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Slow time behavior of the semidiscrete Perona-Malik scheme in dimension one

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Abstract

We consider the long time behavior of the semidiscrete scheme for the Perona-Malik equation in dimension one. We prove that approximated solutions converge, in a slow time scale, to solutions of a limit problem. This limit problem evolves piecewise constant functions by moving their plateaus in the vertical direction according to a system of ordinary differential equations.

Our convergence result is *global-in-time*, and this forces us to face the collision of plateaus when the system singularizes.

The proof is based on energy estimates and gradient-flow techniques, according to the general idea that "the limit of the gradient-flows is the gradient-flow of the limit functional". Our main innovations are a uniform Hölder estimate up to the first collision time included, a well preparation result with a careful analysis of what happens at discrete level during collisions, and renormalizing the functionals after each collision in order to have a nontrivial Gamma-limit for all times.

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1 Introduction

The one dimensional Perona-Malik equation is the partial differential equation

$$u_t = \left(\frac{u_x}{1+u_x^2}\right)_x = \frac{1-u_x^2}{(1+u_x^2)^2}\, u_{xx} \qquad (x,t) \in (0,1) \times (0,+\infty), \tag{1.1}$$

which is usually coupled with Neumann boundary conditions

$$u_x(0,t) = u_x(1,t) = 0 \qquad \forall t > 0, \tag{1.2}$$

and an initial condition

$$u(x,0) = u_0(x) \qquad \forall x \in (0,1). \tag{1.3}$$

Problem (1.1), (1.2), (1.3) is the formal gradient-flow of the functional

$$PM(u) := \frac{1}{2}\int_0^1 \log\left(1+u_x^2\right)\, dx. \tag{1.4}$$

The convex-concave behavior of the integrand in (1.4) makes (1.1) a parabolic equation of forward-backward type, with forward (or subcritical) regime in the region where $|u_x| < 1$, and backward (or supercritical) regime in the region where $|u_x| > 1$.

The analogous problem in two space dimensions was introduced by P. Perona and J. Malik [18] in the context of image denoising. The rough idea is that small disturbances, corresponding to small values of the gradient, are expected to be smoothed out by the diffusion in forward regions. On the contrary, sharp edges should be enhanced by the backward nature of the equation in regions where the gradient is large.

This intuition has actually been confirmed by numerical experiments. There are some well known shortcomings, such as the staircasing effect observed in supercritical regions, but nevertheless the method reveals some stability, and in any case much more stability than expected from a backward diffusion process.

Equation (1.1) is the prototype of all forward-backward parabolic equations such as $u_t = (\varphi'(u_x))_x$, where φ is a nonconvex integrand. It is also strongly related to forward-backward parabolic equations of the form $u_t = (\phi(u))_{xx}$, where ϕ is a nonmonotone response function (indeed this is the equation solved by the derivative u_x of solutions u of (1.1)). Such equations attracted a considerable attention in the last years because they are involved in several models, from phase transitions to population dynamic (see [19] and the references quoted therein).

A natural approach to an ill-posed problem is to approximate it by more stable ones. Following this idea, several authors proved well posedness results for approximations of (1.1) obtained via space discretization [12] or convolution [9], time delay [2], fractional derivatives [16, 17], fourth order regularization [3], simplified nonlinearities [4]. A satisfactory understanding of what happens as the suitable parameter vanishes still seems to be out of reach.

Several papers reported numerical experiments on the Perona-Malik equation in dimension one or two. We refer in particular to [2, 6, 10, 11, 16, 17]. All these experiments, although obtained through different approximation methods, seem to reveal some common *qualitative* features. In particular, the evolution seems to happen in three different times scales. We call them "fast time", "standard time", and "slow time", according to the terminology introduced in [3].

- *Fast time.* In a time interval of order $o(1)$, solutions of approximated problems tend to develop microstructures in the concave region, with fast oscillations between very small and very large values of the derivative. This is the staircasing effect, which causes an instantaneous drastic reduction of the energy in the backward regime. From the variational point of view this is hardly surprising, due to the concavity of the integrand in that region. More surprising is that this effect does not extend immediately to the forward regime, as it could be expected after remarking that the relaxation of (1.4) is trivially zero.

  Up to our knowledge, there is no rigorous treatment of this phenomenon. On the other hand, the existence of dense classes of smooth solutions of (1.1) (see [13, 14]) suggests that it is not reasonable to expect the staircasing effect for *all* initial data with both subcritical and supercritical regions, but at most for "generic" such data. This remains a challenging open problem.

- *Standard time.* In a time interval of order $O(1)$, solutions of approximated problems evolve in order to reduce the energy in the convex region. Rigorous results in this time scale are known only for the semidiscrete scheme in dimension one (in this paper semidiscrete means discrete with respect to space, and continuous with respect to time). In [12] a compactness result was proven, according to which solutions of approximated problems converge to something (as the size of the grid goes to zero), and all possible limits are classical solutions of (1.1) in the subcritical region of u_0.

  The characterization of such limits in supercritical regions remains an open problem, as well as any compactness result for different approximation methods or in more space dimensions.

- *Slow time.* After the second phase of the evolution, the energy has been reduced almost to zero, and the solution is close to a piecewise constant function. This is consistent with the intuitive idea that piecewise constant functions are stationary points of $PM(u)$. Since there is almost no energy left, the evolution slows down.

  Nevertheless, in a slower time scale the plateaus of this piecewise constant function tend to move in the vertical direction, with jump points which remain fixed in space. The vertical dynamic is nontrivial because neighboring plateaus can collide, and actually do collide in a finite time. After each collision at least one discontinuity point disappears, and the evolution proceeds as soon as the solution becomes a constant and there is nothing else to evolve.

The aim of this paper is a rigorous analysis of the slow time for the semidiscrete scheme in dimension one (we refer to section 2.3 for precise definitions). The first three steps in this direction were done by G. Bellettini, M. Novaga and M. Paolini in [5]. First of all, they identified the right time-scale, which turns out to be of the same order of the inverse of the grid size (namely $O(n)$ if the grid size is $1/n$). Secondly, they identified the system of ordinary differential equations describing the evolution of the plateau heights in the limit problem. Finally, they proved that the rescaled solutions of the semidiscrete scheme converge to the limit evolution described by that system in the half-open interval $[0, T_{\text{sing}})$, where T_{sing} is the life span of the solution of the system.

The proof of their convergence result is based on the construction of suitable subsolutions and supersolutions, suggested by a formal development of approximating solutions. This method reveals some drawbacks. First of all, it requires some heavy computations, which in [5] are carried out at the expenses of choosing a simplified form of the nonlinearity, a very special sequence of initial data, and Dirichlet boundary conditions. More important, it seems quite hard to extend these arguments beyond T_{sing}, namely when the interaction of plateaus makes the dynamic highly nontrivial.

In this paper we overcome these difficulties, and we prove a *global-in-time* convergence result (Theorem 3.1). Of course the limit problem is defined by restarting the evolution after each collision according to the same rule applied to the new (smaller) set of plateaus.

Before restarting the evolution, it is however necessary to prove that it can be extended up to T_{sing} (included). This fact is quite intuitive, and indeed it has been implicitly mentioned (but not proved) in [5], when the authors say that the system singularizes due to collisions, and not to more strange phenomena. On the other hand, the possibility that more than two neighboring plateaus collide in the same time makes this issue nontrivial. We overcome this difficulty by proving a $1/4$-Hölder estimate up to T_{sing} (see Proposition 2.1, and Proposition 4.3 for the corresponding estimate at discrete level).

Then we pass to our convergence result, inspired by the general principle that "the limit of gradient-flows is the gradient-flow of the limit". Since approximating solutions are gradient-flows of rescaled approximations of (1.4), it is reasonable to expect the limit of the evolutions to be the gradient-flow of the limit energy (in the sense of Gamma-convergence). Unfortunately, if we want the limit energy to be finite, we are forced to fix a priori the number of discontinuities, and we are back to the interval $[0, T_{\text{sing}})$.

The first idea is therefore to renormalize the energy after each collision. If we add a constant (depending on the grid size) to each approximated energy, then the approximated gradient-flows do not change, but the limit energy can be different. This allows to iterate the convergence result after each collision, provided that we arrive up to T_{sing} (included) with approximating solutions which remain "well prepared", namely close enough in many senses to the continuous limit.

To this end we develop two main tools. The first one is a well preparation result (Proposition 4.1). When a discontinuity disappears in the continuous limit, then in the corresponding interval of the grid the discrete derivative of approximating solutions crosses the critical threshold, switching from the concave region to the convex one, and instantaneously the discrete solution becomes a well prepared approximation of the new piecewise constant function with a smaller set of plateaus. The second tool (Proposition 4.2) is a convergence result up to the first collision time (included), which by itself improves the convergence result of [5]. We prove it by rewriting both the approximating problems, and the limit problem, in terms of *integral inequalities* instead of differential equations. This formulation, inspired by the theory of *maximal slope curves* (introduced in [8], see [1] for a modern presentation), happens to be much more stable when passing to the limit.

Our techniques work with general nonlinearities (we only need the convex-concave behavior of the integrand), general sequences of initial data (we do not even assume the boundedness of the energy), and general boundary conditions (we work with Neumann boundary conditions because this is the natural choice in applications, but the same arguments apply to Dirichlet or periodic boundary conditions).

Of course several problems remain open. Apart from the notorious questions concerning fast time and standard time, it could be interesting to prove similar results for the slow time in higher dimension, or again in dimension one but with different approximation methods.

A partial contribution in this direction is due to G. Bellettini and A. Fusco [3]. They considered a fourth order regularization of (1.1), corresponding to adding a vanishing second order term to (1.4). They identified the time-scale of slow time, they computed the Gamma-limit of the rescaled energies, and they conjectured that the limit problem is the gradient-flow of the limit energy. Unfortunately in that case this remains a conjecture, since up to now no convergence result (even before collisions) is known.

The limit conjectured in [3] evolves once again piecewise constant functions, but the law of the vertical motion is in their case different (the system of ordinary differential equations is similar, but with different exponents). This suggests two remarks. On the one hand the existence of a slow time vertical motion is a qualitative feature which is intrinsic in the nature of (1.1). On the other hand, what exactly happens in the slow time from the quantitative point of view does depend on the approximation method.

This paper is organized as follows. In Section 2 we introduce the rescaled semidiscrete scheme, the variational setting, and the limit problem. We also recall the previous results which are needed throughout this paper. In Section 3 we state our main results. In Section 4 we present the basic tools of our analysis. In Section 5 we collect all proofs.

2 Notation and definitions

2.1 Functional spaces

Continuous setting The more general ambient space we consider is $L^2((0,1))$, shortened to L^2 when it is clear that we are working in the interval $(0,1)$. We write $\|u\|_{L^p((0,1))}$, or simply $\|u\|_p$, to denote the p-norm ($p \in [0,+\infty]$) of a function u, and $\langle u, v\rangle$ to denote the scalar product of the functions u and v in the appropriate L^2 space.

Let $D \subseteq (0,1)$ be a finite set, and let $k := |D|$. The elements of D divide $(0,1)$ into

$(k+1)$ subintervals.

We call PC_D the space of functions which are constant in each subinterval, with the agreement that the constant values in any two neighboring subintervals are different. In other words, elements of PC_D are *piecewise constant* functions with exactly k jump points located in the discontinuity set D.

We call PS_D the space of functions which are Lipschitz continuous in each subinterval, with Lipschitz constant less than or equal to 1, with the agreement that for each $d \in D$ the limit as $x \to d^-$ is different from the limit as $x \to d^+$. In other words, elements of PS_D are *piecewise subcritical* functions with exactly k jump points located in the discontinuity set D. For every $u \in PS_D$, and every $d \in D$, the *jump height* of u in d is defined as

$$J_d(u) := \lim_{x \to d^+} u(x) - \lim_{x \to d^-} u(x). \tag{2.1}$$

It is easy to see that $PC_D \subseteq PS_D \subseteq L^2$. Every element of PC_D is uniquely determined by the heights of its $(k+1)$ plateaus. This correspondence defines an isometry between PC_D and an open subset of a Euclidean space of dimension $(k+1)$. When needed, we assume that elements of PC_D and PS_D are defined in the jump points in such a way that they are right-continuous.

Discrete setting Given a positive integer n, we divide $[0,1]$ into n intervals of length $1/n$, and we consider the space PC_n of all functions which are constant in each subinterval (in this case constants in neighboring subintervals may be equal). The space PC_n, when endowed with the L^2-norm inherited as a subset of L^2, becomes a Euclidean space isomorphic to $\mathbb{R}^n$. Since elements of PC_n are thought as L^2 functions, it is not so essential to define them also in points of the form i/n (with $i = 0, 1, \ldots, n$). In any case, when needed we assume that the value in any of these points is the same as in the interval on its right (on its left in the case $i = n$).

Given $u \in PC_n$, the *discrete derivatives* $D^{1/n}u$ and $D^{-1/n}u$ are defined as the incremental quotients

$$D^{\pm 1/n}u(x) := \frac{u(x \pm 1/n) - u(x)}{\pm 1/n} \qquad \forall x \in [0,1],$$

with the agreement that u has been extended previously to the whole real line (or at least to a neighborhood of $[0,1]$ of width $1/n$) by setting $u(x) = u(0)$ for every $x \leq 0$, and $u(x) = u(1)$ for every $x \geq 1$.

Given a finite set $D \subseteq (0,1)$ with k elements, we set

$$D_n := \bigcup_{d \in D} \left[\frac{\lceil nd \rceil - 1}{n}, \frac{\lceil nd \rceil}{n} \right) \subseteq [0,1]. \tag{2.2}$$

In other words, D_n is the union of all subintervals which intersect D (when d is of the form i/n we take the subinterval on its left). It is easy to see that, when n is large enough, D_n is the union of k disjoint intervals, and $(0,1) \setminus D_n$ has $(k+1)$ connected components. Since we are interested in passing to the limit as $n \to +\infty$, we can always work under this assumption.

We call $PS_{D,n}$ the set of all functions $u \in PC_n$ such that

$$\left|D^{1/n}u(x)\right| \leq 1 \Longleftrightarrow x \in [0,1] \setminus D_n. \tag{2.3}$$

The space $PS_{D,n}$ is obviously the discrete counterpart of PS_D. In analogy with (2.1), the *discrete jump height* of a function $u \in PS_{D,n}$ in a point $d \in D$ is defined as

$$J_{d,n}(u) := u\left(\frac{\lceil nd \rceil}{n}\right) - u\left(\frac{\lceil nd \rceil - 1}{n}\right). \tag{2.4}$$

This is equivalent to say that $J_{d,n}(u) := u(x+1/n) - u(x)$, where x is any point of the subinterval containing d, or of the subinterval on the left of d if $d = i/n$ for some $i = 1, \ldots, n-1$. We point out that (2.4) makes sense for every $d \in [0,1]$, and not only for $d \in D$. Of course we have that $|u(x+1/n) - u(x)| > 1/n$ if and only if $x \in D_n$.

