Additive groups and semigroups of matrices on which the exponential is a homomorphism

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December 21, 2010

Abstract

Let G be a subgroup of $(\mathrm{M}_n(\mathbb{C}),+)$. We show that $M \mapsto \exp(M)$ is a group homomorphism from G to $(\mathrm{GL}_n(\mathbb{C}),\times)$ if and only if G consists of commuting matrices. We also prove that if S is a sub-semigroup of $(\mathrm{M}_n(\mathbb{C}),+)$ such that $M \mapsto \exp(M)$ is a homomorphism from S to $(\mathrm{GL}_n(\mathbb{C}),\times)$, then the linear subspace $\mathrm{Span}(S)$ has property L of Motzkin and Taussky.

AMS Classification: 15A16; 15A22

Keywords: matrix pencils, commuting exponentials, property L.

1 Introduction

We denote by $\mathrm{M}_n(\mathbb{C})$ the algebra of square matrices of order n with entries in the field of complex numbers. For $M \in \mathrm{M}_n(\mathbb{C})$, we denote by e^M or $\exp(M)$ its exponential. It is folklore that exp is not a group homomorphism from $(\mathrm{M}_n(\mathbb{C}),+)$ from $(\mathrm{GL}_n(\mathbb{C}),\times)$ if $n \geq 2$. However, when A and B are commuting matrices of $\mathrm{M}_n(\mathbb{C})$, then

$$e^{A+B} = e^A e^B = e^B e^A.$$

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The converse does not hold, moreover it is not necessary for A and B to be simultaneously triangularizable for this condition to hold. If however

$$\forall t \in \mathbb{R},\ e^{tA}e^{tB} = e^{tB}e^{tA}, \tag{1}$$

then a power series expansion at $t = 0$ shows that $AB = BA$. In the 1950's, pairs of matrices (A, B) of small size such that $e^{A+B} = e^A e^B$ have been under extensive scrutiny [3, 4, 6, 7, 9, 10]. More recently, E. Wermuth [16, 17] and Schmoeger [14, 15] have studied the problem of adding extra conditions on the matrices A and B for the commutation $e^A e^B = e^B e^A$ to imply the commutation of A with B. A few years ago [1] G. Bourgeois investigated, for small n, the pairs $(A, B) \in \mathrm{M}_n(\mathbb{C})^2$ which satisfy:

$$\forall k \in \mathbb{N}, \quad e^{kA+B} = e^{kA}e^B = e^B e^{kA}. \tag{2}$$

The main interest in this condition lies in the fact that, *contra* (1), it is not possible to use it to obtain information on A and B based on the sole local behavior of exp around 0. Bourgeois showed that (2) implies that A and B are simultaneously triangularizable when $n = 2$, and produced a proof that this also holds when $n = 3$. This last result is however false, as the following counterexample - communicated to us by Jean-Louis Tu - shows: consider the matrices

$$A_1 := 2i\pi \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B_1 := 2i\pi \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & -2 \\ 1 & 1 & 0 \end{bmatrix}.$$

Notice that A_1 and B_1 are not simultaneously triangularizable since they share no eigenvector (indeed, the eigenspaces of A_1 are the lines generated by the three vectors of the canonical basis, none of which is stabilized by B_1). However, for every $t \in \mathbb{C}$, a straightforward computation shows that the characteristic polynomial of $tA + B$ is

$$X\big(X - 2i\pi(t+2)\big)\big(X - 2i\pi(2t+3)\big).$$

Then for every $t \in \mathbb{N}$, the matrix $tA + B$ has three distinct eigenvalues in $2i\pi\mathbb{Z}$, hence is diagonalisable with $e^{tA+B} = I_3$. In particular $e^B = I_3$, and on the other hand $e^A = I_3$ which shows that condition (2) holds.

It then appears that one should strengthen Bourgeois' condition as follows in order to obtain at least the simultaneous triangularizability of A and B:

$$\forall (k, l) \in \mathbb{Z}^2, \quad e^{kA+lB} = e^{kA}e^{lB}. \tag{3}$$

Notice immediately that this condition implies that e^A and e^B commute. If indeed it holds, then

$$e^B e^A = \left(e^{-A}e^{-B}\right)^{-1} = \left(e^{-A-B}\right)^{-1} = e^{A+B} = e^A e^B.$$

Therefore (3) is equivalent to

$$\forall (k,l) \in \mathbb{Z}^2, \quad e^{kA+lB} = e^{kA}e^{lB} = e^{lB}e^{kA}. \tag{4}$$

Here is our main result:

