PATH SUBCOALGEBRAS, FINITENESS PROPERTIES AND QUANTUM GROUPS

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ABSTRACT. We study subcoalgebras of path coalgebras that are spanned by paths (called path subcoalgebras) and subcoalgebras of incidence coalgebras, and propose a unifying approach for these classes. We discuss the left quasi-co-Frobenius and the left co-Frobenius properties for these coalgebras. We classify the left co-Frobenius path subcoalgebras, showing that they are direct sums of certain path subcoalgebras arising from the infinite line quiver or from cyclic quivers. We also discuss the coreflexive property for the considered classes of coalgebras. Finally, we investigate which of the co-Frobenius path subcoalgebras can be endowed with Hopf algebra structures, in order to produce some quantum groups with non-zero integrals, and classify all these structures over a field with primitive roots of unity of any order. These turn out to be liftings of quantum lines over certain not necessarily abelian groups.

1. INTRODUCTION AND PRELIMINARIES

Let K be an arbitrary field. A quadratic algebra is a quotient of a free noncommutative algebra $K < x_1, \ldots, x_n >$ in n variables by an ideal I generated by elements of degree 2. For example, the usual commutative polynomial ring is such an example, with I generated by $x_i x_j - x_j x_i$. These are important in many places in mathematics, and one very important class of such algebras consists of Koszul algebras and Koszul duals of quadratic algebras. More generally, one can consider quotients $K < x_1, \ldots, x_n > /I$ for ideals I generated by homogeneous elements. Many examples of algebras important in topology, noncommutative geometry, representation theory, or theoretical physics occur in this way (see for example the examples and references in [7]). For example, such are the cubic Artin-Schreier regular algebras $\mathbb{C} < x, y > /(ay^2x + byxy + axy^2 + cx^3, ax^2y + bxyx + ayx^2 + xy^3)$ in noncommutative projective algebraic geometry (see [3]), the skew-symmetrizer killing algebras $\mathbb{C} < x_1, \ldots, x_n > /(\sum_{\sigma \in \Sigma_p} \text{sgn}(\sigma) x_{i_{\sigma(1)}} \ldots x_{i_{\sigma(p)}})$ (the ideal we factor by has $\binom{n}{p}$ generators, each one corresponding to some fixed $1 \le i_1 < \ldots < i_p \le n$) for a fixed $2 \le p \le n$, in representation theory (see [6]), or the Yang-Mills algebras $\mathbb{C} < \nabla_0, \ldots, \nabla_n > /(\sum_{\lambda,\mu} g^{(\lambda,\mu)} [\nabla_\lambda [\nabla_\mu, \nabla_\nu]])$ (with $(g^{(\lambda,\mu)})_{\lambda,\mu}$ an invertible symmetric real matrix, and the ideal we factor by has $n + 1$ generators, as $0 \le \nu \le n$) in theoretical physics (see [13]), to name a few. More generally, one could start with a quiver Γ, and define path algebras with relations by taking quotients of the path algebra $K[\Gamma]$ by an ideal (usually) generated by homogeneous

elements, which are obtained as linear combinations of paths of the same length. Note that the examples above are if this type: the free algebra with n elements can be thought as the path algebra of the quiver Γ with one vertex 1 (which becomes the unit in the algebra) and n arrows $x_1, \ldots, x_n$ starting and ending at 1; the relations are then given by linear combinations of paths of the same length. This approach, for example, allows the generalization of N-Koszulity to quiver algebras with relations, see [17].