The *subcritical incremental quotient* of a function $u \in PS_{D,n}$ is defined as

$$SQ_n(u) := \left\|D^{1/n}u(x)\right\|_{L^\infty((0,1)\setminus D_n)}. \tag{2.5}$$

Due to (2.3) we have that $0 \leq SQ_n(u) \leq 1$ for every $u \in PS_{D,n}$. The subcritical incremental quotient is the discrete counterpart of the Lipschitz constant in the intervals between discontinuities.

2.2 Functionals

A discrete approximation of (1.4) is obtained by replacing the derivative with discrete derivatives. Thus we introduce the functionals $PM_n : PC_n \to \mathbb{R}$ defined by

$$PM_n(u) = \frac{1}{2}\int_0^1 \log\left(1 + |D^{1/n}u(x)|^2\right) dx \qquad \forall u \in PC_n. \tag{2.6}$$

The time rescaling due to the "slow time" leads us to consider also the functionals $nPM_n(u)$. More generally, for each nonnegative integer k we consider the sequence of functionals $G_n^{(k)} : PC_n \to \mathbb{R}$ defined by

$$G_n^{(k)}(u) := \frac{n}{2}\int_0^1 \log\left(1 + |D^{1/n}u(x)|^2\right) dx - k\log n \qquad \forall u \in PC_n. \tag{2.7}$$

It is clear that $G_n^{(k)}(u)$ (which sometimes we call *k-energy* of u) coincides with $nPM_n(u)$ up to an additive constant, and in particular these functionals have the same gradient, hence also the same gradient-flow.

Computing the gradient is a simple exercise in finite dimension. It turns out that

$$\nabla G_n^{(k)}(u) = n\nabla PM_n(u) = -n\, D^{-1/n}\left[\frac{D^{1/n}u}{1 + |D^{1/n}u|^2}\right] \qquad \forall u \in PC_n. \tag{2.8}$$

The "second order discrete operator" in the right-hand side of (2.8) needs some interpretation in the two extremal subintervals, where it requires to compute values of u outside $[0, 1]$. Once again this is done after extending u by setting $u(x) = u(0)$ for every $x \leq 0$, and $u(x) = u(1)$ for every $x \geq 1$. This agreement is the discrete counterpart of the Neumann boundary condition, which in this sense is now included in the right-hand side of (2.8).

Finally, for every finite set $D \subseteq (0, 1)$ with $|D| =: k$, and every $u \in PS_D$, we set

$$G_\infty^{(k)}(u) := \sum_{d \in D} \log |J_d(u)|. \tag{2.9}$$

The following result justifies the notation used for $G_\infty^{(k)}$, and shows that the sequence $G_n^{(k)}(u)$ has a less trivial limit as $n \to +\infty$ with respect to the sequence $nPM_n(u)$. A proof of this result can be found in [7], or simply deduced from the theory of convex-concave integrands developed in [15].

Theorem A (Gamma-limit of discrete functionals) *Let k be a nonnegative integer. Let us extend PM_n and $G_n^{(k)}$ by setting them equal to $+\infty$ for every $u \in L^2 \setminus PC_n$.*

Then we have that (all Γ-limits are intended with respect to L^2-metric)

$$\Gamma\!-\lim_{n \to +\infty} nPM_n(u) = \begin{cases} 0 & \textit{if } u \textit{ is constant,} \\ +\infty & \textit{otherwise,} \end{cases}$$

and

$$\Gamma\!-\lim_{n \to +\infty} G_n^{(k)}(u) = \begin{cases} -\infty & \textit{if } u \in PC_D \textit{ for some } D \subseteq (0,1) \textit{ with } |D| < k, \\ G_\infty^{(k)}(u) & \textit{if } u \in PC_D \textit{ for some } D \subseteq (0,1) \textit{ with } |D| = k, \\ +\infty & \textit{otherwise.} \end{cases}$$

2.3 The semidiscrete scheme

The semidiscrete scheme for the one dimensional Perona-Malik equation is the gradient-flow of (2.6). This leads to the problem

$$v_n'(t) = -\nabla PM_n(v_n(t)) \qquad \forall t \geq 0, \tag{2.10}$$

$$v_n(0) = u_{0n}, \tag{2.11}$$

where $\{u_{0n}\}$ is a suitable sequence of initial conditions with $u_{0n} \in PC_n$ for every $n \geq 1$. The behavior of $v_n(t)$ as $n \to +\infty$ is the subject of the "standard time" theory.

In the "slow time" theory we speed up the evolution by considering the sequence $u_n(t) := v_n(nt)$. It is very simple to show that this sequence solves the rescaled problems

$$u_n'(t) = -n\nabla PM_n(u_n(t)) \qquad \forall t \geq 0, \tag{2.12}$$

$$u_n(0) = u_{0n}. \tag{2.13}$$

Thanks to (2.8), the differential equation (2.12) is equivalent to

$$u_n'(t) = -\nabla G_n^{(k)}(u_n(t)) \qquad \forall t \geq 0. \tag{2.14}$$

All these problems admit a unique solution defined for every $t \geq 0$. Indeed PC_n is a finite dimensional vector space, the functionals we consider are of class C^∞, and their gradient (2.8) is globally Lipschitz continuous. It is worthwhile to notice that formula (2.8) makes (2.10) the discrete counterpart of (1.1).

Throughout this paper we consider u_n both as a function $u_n(t)$ of the time variable with values in L^2, and as a function $u_n(x,t)$ of (x,t) with real values.

The following properties of u_n are used several times. The proof can be deduced from the corresponding properties of v_n stated in [12].

Theorem B (Properties of approximating solutions) *Let n be a fixed positive integer, let $u_{0n} \in PC_n$, and let $u_n : [0,+\infty) \to L^2$ be the solution of problem (2.12), (2.13).*

Then the following properties hold true.

(1) (Regularity) *We have that*

$$u_n \in C^\infty\left([0,+\infty); PC_n\right) \subseteq C^\infty\left([0,+\infty); L^2\right).$$

(2) (Standard gradient-flow estimate) *Let k be any nonnegative integer. Then the function $t \to G_n^{(k)}(u_n(t))$ is nonincreasing, and for every $0 \leq s \leq t$ we have that*

$$\|u_n(t) - u_n(s)\|_2 \leq \left\{G_n^{(k)}(u_n(s)) - G_n^{(k)}(u_n(t))\right\}^{1/2} |t-s|^{1/2}.$$

(3) (L^p estimate) *The function $t \to \|u_n(x,t)\|_{L^p((0,1))}$ is nonincreasing for every $p \geq 1$ (including $p = \infty$).*

(4) (Total variation estimate) *The function $t \to \|D^{1/n}u_n(x,t)\|_{L^1((0,1))}$ is nonincreasing. The same is not necessarily true for p-norms with $p > 1$.*

(5) (Asymptotic behavior) *The function $u_n(t)$ tends, as $t \to +\infty$, to the constant function equal to the average of u_{0n}.*

(6) (Monotonicity and extinction of supercritical regions) *Supercritical regions are nonincreasing (as set valued maps), and they disappear after a finite time.*

In other words, if $u_{0n} \in PS_{D,n}$ for some finite set $D \subseteq (0,1)$, then there exist $j \in \mathbb{N}$, and a finite sequence of times

$$0 = T_0 < T_1 < \ldots < T_j < T_{j+1} = +\infty,$$

and a finite sequence of subsets

$$D = D(0) \supset D(1) \supset \ldots \supset D(j) = \emptyset$$

(with strict inclusions) such that

$$u_n(t) \in PS_{D(i),n} \qquad \forall t \in [T_i, T_{i+1}),\ \forall i \in \{0, 1, \ldots, j\}.$$

The fact that supercritical regions disappear in a finite time (which of course depends on n) has probably never been stated in the literature, but it is a simple consequence of statement (5). In turn, statement (5) follows from three general facts: the average of $u_n(t)$ is invariant during the evolution, the limit of a gradient-flow is a steady state solution, and (2.8) is zero if and only if u is constant.

2.4 The limit problem

Let k be a nonnegative integer, and let $D \subseteq (0, 1)$ be a finite set with $|D| = k$. Given an initial condition $u_0 \in PC_D$, we define an evolution $u(t)$ starting from u_0 according to the following algorithm.

If $k = 0$, then the initial datum u_0 is constant, and we define $u(t)$ as the stationary solution $u(t) \equiv u_0$.

If $k > 0$, let $D = \{d_1, \ldots, d_k\}$ with $0 < d_1 < \ldots < d_k < 1$, let $d_0 := 0$, $d_{k+1} := 1$, and let a_{0i} (with $i = 0, \ldots, k$) denote the constant value of u_0 in the interval (d_i, d_{i+1}). Let $(a_0(t), a_1(t), \ldots, a_k(t))$ be the (unique) solution of the system of $(k + 1)$ ordinary differential equations

$$\begin{aligned}
a_0'(t) &= \frac{1}{d_1 - d_0} \cdot \frac{1}{a_1(t) - a_0(t)}, \\
a_i'(t) &= \frac{1}{d_{i+1} - d_i} \left(\frac{1}{a_{i+1}(t) - a_i(t)} - \frac{1}{a_i(t) - a_{i-1}(t)} \right) \qquad i = 1, \ldots, k-1, \\
a_k'(t) &= -\frac{1}{d_{k+1} - d_k} \cdot \frac{1}{a_k(t) - a_{k-1}(t)},
\end{aligned}$$

with initial conditions $a_i(0) = a_{0i}$ for every $i = 0, \ldots, k$.

Let $u(t)$ be the piecewise constant function whose value in (d_i, d_{i+1}) is $a_i(t)$, defined as soon as the solution of the system exists. To this end, we have the following result (the proof is given is section 5.3).

Proposition 2.1 *Let k, D, u_0, $u(t)$ be as above. Then we have the following conclusions.*

(1) (Local but not global existence) *The system of ordinary differential equations has a local solution defined on a maximal interval $[0, T_{\text{sing}})$ with $T_{\text{sing}} \in (0, +\infty)$.*

(2) (L^∞ estimate) *We have that*

$$\|u(t)\|_\infty \leq \|u_0\|_\infty \qquad \forall t \in [0, T_{\text{sing}}). \tag{2.15}$$

(3) (1/4-Hölder continuity up to collision) *For every* $(s,t) \in [0, T_{\text{sing}})^2$ *we have that*

$$\|u(t) - u(s)\|_2 \leq (3k)^{3/4} \exp\left(\frac{1}{2k} G_\infty^{(k)}(u_0)\right) |t-s|^{1/4}. \tag{2.16}$$

Thanks to the Hölder continuity (2.16), we can define $u(t)$ up to T_{sing} (included). This extension fulfils (2.16) and (2.15) in the closed interval.

Moreover, $u(t) \in PC_{D'}$ for some $D' \subseteq D$. If $D' = D$ we can continue the solution of the system of ordinary differential equations beyond T_{sing}, but this contradicts the maximality of T_{sing}. Thus D' is strictly contained in D, which means that at time $t = T_{\text{sing}}$ we have a collision between at least two adjacent plateaus. At this point we restart the construction of $u(t)$ from $u(T_{\text{sing}})$, which has a smaller set of jump points.

This procedure defines a function $u \in C^0([0, +\infty); L^2)$ with $u(0) = u_0$. For each $t \geq 0$ the function $u(t)$ is piecewise constant in the space variable, and its discontinuity set is contained in the discontinuity set of u_0. There is a finite set of singular times when two or more adjacent plateaus collide, hence one or more discontinuities disappear. After each collision, the involved plateaus remain attached forever, and the evolution goes on according to the same rule applied to the new set of plateaus. After the last collision $u(t)$ becomes constant, and it does not move any more. This constant is actually the average of u_0 (indeed the average of $u(t)$ is invariant during the evolution). The function u is of class C^∞ with respect to the time variable outside the finite set of collision times, and uniformly continuous as a function from $[0, +\infty)$ to L^2.

3 Main results

The main result of this paper is the following convergence result.

Theorem 3.1 (Global-in-time convergence) *Let* $D' \subseteq D \subseteq (0,1)$ *be two finite sets. Let* $u_0 \in PC_{D'}$*, and let* $\{u_{0n}\}$ *be a sequence such that*

$$u_{0n} \in PS_{D,n} \qquad \forall n \geq 1, \tag{3.1}$$

$$u_{0n} \to u_0 \quad \text{in } L^2((0,1)). \tag{3.2}$$

For every $n \geq 1$*, let* $u_n(t)$ *be the solution of the approximating problem (2.12), (2.13). Let* $u(t)$ *be the solution of the limit problem with initial condition* u_0*, as defined in section 2.4.*

Then we have the following conclusions.

(1) (Global-in-time L^2-convergence) *We have that* $u_n(t) \to u(t)$ *in* $C^0([0, +\infty); L^2)$*, namely*

$$\lim_{n\to+\infty} \sup_{t\geq 0} \|u_n(t) - u(t)\|_{L^2((0,1))} = 0. \tag{3.3}$$

(2) (Global-in-time "uniform" convergence) *For every $t \geq 0$, let $D(t)$ be the discontinuity set of $u(t)$, and let $D_n(t)$ be the union of all subintervals containing elements of $D(t)$, defined according to (2.2) with $D(t)$ instead of D. Let us set*

$$\mathcal{K}_n := \{(x,t) \in [0,1] \times [0,+\infty) : x \notin D_n(t)\}. \tag{3.4}$$

Then we have that

$$\lim_{n\to+\infty} \|u_n(x,t) - u(x,t)\|_{L^\infty(\mathcal{K}_n)} = 0. \tag{3.5}$$

We considered a piecewise constant initial datum u_0 because this is the natural space where the Gamma-limit of the renormalized functionals is finite. On the other hand, we emphasize that the approximating sequence $\{u_{0n}\}$ is quite general. In particular, we did not assume that it is a recovery sequence, or that its energy is bounded, and the set of discrete jump points of u_{0n} can be larger than the set of jump points of u_0. What is essential is that all discrete jump points of approximating functions are contained in a *fixed* finite set D (if not, there are counterexamples even to local-in-time convergence).

Another reason for looking at piecewise constant data is that they are expected to be the limit as $t \to +\infty$ of evolutions in "standard time". Up to our knowledge, this has been proved rigorously only for "generic" piecewise subcritical data (see [6, 12]).

Whenever the "standard time" evolution admits a piecewise constant limit, we can start our "slow time" analysis from that limit. We state this idea formally in the next result. We point out that in this case we do not assume that the sequence $\{u_{0n}\}$ of initial data has a piecewise constant or piecewise subcritical limit (actually we do not even assume that it has a limit).