Theorem 1. *Let $(A,B) \in M_n(\mathbb{C})^2$ satisfying (3). Then $AB = BA$.*

Note that the converse is trivial. The following corollary is straightforward:

Theorem 2. *Let G be a subgroup of $(M_n(\mathbb{C}),+)$ and assume that $M \mapsto \exp(M)$ is a homomorphism from $(G,+)$ to $(GL_n(\mathbb{C}),\times)$. Then $\forall (A,B) \in G^2,\ AB = BA$.*

Again, the converse is trivial. A special case will be an important step in our proof: recall (see e.g. Theorem 1.27 [5]) that the solutions of the equation $e^M = I_n$ are the diagonalisable matrices M such that $\operatorname{Sp}(M) \subset 2i\pi\mathbb{Z}$ (where $\operatorname{Sp}(M)$ denotes the set of eigenvalues of M). The case $e^A = e^B = I_n$ in Theorem 1 is thus obviously equivalent to the following result:

Proposition 3. *Let $(A,B) \in M_n(\mathbb{C})^2$. Assume that, for every $(k,l) \in \mathbb{Z}^2$, the matrix $kA + lB$ is diagonalisable and $\operatorname{Sp}(kA+lB) \subset \mathbb{Z}$. Then $AB = BA$.*

For sub-semigroups of $(\mathrm{M}_n(\mathbb{C}),+)$, the above results surely fail. A very simple counterexample is indeed given by the semigroup generated by

$$A := \begin{bmatrix} 0 & 0 \\ 0 & 2i\pi \end{bmatrix} \quad \text{and} \quad B := \begin{bmatrix} 0 & 1 \\ 0 & 2i\pi \end{bmatrix}.$$

One may however wonder if a sub-semigroup S on which exp is a homomorphism must be simultaneously triangularizable. Alas the additive semigroup generated by the matrices A_1 and B_1 above is a counterexample. Nevertheless, we will prove a weaker property. Before stating it, we need a few notations and definitions.

We denote by Σ_n the group of permutations of $\{1,\dots,n\}$, make it act on $\mathbb{C}^n$ by $\sigma.(z_1,\dots,z_n) := (z_{\sigma(1)},\dots,z_{\sigma(n)})$, and consider the quotient set $\mathbb{C}^n/\Sigma_n$.

The class of a list $(z_1,\dots,z_n)\in\mathbb{C}^n$ in this quotient space will be denoted by $[z_1,\dots,z_n]$. For $M\in\mathrm{M}_n(\mathbb{C})$, denote by $\chi_M(X)$ its characteristic polynomial,

$$\mathrm{OSp}(M):=[z_1,\dots,z_n],\quad \text{where } \chi_M(X)=\prod_{k=1}^{n}(X-z_k).$$

Definition 1. A pair $(A,B)\in\mathrm{M}_n(\mathbb{C})^2$ has **property L** when there are n linear forms $f_1,\dots,f_n$ on $\mathbb{C}^2$ such that

$$\forall(x,y)\in\mathbb{C}^2,\ \mathrm{OSp}(xA+yB)=\bigl[f_k(x,y)\bigr]_{1\le k\le n}.$$

A linear subspace V of $\mathrm{M}_n(\mathbb{C})$ has property L when there are n linear forms $f_1,\dots,f_n$ on V such that

$$\forall M\in V,\ \mathrm{OSp}(M)=\bigl[f_k(M)\bigr]_{1\le k\le n}.$$

Theorem 4. *Let S be a sub-semigroup of $(M_n(\mathbb{C}),+)$ and assume that $M\mapsto$* $\exp(M)$ *is a homomorphism from $(S,+)$ to $(GL_n(\mathbb{C}),\times)$. Then* $\mathrm{Span}(S)$ *has property L.*

Note that the converse is obviously false. We shall derive this last theorem from a more precise result on pairs satisfying condition (2):

Proposition 5. *Let $(A,B)\in M_n(\mathbb{C})^2$ satisfying* (2)*. Then (A,B) has property L.*

Structure of the paper: The proofs of Theorem 1 and of Proposition 5 have largely similar parts, so they will be tackled simultaneously. There are three main steps:

- We will prove Proposition 5 in the special case where $\mathrm{Sp}(A)\subset 2i\pi\mathbb{Z}$ and $\mathrm{Sp}(B)\subset 2i\pi\mathbb{Z}$. This will involve a study of the matrix pencil $A+zB$. We will then easily derive Proposition 3 using a refinement of the Motzkin-Taussky theorem.
- We will tackle the more general case $\mathrm{Sp}(A)\subset 2i\pi\mathbb{Z}$ and $\mathrm{Sp}(B)\subset 2i\pi\mathbb{Z}$ in Theorem 1 by using the Dunford decompositions of A and B together with Proposition 3.
- In the general case, we use an induction on n to reduce the situation to the previous one, both for Theorem 1 and Proposition 5.