We aim to study a general situation which is dual to the ones above, but is also directly connected to it. If Γ is a quiver, the path algebra $K[\Gamma]$ of Γ plays an important role in the representation theory of Γ. The underlying vector space of the path algebra also has a coalgebra structure, which we denote by $K\Gamma$ and call the path coalgebra of Γ. One motivation for replacing path algebras by path coalgebras is the following: given an algebra A, and its category of finite dimensional representations, one is often lead to considering the category $Ind(A)$ generated by all these finite dimensional representations (direct limits of finite dimensional representations). $Ind(A)$ is well understood as the category of comodules over the finite dual coalgebra A^0 of A (also called the algebra of representative functions on A), and it cannot be regarded as a full category of modules over a ring unless A is finite dimensional. Such situations extend beyond the realm of pure algebra, encompassing representations of compact groups, affine algebraic groups or group schemes, differential affine groups, Lie algebras and Lie groups, infinite tensor categories etc. In general, one has an embedding of the path coalgebra $K\Gamma$ into the finite dual $K[\Gamma]^0$, and this embedding is an isomorphism if, for example, there are only finitely many paths between any two vertices; quiver representations that are also comodules over $K\Gamma$ provide a particularly interesting class of representations.

Another reason for which the study of path coalgebras is interesting is that any pointed coalgebra embeds in the path coalgebra of the associated Gabriel quiver, see [24], [11]. Thus, finite dimensional modules over a path algebra $K[\Gamma]$, which are the finite dimensional comodules over the finite dual $K[\Gamma]^0$, are in fact captured by the study of subcoalgebras of path coalgebras when K is algebraically closed, since $K[\Gamma]^0$ is pointed in that case. On the other hand, if X is a locally finite partially ordered set, the incidence coalgebra KX provides a good framework for interpreting several combinatorial problems in terms of coalgebras, as explained by Joni and Rota [22]. There are several features common to path coalgebras and incidence coalgebras. They are both pointed, the group-like elements recover the vertices of the quiver, respectively the points of the ordered set, the injective envelopes of the simple comodules have similar descriptions, etc. Moreover, as we show later in Section 6, Proposition 6.1, any incidence coalgebra embeds in a path coalgebra, and in fact, it has a basis where each element is a sum of paths of *the same length*. Another example of such a coalgebra can be obtained by considering a subcoalgebra

C of a path coalgebra Γ which has a basis of paths. We note that this is precisely the dual situation to that considered above: for algebras, one considers a path algebra with homogeneous relations, that is $K[\Gamma]$ quotient out by an ideal generated by homogeneous elements, i.e. sums of paths of the same length, with coefficients. For a coalgebra, one considers subcoalgebras of the path coalgebra of Γ such that the coalgebra has a basis consisting of linear combinations of paths of the same length ("homogeneous" elements; more generally, a coalgebra generated by such elements).

The main interest of this paper is in studying Frobenius type properties for path coalgebras, incidence coalgebras and certain subcoalgebras of them. Recall that a coalgebra C is called left co-Frobenius if C embeds in C^* as a left C^*-module. Also, C is called left quasi-co-Frobenius if C embeds in a free module as a left C^*-module. The (quasi)-co-Frobenius properties are interesting for at least three reasons. Firstly, coalgebras with such properties have rich representation theories. Secondly, for a Hopf algebra H, it is true that H is left quasi-co-Frobenius if and only if H is left co-Frobenius, and this is also equivalent to H having non-zero left (or right) integrals. This is one of the most important classes of infinite dimensional Hopf algebras, for the following reason: it generalizes the algebra of representative functions $R(G)$ on a compact group. If G is a compact group, then the left Haar integral of G provides an integral of the Hopf algebra $R(G)$. Moreover, more recent generalizations of these have been made to compact and locally compact quantum groups (whose representation categories are not necessarily semisimple). Thus co-Frobenius coalgebras may be the underlying coalgebras for interesting quantum groups with non-zero integrals. Thirdly, by keeping in mind the duality with Frobenius algebras in the finite dimensional case, co-Frobenius coalgebras have connections to topological quantum field theory.

