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DISPLAYED EQUATIONS FOR GALOIS REPRESENTATIONS

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Abstract. The Galois representation associated to a p-divisible group over a noetherian complete local domain with perfect residue field is described in terms of its Dieudonné display. As a corollary we deduce in arbitrary characteristic Kisin's description of the Galois representation associated to a commutative finite flat p-group scheme over a p-adic discrete valuation ring in terms of its Breuil-Kisin module. This was obtained earlier by W. Kim by a different method.

Introduction

Let R be a noetherian complete local domain with perfect residue field k of positive characteristic p and with fraction field K of characteristic zero. For a p-divisible group G over R, the Tate module $T_p(G)$ is a free $\mathbb{Z}_p$-module of finite rank with a continuous action of the absolute Galois group $\mathcal{G}_K$. We want to describe the Tate module in terms of the Dieudonné display $\mathscr{P} = (P, Q, F, F_1)$ associated to G in [Zi2] and [La3], and relate this to other descriptions of the Tate module when R is a discrete valuation ring.

Let us recall that the Zink ring $\mathbb{W}(R)$ is a subring of the ring of Witt vectors $W(R)$ which is stable under the Frobenius endomorphism f of $W(R)$. The components of $\mathscr{P}$ are a finite free $\mathbb{W}(R)$-module P, a submodule Q such that P/Q is a free R-module, and f-linear maps $F : P \to P$ and $F_1 : Q \to P$, such that the image of F_1 generates P, and $F_1(v(u_0a)x) = aF(x)$ for $x \in P$ and $a \in \mathbb{W}(R)$, where v is the Verschiebung of $W(R)$, and u_0 is the unit of $W(R)$ defined by $u_0 = 1$ if p is odd and by $v(u_0) = p - [p]$ if $p = 2$. The twist by u_0 is necessary since v does not stabilise $\mathbb{W}(R)$ when $p = 2$.

To state the general result we need some notation. Let $\hat{R}^{\rm nr}$ be the completion of the strict henselisation of R, let $\tilde{K}$ be an algebraic closure of its fraction field $\hat{K}^{\rm nr}$, let $\tilde{R} \subset \tilde{K}$ be the integral closure of $\hat{R}^{\rm nr}$, and let $\hat{\tilde{R}}$ be its p-adic completion. Let

$$\mathbb{W}(\tilde{R}) = \varinjlim_{E} \mathbb{W}(R_E)$$

where E runs through the finite extensions of $\hat{K}^{\rm nr}$ contained in $\tilde{K}$ and where $R_E \subset E$ is the integral closure of $\hat{R}^{\rm nr}$. Let $\hat{\mathbb{W}}(\tilde{R})$ be the p-adic completion of $\mathbb{W}(\tilde{R})$. We define:

$$\hat{P}_{\tilde{R}} = \hat{\mathbb{W}}(\tilde{R}) \otimes_{\mathbb{W}(R)} P$$

$$\hat{Q}_{\tilde{R}} = \operatorname{Ker}(\hat{P}_{\tilde{R}} \to \hat{\tilde{R}} \otimes_R P/Q)$$

Let $\bar{K} \subset \tilde{K}$ be the algebraic closure of K and let $\tilde{\mathcal{G}}_K$ be the group of automorphisms of $\tilde{K}$ whose restriction to $\bar{K}\hat{K}^{\rm nr}$ is induced by an element of $\mathcal{G}_K$. The natural map $\tilde{\mathcal{G}}_K \to \mathcal{G}_K$ is surjective, and bijective when R is one-dimensional since then $\tilde{K} = \bar{K}\hat{K}^{\rm nr}$.

Our description of $T_p(G)$ is an exact sequence of $\tilde{\mathcal{G}}_K$-modules

$$0 \to T_p(G) \to \hat{Q}_{\tilde{R}} \xrightarrow{F_1 - 1} \hat{P}_{\tilde{R}} \to 0. \tag{1}$$

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If G is connected, a similar description of $T_p(G)$ in terms of the nilpotent display of G is part of Zink's theory of displays. In this case k need not be perfect; see [Me, Proposition 4.4]. The proof is recalled in Proposition 1.1 below. The exact sequence (1) is proved in Proposition 3.1 using the formula for the p-divisible group associated to a Dieudonné display given in [La3].

Assume now in addition that R is a discrete valuation ring. Then the exact sequence (1) can be related with the descriptions of $T_p(G)$ in terms of p-adic Hodge theory and in terms of Breuil-Kisin modules as follows.

First, let $M_{\rm cris}$ be the value of the covariant Dieudonné crystal of G over $A_{\rm cris}(R)$. It carries a filtration and a Frobenius, and by [Fa] there is a period homomorphism

$$T_p(G) \to \operatorname{Fil} M_{\rm cris}^{F=p}$$

which is bijective if p is odd, and injective with cokernel annihilated by p if $p = 2$. The v-stabilised Zink ring $\mathbb{W}^+(R) = \mathbb{W}(R)[v(1)]$ induces an extension $\hat{\mathbb{W}}^+(\tilde{R})$ of the ring $\hat{\mathbb{W}}(\tilde{R})$ defined above; the extension is trivial if p is odd. Since the v-stabilised Zink ring carries divided powers, the universal property of $A_{\rm cris}$ gives a homomorphism

$$\varkappa_{\rm cris} : A_{\rm cris}(R) \to \hat{\mathbb{W}}^+(\tilde{R}).$$

Using the relation between Dieudonné displays and Dieudonné crystals, $\varkappa_{\rm cris}$ induces a map

$$M_{\rm cris} \xrightarrow{\tau} \hat{\mathbb{W}}^+(\tilde{R}) \otimes_{\hat{\mathbb{W}}(\tilde{R})} \hat{P}_{\tilde{R}}$$

compatible with Frobenius and filtration. We will show that τ induces the identity on $T_p(G)$, viewed as a submodule of $\operatorname{Fil} M_{\rm cris}$ by the period homomorphism and as a submodule of $\hat{Q}_{\tilde{R}} \subset \hat{P}_{\tilde{R}}$ by (1); see Proposition 5.1.

Let us turn to Breuil-Kisin modules. Choose a generator π of the maximal ideal of R. Let $\mathfrak{S} = W(k)[[t]]$ and let $\sigma : \mathfrak{S} \to \mathfrak{S}$ extend the Frobenius automorphism of $W(k)$ by $t \mapsto t^p$; the case of more general Frobenius lifts is discussed below. We consider pairs $M = (M, \phi)$ where M is a finite $\mathfrak{S}$-module and where $\phi : M \to M^{(\sigma)}$ is an $\mathfrak{S}$-linear map with cokernel annihilated by the kernel of the map $\mathfrak{S} \to R$ given by $t \mapsto \pi$. Following [VZ], M is called a Breuil window if M is free over $\mathfrak{S}$, and M is called a Breuil module if M is a p-torsion $\mathfrak{S}$-module of projective dimension at most one.

It is known that p-divisible groups over R are equivalent to Breuil windows. This was conjectured by Breuil [Br] and proved by Kisin [Ki1, Ki2] if p is odd, and for connected groups if $p = 2$. The general case is proved in [La3] by showing that Breuil windows are equivalent to Dieudonné displays; here R can be regular of arbitrary dimension. (For odd p the last equivalence is already proved in [VZ] for some regular rings, including all discrete valuation rings.) As a corollary, commutative finite flat p-group schemes over R are equivalent to Breuil modules. Another proof for $p = 2$, related more closely to Kisin's methods, was obtained independently by W. Kim [K].

Let K_∞ be the extension of K generated by a chosen system of successive p-th roots of π. For a p-divisible group G over R let $T(G)$ be its Tate module, and for a commutative finite flat p-group scheme G over R let $T(G) = G(\bar{K})$. Kisin's and Kim's results include a description of $T(G)$ as a $\mathcal{G}_{K_\infty}$-representation in terms of the Breuil window or Breuil module (M, ϕ) associated to G. In the covariant theory it takes the following form:

$$T(G) = \{x \in M^{\rm nr} \mid \phi(x) = 1 \otimes x \text{ in } \mathfrak{S}^{\rm nr} \otimes_{\sigma, \mathfrak{S}^{\rm nr}} M^{\rm nr}\} \tag{2}$$

Here $M^{\rm nr} = \mathfrak{S}^{\rm nr} \otimes_{\mathfrak{S}} M$, and the ring $\mathfrak{S}^{\rm nr}$ is recalled in section 6 below.

We will show how (2) can be deduced from (1). It suffices to consider the case where G is a p-divisible group. The equivalence between Breuil windows and Dieudonné displays over R is induced by a homomorphism $\varkappa : \mathfrak{S} \to \mathbb{W}(R)$. It can be extended to

$$\varkappa^{\mathrm{nr}} : \mathfrak{S}^{\mathrm{nr}} \to \hat{\mathbb{W}}(\tilde{R}),$$

which allows to define a map of $\mathcal{G}_{K_\infty}$-modules

$$\{x \in M^{\mathrm{nr}} \mid \phi(x) = 1 \otimes x\} \xrightarrow{\tau} \{x \in \hat{Q}_{\tilde{R}} \mid F_1(x) = x\}.$$

Since the target is isomorphic to $T(G)$ by (1), the proof of (2) is reduced to showing that τ is bijective; see Proposition 7.2. The verification is easy if G is étale; the general case follows quite formally using a duality argument.