In the last section, we derive Theorem 4 from Proposition 5.

2 Additive groups or semigroups of matrices with an integral spectrum

2.1 Property L for pairs of matrices with an integral spectrum

Our aim here is to prove the following proposition:

Proposition 6. *Let* $(A,B) \in M_n(\mathbb{C})^2$. *Assume that* $\mathrm{Sp}(kA+B) \subset \mathbb{Z}$ *for every* $k \in \mathbb{N}$. *Then* (A,B) *has property L.*

Notice that a pair (A,B) has property L if and only if there are affine maps $f_1,\dots,f_n$ from $\mathbb{C}$ to $\mathbb{C}$ such that

$$\forall z \in \mathbb{C},\ \mathrm{OSp}(A+zB) = \bigl[f_k(z)\bigr]_{1 \leq k \leq n}.$$

Before proving Proposition 6, let us first recall a few well-known facts on matrix pencils with complex entries. Denote by $\mathcal{K}(\mathbb{C})$ the quotient field of the integral domain $H(\mathbb{C})$ of entire functions (i.e. analytic functions from $\mathbb{C}$ to $\mathbb{C}$). Let $(A,B) \in \mathrm{M}_n(\mathbb{C})^2$. The **generic number** p of eigenvalues of the pencil $z \mapsto A+zB$ is defined as the number of the roots of $\chi_{A+\mathrm{id}_{\mathbb{C}}B}(X)$ in an algebraic closure of $\mathcal{K}(\mathbb{C})$. A complex z is called **regular** when $A+zB$ has exactly p eigenvalues, and **exceptional** otherwise. In a neighborhood of 0, the spectrum of $A+zB$ may be classically described with Puiseux series as follows (see chapter 7 of [2]): there exists a radius $r>0$, an integer $q \in \{1,\dots,n\}$, positive integers $d_1,\dots,d_q$, and analytic functions $f_1,\dots,f_q$ defined on a neighborhood of 0 such that,

$$\forall z \in \mathbb{C} \smallsetminus \{0\}, \quad |z|<r \;\Rightarrow\; \chi_{A+zB}(X) = \prod_{k=1}^{q} \prod_{\zeta \in \mathbb{U}_{d_k}(z)} \bigl(X - f_k(\zeta)\bigr),$$

where, for $N \geq 1$, we write $\mathbb{U}_N(z) := \bigl\{\zeta \in \mathbb{C} :\ \zeta^N = z\bigr\}$.

Assume now that $\mathrm{Sp}(kA+B) \subset \mathbb{Z}$ for every non-negative integer k. We then prove that $f_1,\dots,f_q$ are polynomial functions. Consider f_1 for example, and its power series expansion

$$f_1(z) = \sum_{k=0}^{+\infty} a_k z^k.$$

Set $N := p_1$ for convenience. Let k_0 be a positive integer such that $\frac{1}{k_0} < r$. For any integer $k \geq k_0$, notice that $k f_1(k^{-1/N})$ is an eigenvalue of $kA+B$ and is therefore an integer. It follows that

$$\forall k \in \mathbb{Z},\ k \geq k_0 \Rightarrow\ (k+1) f_1\big((k+1)^{-1/N}\big) - k f_1\big(k^{-1/N}\big) \in \mathbb{Z}.$$

Notice that

$$\forall x \in \mathbb{R}_+^*,\ x > \frac{1}{r} \Rightarrow\ x f_1(x^{-1/N}) = a_0 x + \sum_{k=1}^{+\infty} a_k x^{1-k/N}.$$

Hence, for any integer $k \geq k_0$,

$$(k+1) f_1\big((k+1)^{-1/N}\big) - k f_1\big(k^{-1/N}\big) = a_0 + \sum_{j \in \mathbb{N} \smallsetminus \{0,N\}} a_j \big((k+1)^{1-j/N} - k^{1-j/N}\big).$$

If $a_j \neq 0$ for some $j \geq 1$ with $j \neq N$, define r as the smallest such j, and notice that

$$\sum_{j \in \mathbb{N} \smallsetminus \{0,N\}} a_j \big((k+1)^{1-j/N} - k^{1-j/N}\big) \underset{k \to +\infty}{\sim} a_r\,(1 - r/N)\,k^{-r/N}.$$