We propose an approach leading to similar results for path coalgebras and incidence coalgebras, and which also points out the similarities between these as mentioned above. In Section 2 we discuss when a path coalgebra (or an incidence coalgebra) is left (quasi)-co-Frobenius. It turns out that this happens only very rarely, i.e. when the quiver consists only of isolated points, i.e. the quiver does not have arrows (respectively the order relation is the equality). Thus the left co-Frobenius coalgebras arising from path coalgebras or incidence coalgebras are just grouplike coalgebras. The result for incidence coalgebras was previously proved in [14], with different methods. Our approach uses in a similar way representation theory for incidence coalgebras and path coalgebras. Although this section can be obtained as a consequence of our much more general results later on, we include it for a short and unifying approach to these results. In order to discover more interesting classes of left co-Frobenius coalgebras, we focus our attention to larger classes of coalgebras. On one hand we consider subcoalgebras of path coalgebras which have a linear basis consisting of paths. We call these *path subcoalgebras*. On the other hand,

we look at subcoalgebras of incidence coalgebras; any such coalgebra has a basis consisting of segments. In Section 3 we apply a classical approach to the (quasi)-co-Frobenius property. It is known that a coalgebra C is left co-Frobenius if and only if there exists a left non-degenerate C^*-balanced bilinear form on C. Also, C is left quasi-co-Frobenius if and only if there exists a family $(\beta_i)_{i \in I}$ of C^*-balanced bilinear forms on C such that for any non-zero $x \in C$ there is $i \in I$ with $\beta_i(x, C) \neq 0$. We describe the balanced bilinear forms on path subcoalgebras and subcoalgebras of incidence coalgebras. Such a description was given in [14] for the full incidence coalgebra, and in [5] for certain matrix-like coalgebras. In Section 4 we use this description and an approach using the injective envelopes of the simple comodules to show that for any coalgebra C in one of the two classes, we have that C is left quasi-co-Frobenius if and only if C is left co-Frobenius, and to give several equivalent conditions including combinatorial ones (just in terms of paths of the quiver, or segments of the ordered set). In Section 5 we classify all possible left co-Frobenius path subcoalgebras. We construct some classes of left co-Frobenius coalgebras $K[\mathbb{A}_\infty, r]$ and $K[\mathbb{A}_{0,\infty}, r]$ starting from the infinite line path $\mathbb{A}_\infty$, and a class of left co-Frobenius coalgebras $K[\mathbb{C}_n, s]$ starting from cyclic paths $\mathbb{C}_n$. Our result says that any left co-Frobenius path subcoalgebra is isomorphic to a direct sum of coalgebras of types $K[\mathbb{A}_\infty, r]$, $K[\mathbb{A}_{0,\infty}, r]$, $K[\mathbb{C}_n, s]$ or K, with special quivers $\mathbb{A}_\infty$, $A_{0,\infty}$, $\mathbb{C}_n$ and r, s being certain general types of functions on these quivers. For subcoalgebras of incidence coalgebras we do not have a complete classification in the left co-Frobenius case. We show in Section 6 that more complicated examples than the ones in the path subcoalgebra case can occur for subcoalgebras of incidence coalgebras, and a much larger class of such coalgebras is to be expected. Also, we give several examples of co-Frobenius subcoalgebras of path coalgebras, which are not path subcoalgebras, and moreover, examples of pointed co-Frobenius coalgebras which are not isomorphic to any one of the above mentioned classes. In Section 7 we discuss the property of being coreflexive for path subcoalgebras and subcoalgebras of incidence coalgebras. We prove that such a coalgebra is coreflexive if and only if its coradical is coreflexive. Although this subject is not immediately connected to the main subject of the article, we stress its importance for at least two reasons. One would be the obvious duality between path algebras and path coalgebras, and the coreflexivity of coalgebras is a duality flavored notion: it tests the possibility of recovering a coalgebra C from the dual algebra C^*. On the other hand, this study will again show how a similar unified approach for these classes of coalgebras can be made. In particular, we approach an old question of E. Taft in coalgebra theory: given coreflexive coalgebras C and D, is their tensor product coreflexive? We answer this for a fairly large class of pointed coalgebras (in fact, the general question can be reduced to the pointed case at least for algebraically closed fields). We derive that the tensor product of two coreflexive incidence coalgebras is coreflexive, and that the tensor product of two coreflexive subcoalgebras of path coalgebras associated to quivers with only finitely many paths between any two vertices is again coreflexive. In Section 8 we discuss the possibility of