Finally we recall that the equivalence between Breuil windows and p-divisible groups requires only a Frobenius lift $\sigma : \mathfrak{S} \to \mathfrak{S}$ which stabilises the ideal $t\mathfrak{S}$ such that p^2 divides the linear term of the power series $\sigma(t)$. Let K_∞ be the extension of K generated by a chosen system of successive $\sigma(t)$-roots of π. If the linear term of $\sigma(t)$ is zero, which guarantees that $\varkappa^{\mathrm{nr}}$ is well-defined, we obtain an isomorphism (2) of $\mathcal{G}_{K_\infty}$-modules as before.

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1. The case of connected p-divisible groups

Let R be a complete noetherian local domain with residue field k of characteristic p, with fraction field K of characteristic zero, and with maximal ideal $\mathfrak{m}$. In this section we recall how the Tate module of a connected p-divisible group over R is expressed in terms of its nilpotent display.

Fix an algebraic closure $\bar{K}$ of K and let $\mathcal{G}_K = \mathrm{Gal}(\bar{K}/K)$. Let $\bar{R} \subset \bar{K}$ be the integral closure of R and let $\bar{\mathfrak{m}} \subset \bar{R}$ be the maximal ideal. For a finite extension E of K contained in $\bar{K}$ let $R_E = \bar{R} \cap E$, which is a complete noetherian local ring, and let $\mathfrak{m}_E \subset R_E$ be the maximal ideal. We write

$$\hat{W}(\mathfrak{m}_E) = \varprojlim_n \hat{W}(\mathfrak{m}_E/\mathfrak{m}_E^n); \qquad \hat{W}(\bar{\mathfrak{m}}) = \varinjlim_E \hat{W}(\mathfrak{m}_E).$$

Let $\bar{W}(\bar{\mathfrak{m}})$ be the p-adic completion of $\hat{W}(\bar{\mathfrak{m}})$ and let $\hat{\bar{\mathfrak{m}}}$ be the p-adic completion of $\bar{\mathfrak{m}}$. For a display $\mathscr{P} = (P, Q, F, F_1)$ over R we set

$$\bar{P}_{\bar{\mathfrak{m}}} = \bar{W}(\bar{\mathfrak{m}}) \otimes_{W(R)} P; \qquad \bar{Q}_{\bar{\mathfrak{m}}} = \mathrm{Ker}(\bar{P}_{\bar{\mathfrak{m}}} \to \hat{\bar{\mathfrak{m}}} \otimes_R P/Q).$$

The functor BT of [Zi1] induces an equivalence of categories between nilpotent displays over R and connected p-divisible groups over R; here $\mathscr{P}$ is called nilpotent if $\mathscr{P} \otimes_R k$ is V-nilpotent in the usual sense. The following is stated in [Me, Proposition 4.4].

Proposition 1.1 (Zink). *Let $\mathscr{P}$ be a nilpotent display over R and let G be the associated connected p-divisible group over R. There is a natural exact sequence of $\mathcal{G}_K$-modules*

$$0 \to T_p(G) \to \bar{Q}_{\bar{\mathfrak{m}}} \xrightarrow{F_1 - 1} \bar{P}_{\bar{\mathfrak{m}}} \to 0.$$

Here $T_p(G) = \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, G(\bar{K}))$ is the Tate module of G.

The proof of Proposition 1.1 uses the following well-known facts.

Lemma 1.2. *Let A be an abelian group.*

(i) *If A has no p-torsion then* $\mathrm{Ext}^1(\mathbb{Q}_p/\mathbb{Z}_p, A) = \varprojlim A/p^n A$.

(ii) *If $pA = A$ then* $\mathrm{Ext}^1(\mathbb{Q}_p/\mathbb{Z}_p, A)$ *is zero.*

Proof. The group $\mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, A)$ is isomorphic to $\varprojlim \mathrm{Hom}(\mathbb{Z}/p^n\mathbb{Z}, A)$ with transition maps induced by $p : \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^{n+1}\mathbb{Z}$. The corresponding system $\mathrm{Ext}^1(\mathbb{Z}/p^n\mathbb{Z}, A)$ is isomorphic to A/p^nA with transition maps induced by id_A. Thus there is an exact sequence

$$0 \to \varprojlim{}^1 A[p^n] \to \mathrm{Ext}^1(\mathbb{Q}_p/\mathbb{Z}_p, A) \to \varprojlim A/p^nA \to 0.$$

Both assertions of the lemma follow easily. □

For a p-divisible group G over R and for E as above we write

$$\hat{G}(R_E) = \varprojlim_n G(R_E/\mathfrak{m}_E^n); \qquad \hat{G}(\bar{R}) = \varinjlim_E \hat{G}(R_E).$$

Lemma 1.3. *Multiplication by p is surjective on $\hat{G}(\bar{R})$.*

Proof. Let $x \in \hat{G}(R_E)$ be given. The inverse image of x under p is a compatible system of $G[p]$-torsors Y_n over $R_E/\mathfrak{m}_E^n$. They define a $G[p]$-torsor Y over R_E. For some finite extension F of E the set $Y(F) = Y(R_F)$ is non-empty, and x becomes divisible by p in $\hat{G}(R_F)$. □

Proof of Proposition 1.1. Let E be a finite Galois extension of K in $\bar{K}$. Let

$$\hat{P}_{E,n} = \hat{W}(\mathfrak{m}_E/\mathfrak{m}_E^n) \otimes_{W(R)} P; \quad \hat{Q}_{E,n} = \mathrm{Ker}(\hat{P}_{E,n} \to \mathfrak{m}_E/\mathfrak{m}_E^n \otimes_R P/Q).$$

Recall that P is a finite free $W(R)$-module, and P/Q is a finite free R-module. The definition of the functor BT in [Zi1, Thm. 81] gives an exact sequence of $\mathcal{G}_K$-modules

$$0 \to \hat{Q}_{E,n} \xrightarrow{F_1 - 1} \hat{P}_{E,n} \to G(R_E/\mathfrak{m}_E^n) \to 0.$$

Since the modules $\hat{Q}_{E,n}$ form a surjective system with respect to n, applying $\varinjlim_E \varprojlim_n$ gives an exact sequence of $\mathcal{G}_K$-modules

$$0 \to \hat{Q}_{\bar{\mathfrak{m}}} \xrightarrow{F_1 - 1} \hat{P}_{\bar{\mathfrak{m}}} \to \hat{G}(\bar{R}) \to 0 \tag{1.1}$$

with $\hat{P}_{\bar{\mathfrak{m}}} = \hat{W}(\bar{\mathfrak{m}}) \otimes_{W(R)} P$ and $\hat{Q}_{\bar{\mathfrak{m}}} = \mathrm{Ker}(\hat{P}_{\bar{\mathfrak{m}}} \to \bar{\mathfrak{m}} \otimes_R P/Q)$. The p-adic completions of $\hat{P}_{\bar{\mathfrak{m}}}$ and $\hat{Q}_{\bar{\mathfrak{m}}}$ are $\bar{P}_{\bar{\mathfrak{m}}}$ and $\bar{Q}_{\bar{\mathfrak{m}}}$; here we use that $\bar{\mathfrak{m}} \otimes_R P/Q$ has no p-torsion. Moreover $\hat{P}_{\bar{\mathfrak{m}}}$ has no p-torsion since $\hat{W}(\bar{\mathfrak{m}})$ is contained in the $\mathbb{Q}$-algebra $W(\bar{K})$. Using Lemmas 1.3 and 1.2, the Ext-sequence of $\mathbb{Q}_p/\mathbb{Z}_p$ with (1.1) reduces to the short exact sequence

$$0 \to \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, \hat{G}(\bar{R})) \to \bar{Q}_{\bar{\mathfrak{m}}} \xrightarrow{F_1 - 1} \bar{P}_{\bar{\mathfrak{m}}} \to 0.$$

The proposition follows since the p^n-torsion of $\hat{G}(\bar{R})$ and $G(\bar{K})$ coincide. □

2. Some frame formalism

Before we proceed we introduce a formal definition. Let $\mathcal{F} = (S, R, I, \sigma, \sigma_1)$ be a frame in the sense of [La2] such that S is a $\mathbb{Z}_p$-algebra and σ is $\mathbb{Z}_p$-linear. For an $\mathcal{F}$-window $\mathscr{P} = (P, Q, F, F_1)$ we consider the *module of invariants*

$$T(\mathscr{P}) = \{x \in Q \mid F_1(x) = x\};$$

this is a $\mathbb{Z}_p$-module. Let us list some of its formal properties.