The sequence $\Big((k+1) f_1\big((k+1)^{1/N}\big) - k f_1\big(k^{1/N}\big) - a_0\Big)_{k \geq k_0}$ must then both converge to 0, be integral-valued, on non-zero for large k. This is a contradiction therefore $\forall j \in \mathbb{N} \smallsetminus \{0, N\},\ a_j = 0$. The same line of reasoning shows that, for any $k \in \{1, \dots, q\}$, there exists $b_k \in \mathbb{C}$ such that $f_k(z) = f_k(0) + b_k z^{p_k}$ in a neighborhood of 0. It follows that, in a neighborhood of 0,

$$\chi_{A+zB}(X) = \prod_{k=1}^{q} (X - f_k(0) - b_k z)^{p_k}.$$

Therefore we have found affine maps $g_1, \dots, g_n$ from $\mathbb{C}$ to $\mathbb{C}$ such that, in a neighborhood of 0,

$$\chi_{A+zB}(X) = \prod_{k=1}^{n} \big(X - g_k(z)\big).$$

By analytic continuation[1], we deduce that

$$\forall z \in \mathbb{C},\ \chi_{A+zB}(X) = \prod_{k=1}^{n} \big(X - g_k(z)\big)$$

Hence (A, B) has property L, and Proposition 6 is proven.

[1]Note that the coefficients of these polynomials are polynomial functions of z which coincide on a neighborhood of 0.

2.2 Commutativity for subgroups of diagonalisable matrices with an integral spectrum

Here, we derive Proposition 3 from Proposition 6. The key point is that Kato's proof of the Motzkin-Taussky theorem ([8] p.85 Theorem 2.6) entails that Theorem 4 of [12] may be slightly refined as follows:

Theorem 7 (Refined Motzkin-Taussky theorem)**.** *Let $(A,B) \in M_n(\mathbb{C})^2$ be a pair of matrices which satisfies property L. Assume B is diagonalisable. If $A+z_0B$ is diagonalisable for every exceptional point z_0 of the matrix pencil $z \mapsto A+zB$, then $AB = BA$.*

Proof of Proposition 3. Let then $(A,B) \in \mathrm{M}_n(\mathbb{C})^2$ be a pair of diagonalisable matrices such that $kA+lB$ is diagonalisable with $\mathrm{Sp}(kA+lB) \subset \mathbb{Z}$ for every $(k,l) \in \mathbb{Z}^2$. Then Proposition 6 shows that (A,B) has property L. For $k \in [\![1,n]\!]$, choose $f_k : z \mapsto \alpha_k z + \beta_k$ so that

$$\forall z \in \mathbb{C},\ \mathrm{OSp}(A+zB) = \big[f_k(z)\big]_{1\leq k\leq n}.$$

Notice that $\mathrm{Sp}(A) = \{\alpha_1,\dots,\alpha_n\}$ and $\mathrm{Sp}(B) = \{\beta_1,\dots,\beta_n\}$, hence the α_k's and the β_k's are integers. It follows that the exceptional points of the matrix pencil $z \mapsto A+zB$ are rational numbers. However, the assumptions shows that $A+\frac{l}{k}\,B = \frac{1}{k}\,(k\,A+l\,B)$ is diagonalisable for every $(k,l) \in (\mathbb{Z}\smallsetminus\{0\})\times\mathbb{Z}$. The refined Motzkin-Taussky theorem thus shows that $AB = BA$. □

3 The case $\mathrm{Sp}(A) \subset 2i\pi\mathbb{Z}$ and $\mathrm{Sp}(B) \subset 2i\pi\mathbb{Z}$ in Theorem 1

Let A and B be matrices in $\mathrm{M}_n(\mathbb{C})$ satisfying (3) and such that $\mathrm{Sp}(A) \subset 2i\pi\mathbb{Z}$ and $\mathrm{Sp}(B) \subset 2i\pi\mathbb{Z}$. We consider the Dunford decompositions $A = D+N$ and $B = D'+N'$, where D and D' are diagonalisable, N and N' are nilpotent and $DN = ND$ and $D'N' = N'D'$. For every integer k, note that $kA = kD+kN$ (resp. $kB = kD'+kN'$) is the Dunford decomposition of kA (resp. of kB), and $\mathrm{Sp}(kD) = \mathrm{Sp}(kA) = k\,\mathrm{Sp}(A) \subset 2i\pi\mathbb{Z}$ (resp. $\mathrm{Sp}(kD') = \mathrm{Sp}(kB) = k\,\mathrm{Sp}(B) \subset 2i\pi\mathbb{Z}$) which shows that

$$e^{kA} = e^{kN} \quad \text{and} \quad e^{kB} = e^{kN'}.$$

Condition (4) thus translates into:

$$\forall (k,l) \in \mathbb{Z}^2, \quad e^{kA+lB} = e^{kN}e^{lN'} = e^{lN'}e^{kN}.$$