defining Hopf algebra structures on the path subcoalgebras that are left and right co-Frobenius, classified in Section 5. The main reason for asking this question is the interest in constructing quantum groups with non-zero integrals, whose underlying coalgebras are path subcoalgebras. We answer completely this question in the case where K contains primitive roots of unity of any positive order. Thus we determine all possible co-Frobenius path subcoalgebras admitting a Hopf algebra structure. Moreover, we describe up to an isomorphism all such Hopf algebra structures. It turns out that they are liftings of quantum lines over certain not necessarily abelian groups. In particular, this also answers the question of finding the Hopf algebra structures on finite dimensional path subcoalgebras and on quotients of finite dimensional path algebras by ideals spanned by paths. We note that Hopf algebra structures on incidence coalgebras have been of great interest for combinatorics, see for example [27], [1]. We also note that the classification of path coalgebras that admit a graded Hopf algebra structure was done in [12], see also [16] for a different point of view on Hopf algebra structures on path algebras. In particular, some of the examples in the classification have deep connections with homological algebra: the monoidal category of chain s-complexes of vector spaces over K is monoidal equivalent to the category of comodules of $K[\mathbb{A}_\infty|s]$, a subclass of the Hopf algebras classified here ([21, 8]).

We also note that the unifying approach we propose here seems to suggest that in general for pointed coalgebras interesting methods and results could be obtained provided one can find some suitable bases with properties resembling those of paths in quiver algebras or segments in incidence coalgebras.

Throughout the paper $\Gamma = (\Gamma_0, \Gamma_1)$ will be a quiver. Γ_0 is the set of vertices, and Γ_1 is the set of arrows of Γ. If a is an arrow from the vertex u to the vertex v, we denote $s(a) = u$ and $t(a) = v$. A path in Γ is a finite sequence of arrows $p = a_1 a_2 \dots a_n$, where $n \geq 1$, such that $t(a_i) = s(a_{i+1})$ for any $1 \leq i \leq n-1$. We will write $s(p) = s(a_1)$ and $t(p) = t(a_n)$. Also the length of such a p is $\text{length}(p) = n$. Vertices v in Γ_0 are also considered as paths of length zero, and we write $s(v) = t(v) = v$. If q and p are two paths such that $t(q) = s(p)$, we consider the path qp by taking the arrows of q followed by the arrows of p. We denote by $K\Gamma$ the path coalgebra, which is the vector space with a basis consisting of all paths in Γ, and comultiplication Δ defined by $\Delta(p) = \sum_{qr=p} q \otimes r$ for any path p, and counit ϵ defined by $\epsilon(v) = 1$ for any vertex v, and $\epsilon(p) = 0$ for any path of positive length. In particular, the arrows x between two vertices v and w, i.e. $s(x) = v, t(x) = w$, are the nontrivial elements of $P_{w,v}$, the space of (w,v)-skew-primitive elements: $\Delta(x) = v \otimes x + x \otimes w$. When we use Sweedler's sigma notation $\Delta(p) = \sum p_1 \otimes p_2$ for a path p, we always take representations of the sum such that all p_1's and p_2's are paths.

We also consider partially ordered sets $(X, \leq)$ which are locally finite, i.e. the interval $[x, y] = \{z| \ x \leq z \leq y\}$ is finite for any $x \leq y$. The incidence K-coalgebra of X, denoted by KX, is the

K-vector space with basis $\{e_{x,y} | x, y \in X, x \leq y\}$, and comultiplication Δ and counit ϵ defined by

$$\Delta(e_{x,y}) = \sum_{x \leq z \leq y} e_{x,z} \otimes e_{z,y}$$

$$\epsilon(e_{x,y}) = \delta_{x,y}$$

for any $x, y \in X$ with $x \leq y$, where by $\delta_{x,y}$ we denote Kronecker's delta. The elements $e_{x,y}$ are called segments. Again, when we use Sweedler's sigma notation $\Delta(p) = \sum p_1 \otimes p_2$ for a segment p, we always take representations of the sum such that all p_1's and p_2's are segments. Recall that the length of a segment $e_{x,y}$ is the maximum length n of a chain $x = z_0 < z_1 < \cdots < z_n = y$ For basic terminology and notation about coalgebras and Hopf algebras we refer to [15] and [23].