Theorem 3.2 (Convergence for more general initial data) *Let $D \subseteq (0,1)$ be a finite set, and let $\{u_{0n}\}$ be a sequence satisfying (3.1).*

For every $n \geq 1$, let $v_n(t)$ be the solution of problem (2.10), (2.11) (no time rescaling), and let $u_n(t) = v_n(nt)$ be the solution of the rescaled problem (2.12), (2.13). Let us assume that there exist $S \geq 0$, $v \in C^0([S,+\infty), L^2)$, $D' \subseteq D$, and $v_\infty \in PC_{D'}$ such that

$$\lim_{n\to+\infty} v_n(t) = v(t) \qquad \forall t \geq S, \tag{3.6}$$

$$\lim_{t\to+\infty} v(t) = v_\infty, \tag{3.7}$$

where both limits are intended in L^2. Let $u(t)$ be the solution of the limit problem, defined as in section 2.4, with initial condition $u(0) = v_\infty$.

Then for every $T > 0$ we have that $u_n(t) \to u(t)$ in $C^0([T,+\infty); L^2)$, namely

$$\lim_{n\to+\infty} \sup_{t\geq T} \|u_n(t) - u(t)\|_{L^2((0,1))} = 0. \tag{3.8}$$

Moreover, if $\mathcal{K}_n$ is defined as in (3.4), we have that

$$\lim_{n\to+\infty} \|u_n(x,t) - u(x,t)\|_{L^\infty(\mathcal{K}_n \cap ([0,1]\times[T,+\infty)))} = 0. \tag{3.9}$$

We conclude by mentioning a possible extension of our results.

Remark 3.3 For the sake of simplicity, we devoted this paper to the model case of the Perona-Malik equation, in which the integrand is $\varphi(\sigma) := 2^{-1}\log(1+\sigma^2)$. Similar techniques apply to larger classes of convex-concave integrands, for example the case where $\varphi(\sigma) := \alpha^{-1}(1+\sigma^2)^{\alpha/2}$ for some $\alpha \in (0,1)$. In this case the "slow time" is of order $O(n^{1-\alpha})$, the limit energy is

$$G_\alpha(u) := \sum_{d\in D} |J_d(u)|^\alpha, \tag{3.10}$$

and the system of ordinary differential equations governing the evolution of the plateau heights is

$$a_i'(t) = \frac{1}{d_{i+1}-d_i}\left(\frac{a_{i+1}(t)-a_i(t)}{|a_{i+1}(t)-a_i(t)|^{2-\alpha}} - \frac{a_i(t)-a_{i-1}(t)}{|a_i(t)-a_{i-1}(t)|^{2-\alpha}}\right),$$

suitably modified for $i = 0$ and $i = k$.

There are, however, some remarkable differences. On the one hand, this situation is simpler because the limit energy (3.10) is bounded from below, hence there is no need to renormalize it after each collision. On the other hand, the limit energy can be finite even if u has infinitely many jump points.

4 Fundamental tools

In this section we state the main ingredients needed in the proof of our main results.

The first one is a well preparation result, which plays its role at the beginning of the evolution and during each collision. In input we have a sequence satisfying (3.1) and (3.2) as in the assumptions of Theorem 3.1. In particular, some of the jump points might disappear in the limit (this happens if and only if D' is strictly contained in D), and there is no information on the k-energy or the k'-energy of the sequence (where $k := |D|$ and $k' := |D'|$). We prove the existence of a sequence of times $S_n \to 0$ such that $u_n(S_n)$ is a "well prepared" sequence, namely it still converges to u_0, all its elements lie in the corresponding space $PS_{D',n}$, and their k'-energies converge to the k'-energy of u_0.

Proposition 4.1 (Well preparation) *Let D, D', $\{u_{0n}\}$, u_0, $u_n(t)$ be as in Theorem 3.1, and let $k' := |D'|$.*

Then there exists a sequence $S_n \to 0$ of positive real numbers such that

$$u_n(S_n) \in PS_{D',n} \quad \textit{for every } n \textit{ large enough}, \tag{4.1}$$

$$\lim_{n\to+\infty} G_n^{(k')}(u_n(S_n)) = G_\infty^{(k')}(u_0), \tag{4.2}$$

$$\lim_{n\to+\infty} \max_{t\in[0,S_n]} \|u_n(t)-u_0\|_2 = 0. \tag{4.3}$$

The second tool is a convergence result up to the first jump extinction. It plays its role in the time intervals between collisions. Now in input we have a "well prepared" sequence, or at least a sequence of initial data with bounded energy, and without vanishing jump points. We prove some sort of uniform convergence on an increasing sequence of time intervals (depending on n). As $n \to +\infty$ these intervals invade the whole time interval up to the first collision.

Proposition 4.2 (Convergence up to first jump extinction) *Let k be a positive integer, and let $D \subseteq (0,1)$ be a finite set with $|D| = k$. Let $u_0 \in PC_D$, and let $\{u_{0n}\}$ be a sequence satisfying (3.1), (3.2), and*

$$\sup_{n \geq 1} G_n^{(k)}(u_{0n}) < +\infty. \tag{4.4}$$

For every $n \geq 1$, let $u_n(t)$ be the solution of the approximating problem (2.12), (2.13), and let

$$T_{\mathrm{sing},n} := \sup \{t \geq 0 : u_n(t) \in PS_{D,n}\} \tag{4.5}$$

be the first time when a discrete jump disappears (it is the time T_1 in statement (6) of Theorem B). Let $u(t)$ and T_{sing} be defined as in section 2.4.

Then there exists a sequence $\{T_n\}$ of real numbers such that

$$0 < T_n < T_{\mathrm{sing},n} \qquad \forall n \geq 1, \tag{4.6}$$

$$\lim_{n \to +\infty} T_n = T_{\mathrm{sing}}, \tag{4.7}$$

$$\lim_{n \to +\infty} \max_{t \in [0, T_n]} \|u_n(t) - u(t)\|_2 = 0. \tag{4.8}$$

Moreover we have that

$$\lim_{n \to +\infty} G_n^{(k)}(u_n(t)) = G_\infty^{(k)}(u(t)) \qquad \forall t \in (0, T_{\mathrm{sing}}). \tag{4.9}$$

Finally, we present a qualitative property of approximating solutions which could be interesting in itself. It is the discrete analog of statement (3) of Proposition 2.1. We point out that the standard gradient-flow estimates (statement (2) of Theorem B) control the 1/2-Hölder constant of $u_n(t)$ in terms of the descent of the energy, but such estimates are useless if the energy is not bounded from below independently on n, and this is exactly what happens in this model when t approaches a collision time.

The following 1/4-Hölder estimates overcome this difficulty.

Proposition 4.3 (Uniform 1/4-Hölder continuity) *Let k, D, $\{u_{0n}\}$, $u_n(t)$, $T_{\mathrm{sing},n}$ be as in Proposition 4.2.*

Then for every $n \geq 1$ and every $(s,t) \in [0, T_{\mathrm{sing},n}]^2$ we have that

$$\|u_n(t) - u_n(s)\|_2 \leq (3k)^{3/4} \exp\left(\frac{1}{2k} G_n^{(k)}(u_{0n})\right) |t - s|^{1/4}.$$

5 Proofs

5.1 Basic estimates

In this section we collect some general facts, which are going to be used several times in the proof of our main results and basic tools. The first one concerns "double limits" (we omit the simple proof).

Lemma 5.1 (Double index sequence) *Let $\{A_{m,n}\}$ (with $(m,n) \in \mathbb{N}^2$) be a double indexed sequence with values in a metric space X. Let us assume that for every $m \in \mathbb{N}$ there exists*

$$A_{m,\infty} := \lim_{n \to +\infty} A_{m,n},$$

and that there exists

$$A_{\infty,\infty} := \lim_{m \to +\infty} A_{m,\infty}.$$

Then we have the following conclusions.

(1) (Standard conclusion) *There exists a sequence $m_k \to +\infty$ of nonnegative integers such that*

$$\lim_{k \to +\infty} A_{m_k,k} = A_{\infty,\infty}. \tag{5.1}$$

(2) (Refined conclusion) *For every sequence of real numbers $r_k \to +\infty$ there exists a sequence $m_k \to +\infty$ of nonnegative integers such that $m_k \leq r_k$ for every k large enough, and such that (5.1) holds true.* $\square$

In the next result we estimate from below the norm of a discrete derivative. In the continuous setting, when we know the values of some function $f \in H_0^1((0,1))$ in some given points, then we can estimate $\|f_x\|_2$ from below. The conclusions of the following lemma are the discrete counterpart of such estimates.

Lemma 5.2 (Discrete derivative estimates) *Let n be a positive integer. Let $f \in PC_n$ be a function which is equal to 0 in the last subinterval $(1-1/n, 1)$. Let us consider the discrete derivative $D^{-1/n}f(x)$, defined after setting $f(x) = 0$ in $(-1/n, 0)$.*

Then the following estimates hold true.

(1) We have that

$$\left\|D^{-1/n}f(x)\right\|_2 \geq 2\|f(x)\|_\infty. \tag{5.2}$$

(2) Let k be a positive integer, and let $0 < d_1 < \ldots < d_k < 1$. Then we have that

$$\left\|D^{-1/n}f(x)\right\|_2^2 \geq \sum_{h=0}^{k} \frac{1}{d_{h+1} - d_h + 1/n}|f(d_{h+1}) - f(d_h)|^2, \tag{5.3}$$

with the agreement that $d_0 = 0$, $d_{k+1} = 1$, and $f(d_0) = f(d_{k+1}) = 0$.

Proof Let f_i (with $i = 1, \ldots, n$) denote the value of f in the interval $((i-1)/n, i/n)$. Let us set $f_0 := 0$ (this choice is consistent with our extension of $f(x)$ in $(-1/n, 0)$), and let us recall that $f_n = 0$ due to our assumption on f. Let $j \in \{1, \ldots, n\}$ be the index (or one of the indices) such that $\|f(x)\|_\infty = |f_j|$.

Then we have that

$$\left\|D^{-1/n} f(x)\right\|_2 \;\geq\; \left\|D^{-1/n} f(x)\right\|_1 \;=\; \sum_{i=1}^{j} |f_i - f_{i-1}| + \sum_{i=j+1}^{n} |f_i - f_{i-1}|$$

$$\geq \left|\sum_{i=1}^{j} (f_i - f_{i-1})\right| + \left|\sum_{i=j+1}^{n} (f_i - f_{i-1})\right| \;=\; |f_j - f_0| + |f_n - f_j| \;=\; 2|f_j|,$$

which proves (5.2).

Let us consider now the second statement. Let us set $i_0 := 0$, $i_{k+1} := n$, and $i_h := \lfloor n d_h \rfloor + 1$ for every $h = 1, \ldots, k$. With this notation we have that $f(d_h) = f_{i_h}$ for every $h = 0, 1, \ldots, k+1$. Moreover it is easy to see that

$$\frac{i_{h+1}}{n} - \frac{i_h}{n} = \frac{\lfloor n d_{h+1} \rfloor + 1}{n} - \frac{\lfloor n d_h \rfloor + 1}{n} \leq d_{h+1} - d_h + \frac{1}{n}.$$

Thus from Hölder's inequality it follows that

$$\begin{aligned}
\left\|D^{-1/n} f(x)\right\|^2_{L^2((0,1))} &= \sum_{h=0}^{k} \left\|D^{-1/n} f(x)\right\|^2_{L^2((i_h/n, i_{h+1}/n))} \\
&\geq \sum_{h=0}^{k} \frac{1}{(i_{h+1}/n) - (i_h/n)} \|D^{-1/n} f(x)\|^2_{L^1((i_h/n, i_{h+1}/n))} \\
&\geq \sum_{h=0}^{k} \frac{1}{d_{h+1} - d_h + 1/n} \|D^{-1/n} f(x)\|^2_{L^1((i_h/n, i_{h+1}/n))}. \qquad (5.4)
\end{aligned}$$

To be precise, the first inequality requires that all indices i_h are distinct, and this is true only when n is large enough. On the other hand, the final result is true in any case, because it is enough to ignore the intervals of length zero in the first sum.

Finally we have that

$$\left\|D^{-1/n} f(x)\right\|_{L^1((i_h/n, i_{h+1}/n))} = \sum_{i=i_h+1}^{i_{h+1}} |f_i - f_{i-1}| \geq \left|\sum_{i=i_h+1}^{i_{h+1}} (f_i - f_{i-1})\right| =$$

$$= |f_{i_{h+1}} - f_{i_h}| = |f(d_{h+1}) - f(d_h)|.$$

From the last estimate and (5.4) we obtain (5.3). $\square$

In the next statement n is fixed, and we present several estimates relating k-energies, the norm of their gradient (the slope), jump heights, and subcritical incremental quotients. These estimates are the technical core of our analysis.

Lemma 5.3 (Fundamental estimates) *Let k and n be positive integers, let $D \subseteq (0,1)$ be a finite set with $|D| = k$, and let $v \in PS_{D,n}$. Let D_n be defined according to (2.2), and let us assume that n is big enough so that $(0,1) \setminus D_n$ has $(k+1)$ connected components.*

Let $G_n^{(k)}(v)$ be the functional defined in (2.7), let $\nabla G_n^{(k)}(v)$ be its gradient, let $J_{d,n}(v)$ be the discrete jump heights of v defined in (2.4), and let $SQ_n(v)$ be the subcritical incremental quotient of v defined in (2.5).

Then we have that

$$G_n^{(k)}(v) \geq k \log \left(\min_{d \in D} |J_{d,n}(v)| \right), \tag{5.5}$$

$$G_n^{(k)}(v) \leq \frac{n}{2} \log \left(1 + [SQ_n(v)]^2\right) + \frac{1}{2} \sum_{d \in D} \log \left(\frac{1}{n^2} + [J_{d,n}(v)]^2 \right), \tag{5.6}$$

$$\left\| \nabla G_n^{(k)}(v) \right\|_2 \geq n \, |SQ_n(v)|, \tag{5.7}$$

$$\left\| \nabla G_n^{(k)}(v) \right\|_2 \geq \left(\min_{d \in D} |J_{d,n}(v)| \right)^{-1}. \tag{5.8}$$

Finally, if $D = \{d_1, \ldots, d_k\}$ with $0 < d_1 < \ldots < d_k < 1$, then we have that

$$\left\| \nabla G_n^{(k)}(v) \right\|_2^2 \geq \sum_{i=0}^{k} \frac{1}{d_{i+1} - d_i + n^{-1}} \left(\frac{J_{d_{i+1},n}(v)}{n^{-2} + [J_{d_{i+1},n}(v)]^2} - \frac{J_{d_i,n}(v)}{n^{-2} + [J_{d_i,n}(v)]^2} \right)^2, \tag{5.9}$$

with the agreement that $d_0 = 0$, $d_{k+1} = 1$, and $J_{d_0,n}(v) = J_{d_{k+1},n}(v) = 0$.