Functoriality in $\mathcal{F}$: Let $\alpha : \mathcal{F} \to \mathcal{F}' = (S', I', R', \sigma', \sigma_1')$ be a u-homomorphism of frames, thus $u \in S'$ is a unit, and we have $\sigma_1'\alpha = u \cdot \alpha\sigma_1$ on I. Assume that a unit $c \in S'$ with $c\sigma'(c)^{-1} = u$ is given. For an $\mathcal{F}$-window $\mathscr{P}$ as above, the S-linear map $P \to S' \otimes_S P$, $x \mapsto c \otimes x$ induces a $\mathbb{Z}_p$-linear map

$$\tau(\mathscr{P}) = \tau_c(\mathscr{P}) : T(\mathscr{P}) \to T(\alpha_*\mathscr{P}).$$

Duality: Recall that a bilinear form of $\mathcal{F}$-windows $\gamma : \mathscr{P} \times \mathscr{P}' \to \mathscr{P}''$ is an S-bilinear map $\gamma : P \times P' \to P''$ with $Q \times Q' \to Q''$ such that for $x \in Q$ and $x' \in Q'$ we have $\gamma(F_1x, F_1'x') = F_1''(\gamma(x, x'))$. It induces a bilinear map of $\mathbb{Z}_p$-modules

$T(\mathscr{P}) \times T(\mathscr{P}') \to T(\mathscr{P}'')$. Let us denote the $\mathcal{F}$-window (S, I, σ, σ_1) by $\mathcal{F}$ again. For each $\mathcal{F}$-window $\mathscr{P}$ there is a well-defined dual $\mathcal{F}$-window $\mathscr{P}^t$ together with a perfect bilinear form $\mathscr{P} \times \mathscr{P}^t \to \mathcal{F}$. It gives a bilinear map $T(\mathscr{P}) \times T(\mathscr{P}^t) \to T(\mathcal{F})$. In our applications, $T(\mathcal{F})$ will be free of rank one, and the bilinear map will turn out to be perfect.

Functoriality of duality: For a u-homomorphism of frames $\alpha : \mathcal{F} \to \mathcal{F}'$ with c as above and for a bilinear form of $\mathcal{F}$-windows $\gamma : \mathscr{P} \times \mathscr{P}' \to \mathscr{P}''$, the base change of γ multiplied by c^{-1} is a bilinear form of $\mathcal{F}'$-windows $\alpha_*\mathscr{P} \times \alpha_*\mathscr{P}' \to \alpha_*\mathscr{P}''$, which we denote by $\alpha_*(\gamma)$; see [La2, Lemma 2.14]. By passing to the modules of invariants we obtain a commutative diagram

$$\begin{array}{ccc} T(\mathscr{P}) \times T(\mathscr{P}') & \xrightarrow{\quad\gamma\quad} & T(\mathscr{P}'') \\ {\scriptstyle \tau(\mathscr{P})\times\tau(\mathscr{P}')}\big\downarrow & & \big\downarrow{\scriptstyle \tau(\mathscr{P}'')} \\ T(\alpha_*\mathscr{P}) \times T(\alpha_*\mathscr{P}') & \xrightarrow{\alpha_*(\gamma)} & T(\alpha_*\mathscr{P}''). \end{array}$$

This will be applied to the bilinear form $\mathscr{P} \times \mathscr{P}^t \to \mathcal{F}$.

3. The case of perfect residue fields

Let $R, K, k, \mathfrak{m}$ be as in section 1. Assume that the residue field k is perfect. As in [La3, Sections 2.3 and 2.8] we consider the frame

$$\mathscr{D}_R = \varprojlim_n \mathscr{D}_{R/\mathfrak{m}^n} = (\mathbb{W}(R), \mathbb{I}_R, R, f, f_1).$$

Windows over $\mathscr{D}_R$, called Dieudonné displays over R, are equivalent to p-divisible groups G over R by [Zi2] if p is odd and by [La3, Proposition 5.7] in general. The Tate module $T_p(G)$ can be expressed in terms of the associated Dieudonné display by a variant of Proposition 1.1 as follows.

Let R^{nr} be the strict henselisation of R. This is an excellent normal domain by [Gre] or [Se], so its completion $\hat{R}^{\mathrm{nr}}$ is a normal domain again. Let $K^{\mathrm{nr}} \subset \hat{K}^{\mathrm{nr}}$ be the fraction fields of $R^{\mathrm{nr}} \subset \hat{R}^{\mathrm{nr}}$, let $\tilde{K}$ be an algebraic closure of $\hat{K}^{\mathrm{nr}}$, and let $\tilde{R} \subset \tilde{K}$ be the integral closure of $\hat{R}^{\mathrm{nr}}$. We define a frame

$$\mathscr{D}_{\tilde{R}} = \varinjlim_E \varprojlim_n \mathscr{D}_{R_E/\mathfrak{m}_E^n} = (\mathbb{W}(\tilde{R}), \mathbb{I}_{\tilde{R}}, \tilde{R}, f, f_1)$$

where E runs through the finite extensions of $\hat{K}^{\mathrm{nr}}$ in $\tilde{K}$ and where $R_E \subset E$ is the integral closure of $\hat{R}^{\mathrm{nr}}$. Since $\tilde{R}$ has no p-torsion, the component-wise p-adic completion of $\mathscr{D}_{\tilde{R}}$ is a frame again, which we denote by

$$\hat{\mathscr{D}}_{\tilde{R}} = (\hat{\mathbb{W}}(\tilde{R}), \hat{\mathbb{I}}_{\tilde{R}}, \hat{\tilde{R}}, f, f_1).$$

Let $\bar{K} \subset \tilde{K}$ be the algebraic closure of K and let $\mathcal{G}_K = \mathrm{Gal}(\bar{K}/K)$. The tensor product $\bar{K} \otimes_{K^{\mathrm{nr}}} \hat{K}^{\mathrm{nr}}$ is a subfield of $\tilde{K}$, with equality if R is one-dimensional; here we use that in any case the etale coverings of the complements of the maximal ideals in $\operatorname{Spec} R^{\mathrm{nr}}$ and $\operatorname{Spec} \hat{R}^{\mathrm{nr}}$ coincide by [El, Th. 5] or by [Ar, II 2.1]. Let $\tilde{\mathcal{G}}_K$ be the group of automorphisms of $\tilde{K}$ whose restriction to $\bar{K}\hat{K}^{\mathrm{nr}}$ is induced by an element of $\mathcal{G}_K$. This group acts naturally on $\mathscr{D}_{\tilde{R}}$ and on $\hat{\mathscr{D}}_{\tilde{R}}$. The projection $\tilde{\mathcal{G}}_K \to \mathcal{G}_K$ is surjective, and bijective if R is one-dimensional.

Proposition 3.1. *Let $\mathscr{P}$ be a Dieudonné display over R and let G be the associated p-divisible group over R. Let $\hat{\mathscr{P}}_{\tilde{R}} = (\hat{P}_{\tilde{R}}, \hat{Q}_{\tilde{R}}, F, F_1)$ be the base change of $\mathscr{P}$ to $\hat{\mathscr{D}}_{\tilde{R}}$. There is a natural exact sequence of $\tilde{\mathcal{G}}_K$-modules*

$$0 \to T_p(G) \to \hat{Q}_{\tilde{R}} \xrightarrow{F_1 - 1} \hat{P}_{\tilde{R}} \to 0.$$

In particular we have an isomorphism of $\mathcal{G}_K$-modules

$$\mathrm{per} : T_p(G) \xrightarrow{\sim} T(\hat{\mathscr{P}}_{\tilde{R}})$$

which we call the period isomorphism is display theory.

Proof of Proposition 3.1. For a p-divisible group G over R and for finite extensions E of $\hat{K}^{\mathrm{nr}}$ in $\tilde{K}$ we set

$$\hat{G}(\hat{R}_E) = \varprojlim_n G(R_E/\mathfrak{m}_E^n); \qquad \hat{G}(\tilde{R}) = \varinjlim_E \hat{G}(\hat{R}_E).$$

Multiplication by p is surjective on $\hat{G}(\tilde{R})$ by Lemma 1.3 applied over $\hat{R}^{\mathrm{nr}}$. Suppose E is a normal extension of $\hat{K}^{\mathrm{nr}}$ and thus stable under $\tilde{\mathcal{G}}_K$. The rings $R_{E,n} = R_E/\mathfrak{m}_E^n$ are local Artin rings with residue field $\bar{k}$. Thus $R_{E,n}$ lies in the category $\mathcal{J}_{R/\mathfrak{m}^n}$ used in [La3, Section 5]. Let $\mathscr{P}_{E,n} = (P_{E,n}, Q_{E,n}, F, F_1)$ be the base change of $\mathscr{P}$ to $R_{E,n}$. Since every ind-étale covering of $\operatorname{Spec} R_{E,n}$ has a section, the definition of the functor BT in [La3, Proposition 5.4] as an ind-étale cohomology sheaf shows that $G(R_{E,n}) = \mathrm{BT}(\mathscr{P}_{E,n})$ is quasi-isomorphic to the complex of $\tilde{\mathcal{G}}_K$-modules in degrees $-1, 0, 1$

$$C_{E,n} = [Q_{E,n} \xrightarrow{F_1 - 1} P_{E,n}] \otimes [\mathbb{Z} \to \mathbb{Z}[1/p]].$$