Note in particular that e^N and $e^{N'}$ commute. However, N is nilpotent hence $N = \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k}\left(e^N - I_n\right)^k$ which shows that N is a polynomial of e^N. Similarly N' is a polynomial of $e^{N'}$ therefore:

$$NN' = N'N.$$

With that in mind, the above condition yields:

$$\forall (k,l) \in \mathbb{Z}^2,\ e^{kA+lB} = e^{kN+lN'}.$$

Fix $(k,l) \in \mathbb{Z}^2$. Then $kN + lN'$ is nilpotent since N and N' are commuting nilpotent matrices, hence $kN + lN'$ is a polynomial of $e^{kN+lN'}$. Since $kA + lB$ commutes with e^{kA+lB}, it thus commutes with $kN + lN'$. However

$$kA + lB = (kD + lD') + (kN + lN')$$

hence

$$e^{kD+lD'} = e^{kA+lB} e^{-kN-lN'} = I_n.$$

In particular, this yields that $kD + lD'$ is diagonalisable with $\mathrm{Sp}(kD + lD') \subset 2i\pi\mathbb{Z}$, and the Dunford decomposition of $kA + lB$ is therefore $kA + lB = (kD + lD') + (kN + lN')$ since $kN + lN'$ commutes with $kA + lB$.

Applying Proposition 3 to the pair $\left(\frac{1}{2i\pi}D, \frac{1}{2i\pi}D'\right)$, we then find that D and D' commute. In particular (D, D') has property L, which yields affine maps $f_1, \dots, f_n$ from $\mathbb{C}$ to $\mathbb{C}$ such that

$$\forall z \in \mathbb{C},\ \mathrm{OSp}(D + zD') = \left[f_k(z)\right]_{1\leq k\leq n}.$$

The set $E := \left\{k \in \mathbb{Z} :\ \exists (i,j) \in \{1,\dots,n\}^2 :\ \ f_i \neq f_j \text{ and } f_i(k) = f_j(k)\right\}$ is clearly finite. We may then choose two distinct elements a and b in $\mathbb{Z} \smallsetminus E$. Notice then that

$$\forall (i,j) \in [\![1,n]\!]^2,\ f_i(a) = f_j(a) \Leftrightarrow f_i = f_j \Leftrightarrow f_i(b) = f_j(b).$$

Since D and D' are simultaneously diagonalisable, it easily follows that $D + aD'$ is a polynomial of $D + bD'$ and vice versa. Hence $N + aN'$ and $N + bN'$ both commute with $D + aD'$ and $D + bD'$. Since $N + aN'$ and $N + bN'$ both commute with one another, we deduce that $A + aB = (D + aD') + (N + aN')$ commutes with $A + bB = (D + bD') + (N + bN')$. Finally both A and B belong to $\mathrm{Span}(A + aB, A + bB)$ since $a \neq b$, therefore $AB = BA$.

4 Reduction to the situation where $\mathrm{Sp}(A) \subset 2i\pi\mathbb{Z}$ and $\mathrm{Sp}(B) \subset 2i\pi\mathbb{Z}$

Here, we use an induction on n to prove Theorems 1 and 5 in the general case. Both theorems are obviously true for $n = 1$, so we fix $n \geq 2$ and assume that they hold for any pair $(A, B) \in \mathrm{M}_k(\mathbb{C})^2$ with $k \in \{1, \ldots, n-1\}$. Let $(A, B) \in \mathrm{M}_n(\mathbb{C})^2$ satisfying (3) (respectively (2)).

Assume first that (A, B) is not irreducible, i.e. that there exists a non-trivial decomposition $\mathbb{C}^n = F \oplus G$ such that F and G are invariant linear subspaces for both A and B. Then there exists $p \in \{1, \ldots, n-1\}$, a non-singular matrix $P \in \mathrm{GL}_n(\mathbb{C})$ and square matrices A_1, B_1, A_2, B_2 respectively in $\mathrm{M}_p(\mathbb{C})$, $\mathrm{M}_p(\mathbb{C})$, $\mathrm{M}_{n-p}(\mathbb{C})$ and $\mathrm{M}_{n-p}(\mathbb{C})$ such that

$$A = P \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} P^{-1} \quad \text{and} \quad B = P \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} P^{-1}.$$

Since the pair (A, B) satisfies (3) (resp. (2)), it easily follows that this is also the case of (A_1, B_1) and (A_2, B_2), hence the induction hypothesis yields that (A_1, B_1) and (A_2, B_2) are commuting pairs (resp. satisfy property L), hence (A, B) is also a commuting pair (resp. satisfies property L).