Proof of estimates on the functional From the definition of jump heights we have that

$$\begin{aligned} \frac{n}{2} \int_{D_n} \log \left(1 + |D^{1/n} v(x)|^2\right) dx - k \log n &= \frac{1}{2} \sum_{d \in D} \log \left(1 + n^2 \left[J_{d,n}(v)\right]^2\right) - \frac{k}{2} \log n^2 \\ &= \frac{1}{2} \sum_{d \in D} \log \left(\frac{1}{n^2} + [J_{d,n}(v)]^2 \right). \end{aligned} \tag{5.10}$$

In order to prove (5.5), we estimate the right-hand side of (2.7) from below by considering only the integration over D_n. From (5.10) we deduce that

$$G_n^{(k)}(v) \geq \frac{n}{2} \int_{D_n} \log \left(1 + |D^{1/n} v(x)|^2\right) dx - k \log n = \frac{1}{2} \sum_{d \in D} \log \left(\frac{1}{n^2} + [J_{d,n}(v)]^2 \right)$$

$$\geq \sum_{d \in D} \log |J_{d,n}(v)| \geq k \log \left(\min_{d \in D} |J_{d,n}(v)| \right),$$

which proves (5.5).

On the other hand, from (2.5) we have that

$$\frac{n}{2}\int_{[0,1]\setminus D_n}\log\left(1+|D^{1/n}v(x)|^2\right)\,dx \leq \frac{n}{2}\log\left(1+[SQ_n(v)]^2\right). \tag{5.11}$$

Summing (5.11) and (5.10) we obtain (5.6).

Proof of estimates on the slope Let us consider the function

$$f(x) := n\,\frac{D^{1/n}v(x)}{1+\left[D^{1/n}v(x)\right]^2}. \tag{5.12}$$

It turns out that $f(x)=0$ in the last subinterval $(1-1/n,1)$. Moreover $\nabla G_n^{(k)}(v)$, whose expression has been computed in (2.8), coincides (up to the sign) with the discrete derivative $D^{-1/n}f(x)$, computed after setting $f(x)=0$ in the interval $(-1/n,0)$.

Therefore, applying (5.2) to the function $f(x)$ defined in (5.12), we obtain that

$$\left\|\nabla G_n^{(k)}(v)\right\|_2 = \left\|D^{-1/n}f(x)\right\|_2 \geq 2\|f(x)\|_\infty. \tag{5.13}$$

Let us estimate the right-hand side of (5.13) from below by restricting the L^∞-norm to $[0,1]\setminus D_n$. Since the function $\sigma\to\sigma(1+\sigma^2)^{-1}$ is increasing in $[-1,1]$, and since $0\leq SQ_n(v)\leq 1$, we obtain that

$$\|f(x)\|_\infty \geq \max\left\{\frac{n\,|D^{1/n}v(x)|}{1+\left[D^{1/n}v(x)\right]^2} : x\in[0,1]\setminus D_n\right\} = \frac{n\,SQ_n(v)}{1+\left[SQ_n(v)\right]^2} \geq \frac{n}{2}\,SQ_n(v),$$

which proves (5.7).

Now let us estimate the right-hand side of (5.13) from below by restricting the L^∞-norm to D_n. We obtain that

$$\|f(x)\|_\infty \geq \max\left\{\frac{n\,|D^{1/n}v(x)|}{1+\left[D^{1/n}v(x)\right]^2} : x\in D_n\right\} = \max_{d\in D}\frac{n^2\,|J_{d,n}(v)|}{1+n^2\left[J_{d,n}(v)\right]^2}.$$

Since $n|J_{d,n}(v)|\geq 1$ for every $d\in D$, and since the function $\sigma\to\sigma(1+\sigma^2)^{-1}$ is decreasing for $\sigma\geq 1$, we have that the maximum is achieved when the argument is minimum, hence

$$\left\|\nabla G_n^{(k)}(v)\right\|_2 \geq 2\|f(x)\|_\infty \geq 2\max_{d\in D}\frac{n^2\,|J_{d,n}(v)|}{1+n^2\left[J_{d,n}(v)\right]^2} = \frac{2n^2\min\limits_{d\in D}|J_{d,n}(v)|}{1+n^2\left[\min\limits_{d\in D}|J_{d,n}(v)|\right]^2}.$$

Recalling once more that the minimum is greater than or equal to $1/n$, estimate (5.8) follows.

A more refined estimate, keeping into account all jump heights, follows from (5.3) applied to the function $f(x)$ defined in (5.12). Since $D^{1/n}v(d) = nJ_{d,n}(v)$ for every $d \in D$, we obtain exactly (5.9). $\square$

In the next lemma we consider the difference between two piecewise constant functions v and w. The main idea is the following. If the number of discrete jump points is finite, and their location is fixed, then the L^2-norm of $v-w$ estimates both the L^∞-norm of $v-w$, and the difference between jump heights.

Lemma 5.4 (Uniform and jump-height estimates) *Let $D \subseteq (0,1)$ be a finite set, let K_0 be the minimum of the lengths of the $(|D|+1)$ intervals into which $(0,1)$ is divided by D, and let $n \geq 3K_0^{-1}$ be a positive integer.*

Let $D' \subseteq D$ and $D'' \subseteq D$ be two subsets, and let $v \in PS_{D',n}$ and $w \in PS_{D'',n}$ be two piecewise constant functions with discrete jump heights $J_{d,n}(v)$ and $J_{d,n}(w)$, respectively (we can think both jump heights as defined for every $d \in D$).

Then we have that

$$\min\left\{K_0, \|v-w\|_\infty\right\} \leq 3\|v-w\|_2^{2/3}, \tag{5.14}$$

$$\min\left\{K_0, |J_{d,n}(v) - J_{d,n}(w)|\right\} \leq 6\|v-w\|_2^{2/3} \qquad \forall d \in D. \tag{5.15}$$

Proof Let $f(x) := |v(x) - w(x)|$, and let f_i (with $i = 1, \ldots, n$) denote the value of f in the i-th subinterval. Let $j \in \{1, \ldots, n\}$ be the index (or one of the indices) such that $\|f(x)\|_\infty = f_j$.

For every $x \in [0,1] \setminus D_n$ we have that $|D^{1/n}v(x)| \leq 1$ and $|D^{1/n}w(x)| \leq 1$, hence

$$|D^{1/n}f(x)| \leq 2 \qquad \forall x \in [0,1] \setminus D_n. \tag{5.16}$$

Let us set $H := \min\{K_0, f_j\}$, and let us consider the two intervals

$$I := \left(\frac{j}{n} - \frac{H}{3}, \frac{j}{n}\right), \qquad I' := \left(\frac{j-1}{n}, \frac{j-1}{n} + \frac{H}{3}\right).$$

Since $n \geq 3K_0^{-1}$, estimate (5.16) implies that in at least one of these intervals the difference between the values of f in any two neighboring subintervals is always less than or equal to $2/n$. Let us assume, without loss of generality, that this happens in I (the other case is specular). Then we have that

$$f(x) \;\geq\; f_j - 2\left(\frac{j}{n} - x\right) \;\geq\; H - 2\frac{H}{3} \;\geq\; \frac{H}{3} \qquad \forall x \in I,$$

so that

$$\|v-w\|_{L^2(I)}^2 = \int_I [f(x)]^2\, dx \geq \frac{H^3}{27},$$

and therefore

$$\min\{K_0, \|v-w\|_\infty\} = H \leq 3\|v-w\|_{L^2(I)}^{2/3} \leq 3\|v-w\|_{L^2((0,1))}^{2/3}.$$

This proves (5.14). Now we have that

$$|J_{d,n}(v) - J_{d,n}(w)| = |J_{d,n}(v-w)| \leq 2\|v-w\|_\infty \qquad \forall d \in D,$$

hence

$$\min\left\{K_0, |J_{d,n}(v) - J_{d,n}(w)|\right\} \leq 2\min\{K_0, \|v-w\|_\infty\} \leq 6\|v-w\|_2^{2/3}$$

for every $d \in D$, which is exactly (5.15). $\square$

In the last lemma we consider a sequence $v_n \to v$. We point out that this sequence is allowed to loose jump points in the limit.

Lemma 5.5 (Jump convergence and BV estimate) *Let $D' \subseteq D \subseteq (0,1)$ be two finite sets. Let $v \in PS_{D'}$, and let $\{v_n\}$ be a sequence such that $v_n \to v$ in L^2. Let us assume that for every $n \geq 1$ we have that $v_n \in PS_{D''(n),n}$ for some finite set $D''(n) \subseteq D$.*

Then we have the following conclusions.

(1) (Jump convergence) *Let $J_d(v)$ be the jump heights of v, and let $J_{d,n}(v_n)$ be the discrete jump heights of v_n (we can think both jump heights as defined for every $d \in D$, with the agreement that $J_d(v) = 0$ when $d \in D \setminus D'$). Then we have that*

$$\lim_{n\to+\infty} J_{d,n}(v_n) = J_d(v) \qquad \forall d \in D. \tag{5.17}$$

As a consequence, $D' \subseteq D''(n)$ for every n large enough.

(2) (Uniform BV estimates) *We have that*

$$\sup_{n\geq 1}\left\{\left\|D^{1/n}v_n\right\|_1 + \|v_n\|_\infty\right\} < +\infty. \tag{5.18}$$

Proof of statement (1) Let $w_n \in PC_n$ be the piecewise constant approximation of v defined by

$$w_n(x) := v\left(\frac{\lfloor nx\rfloor}{n}\right) \qquad \forall x \in [0,1],\ \forall n \geq 1.$$

Let K_0 be as in Lemma 5.4, and let $n \geq 3K_0^{-1}$ as in that lemma. It is not difficult to see that

$$|J_d(v) - J_{d,n}(w_n)| \leq 2n^{-1} \qquad \forall d \in D,$$

hence

$$\begin{aligned} |J_d(v) - J_{d,n}(v_n)| &\leq |J_d(v) - J_{d,n}(w_n)| + |J_{d,n}(w_n) - J_{d,n}(v_n)| \\ &\leq 2n^{-1} + |J_{d,n}(w_n) - J_{d,n}(v_n)|. \end{aligned}$$

Therefore, applying (5.15) with $v = v_n$ and $w = w_n$, we obtain that

$$\begin{aligned}\min\left\{K_0, |J_d(v) - J_{d,n}(v_n)|\right\} &\leq \min\left\{K_0, |J_{d,n}(w_n) - J_{d,n}(v_n)| + 2n^{-1}\right\} \\ &\leq \min\left\{K_0, |J_{d,n}(w_n) - J_{d,n}(v_n)|\right\} + 2n^{-1} \\ &\leq 6\|w_n - v_n\|_2^{2/3} + 2n^{-1} \\ &\leq 6\left(\|w_n - v\|_2 + \|v - v_n\|_2\right)^{2/3} + 2n^{-1}\end{aligned}$$

for every $d \in D$. All the terms in the right-hand side tend to 0 as $n \to +\infty$. This proves (5.17).

Proof of statement (2) Let D_n be defined by (2.2). For every $d \in D$ we have that $D^{1/n}v_n(x) = nJ_{d,n}(v_n)$ for every x in the corresponding subinterval, hence

$$\left\|D^{1/n}v_n\right\|_1 = \int_{D_n} |D^{1/n}v_n(x)|\,dx + \int_{[0,1]\setminus D_n} |D^{1/n}v_n(x)|\,dx \leq \sum_{d\in D} |J_{d,n}(v_n)| + 1.$$

Due to (5.17) we have that

$$\lim_{n\to+\infty} \sum_{d\in D} |J_{d,n}(v_n)| + 1 = \sum_{d\in D'} |J_d(v)| + 1 < +\infty,$$

which is enough to prove the equi-boundedness of the total variations $\|D^{1/n}v_n\|_1$.

The uniform bound on $\|v_n\|_\infty$ follows from the uniform bound on total variations, and from the fact that the average of v_n tends to the average of v (hence averages are equi-bounded). $\square$

5.2 Evolution problems as maximal slope curves

Both the differential equation (2.12), and the system of ordinary differential equations introduced in section 2.4, are equivalent to suitable integral (in)equalities. This equivalence is the key point in the theory of maximal slope curves, for which we refer to [1].

For the sake of simplicity, we want to keep this paper as independent as possible of the general abstract theory of maximal slope curves. For this reason, in Proposition 5.7 we state the two implications we need throughout this paper, and we provide a self-contained and almost elementary proof of them. Before stating these implications, we need the following definition.

Definition 5.6 (Slope of limit functional) Let $D \subseteq (0,1)$ be a finite set with $|D| = k$. Let us assume that $D = \{d_1, \ldots, d_k\}$, with $0 < d_1 < \ldots < d_k < 1$. Let $v \in PC_D$, and let $J_d(v)$ be its jump heights. Let $G_\infty^{(k)}$ be the functional defined in (2.9).

The *slope* of $G_\infty^{(k)}$ in the point v with respect to the L^2-metric is the nonnegative real number whose square is given by

$$\begin{aligned}\left\|\nabla G_\infty^{(k)}(v)\right\|_2^2 \;:=\; & \frac{1}{d_1}\cdot\frac{1}{[J_{d_1}(v)]^2}+\frac{1}{1-d_k}\cdot\frac{1}{[J_{d_k}(v)]^2}\\ & +\sum_{i=1}^{k-1}\frac{1}{d_{i+1}-d_i}\left(\frac{1}{J_{d_{i+1}}(v)}-\frac{1}{J_{d_i}(v)}\right)^2.\end{aligned}$$

There are several interpretations of the slope. In this case the domain where the functional is finite is isometric to an open subset of $\mathbb{R}^{k+1}$, and under this isometry the functional can be identified with a function of $(k+1)$ real variables. The slope coincides with the norm of the gradient of this function.

The approximated solution $u_n(t)$ is the gradient-flow in L^2 of the functional $G_n^{(k)}$ for every $t \geq 0$, while the function $u(t)$ defined in section 2.4 is the gradient-flow in L^2 of the functional $G_\infty^{(k)}$ up to T_{sing}. In any case, what we need in this paper (and in particular in the proof of Proposition 4.2) are the following two facts.

Proposition 5.7 (Differential equations vs maximal slope curves) *The approximating problems and the limit problem can be reformulated as follows.*

(1) (Approximating problems) *Let n be a positive integer, and let $u_{0n} \in PC_n$. Let $u_n \in C^1\left([0,+\infty);L^2\right)$ be the solution of (2.12), (2.13). Then we have that*

$$G_n^{(k)}(u_n(s)) - G_n^{(k)}(u_n(t)) = \frac{1}{2}\int_s^t \|u_n'(\tau)\|_2^2\,d\tau + \frac{1}{2}\int_s^t \left\|\nabla G_n^{(k)}(u_n(\tau))\right\|_2^2\,d\tau \tag{5.19}$$

for every $0 \leq s \leq t$ and every $k \in \mathbb{N}$.

(2) (Limit problem) *Let k be a positive integer, let $D \subseteq (0,1)$ be a finite set with $|D| = k$, and let $u_0 \in PC_D$. Let $T_0 > 0$, and let $v \in H^1\left((0,T_0);L^2\right)$ be a function such that $v(0) = u_0$, $v(t) \in PC_D$ for every $t \in [0,T_0]$, and*

$$G_\infty^{(k)}(v(s)) - G_\infty^{(k)}(v(t)) \geq \frac{1}{2}\int_s^t \|v'(\tau)\|_2^2\,d\tau + \frac{1}{2}\int_s^t \left\|\nabla G_\infty^{(k)}(v(\tau))\right\|_2^2\,d\tau \tag{5.20}$$

for every $0 < s \leq t < T_0$. Then $v(t)$ coincides in $[0,T_0]$ with the function $u(t)$ defined in section 2.4.