Let

$$C_E = \varprojlim_n C_{E,n}; \qquad C = \varinjlim_E C_E$$

where E runs through the finite extensions of K^{nr} in $\bar{K}$, or equivalently the finite normal extensions. Since $G(R_{E,n})$ and the components of $C_{E,n}$ form surjective systems with respect to n, the complex C is quasi-isomorphic to $\hat{G}(\tilde{R})$. We will verify the following chain of isomorphisms (denoted $\cong$) and quasi-isomorphisms (denoted $\simeq$) of complexes of $\tilde{\mathcal{G}}_K$-modules, which proves the proposition. Here Ext^1 is taken component-wise in the second argument.

$$\begin{aligned} T_p(G) &\overset{(1)}{\cong} \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, \hat{G}(\tilde{R})) \overset{(2)}{\simeq} R\,\mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, \hat{G}(\tilde{R})) \\ &\overset{(3)}{\simeq} R\,\mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, C) \overset{(4)}{\simeq} \mathrm{Ext}^1(\mathbb{Q}_p/\mathbb{Z}_p, C[-1]) \overset{(5)}{\cong} [\hat{Q}_{\tilde{R}} \xrightarrow{F_1-1} \hat{P}_{\tilde{R}}]. \end{aligned}$$

Since the torsion subgroups of $G(\bar{K})$ and of $\hat{G}(\tilde{R})$ coincide, we have (1). For (2) we need that $\mathrm{Ext}^1(\mathbb{Q}_p/\mathbb{Z}_p, \hat{G}(\tilde{R}))$ vanishes, which is true since p is surjective on $\hat{G}(\tilde{R})$; see Lemma 1.2. The quasi-isomorphism between $\hat{G}(\tilde{R})$ and C gives (3). Let $(P_{\tilde{R}}, Q_{\tilde{R}}, F, F_1)$ be the base change of $\mathscr{P}$ to $\mathscr{D}_{\tilde{R}}$ and let $P_{\bar{k}} = W(\bar{k}) \otimes_{\mathbb{W}(R)} P$. The complex C can be identified with the cone of the map of complexes

$$[Q_{\tilde{R}} \xrightarrow{F_1-1} P_{\tilde{R}}] \to [P_{\bar{k}}[1/p] \xrightarrow{F_1-1} P_{\bar{k}}[1/p]].$$

Since $\tilde{R}$ is a domain of characteristic zero, the rings $\mathbb{W}(\tilde{R}) \subset W(\tilde{R})$ have no p-torsion, and thus the components of C have no p-torsion either. In particular, $\mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, C)$ vanishes, which proves (4). The p-adic completions of $P_{\tilde{R}}$ and $Q_{\tilde{R}}$ are $\hat{P}_{\tilde{R}}$ and $\hat{Q}_{\tilde{R}}$. Thus Lemma 1.2 gives (5). □

4. A variant for the prime 2

We keep the notation of section 3 and assume that $p = 2$. One may ask what the preceding constructions give if $\mathbb{W}$ and $\mathscr{D}$ are replaced by their v-stabilised variants $\mathbb{W}^+$ and $\mathscr{D}^+$. Recall that $\mathbb{W}^+(R) = \mathbb{W}(R)[v(1)]$ as a subring of $W(R)$, and we have a frame $\mathscr{D}_R^+ = (\mathbb{W}^+(R), \mathbb{I}_R^+, R, f, f_1)$ where f_1 is the inverse of v. The $\mathbb{W}(R)$-module $\mathbb{W}^+(R)/\mathbb{W}(R)$ is a one-dimensional k-vector space generated by $v(1)$; see [La3, Sections 1.4 and 2.5]. We put

$$\mathscr{D}_{\tilde{R}}^+ = \varinjlim_{E} \varprojlim_{n} \mathscr{D}_{R_E/\mathfrak{m}_E^n}^+ = (\mathbb{W}^+(\tilde{R}), \mathbb{I}_{\tilde{R}}^+, \tilde{R}, f, f_1)$$

with E as in section 3, and denote the p-adic completion of $\mathscr{D}_{\tilde{R}}^+$ by

$$\hat{\mathscr{D}}_{\tilde{R}}^+ = (\hat{\mathbb{W}}^+(\tilde{R}), \hat{\mathbb{I}}_{\tilde{R}}^+, \hat{\tilde{R}}, f, f_1).$$

For a p-divisible group G over R let G^m be the multiplicative part of G and define G^+ by the following homomorphism of exact sequences.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & G^m & \longrightarrow & G & \longrightarrow & G^u & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle p} & & \downarrow & & \| & & \\
0 & \longrightarrow & G^m & \longrightarrow & G^+ & \longrightarrow & G^u & \longrightarrow & 0
\end{array}$$

Proposition 4.1. *Let $\mathscr{P}$ be a Dieudonné display over R and let G be the associated p-divisible group over R. Let $\hat{\mathscr{P}}_{\tilde{R}}^+ = (\hat{P}_{\tilde{R}}^+, \hat{Q}_{\tilde{R}}^+, F, F_1^+)$ be the base change of $\mathscr{P}$ to $\hat{\mathscr{D}}_{\tilde{R}}^+$. There is a natural exact sequence of $\tilde{\mathcal{G}}_K$-modules*

$$0 \to T_p(G^+) \to \hat{Q}_{\tilde{R}}^+ \xrightarrow{F_1^+ - 1} \hat{P}_{\tilde{R}}^+ \to 0.$$

In particular we have an isomorphism of $\mathcal{G}_K$-modules

$$\mathrm{per}^+ : T_p(G^+) \xrightarrow{\sim} T(\hat{\mathscr{P}}_{\tilde{R}}^+).$$

Proof. Let $\bar{P}_{\bar{k}} = \bar{k} \otimes_{\mathbb{W}(R)} P$. We will construct the following commutative diagram with exact rows, where $\bar{F}$ is induced by F.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \hat{Q}_{\tilde{R}} & \longrightarrow & \hat{Q}_{\tilde{R}}^+ & \longrightarrow & \bar{P}_{\bar{k}} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle F_1 - 1} & & \downarrow{\scriptstyle F_1^+ - 1} & & \downarrow{\scriptstyle \bar{F} - 1} & & \\
0 & \longrightarrow & \hat{P}_{\tilde{R}} & \longrightarrow & \hat{P}_{\tilde{R}}^+ & \longrightarrow & \bar{P}_{\bar{k}} & \longrightarrow & 0
\end{array}$$

Here the Frobenius linear endomorphism $\bar{F}$ is nilpotent if G is unipotent, and is given by an invertible matrix if G is of multiplicative type. Thus $\bar{F} - 1$ is surjective with kernel an $\mathbb{F}_p$-vector space of dimension equal to the height of G^m, and Proposition 4.1 follows from Proposition 3.1.

The natural homomorphism $\hat{\mathbb{W}}(\tilde{R}) \to \hat{\mathbb{W}}^+(\tilde{R})$ is injective and defines a u_0-homomorphism of frames $\iota : \hat{\mathscr{D}}_{\tilde{R}} \to \hat{\mathscr{D}}_{\tilde{R}}^+$ where the unit $u_0 \in \mathbb{W}^+(\mathbb{Z}_2)$ is defined by $v(u_0) = p - [p]$; see [La3, Section 2.5]. Since u_0 maps to 1 in $W(\mathbb{F}_2)$ there is a unique unit c_0 of $\mathbb{W}^+(\mathbb{Z}_2)$ which maps to 1 in $W(\mathbb{F}_2)$ such that $c_0 f(c_0^{-1}) = u_0$, namely $c_0 = u_0 f(u_0) f^2(u_0) \cdots$; see the proof of [La2, Proposition 8.7].

The cokernel of ι is given by

$$\hat{\mathbb{I}}_{\tilde{R}}^+ / \hat{\mathbb{I}}_{\tilde{R}} = \hat{\mathbb{W}}^+(\tilde{R}) / \hat{\mathbb{W}}(\tilde{R}) = \bar{k} \cdot v(1); \tag{4.1}$$

see [La3, Le. 1.10]. We extend the operator $\mathbb{f}_1$ of $\hat{\mathscr{D}}_{\tilde{R}}$ to $\hat{\mathscr{D}}_{\tilde{R}}^+$ by $\mathbb{f}_1 = u_0^{-1} f_1$. Then $\mathbb{f}_1$ induces an f-linear endomorphism $\bar{\mathbb{f}}_1$ of $\bar{k} \cdot v(1)$. We claim that $\bar{\mathbb{f}}_1(v(1)) = v(1)$. It suffices to prove this formula in $\mathbb{W}^+(\mathbb{Z}_2)/\mathbb{W}(\mathbb{Z}_2) \cong \mathbb{F}_2$, and thus it suffices to show that $\mathbb{f}_1(v(1))$ does not lie in $\mathbb{W}(\mathbb{Z}_2)$. But $\mathbb{W}(\mathbb{Z}_2)$ is stable under the operator $x \mapsto \mathbb{v}(x) = v(u_0 x)$, and $\mathbb{v}(\mathbb{f}_1(v(1))) = v(1)$ does not lie in $\mathbb{W}(\mathbb{Z}_2)$. This proves the claim.

