [things].

I don’t say that two points of Space are one same point, neither that two instants of time are one same instant as it seems that one imputes to me ...

On *uncountable* sets the situation is highly problematic: it is proved in [9] that Hausdorff ultrafilters on sets of size not less than \mathfrak{u} cannot be *regular*.¹⁰ In particular, the existence of a hyperextension satisfying **Ind** with uniform ultrafilters, even on \mathbb{R} , would imply that of inner models with measurable cardinals. (To be sure, such ultrafilters have been obtained only by much stronger hypotheses, see [12]).

Be it as it may, as far as we do not abide **ZFC** as our foundational theory, *we cannot prove that hyperextensions without indiscernibles exist at all.*

3.2. Some open questions

We conclude this paper with a few open questions that involve special ultrafilters, and so should be of independent set theoretic interest.

1. Is the existence of topological hyperextensions of \mathbb{N} without indiscernibles provable in **ZFC**, or at least derivable from set-theoretic hypotheses weaker than those providing selective ultrafilters? *E.g.* from $\mathfrak{r} = \mathfrak{c}$, where \mathfrak{r} is a cardinal invariant of the continuum not dominated by $\mathfrak{cov}(\mathcal{B})$?
2. Is it consistent with **ZFC** that there are nonstandard real lines ${}^*\mathbb{R}$ without indiscernibles where all ultrafilters are uniform?
3. Is the existence of countably compact hyperextensions consistent with **ZFC**? (These extensions would be of great interest, because they would verify **Ind**, **Tran**, and the weakened version of **Poss** that considers only *sequences* of properties.)

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¹⁰ \mathfrak{u} is the least size of an ultrafilter basis on \mathbb{N} . All what is provable in **ZFC** about the size of \mathfrak{u} is that $\aleph_1 \leq \mathfrak{u} \leq 2^{\aleph_0}$ (see e.g. [5]).

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