Proof From equation (2.12), which is the same as (2.14), we have that

$$-\frac{d}{dt}G_n^{(k)}(u_n(t)) = -\langle\nabla G_n^{(k)}(u_n(t)), u_n'(t)\rangle = \frac{1}{2}\|u_n'(t)\|_2^2 + \frac{1}{2}\left\|G_n^{(k)}(u_n(t))\right\|_2^2.$$

Integrating in $[s,t]$ we obtain (5.19).

Let us consider now inequality (5.20). Let $D = \{d_1, \ldots, d_k\}$ with $0 < d_1 < \ldots < d_k < 1$, let $d_0 := 0$ and $d_{k+1} := 1$, and let us identify $v(t)$ with the vector of plateau heights $(a_0(t), a_1(t), \ldots, a_k(t))$, where $a_i(t)$ is the constant value of $v(t)$ for $x \in (d_i, d_{i+1})$.

The H^1 regularity of $v(t)$ implies H^1 regularity of all components. Let us compute the time derivative of the function $t \to G_\infty^{(k)}(v(t))$. Using the chain rule, and rearranging the terms, for almost every $t \in (0, T_0)$ we obtain that

$$
\begin{aligned}
-\frac{d}{dt} G_\infty^{(k)}(v(t)) &= -\frac{d}{dt} \sum_{i=1}^{k} \log |a_i(t) - a_{i-1}(t)| \\
&= \sum_{i=0}^{k} a_i'(t) \left(\frac{1}{a_{i+1}(t) - a_i(t)} - \frac{1}{a_i(t) - a_{i-1}(t)} \right) \\
&= \sum_{i=0}^{k} \sqrt{d_{i+1} - d_i}\, a_i'(t) \cdot \frac{1}{\sqrt{d_{i+1} - d_i}} \left(\frac{1}{a_{i+1}(t) - a_i(t)} - \frac{1}{a_i(t) - a_{i-1}(t)} \right),
\end{aligned}
$$

with the agreement to neglect the two fractions involving indices less than 0 or larger than k (which appear in the terms of the sum corresponding to $i = 0$ and $i = k$).

Applying the inequality $xy \leq 2^{-1}(x^2 + y^2)$ to each term of the sum, we find that (for shortness' sake we drop the dependence on t in the right-hand side of the first line)

$$
\begin{aligned}
-\frac{d}{dt} G_\infty^{(k)}(v(t)) &\leq \frac{1}{2} \sum_{i=0}^{k} (d_{i+1} - d_i) \left[a_i'\right]^2 + \frac{1}{2} \sum_{i=0}^{k} \frac{1}{d_{i+1} - d_i} \left(\frac{1}{a_{i+1} - a_i} - \frac{1}{a_i - a_{i-1}} \right)^2 \\
&= \frac{1}{2} \|v'(t)\|_2^2 + \frac{1}{2} \left\| \nabla G_\infty^{(k)}(v(t)) \right\|_2^2
\end{aligned}
\tag{5.21}
$$

for almost every $t \in (0, T_0)$. On the other hand, from (5.20) we easily obtain that

$$
-\frac{d}{dt} G_\infty^{(k)}(v(t)) \geq \frac{1}{2} \|v'(t)\|_2^2 + \frac{1}{2} \left\| \nabla G_\infty^{(k)}(v(t)) \right\|_2^2 \tag{5.22}
$$

for almost every $t \in (0, T_0)$. Comparing (5.21) and (5.22) we deduce that there is equality for almost every $t \in (0, T_0)$. But in the inequality used to deduce (5.21) we have equality if and only if

$$
\sqrt{d_{i+1} - d_i}\; a_i'(t) = \frac{1}{\sqrt{d_{i+1} - d_i}} \left(\frac{1}{a_{i+1}(t) - a_i(t)} - \frac{1}{a_i(t) - a_{i-1}(t)} \right)
$$

for every $i = 0, \ldots, k$ and for almost every $t \in (0, T_0)$, which is equivalent to the system of ordinary differential equations introduced in section 2.4. Since the right-hand side is continuous, we can conclude that actually $a_i(t)$ is of class C^1, and we have equality for every t in the closed interval $[0, T_0]$. □

The formulation in terms of differential inequalities is very stable when passing to the limit. In the proof of Proposition 4.2 we obtain (5.20) by passing to the limit in (5.19) as $n \to +\infty$. The following result is fundamental in that stage.

Proposition 5.8 (Bounded slope sequences) *Let k be a nonnegative integer, and let $D \subseteq (0,1)$ be a finite set with $|D| = k$. Let $v \in L^2$, and let $\{v_n\}$ be a sequence such that $v_n \in PS_{D,n}$ for every $n \geq 1$, and such that $v_n \to v$ in L^2.*

Let $J_{d,n}(v_n)$ denote the discrete jump heights of v_n, and let us suppose that there exist $c_0 > 0$ and $n_0 \geq 1$ such that

$$|J_{d,n}(v_n)| \geq c_0 \qquad \forall d \in D,\ \forall n \geq n_0. \tag{5.23}$$

Then we have the following conclusions.

(1) (Gamma-liminf inequality for slopes) *We have that*

$$\liminf_{n\to+\infty} \left\|\nabla G_n^{(k)}(v_n)\right\|_2 \geq \left\|\nabla G_\infty^{(k)}(v)\right\|_2, \tag{5.24}$$

where the right-hand side is intended to be $+\infty$ if $v \notin PC_D$.

(2) (Bounded slope sequences are recovery sequences) *If in addition we assume that*

$$\sup_{n\geq 1} \left\|\nabla G_n^{(k)}(v_n)\right\|_2 < +\infty, \tag{5.25}$$

then we have that $v \in PC_D$, and moreover

$$|J_d(v)| \geq c_0 \qquad \forall d \in D, \tag{5.26}$$

$$\lim_{n\to+\infty} G_n^{(k)}(v_n) = G_\infty^{(k)}(v). \tag{5.27}$$

Proof We prove the two statements in reverse order.

Statement (2) Let c_1 denote the supremum in (5.25). From (5.7) we have that

$$c_1 \geq \left\|\nabla G_n^{(k)}(v_n)\right\|_2 \geq n\, SQ_n(v_n) \qquad \forall n \geq 1. \tag{5.28}$$

Let $\delta > 0$, and let D_δ denote the neighborhood of D with width δ. Then (5.28) implies that

$$D^{1/n} v_n(x) \to 0 \qquad \text{uniformly in } [0,1] \setminus D_\delta,$$

which in turn implies that $v(x)$ is constant in each connected component of $[0,1] \setminus D_\delta$.

Since δ is arbitrary, this proves that $v \in PC_{D'}$ for some $D' \subseteq D$. On the other hand, assumption (5.23) and the convergence of jump heights (5.17) imply (5.26), which proves also that actually $D' = D$.

In order to prove (5.27), we split the integral in (2.7) into an integral in D_n, and an integral in $[0,1] \setminus D_n$. For the second one we apply (5.28) and we deduce that

$$0 \leq \frac{n}{2} \int_{[0,1]\setminus D_n} \log\left(1 + |D^{1/n} v_n(x)|^2\right) dx \leq \frac{n}{2} \log\left(1 + |SQ_n(v_n)|^2\right) \leq \frac{n}{2} \log\left(1 + \frac{c_1^2}{n^2}\right),$$

which proves that the integral in $[0,1] \setminus D_n$ tends to 0.

For the integral in D_n we apply (5.10) to the function v_n, and we deduce that

$$\frac{n}{2}\int_{D_n} \log\left(1+|D^{1/n}v_n(x)|^2\right)\,dx - k\log n = \frac{1}{2}\sum_{d\in D}\log\left(\frac{1}{n^2}+[J_{d,n}(v_n)]^2\right).$$

From the jump convergence (5.17) it follows that this expression tends to

$$\sum_{d\in D}\log|J_d(v)| = G_\infty^{(k)}(v),$$

which completes the proof of (5.27).

Statement (1) Let us take any subsequence (not relabeled) which realizes the lim inf in (5.24). We can assume that (5.25) holds true on this subsequence (otherwise the conclusion is trivial). As we have seen in the proof of statement (2), this implies in particular that $v \in PC_D$.

Now let us apply estimate (5.9) to the function v_n. We obtain that

$$\begin{aligned}
\left\|\nabla G_n^{(k)}(v_n)\right\|_2^2 \;\geq\;& \frac{1}{d_1+n^{-1}}\left(\frac{J_{d_1,n}(v_n)}{n^{-2}+[J_{d_1,n}(v_n)]^2}\right)^2 \\
&+\frac{1}{1-d_k+n^{-1}}\left(\frac{J_{d_k,n}(v_n)}{n^{-2}+[J_{d_k,n}(v_n)]^2}\right)^2 \\
&+\sum_{i=1}^{k-1}\frac{1}{d_{i+1}-d_i+n^{-1}}\left(\frac{J_{d_{i+1},n}(v_n)}{n^{-2}+[J_{d_{i+1},n}(v_n)]^2}-\frac{J_{d_i,n}(v_n)}{n^{-2}+[J_{d_i,n}(v_n)]^2}\right)^2.
\end{aligned}$$

Let us finally pass to the limit as $n \to +\infty$. Thanks to the jump convergence (5.17), the right-hand side tends to $\|\nabla G_\infty^{(k)}(v)\|_2^2$ (assumption (5.23) guarantees that all fractions have a finite limit), as defined in Definition 5.6. □

5.3 Hölder continuity of approximating and limit problems

Proof of Proposition 2.1

The existence of a local solution to the system of ordinary differential equations is trivial. From now on we identify the vector $(a_0(t),\ldots,a_k(t))$ with the function $u(t)$, and we define the points d_i $(i=0,1,\ldots,k,k+1)$ as in section 2.4.

L^∞ estimate Let $\psi \in C^1(\mathbb{R})$ be an even convex function, and let us set

$$\Psi(t) := \int_0^1 \psi(u(t))\,dx = \sum_{i=0}^{k}(d_{i+1}-d_i)\psi(a_i(t)).$$

Since ψ' is nondecreasing, with some computations it turns out that

$$\Psi'(t) = -\sum_{i=1}^{k} \frac{\psi'(a_i(t)) - \psi'(a_{i-1}(t))}{a_i(t) - a_{i-1}(t)} \leq 0. \tag{5.29}$$

Let us assume now in addition that $\psi(\sigma) = 0$ if and only if $|\sigma| \leq \|u_0\|_\infty$. Then we have that $\Psi(0) = 0$, $\Psi(t) \geq 0$ as soon as it is defined, and Ψ is nonincreasing because of (5.29). It follows that $\Psi(t) = 0$ as soon as it is defined, which proves (2.15).

Finite time break-down Let us consider the function

$$S(t) := \sum_{d \in D} |J_d(u(t))| = \sum_{i=1}^{k} |a_i(t) - a_{i-1}(t)|.$$

The sign of all jump heights is constant as soon as the solution is defined. This implies that $S(t)$ is smooth. Computing the time derivative, and rearranging the terms, we obtain that (for shortness's sake we drop the dependence on t in the second line)

$$\begin{aligned}
S'(t) &= -\frac{1}{d_1} \frac{1}{|a_1(t) - a_0(t)|} - \frac{1}{1 - d_k} \frac{1}{|a_k(t) - a_{k-1}(t)|} \\
&\quad - \sum_{i=1}^{k-1} \frac{1}{d_{i+1} - d_i} \left(\frac{1}{|a_{i+1} - a_i|} + \frac{1}{|a_i - a_{i-1}|} \right) \left(1 - \frac{a_i - a_{i-1}}{|a_i - a_{i-1}|} \cdot \frac{a_{i+1} - a_i}{|a_{i+1} - a_i|} \right) \\
&\leq -\frac{1}{d_1} \frac{1}{|a_1(t) - a_0(t)|} - \frac{1}{1 - d_k} \frac{1}{|a_k(t) - a_{k-1}(t)|} \\
&\leq -\frac{1}{\|u_0\|_\infty}.
\end{aligned}$$

Since $S(t)$ is clearly nonnegative, this implies that the solution cannot be global, and also provides an estimate on the life span.

Hölder continuity up to collision Let us compute the time derivative of the function $t \to G_\infty^{(k)}(u(t))$. Rearranging the terms we obtain that

$$\begin{aligned}
-\frac{d}{dt} G_\infty^{(k)}(u(t)) &= \sum_{i=0}^{k} a_i'(t) \left(\frac{1}{a_{i+1}(t) - a_i(t)} - \frac{1}{a_i(t) - a_{i-1}(t)} \right) \\
&= \sum_{i=0}^{k} \sqrt{d_{i+1} - d_i}\, a_i'(t) \cdot \frac{1}{\sqrt{d_{i+1} - d_i}} \left(\frac{1}{a_{i+1}(t) - a_i(t)} - \frac{1}{a_i(t) - a_{i-1}(t)} \right),
\end{aligned}$$

with the usual agreement to neglect the two fractions involving terms $a_i(t)$ with indices less than 0 or larger than k.