Assume, for the rest of the section, that (A, B) is irreducible. Note that we lose no generality assuming furthermore that A satisfies:

$$\forall (\lambda, \mu) \in \mathrm{Sp}(A)^2,\ \lambda - \mu \in 2i\pi\mathbb{Q} \Rightarrow \lambda - \mu \in 2i\pi\mathbb{Z}. \tag{5}$$

Indeed, consider in general the finite set $\mathcal{E} := \mathbb{Q} \cap \frac{1}{2i\pi}\left\{\lambda - \mu \mid (\lambda, \mu) \in \mathrm{Sp}(A)^2\right\}$. Since it consists entirely of rational numbers, we may find some integer $p > 0$ such that $p\mathcal{E} \subset \mathbb{Z}$. Replacing A with pA, we notice that (pA, B) still satisfies (3) (resp. (2)) and is a commuting pair (resp. satisfies property L) if and only if (A, B) is a commuting pair (resp. satisfies property L).

Assume now that A satisfies (5) on top of all the previous assumptions, i.e. (A, B) is irreducible and satisfies (3) (resp. (2)). Let now $k \in \mathbb{N}$. Notice that e^A and e^B commute hence are simultaneously triangularizable (see Theorem 1.1.5 of [13]), which shows that the range of the map

$$\gamma_k : \begin{cases} \mathrm{Sp}(e^A) \times \mathrm{Sp}(e^B) & \longrightarrow \mathbb{C} \\ (\lambda, \mu) & \longmapsto \lambda^k \mu \end{cases}$$

contains $\mathrm{Sp}(e^{kA}e^B)$.

Claim 1. *With the above assumptions, there exists $k \in \mathbb{N} \smallsetminus \{0\}$ such that γ_k is one-to-one.*

Proof. Assume that for every $k \in \mathbb{N} \smallsetminus \{0\}$, there are distinct pairs (λ, μ) and (λ', μ') in $\mathrm{Sp}(e^A) \times \mathrm{Sp}(e^B)$ such that $\lambda^k \mu = (\lambda')^k \mu'$. Since $\mathrm{Sp}(e^A) \times \mathrm{Sp}(e^B)$ is finite and $\mathbb{N} \smallsetminus \{0\}$ is infinite, we may then find distinct pairs (λ, μ) and (λ', μ') in $\mathrm{Sp}(e^A) \times \mathrm{Sp}(e^B)$ and distinct non-zero integers a and b such that

$$\lambda^a \mu = (\lambda')^a \mu' \quad \text{and} \quad \lambda^b \mu = (\lambda')^b \mu'.$$

All those eigenvalues are non-zero hence $(\lambda/\lambda')^{a-b} = 1$ with $a \neq b$. It follows that $\frac{\lambda}{\lambda'}$ is a root of unity. However $\lambda = e^\alpha$ and $\lambda' = e^\beta$ for some $(\alpha, \beta) \in \mathrm{Sp}(A)^2$, which shows that $(a-b)(\alpha - \beta) \in 2i\pi\mathbb{Z}$. Condition (5) then yields $\alpha - \beta \in 2i\pi\mathbb{Z}$, hence $\lambda = \lambda'$. It then follows that $\mu = \mu'$, contradicting $(\lambda, \mu) \neq (\lambda', \mu')$. □

Choose finally $k \in \mathbb{N} \smallsetminus \{0\}$ such that γ_k is one-to-one. Notice that we lose no generality replacing A with kA, so we may assume, on top of the previous assumptions, that the map

$$\begin{cases} \mathrm{Sp}(e^A) \times \mathrm{Sp}(e^B) & \longrightarrow \mathbb{C} \\ (\lambda, \mu) & \longmapsto \lambda\mu \end{cases}$$

is one-to-one. For $M \in \mathrm{M}_n(\mathbb{C})$ and $\lambda \in \mathbb{C}$, denote by $C_\lambda(M)$ the characteristic subspace of M with respect to M, i.e. $C_\lambda(M) = \mathrm{Ker}(M - \lambda I_n)^n$. We now prove:

Claim 2. *The characteristic subspaces of e^A and e^B are all stabilized by A and B.*

Proof. Notice that $A+B$ commutes with e^{A+B} hence with $e^A e^B$. It thus stabilizes the characteristic subspaces of $e^A e^B$. Let us show that:

$$\forall \mu \in \mathrm{Sp}(e^B),\ C_\mu(e^B) = \bigoplus_{\lambda \in \mathrm{Sp}(e^A)} C_{\lambda\mu}(e^A e^B). \tag{6}$$