Let us extend the operator F_1 of $\hat{\mathscr{P}}_{\tilde{R}}$ to $\hat{\mathscr{P}}^+_{\tilde{R}}$ by $F_1 = u_0^{-1}F_1^+$. Since we have $c_0(F_1 - 1) = (F_1^+ - 1)c_0$ as a homomorphism $\hat{Q}^+_{\tilde{R}} \to \hat{P}^+_{\tilde{R}}$, it suffices to construct the above diagram with F_1 in place of F_1^+. Now (4.1) implies that $\hat{Q}^+_{\tilde{R}}/\hat{Q}_{\tilde{R}} = \hat{P}^+_{\tilde{R}}/\hat{P}_{\tilde{R}} = \bar{P}_{\bar{k}} \cdot v(1)$, which gives the exact rows. Clearly the left hand square commutes. The relation $F_1(ax) = \mathbb{f}_1(a)F(x)$ for $x \in \hat{P}^+_{\tilde{R}}$ and $a \in \hat{\mathbb{I}}^+_{\tilde{R}}$ applied with $a = v(1)$ shows that the right hand square commutes. □

Remark 4.2. The period isomorphisms per and per$^+$ satisfy per$^+ = \tau_{c_0}$ per, where $\tau_{c_0} : T(\hat{\mathscr{P}}_{\tilde{R}}) \to T(\hat{\mathscr{P}}^+_{\tilde{R}})$ is the homomorphism defined in section 2.

5. The relation with A_{cris}

Let R be a complete discrete valuation ring with perfect residue field k of characteristic p and fraction field K of characteristic zero. In this case our ring $\hat{\tilde{R}}$ is equal to $\hat{\bar{R}}$, the p-adic completion of the integral closure of R in $\bar{K}$. Let $A_{\text{cris}} = A_{\text{cris}}(R)$ and consider the frame

$$\mathcal{A}_{\text{cris}} = (A_{\text{cris}}, \operatorname{Fil} A_{\text{cris}}, \hat{\bar{R}}, \sigma, \sigma_1)$$

with $\sigma_1 = p^{-1}\sigma$. [1] For a p-divisible group G over R let $\mathbb{D}(G)$ be its covariant Dieudonné crystal. The free A_{cris}-module $M = \mathbb{D}(G_{\hat{\bar{R}}})_{A_{\text{cris}}}$ carries a filtration $\operatorname{Fil} M$ and a σ-linear endomorphism F. The operator $F_1 = p^{-1}F$ is well-defined on $\operatorname{Fil} M$, and we get an $\mathcal{A}_{\text{cris}}$-window $\mathcal{M} = (M, \operatorname{Fil} M, F, F_1)$; see [Ki1, A.2] or [La3, Proposition 3.15]. Faltings [Fa] constructs a period homomorphism

$$\text{per}_{\text{cris}} : T_p(G) \to \operatorname{Fil} M^{F=p} = T(\mathcal{M})$$

which is bijective if p is odd; for $p = 2$ the homomorphism is injective with cokernel annihilated by p. More precisely, for $p = 2$ the cokernel is zero if G is unipotent by [Ki2, Proposition 1.1.10], while the cokernel is non-zero if G is non-zero and of multiplicative type; thus the period homomorphism extends to an isomorphism $T_p(G^+) \cong T(\mathcal{M})$ with G^+ as in section 4.

Let us relate this with the period isomorphisms of sections 3 and 4. For the sake of uniformity, in the following we write $\mathbb{W}^+ = \mathbb{W}$ etc. if p is odd. Then $\hat{\mathbb{W}}^+(\tilde{R}) \to \hat{\bar{R}}$ is a divided power extension of p-adic rings for all p. By the universal property of A_{cris} there is a unique ring homomorphism

$$\varkappa_{\text{cris}} : A_{\text{cris}} \to \hat{\mathbb{W}}^+(\tilde{R})$$

which commutes with the projections to $\hat{\bar{R}}$. The proof of this universal property shows that $\varkappa_{\text{cris}} \circ \sigma = f \circ \varkappa_{\text{cris}}$. Since $\hat{\mathbb{W}}(\tilde{R})$ has no p-torsion, it follows that $\varkappa_{\text{cris}}$ is a $\mathcal{G}_K$-equivariant strict frame homomorphism

$$\varkappa_{\text{cris}} : \mathcal{A}_{\text{cris}} \to \hat{\mathscr{D}}^+_{\tilde{R}}.$$

Let $\mathscr{P}$ be the Dieudonné display associated to G so that $G = \text{BT}(\mathscr{P})$. The Dieudonné crystal $\mathbb{D}(G)$ gives rise to a $\mathscr{D}^+_R$-window $\Phi^+_R(G)$ by [La3, Section 3]. Its base change to $\hat{\mathscr{D}}^+_{\tilde{R}}$ is isomorphic to $\varkappa_{\text{cris}\,*}(\mathcal{M})$ by the functoriality of $\mathbb{D}(G)$. Let $\iota : \mathscr{D}_R \to \mathscr{D}^+_R$ be the inclusion. We have an isomorphism $\iota_*(\mathscr{P}) \cong \Phi^+_R(G)$ by [La3, Proposition 5.7] if p is odd and by [La3, Corollary 6.12] if $p = 2$. Thus we get an isomorphism $\hat{\mathscr{P}}^+_{\tilde{R}} \cong \varkappa_{\text{cris}\,*}(\mathcal{M})$, which induces a homomorphism of $\mathcal{G}_K$-modules

$$\tau : T(\mathcal{M}) \to T(\hat{\mathscr{P}}^+_{\tilde{R}})$$

as explained in section 2.

[1] Here we need that $\sigma_1(\operatorname{Fil} A_{\text{cris}})$ generates A_{cris}. But $\xi = p - [\underline{p}]$ lies in $\operatorname{Fil} A_{\text{cris}}$, and $\sigma_1(\xi) = 1 - [\underline{p}]^p/p$ is a unit because $[\underline{p}]$ lies in the divided power ideal $\operatorname{Fil} A_{\text{cris}} + pA_{\text{cris}}$.

Proposition 5.1. *The following diagram of $\mathcal{G}_K$-modules commutes up to multiplication by a p-adic unit which is independent of G.*

$$\begin{array}{ccc} T_p(G) & \xrightarrow{\mathrm{per}_{\mathrm{cris}}} & T(\mathcal{M}) \\ {\scriptstyle \mathrm{per}}\downarrow & & \downarrow{\scriptstyle \tau} \\ T(\hat{\mathscr{P}}_{\tilde{R}}) & \xrightarrow{\tau_{c_0}} & T(\hat{\mathscr{P}}^+_{\tilde{R}}). \end{array}$$

Remark 5.2. The p-adic unit in the statement of the proposition remains indetermined because only the existence of an isomorphism $\iota_*(\mathscr{P}) \cong \Phi^+_R(G)$ is proved in [La3], but a priori this isomorphism and the related homomorphism τ are defined only up to multiplication by a p-adic unit; cf. [La3, Lemma 4.6]. By a suitable choice one can arrange that the diagram commutes.

Remark 5.3. Since per is bijective by Proposition 3.1, the Propositions 3.1 and 4.1 together with Remark 4.2 imply that τ is an isomorphism. In fact, for this conclusion one needs only that the $\mathbb{Q}_p$-dimension of $T(\mathcal{M}) \otimes \mathbb{Q}$ is $\leq$ the height of G and that $\mathrm{per}_{\mathrm{cris}}$ is not bijective if $p = 2$ and G is non-zero of multiplicative type. Thus we recover the isomorphism $T_p(G^+) \cong T(\mathcal{M})$.

Proof of Proposition 5.1. We first consider the case $G = \mathbb{Q}_p/\mathbb{Z}_p$. Then per and τ_{c_0} are isomorphisms by Propositions 3.1 and 4.1. We have $T_p(G) = \mathbb{Z}_p$, and $M = \mathrm{Fil}\, M = A_{\mathrm{cris}}$ with Frobenius $p\sigma$, which implies that $\hat{Q}^+_{\tilde{R}} = \hat{P}^+_{\tilde{R}} = \hat{\mathbb{W}}^+_{\tilde{R}}$ with $F_1 = f$. Thus τ can be identified with the homomorphism $A^{\sigma=1}_{\mathrm{cris}} \to \hat{\mathbb{W}}^+(\tilde{R})^{f=1}$. Since the target is a $\mathbb{Z}_p$-algebra isomorphic to $\mathbb{Z}_p$ as a module, τ is bijective. Thus $\tau_{c_0} \circ \mathrm{per} = \rho \cdot \tau \circ \mathrm{per}_{\mathrm{cris}}$ for a well defined $\rho \in \mathbb{Z}_p^*$.