The two factors in each term of the sum are equal due to the system of ordinary differential equations. Therefore, the sum can be rewritten both in the form

$$\sum_{i=0}^{k}(d_{i+1}-d_i)[a_i'(t)]^2 = \|u'(t)\|_2^2,$$

and in the form

$$\sum_{i=0}^{k}\frac{1}{d_{i+1}-d_i}\left(\frac{1}{a_{i+1}(t)-a_i(t)}-\frac{1}{a_i(t)-a_{i-1}(t)}\right)^2 = \left\|\nabla G_\infty^{(k)}(u(t))\right\|_2^2.$$

As a consequence, we have also that

$$-\frac{d}{dt}G_\infty^{(k)}(u(t)) = \left\|\nabla G_\infty^{(k)}(u(t))\right\|_2^{2/3}\|u'(t)\|_2^{4/3}. \tag{5.30}$$

Now let us consider the function

$$H(t) := 3k\exp\left(\frac{2}{3k}G_\infty^{(k)}(u(t))\right). \tag{5.31}$$

We claim that the descent of $H(t)$ estimates the 1/4-Hölder constant of $u(t)$. To this end, we begin by computing the time derivative of $H(t)$. From (5.30) we have that

$$-H'(t) = 2\exp\left(\frac{2}{3k}G_\infty^{(k)}(u(t))\right)\left\|\nabla G_\infty^{(k)}(u(t))\right\|_2^{2/3}\|u'(t)\|_2^{4/3}. \tag{5.32}$$

Let us estimate the first two terms in the right-hand side. For the first one we have that

$$\exp\left(\frac{2}{3k}G_\infty^{(k)}(u(t))\right) = \left[\prod_{i=1}^{k}|a_i(t)-a_{i-1}(t)|\right]^{2/(3k)} \geq \min_{i=1,\ldots,k}|a_i(t)-a_{i-1}(t)|^{2/3}. \tag{5.33}$$

Let j be the index (or one of the indices) which realizes the minimum. Then from Cauchy-Schwarz inequality we have that

$$\begin{aligned}
\left\|\nabla G_\infty^{(k)}(u(t))\right\|_2^2 &\geq \sum_{i=0}^{j-1}(d_{i+1}-d_i)\cdot\sum_{i=0}^{j-1}\frac{1}{d_{i+1}-d_i}\left(\frac{1}{a_{i+1}(t)-a_i(t)}-\frac{1}{a_i(t)-a_{i-1}(t)}\right)^2\\
&\geq \left[\sum_{i=0}^{j-1}\left(\frac{1}{a_{i+1}(t)-a_i(t)}-\frac{1}{a_i(t)-a_{i-1}(t)}\right)\right]^2\\
&= \left(\frac{1}{a_j(t)-a_{j-1}(t)}\right)^2.
\end{aligned} \tag{5.34}$$

Plugging (5.33) and (5.34) into (5.32) we obtain that

$$-H'(t) \geq 2\|u'(t)\|_2^{4/3} \geq \|u'(t)\|_2^{4/3} \qquad \forall t \in [0, T_{\text{sing}}).$$

Now we integrate in $[s,t]$, and we exploit that $H(t)$ is nonnegative and nonincreasing (which follows from (5.32)). We deduce that

$$\int_s^t \|u'(\tau)\|_2^{4/3} d\tau \leq H(s) - H(t) \leq H(0) = 3k \exp\left(\frac{2}{3k} G_\infty^{(k)}(u_0)\right)$$

for every $0 \leq s \leq t < T_{\text{sing}}$. Finally, from Hölder's inequality we obtain that

$$\|u(t) - u(s)\|_2 \leq \int_s^t \|u'(\tau)\|_2 \, d\tau \leq \left(\int_s^t \|u'(\tau)\|_2^{4/3} d\tau\right)^{3/4} |t-s|^{1/4}$$

for every $0 \leq s \leq t < T_{\text{sing}}$. Combining the last two estimates we obtain (2.16). $\square$

Proof of Proposition 4.3

In analogy with (5.31), let us consider the function

$$H_n(t) := 3k \exp\left(\frac{2}{3k} \, G_n^{(k)}(u_n(t))\right).$$

Exploiting equation (2.14) (which is the same as (2.12)), in analogy with (5.32) we obtain that

$$\begin{aligned} -H_n'(t) &= -2\exp\left(\frac{2}{3k}\, G_n^{(k)}(u_n(t))\right) \langle \nabla G_n^{(k)}(u_n(t)), u_n'(t)\rangle \\ &= 2\exp\left(\frac{2}{3k}\, G_n^{(k)}(u_n(t))\right) \left\|\nabla G_n^{(k)}(u_n(t))\right\|_2^{2/3} \|u_n'(t)\|_2^{4/3}. \end{aligned} \tag{5.35}$$

Let us estimate the first two terms of this product for all $t \in [0, T_{\text{sing},n}]$. Let $J_{d,n}(t) := J_{d,n}(u_n(t))$ denote the discrete jump heights of $u_n(t)$, defined according to (2.4). From (5.5) we have that

$$\exp\left(\frac{2}{3k}\, G_n^{(k)}(u_n(t))\right) \geq \exp\left(\frac{2}{3}\log\left(\min_{d\in D} |J_{d,n}(t)|\right)\right) = \left(\min_{d\in D} |J_{d,n}(t)|\right)^{2/3}. \tag{5.36}$$

Moreover, from (5.8) we have that

$$\left\|\nabla G_n^{(k)}(u_n(t))\right\|_2^{2/3} \geq \left(\min_{d\in D} |J_{d,n}(t)|\right)^{-2/3}. \tag{5.37}$$

Plugging (5.36) and (5.37) into (5.35) we obtain that

$$-H_n'(t) \geq 2\|u_n'(t)\|_2^{4/3} \geq \|u_n'(t)\|_2^{4/3} \qquad \forall t \in [0, T_{\text{sing},n}].$$

Now we can conclude by integrating in $[s,t]$ and then applying Hölder's inequality exactly as in the proof of statement (3) of Proposition 2.1. $\square$

5.4 Well preparation

In this section we prove our well preparation result (Proposition 4.1). To this end, in a time $S_n \to 0$ we have three tasks to accomplish: extinguishing all vanishing jump points, adjusting the energy, and remaining close enough to u_0. In the next three lemmata we examine these three issues separately. Then we make an alternate use of them in order to conclude the proof of Proposition 4.1.

In the following, $J_d(u_0)$ denotes the jump heights of u_0, and $J_{d,n}(t) := J_{d,n}(u_n(t))$ denotes the discrete jump heights of $u_n(t)$, defined according to (2.4).

Lemma 5.9 (Infinitesimal interval convergence) *Let D, D', $\{u_{0n}\}$, u_0, $u_n(t)$ be as in Theorem 3.1. Let $\{S_n\}$ be any sequence of nonnegative real numbers such that $S_n \to 0$ as $n \to +\infty$.*

Then we have that

$$\lim_{n\to+\infty} \max_{t\in[0,S_n]} \|u_n(t) - u_0\|_2 = 0. \tag{5.38}$$

Moreover, for every n large enough we have that $u_n(S_n) \in PS_{D''(n),n}$ for some finite set $D''(n)$ (which may depend on n) such that $D' \subseteq D''(n) \subseteq D$.

Proof Let $w_{0n} \in PC_n$ denote the approximation of u_0 defined by

$$w_{0n}(x) := u_0\left(\frac{\lfloor nx\rfloor}{n}\right) \qquad \forall x \in [0,1],\ \forall n \geq 1.$$

Since u_0 is already a piecewise constant function, we have that w_{0n} coincides with u_0 but for the subintervals corresponding to elements of D'. In other words, one can think of w_{0n} as obtained by moving every jump of u in the point of the grid on its left. In particular, when n is large enough we have that

$$J_{d,n}(w_{0n}) = J_d(u_0) \qquad \forall d \in D'. \tag{5.39}$$

Of course we have also that $w_{0n} \to u_0$ in L^2 as $n \to +\infty$.

Let $k' := |D'|$, let K_0 be the constant defined in Lemma 5.4, and let us set

$$c_0 := \min_{d\in D'} |J_d(u_0)|, \quad c_1 := \max_{d\in D'} |J_d(u_0)|, \quad c_2 := \left(\min\left\{\frac{c_0}{12}, \frac{K_0}{12}\right\}\right)^3, \quad t_0 := \frac{c_0c_2}{8k'c_1}.$$

Let us consider the function

$$y_n(t) := \|u_n(t) - w_{0n}\|^2_{L^2((0,1))},$$

where the norm is intended with respect to the space variable, and let

$$R_n := \sup\{s \geq 0 : y_n(t) \leq c_2 \quad \forall t \in [0,s]\}. \tag{5.40}$$

Since

$$y_n(0) = \|u_{0n} - w_{0n}\|_2^2 \leq \left(\|u_{0n} - u_0\|_2 + \|u_0 - w_{0n}\|_2\right)^2,$$

it is easy to see that $y_n(0) \to 0$ as $n \to +\infty$, hence R_n is the supremum of a nonempty set when n is large enough. Let us fix n_0 big enough so that we can apply Lemma 5.4 for every $n \geq n_0$, and such that

$$y_n(0) \leq \frac{c_2}{2} \qquad \forall n \geq n_0.$$

We claim that for every $n \geq n_0$ we have that $R_n \geq t_0$, and

$$y_n(t) \leq y_n(0) + \frac{4k'c_1}{c_0}t \qquad \forall t \in [0, t_0]. \tag{5.41}$$

If we prove these claims, then (5.38) is proved. Indeed $S_n \to 0$, hence $S_n \leq t_0$ for n large enough, and therefore

$$\|u_n(t) - u_0\|_2 \leq \|u_n(t) - w_{0n}\|_2 + \|w_{0n} - u_0\|_2 \leq \left(y_n(0) + \frac{4k'c_1}{c_0}S_n\right)^{1/2} + \|w_{0n} - u_0\|_2$$

for every $t \in [0, S_n]$. Since all the terms in the right-hand side tend to zero, estimate (5.38) is proved.

So we are left to prove these claims. Let us consider $t \in [0, R_n]$. Let us apply (5.15) with $v = u_n(t)$ and $w = w_{0n}$ (this can be done because the discontinuity sets of $u(t)$ and w_{0n} are contained in the fixed set D). We obtain that

$$\min\left\{K_0, |J_{d,n}(t) - J_{d,n}(w_{0n})|\right\} \leq 6\,[y_n(t)]^{1/3} \leq 6c_2^{1/3} \qquad \forall d \in D'.$$

The right-hand side is less than or equal to $K_0/2$ due to our definition of c_2. Combining with (5.39) it follows that

$$|J_{d,n}(t) - J_d(u_0)| = |J_{d,n}(t) - J_{d,n}(w_{0n})| \leq 6c_2^{1/3} \qquad \forall d \in D'.$$

Therefore, from our definition of c_0 and c_2 it follows that

$$|J_{d,n}(t)| \geq |J_d(u_0)| - |J_{d,n}(t) - J_d(u_0)| \geq c_0 - 6c_2^{1/3} \geq c_0 - \frac{c_0}{2} = \frac{c_0}{2} \tag{5.42}$$

for every $d \in D'$. Now let us compute the time derivative of $y_n(t)$. From equation (2.12) and formula (2.8) we obtain that

$$\begin{aligned}
y_n'(t) &= 2\int_0^1 (u_n(x,t) - w_{0n}(x))\,\frac{d}{dt}u_n(x,t)\,dx \\
&= 2\int_0^1 (u_n(x,t) - w_{0n}(x)) \cdot nD^{-1/n}\left[\frac{D^{1/n}u_n(x,t)}{1 + |D^{1/n}u_n(x,t)|^2}\right]\,dx.
\end{aligned}$$

Now we apply the discrete version of the integration-by-parts formula (which actually is a simple algebraic manipulation of finite sums). We do not have boundary terms because $D^{1/n}u_n(x,t)$ is zero both in the last subinterval $(1-1/n,1)$ and in $(-1/n,0)$. We obtain that

$$\begin{aligned}
y_n'(t) &= -2\int_0^1 \left(D^{1/n}u_n(x,t) - D^{1/n}w_{0n}(x)\right)\cdot n\,\frac{D^{1/n}u_n(x,t)}{1+|D^{1/n}u_n(x,t)|^2}\,dx \\
&\leq 2\int_0^1 |D^{1/n}w_{0n}(x)|\cdot n\,\frac{|D^{1/n}u_n(x,t)|}{1+|D^{1/n}u_n(x,t)|^2}\,dx \\
&= 2\sum_{d\in D'}|J_{d,n}(w_{0n})|\cdot\frac{n^2|J_{d,n}(t)|}{1+n^2|J_{d,n}(t)|^2} \\
&\leq 2\sum_{d\in D'}|J_{d,n}(w_{0n})|\cdot\frac{1}{|J_{d,n}(t)|}.
\end{aligned}$$

The terms of the last sum can be estimated using (5.39), our definition of c_1, and (5.42). We obtain that $y_n'(t)\leq 4k'c_1/c_0$ for every $t\in[0,R_n]$, hence the estimate in (5.41) holds true for every $t\in[0,R_n]$.

Let us assume now that $R_n<t_0$ for some $n\geq n_0$. Due to the maximality of R_n we obtain that

$$c_2 = y_n(R_n)\leq y_n(0)+\frac{4k'c_1}{c_0}R_n < \frac{c_2}{2}+\frac{4k'c_1}{c_0}t_0 = c_2,$$

which is a contradiction. This completes the proof of our claims, hence also of (5.38).

It remains to understand the location of discrete jump points of $u_n(S_n)$. First of all, the creation of new jump points is forbidden by statement (6) of Theorem B, and this proves that $D''(n)\subseteq D$. On the other hand, we have that $u_n(S_n)\to u_0$, hence statement (1) of Lemma 5.5 implies that $D'\subseteq D''(n)$ when n is large enough. □

Lemma 5.10 (Energy adjustment) *Let D, D', $\{u_{0n}\}$, u_0, $u_n(t)$, be as in Theorem 3.1, and let $k:=|D|$. Let $\{S_n\}$ be any sequence such that $S_n\to 0$ and*

$$\lim_{n\to+\infty} ne^{-2nS_n} = 0. \tag{5.43}$$

Let $T_{\mathrm{sing},n}$ be defined as in (4.5), and let us assume that

$$T_{\mathrm{sing},n} > S_n \quad \textit{for every } n \textit{ large enough.} \tag{5.44}$$

Then we have that

$$\lim_{n\to+\infty} G_n^{(k)}(u_n(S_n)) = G_\infty^{(k)}(u_0), \tag{5.45}$$

where of course $G_\infty^{(k)}(u_0)=-\infty$ if D' is strictly contained in D.

Proof Since $S_n \to 0$, from Lemma 5.9 we know that $u_n(S_n) \to u_0$. Therefore, the Gamma-convergence of $G_n^{(k)}$ to $G_\infty^{(k)}$ implies that

$$\liminf_{n\to+\infty} G_n^{(k)}(u_n(S_n)) \geq G_\infty^{(k)}(u_0).$$

So we are left to prove the opposite inequality with the lim sup, which in turn is equivalent to show that

$$\limsup_{n\to+\infty} G_n^{(k)}(u_n(S_n)) \leq M \tag{5.46}$$

for every $M > G_\infty^{(k)}(u_0)$.

Let us fix any such M. To begin with, we claim that

$$\frac{1}{2}\sum_{d\in D} \log\left(\frac{1}{n^2} + J_{d,n}^2(t)\right) \leq M \qquad \forall t \in [0, S_n] \tag{5.47}$$

for every n large enough. Indeed let us assume that this is not the case. Then there exists a sequence $\{t_n\}$, with $t_n \in [0, S_n]$ for every $n \geq 1$, such that

$$\frac{1}{2}\sum_{d\in D} \log\left(\frac{1}{n^2} + J_{d,n}^2(t_n)\right) > M \qquad \text{for infinitely many } n\text{'s.} \tag{5.48}$$

On the other hand, from Lemma 5.9 we deduce that $u_n(t_n) \to u_0$, hence from the jump convergence (5.17) we obtain that $J_{d,n}(t_n) \to J_d(u_0)$ for every $d \in D$, where of course $J_d(u_0) = 0$ if $d \in D \setminus D'$. In particular we have that

$$\lim_{n\to+\infty} \frac{1}{2}\sum_{d\in D} \log\left(\frac{1}{n^2} + J_{d,n}^2(t_n)\right) = G_\infty^{(k)}(u_0) < M,$$

which contradicts (5.48).