2. WHEN IS A PATH COALGEBRA OR AN INCIDENCE COALGEBRA LEFT QUASI-CO-FROBENIUS?

In this section we present a common approach to the study of certain finiteness properties for path coalgebras and incidence coalgebras. If C is a coalgebra, we denote by $C_0 \subseteq C_1 \subseteq \ldots \subseteq C_n \subseteq \ldots$ its coradical filtration. C_0 is the coradical of C, and C is called a pointed coalgebra if C_0 is a grouplike coalgebra, i.e. it is spanned by grouplike elements of C.

Definition 2.1. *(i) Let C be a coalgebra. We say that $\mathcal{B}$ is a coradical basis of C if $\mathcal{B}$ is a K-basis of C and $\mathcal{B} \cap C_n$ is a basis of C_n for all n.*
(ii) A right (or left) coideal M of C will be called $\mathcal{B}$-supported if $\mathcal{B} \cap M$ is a K-basis of M.

We say that a non-zero (co)module is local if it has a unique maximal sub-(co)module. The following result is obvious.

Proposition 2.2. *Let C be a coalgebra with a coradical basis $\mathcal{B}$ and let M be a finite dimensional local right coideal of C which is $\mathcal{B}$-supported. Then M is generated as a right C-comodule by any element in the nonempty set $(\mathcal{B} \cap M) \setminus Jac(M)$ (here, $Jac(M)$ denotes the Jacobson radical of M).*

The path coalgebra $K\Gamma$ and the incidence coalgebra KX are both pointed. The simple left (or right) comodules are the one-dimensional spaces spanned by the vertices of Γ in the first case, respectively by the trivial segments $e_{x,x}$, $x \in X$, in the second one. The injective envelopes of the simple left (right) comodules were described in [28, Lemma 5.1] for incidence coalgebras and in [10, Corollary 6.3] for path coalgebras as follows.

Proposition 2.3. *(i) If $C = K\Gamma$, then for each vertex v of Γ, the injective envelope of the left (right) C-comodule Kv is (the K-span) $E_l(Kv) = < p \mid p$ path and $t(p) = v >$ (and $E_r(Kv) = < p \mid p$ path and $s(p) = v >$, respectively).*
(ii) If $C = KX$, then the injective envelope of the left (right) C-comodule $Ke_{a,a}$ is (the K-span) $E_l(Ke_{a,a}) = < e_{x,a} | x \in X, x \leq a >$ (and $E_r(Ke_{a,a}) = < e_{a,x} | x \in X, a \leq x >$, respectively).

Now we can prove the following.

Proposition 2.4. *If $C = K\Gamma$, a path coalgebra, or $C = KX$, an incidence coalgebra, the following are equivalent*

(i) C is co-semisimple (i.e. Γ has no arrows for $C = K\Gamma$, and the order relation on X is the equality for $C = KX$).

(ii) C is left QcF.

(iii) C is left co-Frobenius.

(iv) C is right QcF.

(v) C is right co-Frobenius.

Proof. (i)$\Rightarrow$(iii)$\Rightarrow$(ii) holds for any coalgebra C. We show that (ii)$\Rightarrow$(i). By [20, Proposition 1.4] and [21, Proposition 1.2], any injective indecomposable right C-comodule is the dual of an injective indecomposable left C-comodule, so it is local by [19, Lemma 1.4]. By Proposition 2.3 in either case the injective indecomposable comodules are $\mathcal{B}$-supported, where $\mathcal{B}$ is the basis consisting of all paths in Γ if $C = K\Gamma$, and of all segments $e_{x,y}$ if $C = KX$, and by Proposition 2.2, they must be generated as C-comodules by an element of the basis $\mathcal{B}$.