Let now G be arbitrary. Since the map $\tau_{c_0} \circ \mathrm{per} = \mathrm{per}^+$ is injective with cokernel annihilated by p, the composition $\gamma = p\rho \cdot (\mathrm{per}^+)^{-1} \circ \tau \circ \mathrm{per}_{\mathrm{cris}}$ is a well-defined functorial endomorphism of T_pG. We have to show that $\gamma = p$. By [Ta, 4.2], γ comes from an endomorphism γ_G of G; moreover γ_G is functorial in G and compatible with finite extensions of the base ring R inside $\bar{K}$. The endomorphisms γ_G induce a functorial endomorphism γ_H of each commutative finite flat p-group scheme H over a finite extension R' of R inside $\bar{K}$ because H can be embedded into a p-divisible group by Raynaud [BBM, 3.1.1]; cf. [Ki1, 2.3.5] or [La3, Proposition 4.1]. Assume that H is annihilated by p^r and let $H_0 = \mathbb{Z}/p^r\mathbb{Z}$. There is a finite extension R'' of R' inside $\bar{K}$ such that $H(\bar{K}) = H(R'') = \mathrm{Hom}_{R''}(H_0, H)$. Since $\gamma_{H_0} = p$ it follows that $\gamma_H = p$, and thus $\gamma_G = p$ for all G. □

6. The ring $\mathfrak{S}^{\mathrm{nr}}$

Let us recall the ring $\mathfrak{S}^{\mathrm{nr}}$ of [Ki1], which is denoted A^+_S in [Fo], and some of this properties. One starts with a two-dimensional complete regular local ring $\mathfrak{S}$ of characteristic zero with perfect residue field k of characteristic p equipped with a Frobenius lift $\sigma : \mathfrak{S} \to \mathfrak{S}$. Let $\delta : \mathfrak{S} \to W(\mathfrak{S})$ be the unique ring homomorphism with $\delta\sigma = f\delta$ and $w_0\delta = \mathrm{id}$. Let t be a generator of the kernel of $\mathfrak{S} \to W(\mathfrak{S}) \to W(k)$. Then $\mathfrak{S} = W(k)[[t]]$ and $\sigma(t) \in t\mathfrak{S}$.

Let $\mathcal{O}_{\mathcal{E}}$ be the p-adic completion of $\mathfrak{S}[t^{-1}]$ and let $\mathbb{E} = k((t))$ be its residue field. Fix a maximal unramified extension $\mathcal{O}_{\mathcal{E}^{\mathrm{nr}}}$ of $\mathcal{O}_{\mathcal{E}}$ and let $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{nr}}}}$ be its p-adic completion. Let $\mathbb{E}^{\mathrm{sep}}$ be the residue field of $\mathcal{E}^{\mathrm{nr}}$, let $\bar{\mathbb{E}}$ be an algebraic closure of $\mathbb{E}^{\mathrm{sep}}$, let $\mathcal{O}_{\mathbb{E}} = \mathfrak{S}/p\mathfrak{S} = k[[t]]$, and let $\mathcal{O}_{\bar{\mathbb{E}}} \subset \bar{\mathbb{E}}$ be its integral closure. The Frobenius lift σ on $\mathfrak{S}$ extends uniquely to $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{nr}}}}$ and induces an embedding

$$\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{nr}}}} \xrightarrow{\delta} W(\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{nr}}}}) \to W(\bar{\mathbb{E}})$$

with δ as above. Let $\mathfrak{S}^{\rm nr} = \mathcal{O}_{\widehat{\mathcal{E}^{\rm nr}}} \cap W(\mathcal{O}_{\bar{\mathbb{E}}})$ and $\mathfrak{S}^{(\rm nr)} = \mathcal{O}_{\mathcal{E}^{\rm nr}} \cap W(\mathcal{O}_{\bar{\mathbb{E}}})$ and $\mathfrak{S}^{\rm nr}_n = \mathcal{O}_{\mathcal{E}^{\rm nr}}/p^n\mathcal{O}_{\mathcal{E}^{\rm nr}} \cap W_n(\mathcal{O}_{\bar{\mathbb{E}}})$. These rings are stabilised by σ.

Suppose a finite extension $\mathbb{E}'$ of $\mathbb{E}$ contained in $\mathbb{E}^{\rm sep}$ is given. Let $\mathcal{O}_{\mathcal{E}'}$ be the étale extension of $\mathcal{O}_{\mathcal{E}}$ contained in $\mathcal{O}_{\mathcal{E}^{\rm nr}}$ with residue field $\mathbb{E}'$. We write $\mathfrak{S}' = \mathcal{O}_{\mathcal{E}'} \cap W(\mathcal{O}_{\bar{\mathbb{E}}})$ and $\mathfrak{S}'_n = \mathcal{O}_{\mathcal{E}'}/p^n\mathcal{O}_{\mathcal{E}'} \cap W_n(\mathcal{O}_{\bar{\mathbb{E}}})$; these are the invariants under $\mathcal{G}_{\mathbb{E}'} = \operatorname{Gal}(\mathbb{E}^{\rm sep}/\mathbb{E}')$ in $\mathfrak{S}^{\rm nr}$ and in $\mathfrak{S}^{\rm nr}_n$. Let us recall the following well-known consequence of [Fo, B 1.8.4].

Lemma 6.1. *We have $\mathfrak{S}^{\rm nr}/p^n\mathfrak{S}^{\rm nr} = \mathfrak{S}^{(\rm nr)}/p^n\mathfrak{S}^{(\rm nr)} = \mathfrak{S}^{\rm nr}_n$, and $\mathfrak{S}^{\rm nr}$ is the p-adic completion of $\mathfrak{S}^{(\rm nr)}$. The ring $\mathfrak{S}'$ is p-adic with $\mathfrak{S}'/p^n\mathfrak{S}' = \mathfrak{S}'_n$.*

Proof. It is easy to see that $\mathfrak{S}^{\rm nr} = \varprojlim \mathfrak{S}^{\rm nr}_n$ and that $\mathfrak{S}^{\rm nr}/p^n \to \mathfrak{S}^{\rm nr}_n$ is injective. The projection $\mathfrak{S}^{\rm nr}_{n+1} \to \mathfrak{S}^{\rm nr}_n$ is surjective by [Fo, B 1.8.4]. It follows that $\mathfrak{S}^{\rm nr}/p^n = \mathfrak{S}^{\rm nr}_n$, and $\mathfrak{S}^{\rm nr}$ is p-adic. The projection $\mathfrak{S}'_{n+1} \to \mathfrak{S}'_n$ is surjective too since $H^1(\mathcal{G}_{\mathbb{E}'}, \mathcal{O}_{\mathbb{E}^{\rm sep}})$ is zero. Again it follows that $\mathfrak{S}'/p^n = \mathfrak{S}'_n$, and $\mathfrak{S}'$ is p-adic. Since $\mathfrak{S}^{(\rm nr)}$ is the union over $\mathbb{E}'$ of $\mathfrak{S}'$, we get $\mathfrak{S}^{\rm nr}/p^n = \mathfrak{S}^{(\rm nr)}/p^n$, and thus $\mathfrak{S}^{\rm nr}$ is the p-adic completion of $\mathfrak{S}^{(\rm nr)}$. □

Since $\mathfrak{S}'/p\mathfrak{S}' = \mathcal{O}_{\mathbb{E}'}$ is a finite free $\mathcal{O}_{\mathbb{E}}$-module and a complete discrete valuation ring, $\mathfrak{S}'$ is a finite free $\mathfrak{S}$-module and a complete regular local ring of dimension two. Let k' be its residue field and let t' generate the kernel of $\mathfrak{S}' \to W(\mathfrak{S}') \to W(k')$. Then $\mathfrak{S}' = W(k')[[t']]$ and $\sigma(t') \in t'\mathfrak{S}'$.

Lemma 6.2. *Let r be minimal with $\sigma(t) \in t^r\mathfrak{S}$ and let r' be minimal with $\sigma(t') \in t'^{r'}\mathfrak{S}'$. Then $r = r'$.*

Proof. We have $t \in t'\mathfrak{S}'$. Let $t \equiv bt'^s$ modulo $t'^{s+1}\mathfrak{S}'$ with non-zero $b \in W(k')$ and $s \geq 1$. If $\sigma(t) \equiv at^r$ modulo $t^{r+1}\mathfrak{S}$ and $\sigma(t') \equiv a't'^{r'}$ modulo $t'^{r'+1}\mathfrak{S}'$ with non-zero $a \in W(k)$ and non-zero $a' \in W(k')$, then

$$\sigma(t) \equiv at^r \equiv ab^r t'^{rs} \mod t'^{rs+1}\mathfrak{S}',$$

$$\sigma(t) \equiv \sigma(b)a'^s t'^{r's} \mod t'^{r's+1}\mathfrak{S}'.$$

It follows that $r's = rs$ and hence $r = r'$. □

7. Breuil-Kisin modules

Let R be a complete discrete valuation ring with perfect residue field k of characteristic p and fraction field K of characteristic zero. Let $\mathfrak{S} = W(k)[[t]]$ and let $\sigma : \mathfrak{S} \to \mathfrak{S}$ be a Frobenius lift that stabilises the ideal $t\mathfrak{S}$. We choose a representation $R = \mathfrak{S}/E\mathfrak{S}$ where E has constant term p. Let $\pi \in R$ be the image of t, so π generates the maximal ideal of R.