From now on we work in the interval $[0, S_n]$, and we assume n to be large enough so that (5.47) and (5.44) hold true. Since $S_n < T_{\text{sing},n}$, in this interval we know that $u_n(t) \in PS_{D,n}$, hence we can use the estimates of Lemma 5.3. Let $SQ_n(t) := SQ_n(u_n(t))$ denote the subcritical incremental quotient of $u_n(t)$, defined according to (2.5). Let us estimate the time derivative of the function $t \to G_n^{(k)}(u_n(t))$. From equation (2.14) and estimate (5.7) we have that

$$\frac{d}{dt} G_n^{(k)}(u_n(t)) \;=\; -\left\|\nabla G_n^{(k)}(u_n(t))\right\|_2^2 \;\leq\; -n^2[SQ_n(t)]^2 \tag{5.49}$$

for every $t \in [0, S_n]$. On the other hand, from (5.6) and (5.47) we have that

$$\begin{aligned} G_n^{(k)}(u_n(t)) &\leq \frac{n}{2}\log\left(1 + [SQ_n(t)]^2\right) + \frac{1}{2}\sum_{d\in D} \log\left(\frac{1}{n^2} + J_{d,n}^2(t)\right) \\ &\leq \frac{n}{2}\log\left(1 + [SQ_n(t)]^2\right) + M, \end{aligned} \tag{5.50}$$

hence

$$[SQ_n(t)]^2 \;\geq\; \exp\left(\frac{2}{n}\left[G_n^{(k)}(u_n(t))-M\right]\right)-1 \;\geq\; \frac{2}{n}\left[G_n^{(k)}(u_n(t))-M\right].$$

Plugging this estimate into (5.49) we obtain that

$$\frac{d}{dt}G_n^{(k)}(u_n(t)) \leq -2n\left[G_n^{(k)}(u_n(t))-M\right] \qquad \forall t\in[0,S_n],$$

hence

$$G_n^{(k)}(u_n(t)) \leq \left[G_n^{(k)}(u_{0n})-M\right]e^{-2nt}+M \qquad \forall t\in[0,S_n].$$

Let us estimate $G_n^{(k)}(u_{0n})$. From (5.50) with $t=0$ we have that

$$G_n^{(k)}(u_{0n}) \;\leq\; \frac{n}{2}\log\left(1+[SQ_n(0)]^2\right)+M \;\leq\; \frac{n}{2}\log 2+M,$$

hence

$$G_n^{(k)}(u_n(S_n)) \leq \frac{\log 2}{2}\cdot ne^{-2nS_n}+M.$$

If S_n satisfies (5.43), then the first term in the right-hand side tends to 0. This completes the proof of (5.46), hence also the proof of (5.45). $\Box$

Lemma 5.11 (Discrete jump extinction) *Let D, D', $\{u_{0n}\}$, u_0, $u_n(t)$ be as in Theorem 3.1. Let us assume that D' is strictly contained in D, and let $T_{\mathrm{sing},n}$ be the first time when a discrete jump disappears, defined according to (4.5).*

Then we have that

$$\lim_{n\to+\infty} T_{\mathrm{sing},n}=0. \tag{5.51}$$

Proof Let us assume by contradiction that (5.51) is false. This is equivalent to say that there exist $\delta>0$ and a subsequence (not relabeled) such that

$$T_{\mathrm{sing},n}\geq\delta>0 \quad \text{for every } n \text{ large enough.} \tag{5.52}$$

Let us take any sequence $S_n\to 0$ satisfying (5.43), for example $S_n=n^{-1/2}$. Due to (5.52) this sequence satisfies also (5.44), hence we can apply Lemma 5.10 to this sequence. Since D' is strictly contained in D, we have that $G_\infty^{(k)}(u_0)=-\infty$, hence (5.45) reads as

$$\lim_{n\to+\infty} G_n^{(k)}(u_n(S_n))=-\infty.$$

Moreover, from (5.44) we have that $u_n(S_n)\in PS_{D,n}$ for every n large enough. Therefore, up to replacing the initial sequence $u_{0n}\to u_0$ with the sequence $u_n(S_n)\to u_0$ (the convergence to u_0 is due to Lemma 5.9), we can always assume that the sequence of initial data satisfies (3.1) and

$$\lim_{n\to+\infty} G_n^{(k)}(u_{0n})=-\infty. \tag{5.53}$$

Thus from now on we work under this assumption. Let us estimate the time derivative of the function $t \to G_n^{(k)}(u_n(t))$ in the interval $[0, T_{\text{sing},n}]$. Using equation (2.14), and estimate (5.8), we find that

$$\frac{d}{dt} G_n^{(k)}(u_n(t)) = -\left\|\nabla G_n^{(k)}(u_n(t))\right\|_2^2 \leq -\left(\min_{d\in D} |J_{d,n}(t)|\right)^{-2}.$$

Combining with (5.5) we deduce that

$$\frac{d}{dt} G_n^{(k)}(u_n(t)) \leq -\exp\left(-\frac{2}{k}\, G_n^{(k)}(u_n(t))\right).$$

Integrating this differential inequality in $[0, T_{\text{sing},n}]$ we obtain that

$$\frac{2}{k}\, T_{\text{sing},n} \leq \exp\left(\frac{2}{k}\, G_n^{(k)}(u_n(0))\right) - \exp\left(\frac{2}{k}\, G_n^{(k)}(u_n(T_{\text{sing},n}))\right) \leq \exp\left(\frac{2}{k}\, G_n^{(k)}(u_{0n})\right).$$

Thanks to assumption (5.53), this contradicts (5.52), hence it proves (5.51). □

Proof of Proposition 4.1

We argue by induction on $k - k'$, where $k := |D|$ and $k' := |D'|$.

Let us assume that $k - k' = 0$, namely $D = D'$. In this case we claim that conclusions (4.1) through (4.3) hold true for every sequence $S_n \to 0$ satisfying (5.43), for example $S_n := n^{-1/2}$. First of all, from Lemma 5.9 we obtain that (4.3) holds true, and $u_n(S_n) \in PS_{D',n}$ when n is large enough (because $D = D'$). This proves (4.1), and implies that assumption (5.44) is satisfied. Therefore, we can apply Lemma 5.10 and deduce (5.45). Since $k = k'$, this proves (4.2).

If $k - k' > 0$, then we begin by applying Lemma 5.11. We obtain the extinction of at least one discrete jump in a time $T_{\text{sing},n} \to 0$. Thus from Lemma 5.9 we have also that

$$u_n(T_{\text{sing},n}) \to u_0 \tag{5.54}$$

and every element of this new sequence belongs to $PS_{D''(n),n}$ for some $D''(n)$ strictly contained in D and such that $D''(n) \supseteq D'$. Up to subsequences we can assume that $D''(n) =: D''$ is independent of n. Since the number of possible choices of D'' is finite, we have only finitely many subsequences to consider. Therefore, it is enough to conclude on all such subsequences.

Setting $k'' := |D''|$, all these subsequences of (5.54) satisfy the same assumptions of the initial sequence with D'' instead of D, and in particular with $k'' - k' < k - k'$. Therefore, the conclusion follows from the inductive assumption. □

5.5 Convergence up to collisions

Proof of Proposition 4.2

"Safe intervals" Let $J_d(u_0)$ be the jump heights of u_0, and let $J_{d,n}(t) := J_{d,n}(u_n(t))$ be the discrete jump heights of $u_n(t)$, defined according to (2.4).

We say that $[0, T_0]$ (with $T_0 > 0$) is a safe interval if there exists a positive real number c_0, and a positive integer n_0 (both may depend on T_0) such that

$$|J_{d,n}(t)| \geq c_0 \qquad \forall d \in D,\ \forall t \in [0, T_0],\ \forall n \geq n_0. \tag{5.55}$$

We claim that safe intervals do exist. Indeed let us assume that this is not the case. Then there exist a sequence $\{d_k\} \subseteq D$, a sequence $n_k \to +\infty$ of positive integers, and a sequence $t_k \to 0$ of positive times such that

$$\lim_{k \to +\infty} J_{d_k,n_k}(t_k) = 0. \tag{5.56}$$

Up to subsequences, we can assume that d_k does not depend on k. Since $t_k \to 0$, Lemma 5.9 implies that $u_{n_k}(t_k) \to u_0$. At this point the jump convergence (5.17) contradicts (5.56). From (5.55) it is also clear that

$$T_{\text{sing},n} \geq T_0 \qquad \text{for every } n \text{ large enough}. \tag{5.57}$$

Boundedness and compactness in safe intervals Let $[0, T_0]$ be a safe interval according to (5.55). In this part of the proof we show that there exist real constants c_1 and c_2 such that

$$c_1 \leq G_n^{(k)}(u_n(t)) \leq c_2 \qquad \forall t \in [0, T_0],\ \forall n \geq 1, \tag{5.58}$$

and that there exists $v \in C^0([0, T_0]; L^2)$ such that (up to subsequences, which we do not relabel)

$$u_n \to v \quad \text{in } C^0\left([0, T_0]; L^2\right). \tag{5.59}$$

Indeed from the monotonicity of the function $t \to G_n^{(k)}(u_n(t))$ we have that

$$G_n^{(k)}(u_n(t)) \leq G_n^{(k)}(u_{0n}) \qquad \forall t \geq 0,$$

and the right-hand side is bounded from above because of (4.4). Moreover, (5.5) and the safe interval assumption (5.55) imply that $G_n^{(k)}(u_n(t)) \geq k \log c_0$ for every $n \geq n_0$. This completes the proof of (5.58). Now we exploit a compactness argument.

- For every $t \in [0, T_0]$ (and actually for every $t \geq 0$) the sequence $\{u_n(t)\}$ is relatively compact in L^2. Indeed from statement (2) of Lemma 5.5 (applied with $D' = D$) we have that
$$\sup_{n \geq 1} \left(\|u_{0n}\|_\infty + \left\|D^{1/n} u_{0n}\right\|_1\right) < +\infty,$$

hence from statements (3) and (4) of Theorem B we deduce that

$$\sup_{n\geq 1}\left(\|u_n(t)\|_\infty + \left\|D^{1/n}u_n(t)\right\|_1\right) < +\infty.$$

In other words, we control the L^∞-norm and the total variation of $u_n(t)$ (as functions of the space variable). This guarantees the required compactness.

- The functions $u_n : [0,T_0] \to L^2$ are 1/2-Hölder continuous, with equi-bounded Hölder constants, because of (5.58) and statement (2) of Theorem B. Alternatively, they are 1/4-Hölder continuous, with equi-bounded Hölder constants, because of (5.57), (4.4), and Proposition 4.3.

Therefore, Ascoli's Theorem implies that the sequence $\{u_n(t)\}$ is relatively compact in $C^0([0,T_0];L^2)$. This proves (5.59).

Passing to the limit in safe intervals Let $[0,T_0]$ be a safe interval according to (5.55), and let $v(t)$ be any limit point of the sequence $u_n(t)$. In this part of the proof we show that $v(t) = u(t)$ in the safe interval $[0,T_0]$. As a consequence, we obtain also that (5.59) holds true for the whole sequence, and not only up to subsequences.

To this end, we write the differential equations in integral form, as in the theory of maximal slope curves. From statement (1) of Proposition 5.7, we know that equation (2.14) implies that (and actually is equivalent to)

$$G_n^{(k)}(u_n(s)) - G_n^{(k)}(u_n(t)) = \frac{1}{2}\int_s^t \|u_n'(\tau)\|_2^2\,d\tau + \frac{1}{2}\int_s^t \left\|\nabla G_n^{(k)}(u_n(\tau))\right\|_2^2\,d\tau \tag{5.60}$$

for every $0 \leq s \leq t \leq T_0$.

From (5.58) we know that the left-hand side of (5.60) is bounded from above. In particular, setting $s = 0$ and $t = T_0$, we obtain that

$$\sup_{n\geq 1}\int_0^{T_0} \|u_n'(\tau)\|_2^2\,d\tau < +\infty, \tag{5.61}$$

$$\sup_{n\geq 1}\int_0^{T_0} \left\|\nabla G_n^{(k)}(u_n(\tau))\right\|_2^2\,d\tau < +\infty. \tag{5.62}$$

From (5.61) we easily deduce that $v \in H^1((0,T_0);L^2)$, and

$$\liminf_{n\to+\infty}\int_s^t \|u_n'(\tau)\|_2^2\,d\tau \geq \int_s^t \|v'(\tau)\|_2^2\,d\tau \tag{5.63}$$

for every $0 \leq s \leq t \leq T_0$.

Let us consider now the second term in the right-hand side of (5.60). Due to the safe interval assumption (5.55) and (5.57), we can apply Proposition 5.8. From (5.24) we deduce that

$$\liminf_{n\to+\infty}\left\|\nabla G_n^{(k)}(u_n(t))\right\|_2 \geq \left\|\nabla G_\infty^{(k)}(v(t))\right\|_2 \qquad \forall t\in[0,T_0].$$

Thus from Fatou's Lemma it follows that

$$\begin{aligned}\liminf_{n\to+\infty}\int_s^t \left\|\nabla G_n^{(k)}(u_n(\tau))\right\|_2^2\,d\tau &\geq \int_s^t \left(\liminf_{n\to+\infty}\left\|\nabla G_n^{(k)}(u_n(\tau))\right\|_2^2\right)\,d\tau \\ &\geq \int_s^t \left\|\nabla G_\infty^{(k)}(v(\tau))\right\|_2^2\,d\tau \end{aligned} \tag{5.64}$$

for every $0 \leq s \leq t \leq T_0$.

Now we consider the left-hand side of (5.60). The functions $t \to G_n^{(k)}(u_n(t))$ are equi-bounded and nonincreasing. By the usual compactness result for monotone functions (known as Helly's Lemma, see [1, Lemma 3.3.3]) there exists a nonincreasing function $\psi : [0, T_0] \to \mathbb{R}$ such that (up to subsequences)

$$\lim_{n\to+\infty} G_n^{(k)}(u_n(t)) = \psi(t) \qquad \forall t \in [0, T_0]. \tag{5.65}$$

Now we can take the liminf of both sides of (5.60). Thanks to (5.63), (5.64), and (5.65) we obtain that

$$\psi(s) - \psi(t) \geq \frac{1}{2}\int_s^t \|v'(\tau)\|_2^2\,d\tau + \frac{1}{2}\int_s^t \left\|\nabla G_\infty^{(k)}(v(\tau))\right\|_2^2\,d\tau. \tag{5.66}$$

It remains to characterize the function $\psi(t)$. Coming back to (5.62), and exploiting once more Fatou's Lemma, we obtain that

$$\int_0^{T_0}\left(\liminf_{n\to+\infty}\left\|\nabla G_n^{(k)}(u_n(\tau))\right\|_2^2\right)\,d\tau \leq \liminf_{n\to+\infty}\int_0^{T_0}\left\|\nabla G_n^{(k)}(u_n(\tau))\right\|_2^2\,d\tau < +\infty.$$

Therefore there exists a set $E \subseteq [0, T_0]$, with Lebesgue measure equal to 0, such that

$$\liminf_{n\to+\infty}\left\|\nabla G_n^{(k)}(u_n(t))\right\|_2 < +\infty \qquad \forall t \in [0, T_0] \setminus E.$$

As a consequence, for every $t \in [0, T_0] \setminus E$ there exists a (t-dependent) sequence $n_h \to +\infty$ such that

$$\sup_{h\in\mathbb{N}}\left\|\nabla G_{n_h}^{(k)}(u_{n_h}(t))\right\|_2 < +\infty.$$

On this subsequence we can apply statement (2) of Proposition 5.8 and deduce that

$$v(t) = \lim_{n\to+\infty} u_n(t) = \lim_{h\to+\infty} u_{n_h}(t) \in PC_D,$$

$J_d(v(t)) \geq c_0$ for every $d \in D$, and

$$\psi(t) = \lim_{n\to+\infty} G_n^{(k)}(u_n(t)) = \lim_{h\to+\infty} G_{n_h}^{(k)}(u_{n_h}(t)) = G_\infty^{(k)}(v(t)).$$

We have thus proved that

$$\psi(t) = G_\infty^{(k)}(v(t)) \qquad \forall t \in [0, T_0] \setminus E. \tag{5.67}$$

The last step is to prove the same equality for every $t \in (0, T_0)$. To this end we remark that $v(t)$ is a continuous function with values in PC_D, and $G_\infty^{(k)}$ is continuous in PC_D. It follows that the right-hand side of (5.67) is a continuous function. Therefore, in (5.67) we have a continuous function and a monotone function which coincide almost everywhere in $[0, T_0]$, hence they coincide everywhere in $(0, T_0)$.