(I) If $C = K\Gamma$, for a vertex v, let $E_r(Kv) = C^*p = <q \mid q$ path and $p = qh$ for some path $h >$ for a path $p \in \mathcal{B}$. This means that any path starting at v is a subpath of p. We have that $s(p) = v$ and let $t(p) = u$. Then it is easy to see that the linear map $\phi : C^*p \to Ku$ taking p to u, and any other q to 0, is a surjective morphism of left C^*-modules. Hence if $E_r(Kv) \cong E_l(Kw)^*$ as left C^*-modules for some vertex w, we have a surjective morphism of left C^*-modules $E_l(Kw)^* \to Ku$, inducing an injective morphism of right C^*-modules $(Ku)^* \to E_l(Kw)$. Since $(Ku)^* \cong Ku$ as right C^*-modules, and the socle of the comodule $E_l(Kw)$ is Kw, we must have $w = u$, showing that $E_r(Kv) \cong E_l(Ku)^*$.

On the other hand there is no arrow starting at u (otherwise we would have a path starting at v, which is not a subpath of p), so $E_r(Ku) = Ku$. By the above considerations for u instead of v, $E_r(Ku)$ can only be isomorphic to the dual of $E_l(Ku)$. Therefore $E_r(Kv) \cong E_r(Ku)$, and then $u = v$ and $p = v$. Thus $E_r(Kv)$ is simple and hence C is cosemisimple.

(II) If $C = KX$, the same argument repeats word by word, but with the elements of the basis $\mathcal{B}$ consisting of all the segments $e_{x,y}$. Here a segment $e_{a,x}$ will be a subsegment of $e_{a,b}$ if $a \leq x \leq b$. $\square$

As an immediate consequence we describe the situations where a finite dimensional path algebra is Frobenius. We note that the path algebra of a quiver Γ (as well as the path coalgebra $K\Gamma$) has finite dimension if and only if Γ has finitely many vertices and arrows, and there are no cycles.

Corollary 2.5. *A finite dimensional path algebra is Frobenius if and only if the quiver has no arrows.*

Proof. It follows from the fact that the dual of a finite dimensional path algebra is a path coalgebra, and by Proposition 2.4. □

3. Balanced bilinear forms for path subcoalgebras and for subcoalgebras of incidence coalgebras

In the rest of the paper we will be interested in two classes of coalgebras more general than path coalgebras and incidence coalgebras. Thus we will study

• Subcoalgebras of the path coalgebra $K\Gamma$ having a basis $\mathcal{B}$ consisting of paths in Γ. Such a coalgebra will be called a path subcoalgebra. Note that if $p \in \mathcal{B}$, then any subpath of p, in particular any vertex involved in p, lies in $\mathcal{B}$.

• Subcoalgebras of the incidence coalgebra KX. By [14, Proposition 1.1], any such subcoalgebra has a basis $\mathcal{B}$ consisting of segments $e_{x,y}$, and moreover, if $e_{x,y} \in \mathcal{B}$ and $x \leq a \leq b \leq y$, then $e_{a,b} \in \mathcal{B}$.

It is clear that for a coalgebra C of one of these two types, the distinguished basis $\mathcal{B}$ consists of all paths (or segments) which are elements of C. Let C be a coalgebra of one of these two types, with basis $\mathcal{B}$ as above. When we use Sweedler's sigma notation $\Delta(p) = \sum p_1 \otimes p_2$ for $p \in \mathcal{B}$, we always consider representations of the sum such that all p_1's and p_2's are in $\mathcal{B}$.

A bilinear form $\beta : C \times C \to K$ is C^*-balanced if

$$\sum \beta(p_2, q)p_1 = \sum \beta(p, q_1)q_2 \qquad \text{for any } p, q \in \mathcal{B} \tag{1}$$

It is clear that (1) is equivalent to the fact that for any $p, q \in \mathcal{B}$, the following three conditions hold.

$$\beta(p_2, q) = \beta(p, q_1) \quad \text{for those of the } p_2\text{'s and the } q_1\text{'s such that } p_1 = q_2 \tag{2}$$

$$\beta(p_2, q) = 0 \quad \text{for those } p_2\text{'s for which } p_1 \text{ is not equal to any } q_2 \tag{3}$$

$$\beta(p, q_1) = 0 \quad \text{for those } q_1\text{'s for which } q_2 \text{ is not equal to any } p_1 \tag{4}$$

In the following two subsections we discuss separately path subcoalgebras and subcoalgebras of incidence coalgebras.