For an $\mathfrak{S}$-module M let $M^{(\sigma)} = \mathfrak{S} \otimes_{\sigma,\mathfrak{S}} M$. We consider pairs (M, ϕ) where M is a finite $\mathfrak{S}$-module and where $\phi : M \to M^{(\sigma)}$ is an $\mathfrak{S}$-linear map with cokernel annihilated by E. Following the [VZ] terminology, (M, ϕ) is called a Breuil window (resp. a Breuil module) relative to $\mathfrak{S} \to R$ if the $\mathfrak{S}$-module M is free (resp. annihilated by a power of p and of projective dimension at most one). We have a frame in the sense of [La2]

$$\mathscr{B} = (\mathfrak{S}, E\mathfrak{S}, R, \sigma, \sigma_1)$$

with $\sigma_1(Ex) = \sigma(x)$ for $x \in \mathfrak{S}$. Windows $\mathscr{P} = (P, Q, F, F_1)$ over $\mathscr{B}$ are equivalent to Breuil windows relative to $\mathfrak{S} \to R$ by the functor $\mathscr{P} \mapsto (Q, \phi)$ where $\phi : Q \to Q^{(\sigma)}$ is the composition of the inclusion $Q \to P$ with the inverse of the isomorphism $Q^{(\sigma)} \cong P$ defined by $a \otimes x \mapsto aF_1(x)$.

Let $\varkappa$ be the ring homomorphism

$$\varkappa : \mathfrak{S} \xrightarrow{\delta} W(\mathfrak{S}) \to W(R).$$

It image lies in $\mathbb{W}(R)$ if and only if the endomorphism of $t\mathfrak{S}/t^2\mathfrak{S}$ induced by σ is divisible by p^2. In this case, $\varkappa : \mathfrak{S} \to \mathbb{W}(R)$ is a $\mathbb{u}$-homomorphism of frames $\mathscr{B} \to \mathscr{D}_R$ for a well-defined unit $\mathbb{u}$ of $\mathbb{W}(R)$, and $\varkappa$ induces an equivalence between $\mathscr{B}$-windows and $\mathscr{D}_R$-windows, which are equivalent to p-divisible groups over R; see [La3, Section 7]. As a corollary, Breuil modules relative to $\mathfrak{S} \to R$ are equivalent to commutative finite flat p-group schemes over R. Since $\mathbb{u}$ maps to 1 under $\mathbb{W}(R) \to W(k)$, there is a unique unit $\mathbb{c} \in \mathbb{W}(R)$ which maps to 1 in $W(k)$ with $\mathbb{c}\sigma(\mathbb{c}^{-1}) = \mathbb{u}$. It is given by $\mathbb{c} = \mathbb{u}\sigma(\mathbb{u})\sigma^2(\mathbb{u})\cdots$; see the proof of [La2, Proposition 8.7].

7.1. **Modules of invariants.** For a Breuil module or Breuil window (M, ϕ) relative to $\mathfrak{S} \to R$ we write $M^{\mathrm{nr}} = \mathfrak{S}^{\mathrm{nr}} \otimes_{\mathfrak{S}} M$ and $M^{\mathrm{nr}}_{\mathcal{E}} = \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{nr}}}} \otimes_{\mathfrak{S}} M$. Consider the $\mathbb{Z}_p$-modules:

$$T^{\mathrm{nr}}(M,\phi) = \{x \in M^{\mathrm{nr}} \mid \phi(x) = 1 \otimes x \text{ in } \mathfrak{S}^{\mathrm{nr}} \otimes_{\sigma,\mathfrak{S}^{\mathrm{nr}}} M^{\mathrm{nr}}\}$$
$$T^{\mathrm{nr}}_{\mathcal{E}}(M,\phi) = \{x \in M^{\mathrm{nr}}_{\mathcal{E}} \mid \phi(x) = 1 \otimes x \text{ in } \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{nr}}}} \otimes_{\sigma,\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{nr}}}}} M^{\mathrm{nr}}_{\mathcal{E}}\}$$

By [Fo, A 1.2], $T^{\mathrm{nr}}_{\mathcal{E}}(M,\phi)$ is finitely generated, and the natural map

$$\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{nr}}}} \otimes_{\mathbb{Z}_p} T^{\mathrm{nr}}_{\mathcal{E}}(M,\phi) \to \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{nr}}}} \otimes_{\mathfrak{S}} M \tag{7.1}$$

is bijective. It is pointed out in [Ki1, Ki2] that the natural map

$$T^{\mathrm{nr}}(M,\phi) \to T^{\mathrm{nr}}_{\mathcal{E}}(M,\phi) \tag{7.2}$$

is bijective as well. If (M, ϕ) is a Breuil window, this follows from the proof of [Fo, 1.8.4]. If (M, ϕ) is a Breuil module, the map (7.2) is injective since the group $X = \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{nr}}}}/\mathfrak{S}^{\mathrm{nr}}$ has no p-torsion and thus $\mathrm{Tor}_1^{\mathfrak{S}}(X, M)$ is zero. One can find a Breuil window (M', ϕ') and a surjective map $(M', \phi') \to (M, \phi)$. Then $T^{\mathrm{nr}}(M', \phi') \cong T^{\mathrm{nr}}_{\mathcal{E}}(M', \phi') \to T^{\mathrm{nr}}_{\mathcal{E}}(M, \phi)$ is surjective, thus (7.2) is surjective.

7.2. **The choice of K_∞.** Let $\hat{\bar{\mathfrak{m}}}$ be the maximal ideal of $\hat{\bar{R}}$. The power series $\sigma(t)$ defines a map $\sigma(t) : \hat{\bar{\mathfrak{m}}} \to \hat{\bar{\mathfrak{m}}}$. This map is surjective, and the inverse images of algebraic elements are algebraic by the Weierstrass preparation theorem. Choose a system of elements $(\pi^{(n)})_{n\ge1}$ of $\bar{K}$ with $\pi^{(0)} = \pi$ and $\sigma(t)(\pi^{(n+1)}) = \pi^{(n)}$, and let K_∞ be the extension of K generated by all $\pi^{(n)}$. The system $(\pi^{(n)})$ corresponds to an element $\underline{\pi} \in \mathcal{R} = \varprojlim \bar{R}/p\bar{R}$, the limit taken with respect to Frobenius.

We embed $\mathcal{O}_{\mathbb{E}} = k[[t]]$ into $\mathcal{R}$ by $t \mapsto \underline{\pi}$, and identify $\mathbb{E}^{\mathrm{sep}}$ and $\bar{\mathbb{E}}$ with subfields of $\operatorname{Frac}\mathcal{R}$; thus $W(\bar{\mathbb{E}}) \subset W(\operatorname{Frac}\mathcal{R})$. Then $\mathfrak{S}^{\mathrm{nr}} = \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{nr}}}} \cap W(\mathcal{R})$, and the unique ring homomorphism $\theta : W(\mathcal{R}) \to \hat{\bar{R}}$ which lifts the projection $W(\mathcal{R}) \to \bar{R}/p\bar{R}$ induces a homomorphism

$$pr^{\mathrm{nr}} : \mathfrak{S}^{\mathrm{nr}} \to \hat{\bar{R}}.$$

Let us verify that its restriction to $\mathfrak{S}$ is the given projection $\mathfrak{S} \to R$.

Lemma 7.1. *We have $pr^{\mathrm{nr}}(t) = \pi$.*

Proof. The lemma is evident if $\sigma(t) = t^p$ since then $\delta(t) = [t]$ in $W(\mathfrak{S})$, which maps to $[\underline{\pi}]$ in $W(\mathcal{R})$, and $\theta([\underline{\pi}]) = \pi$. In general let $\delta(t) = (g_0, g_1, \dots)$ with $g_i \in \mathfrak{S}$; these power series are determined by the relations

$$g_0^{p^n} + p g_1^{p^{n-1}} + \cdots + p^n g_n = \sigma^n(t)$$

for $n \ge 0$. Let $x = (x_0, x_1, \ldots) \in W(\mathcal{R})$ be the image of t, thus $x_i = g_i(\underline{\pi})$. Let $x_i = (x_{i,0}, x_{i,1}, \ldots)$ with $x_{i,j} \in \bar{R}/p\bar{R}$. If $\tilde{x}_{i,j} \in \hat{\bar{R}}$ lifts $x_{i,j}$ we have

$$pr^{\mathrm{nr}}(t) = \theta(x) = \lim_{n\to\infty} [(\tilde{x}_{0,n})^{p^n} + p(\tilde{x}_{1,n}^{p^{n-1}}) + \ldots + p^n \tilde{x}_{n,n}].$$

For $\tilde{x}_{i,n} = g_i(\pi^{(n)})$ the sum in the limit becomes $\sigma^n(t)(\pi^{(n)}) = \pi$, and the lemma is proved. ☐

Since the natural action of $\mathcal{G}_{K_\infty} = \operatorname{Gal}(\bar{K}/K_\infty)$ on $W(\operatorname{Frac}\mathcal{R})$ is trivial on $\mathcal{O}_\mathcal{E}$ it stabilises $\mathcal{O}_{\hat{\mathcal{E}}^{\rm nr}}$ and $\mathfrak{S}^{\rm nr}$ with trivial action on $\mathfrak{S}$. Thus $\mathcal{G}_{K_\infty}$ acts on $T^{\rm nr}(M,\phi)$ for each Breuil window or Breuil module (M,ϕ).