- Since e^B and e^A commute, they both stabilize the characteristic subspaces of e^B which shows that

$$\forall \mu \in \mathrm{Sp}(e^B),\ C_\mu(e^B) = \bigoplus_{\lambda \in \mathrm{Sp}(e^A)} C_\lambda(e^A) \cap C_\mu(e^B).$$

- Let $(\lambda,\mu) \in \mathrm{Sp}(e^A) \times \mathrm{Sp}(e^B)$. Since e^A and e^B commute, they both stabilize $C_\lambda(e^A) \cap C_\mu(e^B)$ and induce simultaneously triangularizable endomorphisms of $C_\lambda(e^A) \cap C_\mu(e^B)$ each with a sole eigenvalue, respectively λ and μ: it follows that
$$C_\lambda(e^A) \cap C_\mu(e^B) \subset C_{\lambda\mu}(e^A e^B).$$
- Finally, that $(\lambda,\mu) \mapsto \lambda\mu$ is one-to-one on $\mathrm{Sp}(e^A) \times \mathrm{Sp}(e^B)$ yields that $C_{\lambda\mu}(e^A e^B) \cap C_{\lambda'\mu'}(e^A e^B) = \{0\}$ for all distinct pairs (λ,μ) and (λ',μ') in $\mathrm{Sp}(e^A) \times \mathrm{Sp}(e^B)$. However
$$\mathbb{C}^n = \bigoplus_{\mu \in \mathrm{Sp}(e^B)} C_\mu(e^B) = \bigoplus_{\mu \in \mathrm{Sp}(e^B)} \bigoplus_{\lambda \in \mathrm{Sp}(e^A)} C_\lambda(e^A) \cap C_\mu(e^B)$$
and $\mathbb{C}^n$ is the sum of all the characteristic subspaces of $\exp(A)\exp(B)$. We deduce that
$$\forall (\lambda,\mu) \in \mathrm{Sp}(e^A) \times \mathrm{Sp}(e^B),\; C_{\lambda\mu}(e^A e^B) = C_\lambda(e^A) \cap C_\mu(e^B).$$
This yields (6).

We deduce that $A+B$ stabilizes every characteristic subspace of e^B, however this is also true of B since B commutes with e^B, hence A and B both stabilizes the characteristic subspaces of e^B. Symmetrically, every characteristic subspace of e^A is stabilized by both A and B. □

We may now conclude: if e^B has several eigenvalues, then the above claim contradicts the assumption that (A,B) is irreducible. It follows that e^B has a sole eigenvalue, and for the same reason this is also true of e^A. Choosing $(\alpha,\beta) \in \mathbb{C}^2$ such that $\mathrm{Sp}(e^A) = \{e^\alpha\}$ and $\mathrm{Sp}(e^B) = \{e^\beta\}$, we find that $\exp(A - \alpha\, I_n)$ and $\exp(B - \beta\, I_n)$ both have 1 as sole eigenvalue, hence $\mathrm{Sp}(A - \alpha\, I_n) \subset 2i\pi\mathbb{Z}$ and $\mathrm{Sp}(B - \beta\, I_n) \subset 2i\pi\mathbb{Z}$. Set $A' := A - \alpha\, I_n$ and $B' := B - \beta\, I_n$. We now conclude the proofs of Theorems 1 and 5 by considering the two cases separately:

- The case (A,B) satisfies (3): then the pair (A',B') clearly satisfies (3) so the proof from Section 3 yields that A' commutes with B', hence $AB = BA$.
- The case (A,B) only satisfies (2): then (A',B') obviously satisfies (2), hence $e^{A'}$ and $e^{B'}$ commute, hence are simultaneously triangularizable, and have 1 as sole eigenvalue. Therefore $e^{kA'+B'} = (e^{A'})^k e^{B'}$ has 1 as sole eigenvalue for every $k \in \mathbb{N}$. Proposition 6 then shows that $\bigl(\frac{A'}{2i\pi}, \frac{B'}{2i\pi}\bigr)$ has property L, which clearly entails that (A,B) has property L.

Thus Theorem 1 and Theorem 5 are proven.