Coming back to (5.66), we have proved that

$$G_\infty^{(k)}(v(s)) - G_\infty^{(k)}(v(t)) \geq \frac{1}{2}\int_s^t \|v'(\tau)\|_2^2\, d\tau + \frac{1}{2}\int_s^t \left\|\nabla G_\infty^{(k)}(v(\tau))\right\|_2^2 d\tau$$

for every $0 < s \leq t < T_0$. From statement (2) of Proposition 5.7, this is equivalent to say that $v(t)$ coincides with $u(t)$ in $[0, T_0]$.

We have also proved the energy convergence (4.9) for every $t \in (0, T_0)$.

Continuation up to first jump extinction Let $T_{0\infty}$ be the supremum of all $T_0 > 0$ such that $[0, T_0]$ is a safe interval according to (5.55). From (5.57), and the convergence results on safe intervals, it is easy to see that

$$\liminf_{n\to+\infty} T_{\mathrm{sing},n} \geq T_{0\infty}, \tag{5.68}$$

$$\lim_{n\to+\infty} u_n(t) = u(t) \in PC_D \qquad \forall t \in [0, T_{0\infty}), \tag{5.69}$$

$$\lim_{n\to+\infty} G_n^{(k)}(u_n(t)) = G_\infty^{(k)}(u(t)) \qquad \forall t \in (0, T_{0\infty}). \tag{5.70}$$

Let $\{R_m\}$ be an increasing sequence of positive real numbers such that $R_m \to T_{0\infty}$ as $m \to +\infty$, and let us set

$$A_{m,n} := \max\left\{\|u_n(t) - u(t)\|_2 : 0 \leq t \leq \min\{R_m, (1 - 1/n)T_{\mathrm{sing},n}\}\right\}.$$

Since $[0, R_m]$ is a safe interval for each m, we have that

$$A_{m,n} \overset{n\to+\infty}{\longrightarrow} 0 \overset{m\to+\infty}{\longrightarrow} 0.$$

Therefore, Lemma 5.1 (standard conclusion) implies the existence of a sequence $m_n \to +\infty$ of positive integers such that $A_{m_n,n} \to 0$ as $n \to +\infty$. We claim that

$$T_n := \min\left\{R_{m_n}, \left(1 - \frac{1}{n}\right) T_{\mathrm{sing},n}\right\}$$

is a sequence which satisfies (4.6) through (4.8). Indeed (4.6) is trivial, and (4.8) is equivalent to say that $A_{m_n,n} \to 0$. It remains to prove (4.7). From (5.68) we easily deduce that $T_n \to T_{0\infty}$. Therefore, proving (4.7) is equivalent to show that $T_{0\infty} = T_{\mathrm{sing}}$.

To this end, from (5.69) we immediately deduce that $T_{0\infty} \leq T_{\mathrm{sing}}$. Moreover, since

$$\|u_n(T_n) - u(T_{0\infty})\|_2 \leq \|u_n(T_n) - u(T_n)\|_2 + \|u(T_n) - u(T_{0\infty})\|_2,$$

from (4.8) and the continuity of u we have in particular that

$$u_n(T_n) \to u(T_{0\infty}). \tag{5.71}$$

Let us assume now by contradiction that $T_{0\infty} < T_{\text{sing}}$, hence $u(T_{0\infty}) \in PC_D$. In this case we claim that there exists $\delta > 0$ such that $[0, T_{0\infty} + \delta]$ is a safe interval, and this contradicts the maximality of $T_{0\infty}$.

In order to prove the claim, we argue as in the first paragraph of the proof. If the claim is false, then there exist a sequence $\{d_k\} \subseteq D$, a sequence $n_k \to +\infty$ of positive integers, and a sequence $\{t_k\}$ of times such that $t_k \in [0, T_{0\infty} + 1/k]$ for every $k \geq 1$, and such that (5.56) holds true. Up to subsequences, we can assume that d_k does not depend on k, and that t_k tends to some limit $t_\infty \in [0, T_{0\infty}]$. We can also assume that either $t_k \in [0, T_{n_k}]$ for every $k \geq 1$, or $t_k \in [T_{n_k}, T_{0\infty} + 1/k]$ for every $k \geq 1$.

In the first case we have that $u_{n_k}(t_k) \to u(t_\infty)$ because of (4.8). In the second case we have that $u_{n_k}(t_k) \to u(t_\infty) = u(T_{0\infty})$ because of Lemma 5.9 applied to the sequence of "initial data" (5.71). In both cases the limit lies in PC_D, hence the jump convergence (5.17) contradicts (5.56).

Therefore, we have proved that $T_{0\infty} = T_{\text{sing},n}$, hence also (4.7). At this point conclusion (4.9) is exactly (5.70). □

5.6 Proof of main results

Proof of Theorem 3.1

Global-in-time L^2-convergence We argue by induction on $k' = |D'|$. If $k' = 0$, then u_0 is a constant function, and $u(t) \equiv u_0$ is the stationary solution. In order to prove (3.3) it is therefore enough to show that the function $t \to \|u_n(x,t) - u_0\|_2$ is nonincreasing. This is true because in this case $u_n(t) - u_0$ is once again a solution of equation (2.12), hence its L^2-norm is a nonincreasing function of time because of statement (3) of Theorem B.

Now let us consider the case where $k' > 0$. In this case we argue in three steps.

First of all we exploit Proposition 4.1 in order to "well prepare" the sequence of initial data. Let $\{S_n\}$ be the sequence of times provided by Proposition 4.1. We have that $u_n(S_n) \to u_0$ is a "well prepared" sequence with respect to the discontinuity set D', in the sense that the elements of this sequence lie in the corresponding space $PS_{D',n}$ when n is large enough, and their k'-energies converge to the k'-energy of u_0. Now let us observe that

$$\max_{t\in[0,S_n]} \|u_n(t) - u(t)\|_2 \leq \max_{t\in[0,S_n]} \|u_n(t) - u_0\|_2 + \max_{t\in[0,S_n]} \|u_0 - u(t)\|_2.$$

The first term in the right-hand side tends to 0 as $n \to +\infty$ because of (4.3). The second term tends to 0 because $S_n \to 0$ and u is continuous. It follows that

$$\lim_{n\to+\infty} \max_{t\in[0,S_n]} \|u_n(t) - u(t)\|_2 = 0. \tag{5.72}$$

The second step is to apply Proposition 4.2 to the "well prepared" sequence of "initial data" $u_n(S_n) \to u_0$. Let $\{T_n\}$ be the sequence of times provided by Proposition 4.2. Then (4.8) reads as

$$\lim_{n\to+\infty} \max_{t\in[0,T_n]} \|u_n(S_n+t)-u(t)\|_2 = 0. \tag{5.73}$$

On the other hand we have that

$$\max_{t\in[S_n,S_n+T_n]} \|u_n(t)-u(t)\|_2 \leq \max_{t\in[0,T_n]} \|u_n(S_n+t)-u(t)\|_2 + \max_{t\in[0,T_n]} \|u(t)-u(S_n+t)\|_2.$$

The first term in the right-hand side tends to 0 as $n\to+\infty$ because of (5.73). The second term tends to 0 because $S_n \to 0$ and u is uniformly continuous. It follows that

$$\lim_{n\to+\infty} \max_{t\in[S_n,S_n+T_n]} \|u_n(t)-u(t)\|_2 = 0. \tag{5.74}$$

From (4.7) we have also that $S_n+T_n \to T_{\mathrm{sing}}$. Therefore, since

$$\|u_n\left(S_n+T_n\right)-u(T_{\mathrm{sing}})\|_2 \leq \|u_n\left(S_n+T_n\right)-u(S_n+T_n)\|_2 + \|u\left(S_n+T_n\right)-u(T_{\mathrm{sing}})\|_2,$$

from (5.74) and the continuity of u it follows that

$$u_n\left(S_n+T_n\right) \to u(T_{\mathrm{sing}}). \tag{5.75}$$

In the last step we consider this sequence. Due to (4.6), all the elements of this sequence belong to $PS_{D',n}$ when n is large enough. On the other hand, the limit $u(T_{\mathrm{sing}})$ lies in $PC_{D''}$ for some finite set D'' strictly contained in D'. Therefore, we can apply the inductive assumption to the sequence of "initial data" (5.75). We obtain that

$$\lim_{n\to+\infty} \sup_{t\geq 0} \|u_n(S_n+T_n+t)-u(T_{\mathrm{sing}}+t)\|_2 = 0. \tag{5.76}$$

Now let us observe that

$$\begin{aligned}\sup_{t\geq S_n+T_n} \|u_n(t)-u(t)\|_2 \;&\leq\; \sup_{t\geq 0} \|u_n(S_n+T_n+t)-u(T_{\mathrm{sing}}+t)\|_2 \\ &\quad + \sup_{t\geq 0} \|u(T_{\mathrm{sing}}+t)-u(S_n+T_n+t)\|_2.\end{aligned}$$

The first term in the right-hand side tends to 0 as $n\to+\infty$ because of (5.76). The second term tends to 0 because $S_n+T_n \to T_{\mathrm{sing}}$ and u is uniformly continuous. It follows that

$$\lim_{n\to+\infty} \sup_{t\geq S_n+T_n} \|u_n(t)-u(t)\|_2 = 0. \tag{5.77}$$

Therefore, (3.3) easily follows from (5.72), (5.74), (5.77).

Global-in-time "uniform" convergence Let $w_n(x,t)$ be defined by

$$w_n(x,t) := u\left(\frac{\lfloor nx \rfloor}{n}, t\right) \qquad \forall (x,t) \in [0,1] \times [0,+\infty),\ \forall n \geq 1.$$

Since $u(t)$ is already piecewise constant with respect to the space variable, we have that w_n and u coincide in $\mathcal{K}_n$. Moreover, both u and w_n are equi-bounded in L^∞-norm, and for every $t \geq 0$ the set $D_n(t)$, where u and w_n could be different, is the union of at most k' subintervals of length $1/n$. It follows that there exists a constant c_0 such that

$$\|u(t) - w_n(t)\|_2 \leq \frac{c_0}{\sqrt{n}} \qquad \forall t \geq 0. \tag{5.78}$$

Now for every $t \geq 0$ we have that

$$\|u_n(t) - w_n(t)\|_2 \leq \|u_n(t) - u(t)\|_2 + \|u(t) - w_n(t)\|_2.$$

Thanks to (3.3) and (5.78), both terms in the right-hand side tend to zero as $n \to +\infty$ independently of t, hence

$$\lim_{n \to +\infty} \sup_{t \geq 0} \|u_n(t) - w_n(t)\|_2 = 0. \tag{5.79}$$

In particular, when n is large enough we have that

$$3\|u_n(t) - w_n(t)\|_2^{2/3} < K_0 \qquad \forall t \geq 0,$$

where K_0 is the constant introduced in Lemma 5.4. Now we exploit once again that u and w_n coincide in $\mathcal{K}_n$, and we apply (5.14) with $v = u_n(t)$ and $w = w_n(t)$ (their discontinuity sets depend on time, but what is important is that they lie inside a fixed finite set D). When n in large enough we deduce that

$$\begin{aligned}
\|u_n(x,t) - u(x,t)\|_{L^\infty(\mathcal{K}_n)} &= \sup_{t \geq 0} \|u_n(x,t) - w_n(x,t)\|_{L^\infty((0,1)\setminus D_n(t))} \\
&\leq \sup_{t \geq 0} \|u_n(x,t) - w_n(x,t)\|_{L^\infty((0,1))} \\
&\leq 3 \sup_{t \geq 0} \|u_n(x,t) - w_n(x,t)\|_{L^2((0,1))}^{2/3}.
\end{aligned}$$

Therefore (3.5) follows from (5.79). □

Proof of Theorem 3.2

Let us consider the double index sequence $A_{m,n} := u_n(m/n)$ with values in L^2. Due to (3.6) and (3.7) we have that

$$A_{m,n} = v_n(m) \overset{n \to +\infty}{\longrightarrow} v(m) \overset{m \to +\infty}{\longrightarrow} v_\infty.$$

Let us apply Lemma 5.1 (refined conclusion) with $r_k := \sqrt{k}$, and let m_k be a sequence such that (5.1) holds true. Let us set $T_n := m_n/n$. Then we have that $T_n \to 0$ and $u_n(T_n) = A_{m_n,n} \to v_\infty$.

For every n large enough we have that the function $u_n(T_n)$ lies in some space $PS_{D''(n),n}$, with $D' \subseteq D''(n) \subseteq D$. Since the number of these subsets is finite, then up to subsequences we can always assume that $D''(n) =: D''$ does not depend on n. At this point we can apply Theorem 3.1, with D'' instead of D, to the sequence of "initial data" $u_n(T_n) \to v_\infty$. We obtain that

$$\lim_{n\to+\infty} \sup_{t\geq 0} \|u_n(T_n + t) - u(t)\|_2 = 0. \tag{5.80}$$

Now let us fix any $T > 0$. Then for every n large enough we have that $T_n \leq T$, hence

$$\begin{aligned}
\sup_{t\geq T} \|u_n(t) - u(t)\|_2 &\leq \sup_{t\geq T_n} \|u_n(t) - u(t)\|_2 \\
&= \sup_{t\geq 0} \|u_n(T_n + t) - u(T_n + t)\|_2 \\
&\leq \sup_{t\geq 0} \|u_n(T_n + t) - u(t)\|_2 + \sup_{t\geq 0} \|u(t) - u(T_n + t)\|_2.
\end{aligned}$$

The first term in the right-hand side tends to 0 as $n \to +\infty$ because of (5.80). The second term tends to 0 because u is uniformly continuous. This proves (3.8).

Finally, (3.9) follows from (3.8) as (3.5) follows from (3.3). □

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