7.3. **From $\mathfrak{S}^{\rm nr}$ to Zink rings.** We assume now that $\sigma(t) \in t^2\mathfrak{S}$. For each finite extension $\mathbb{E}'$ of $\mathbb{E}$ in $\mathbb{E}^{\rm sep}$ the associated ring $\mathfrak{S}' = \mathcal{O}_{\mathcal{E}'} \cap W(\mathcal{O}_{\bar{\mathbb{E}}})$ is a finite $\mathfrak{S}$-module, so its image in $\hat{\bar{R}}$ is contained in a finite extension R' of R. Since $\mathfrak{S}' = W(k')[[t']]$ with $\sigma(t') \in t'^2\mathfrak{S}'$ by Lemma 6.2, the image of

$$\varkappa' : \mathfrak{S}' \xrightarrow{\delta} W(\mathfrak{S}') \to W(R')$$

lies in $\mathbb{W}(R')$ by [La3, Pr. 7.2]. Let us compose $\varkappa'$ with $\mathbb{W}(R') \to \mathbb{W}(\tilde{R})$, where $\mathbb{W}(\tilde{R})$ was defined in section 3, and pass to the direct limit over $\mathbb{E}'$. This gives a homomorphism $\varkappa^{(\rm nr)} : \mathfrak{S}^{(\rm nr)} \to \mathbb{W}(\tilde{R})$. Let

$$\varkappa^{\rm nr} : \mathfrak{S}^{\rm nr} \to \hat{\mathbb{W}}(\tilde{R})$$

be its p-adic completion. This map can be viewed as a frame homomorphism as follows. Let $\mathscr{B}^{\rm nr} = (\mathfrak{S}^{\rm nr}, E\mathfrak{S}^{\rm nr}, \mathfrak{S}^{\rm nr}/E\mathfrak{S}^{\rm nr}, \sigma, \sigma_1)$ with $\sigma_1(Ex) = \sigma(x)$ for $x \in \mathfrak{S}^{\rm nr}$. Then there is a commutative square of frames, where the horizontal arrows are $\mathbb{u}$-homomorphisms and the vertical arrows are strict:

$$\begin{array}{ccc} \mathscr{B} & \xrightarrow{\varkappa} & \mathscr{D}_R \\ \downarrow & & \downarrow \\ \mathscr{B}^{\rm nr} & \xrightarrow{\varkappa^{\rm nr}} & \hat{\mathscr{D}}_{\tilde{R}} \end{array}$$

Here $\mathcal{G}_K$ acts on $\hat{\mathscr{D}}_{\tilde{R}}$ and $\mathcal{G}_{K_\infty}$ acts on $\mathscr{B}^{\rm nr}$ and $\varkappa^{\rm nr}$ is $\mathcal{G}_{K_\infty}$-equivariant.

Let (M,ϕ) be a Breuil window relative to $\mathfrak{S} \to R$ with associated $\mathscr{B}$-window $\mathscr{P}$ and let $\mathscr{P}^{\rm nr}$ be the base change of $\mathscr{P}$ to $\mathscr{B}^{\rm nr}$. By definition we have $T^{\rm nr}(M,\phi) = T(\mathscr{P}^{\rm nr})$ as $\mathcal{G}_{K_\infty}$-modules. Let $\mathscr{P}_\mathscr{D}$ be the base change of $\mathscr{P}$ to $\mathscr{D}_R$ and let $\hat{\mathscr{P}}_{\hat{\mathscr{D}}}$ be the common base change of $\mathscr{P}^{\rm nr}$ and $\mathscr{P}_\mathscr{D}$ to $\hat{\mathscr{D}}_{\tilde{R}}$. As in section 2, multiplication by $\mathbb{c}$ induces a $\mathcal{G}_{K_\infty}$-invariant homomorphism

$$\tau(\mathscr{P}^{\rm nr}) : T(\mathscr{P}^{\rm nr}) \to T(\hat{\mathscr{P}}_{\hat{\mathscr{D}}}).$$

We recall that the $\mathcal{G}_K$-module $T(\hat{\mathscr{P}}_{\hat{\mathscr{D}}})$ is isomorphic to the Tate module of the p-divisible group associated to (M,ϕ); see Proposition 3.1.

Proposition 7.2. *The homomorphism $\tau(\mathscr{P}^{\rm nr})$ is bijective.*

Note that we assume $\sigma(t) \in t^2\mathfrak{S}$; otherwise $\tau(\mathscr{P}^{\rm nr})$ has not been defined.

Proof. Let h be the $\mathfrak{S}$-rank of M. The source and target of $\tau(\mathscr{P}^{\rm nr})$ are free $\mathbb{Z}_p$-modules of rank h which are exact functors of $\mathscr{P}$; this is true for $T(\mathscr{P}^{\rm nr})$ since (7.2) and (7.1) are bijective, and for $T(\hat{\mathscr{D}}_{\tilde{R}})$ by Proposition 3.1.

Consider first the case where the p-divisible group associated to $\mathscr{P}$ is étale, which means that $\mathscr{P} = (P,Q,F,F_1)$ has $P = Q$, and $F_1 : Q \to P$ is a σ-linear isomorphism. Then a $\mathbb{Z}_p$-basis of $T(\mathscr{P}^{\rm nr})$ is an $\mathfrak{S}^{\rm nr}$-basis of $P^{\rm nr}$, and a $\mathbb{Z}_p$-basis of $T(\hat{\mathscr{P}}_{\hat{\mathscr{D}}})$ is a $\hat{\mathbb{W}}(\tilde{R})$-basis of $\hat{P}_{\tilde{R}}$. Since $\mathbb{Z}_p \to \hat{\mathbb{W}}(\tilde{R})$ is a local homomorphism it follows that $\tau(\mathscr{P}^{\rm nr})$ is bijective.

Consider next the case $\mathscr{P} = \mathscr{B}$, which corresponds to the p-divisible group μ_{p^∞}. Assume that the proposition does not hold for $\mathscr{B}$, i.e. that $\tau(\mathscr{B}^{\rm nr})$ is divisible by p. We may replace k be an arbitrary perfect extension since this does not change $\tau(\mathscr{B})$; in particular we may assume that k is uncountable. Let $\mathscr{P}_0$ be the etale $\mathscr{B}$-window that corresponds to $\mathbb{Q}_p/\mathbb{Z}_p$. We consider extensions of $\mathscr{B}$-windows $0 \to \mathscr{B} \to \mathscr{P}_1 \to \mathscr{P}_0 \to 0$, which correspond to extensions in $\operatorname{Ext}^1(\mathbb{Q}_p/\mathbb{Z}_p, \mu_{p^\infty})$.

The image of $\tau(\mathscr{P}_1^{\mathrm{nr}})$ provides a splitting of the reduction modulo p of the exact sequence

$$0 \to T(\hat{\mathscr{D}}_{\tilde{R}}) \to T((\hat{\mathscr{P}}_1)_{\hat{\mathscr{D}}}) \to T((\hat{\mathscr{P}}_0)_{\hat{\mathscr{D}}}) \to 0$$

and thus the natural homomorphism

$$\mathrm{Ext}^1_R(\mathbb{Q}_p/\mathbb{Z}_p, \mu_{p^\infty}) \to \mathrm{Ext}^1_K(\mathbb{Z}/p\mathbb{Z}, \mu_p) \to \mathrm{Ext}^1_{K_\infty}(\mathbb{Z}/p\mathbb{Z}, \mu_p) \tag{7.3}$$

is zero. Now the first arrow in (7.3) can be identified with the obvious homomorphism of multiplicative groups $1 + \mathfrak{m}_R \to K^*/(K^*)^p$; see [La1, Lemma 7.2] and its proof. By our assumption on k its image is uncountable. Since for a finite extension K'/K the homomorphism $H^1(K, \mu_p) \to H^1(K', \mu_p)$ has finite kernel, the kernel of the second map in (7.3) is countable. Thus the composition (7.3) cannot be zero, and the proposition is proved for $\mathscr{P} = \mathscr{B}$.

Finally let $\mathscr{P}$ be arbitrary. Duality gives the following commutative diagram; see section 2.

$$\begin{array}{ccc} T(\mathscr{P}^{\mathrm{nr}}) \times T(\mathscr{P}^{t\,\mathrm{nr}}) & \longrightarrow & T(\mathscr{B}^{\mathrm{nr}}) \\ {\scriptstyle \tau(\mathscr{P}^{\mathrm{nr}}) \times \tau(\mathscr{P}^{t\,\mathrm{nr}})}\Big\downarrow & & \Big\downarrow{\scriptstyle \tau(\mathscr{B}^{\mathrm{nr}})} \\ T(\hat{\mathscr{P}}_{\hat{\mathscr{D}}}) \times T(\hat{\mathscr{P}}^t_{\hat{\mathscr{D}}}) & \longrightarrow & T(\hat{\mathscr{D}}_{\tilde{R}}) \end{array}$$

Since (7.2) and (7.1) are bijective, the upper line of the diagram is a perfect bilinear form of free $\mathbb{Z}_p$-modules of rank h. Proposition 3.1 implies that the lower line is a bilinear form of free $\mathbb{Z}_p$-modules of rank h. We have seen that $\tau(\mathscr{B}^{\mathrm{nr}})$ is bijective. These properties imply that $\tau(\mathscr{P}^{\mathrm{nr}})$ is bijective. □

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