5 Additive semigroups on which the exponential is a homomorphism

In this short section, we derive Theorem 4 from Proposition 5. It obviously suffices to prove the following lemma:

Lemma 8. *Let S be a sub-semigroup of $(M_n(\mathbb{C}),+)$. Assume that every pair $(A,B)\in S^2$ has property L. Then the linear subspace* $\operatorname{Span}(S)$ *has property L.*

Proof. We extract from S a basis $(A_1,\dots,A_r)$ of $\operatorname{Span}(S)$.
For every $j\in\{1,\dots,r\}$, we choose a list $(a_1^{(j)},\dots,a_n^{(j)})\in\mathbb{C}^n$ such that

$$\operatorname{OSp}(A_j)=\bigl[a_k^{(j)}\bigr]_{1\leq k\leq n}.$$

Then, for every list $(p_1,\dots,p_r)$ of non-negative integers, we find a list $(\sigma_1,\dots,\sigma_r)\in\Sigma_n^r$ such that

$$\operatorname{OSp}\biggl(\sum_{j=1}^r p_jA_j\biggr)=\biggl[\sum_{j=1}^r p_j\,a^{(j)}_{\sigma_j(k)}\biggr]_{1\leq k\leq n}:$$

this follows indeed from a trivial induction, using the fact that $\Bigl(\sum_{k=1}^{j-1}p_kA_k,A_j\Bigr)$ has property L for every $j\in\{2,\dots,r\}$.
Multiplying by inverses of positive integers, we readily generalize this as follows: for every $(z_1,\dots,z_r)\in\mathbb{Q}_+^r$ (where $\mathbb{Q}_+$ denotes the set of non-negative rationals), there exists a list $(\sigma_1,\dots,\sigma_r)\in\Sigma_n^r$ such that

$$\operatorname{OSp}\biggl(\sum_{j=1}^r z_jA_j\biggr)=\biggl[\sum_{j=1}^r z_j\,a^{(j)}_{\sigma_j(k)}\biggr]_{1\leq k\leq n}.$$

Now, we prove the following property, depending on $l\in\{0,\dots,r\}$, by downward induction:

> $\mathcal{P}(l)$: There exists a list $(\sigma_{l+1},\dots,\sigma_r)\in\Sigma_n^{r-l}$ such that, for every $(z_1,\dots,z_l)\in\mathbb{Q}_+^l$, there exists a list $(\sigma_1,\dots,\sigma_l)\in\Sigma_n^l$ satisfying:
>
> $$\forall(z_{l+1},\dots,z_r)\in\mathbb{C}^{r-l},\ \operatorname{OSp}\biggl(\sum_{j=1}^r z_jA_j\biggr)=\biggl[\sum_{j=1}^r z_j\,a^{(j)}_{\sigma_j(k)}\biggr]_{1\leq k\leq n}.$$

We already know that $\mathcal{P}(r)$ holds, whilst $\mathcal{P}(0)$ implies property L for $\mathrm{Span}(S)$. Let $l \in \{1, \dots, r\}$ such that $\mathcal{P}(l)$ holds, and choose a corresponding list $(\sigma_{l+1}, \dots, \sigma_r) \in \Sigma_n^{r-l}$. Fix $(z_1, \dots, z_{l-1}) \in \mathbb{Q}_+^{l-1}$. For every $r_l \in \mathbb{Q}_+$, we may then choose permutations $\sigma_1, \dots, \sigma_l$ such that:

$$\forall (z_{l+1}, \dots, z_r) \in \mathbb{C}^{r-l},\ \mathrm{OSp}\left(\sum_{j=1}^{r} z_j A_j\right) = \left[\sum_{j=1}^{r} z_j\, a^{(j)}_{\sigma_j(k)}\right]_{1 \leq k \leq n.}$$

Denote by $(\sigma_1^{z_l}, \dots, \sigma_l^{z_l})$ the chosen list. Since Σ_n^l is finite and $\mathbb{Q}_+ \cap (0,1)$ is infinite, we may find some list $(\sigma_1, \dots, \sigma_l) \in \Sigma_n^l$ which equals $(\sigma_1^{z_l}, \dots, \sigma_l^{z_l})$ for infinitely many z_l's in $\mathbb{Q}_+ \cap (0,1)$. Fixing $(z_{l+1}, \dots, z_r) \in \mathbb{C}^{n-l}$, an analytic continuation argument[2] then shows that

$$\forall z_l \in \mathbb{C}, \quad \chi_{\sum_{j=1}^{r} z_j A_j}(X) = \prod_{k=1}^{n}\left(X - \sum_{j=1}^{r} z_j a^{(j)}_{\sigma_j(k)}\right),$$

hence

$$\forall (z_l, \dots, z_r) \in \mathbb{C}^{r-l+1},\ \mathrm{OSp}\left(\sum_{j=1}^{r} z_j A_j\right) = \left[\sum_{j=1}^{r} z_j\, a^{(j)}_{\sigma_j(k)}\right]_{1 \leq k \leq n.}$$

This proves that $\mathcal{P}(l-1)$ holds, QED. □

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