3.1. Path subcoalgebras. In this subsection we consider the case where C is a path subcoalgebra. We note that if Γ is acyclic, then for any paths p and q there is at most a pair (p_1, q_2) (in (1)) such that $p_1 = q_2$.

Denote by $\mathcal{F}$ the set of all paths d satisfying the following three properties

- $d = qp$ for some $q, p \in \mathcal{B}$.

- For any representation $d = qp$ with $q, p \in \mathcal{B}$, and any arrow $a \in \Gamma_1$, if $ap \in \mathcal{B}$ then q must end with a.

- For any representation $d = qp$ with $q, p \in \mathcal{B}$, and any arrow $b \in \Gamma_1$, if $qb \in \mathcal{B}$ then p starts with b.

Now we are able to describe all balanced bilinear forms on C.

Theorem 3.1. *A bilinear form* $\beta : C \times C \to K$ *is* C^**-balanced if and only if there is a family of scalars* $(\alpha_d)_{d \in \mathcal{F}}$ *such that for any* $p, q \in \mathcal{B}$

$$\beta(p, q) \;=\; \begin{cases} \alpha_d, & \text{if } s(p) = t(q) \text{ and } qp = d \in \mathcal{F} \\ 0, & \text{otherwise} \end{cases}$$

In particular the set of all C^**-balanced bilinear forms on* C *is in bijective correspondence to* $K^{\mathcal{F}}$.

Proof. Assume that β is C^*-balanced. If $p, q \in \mathcal{B}$ and $t(q) \neq s(p)$, then $\beta(p, q)s(p)$ appears in the left-hand side of (1), but $s(p)$ does not show up in the right-hand side, so $\beta(p, q) = 0$. Let $\mathcal{P}$ be the set of all paths in Γ for which there are $p, q \in \mathcal{B}$ such that $d = qp$. Let $d \in \mathcal{P}$ and let $d = qp = q'p'$, $p, q, p', q' \in \mathcal{B}$ be two different decompositions of d, and say that, for example, $\text{length}(p') < \text{length}(p)$. Then there is a path r such that $p = rp'$ and $q' = qr$, and clearly $r \in \mathcal{B}$ since it is a subpath of $q' \in B$. Use (2) for p and q', for which there is an equality $p_1 = q'_2 = r$ (and the corresponding $p_2 = p'$ and $q'_1 = q$), and find that $\beta(p', q') = \beta(p, q)$. Therefore, for any $d \in \mathcal{P}$ (not necessarily in $\mathcal{B}$) and any $p, q \in \mathcal{B}$ such that $d = qp$, we have that $\beta(p, q)$ depends only on d. This shows that there is a family of scalars $(\alpha_d)_{d \in \mathcal{P}}$ such that $\beta(p, q) = \alpha_d$ for any $p, q \in \mathcal{B}$ with $qp = d$.

Let $d \in \mathcal{P}$ such that $d = qp$ for some $p, q \in \mathcal{B}$, and there is an arrow $a \in \Gamma_1$ with $ap \in \mathcal{B}$, but q does not end with a. That is, q is not of the form $q = ra$ for some path $r \in \mathcal{B}$. We use (3) for the paths $ap \in \mathcal{B}$ and $q \in \mathcal{B}$, more precisely, for the term $(ap)_1 = a$, which cannot be equal to any of the q_2's (otherwise q would end with a), and we see that $\beta(p, q) = 0$, i.e. $\alpha_d = 0$.

Similarly, if $d \in \mathcal{P}$, $d = qp$ with $p, q \in \mathcal{B}$ and there is $b \in \Gamma_1$ with $qb \in B$ and p not of the form br for some path r (i.e. p does not start with b), then we use (4) for p and qb, and $(qb)_2 = b$, and we find that $\beta(p, q) = 0$, i.e. $\alpha_d = 0$. In conclusion, α_d may be non-zero only for $d \in \mathcal{F}$.

Conversely, assume that β is of the form indicated in the statement. We show that (2), (3) and (4) are satisfied. Let $p, q \in \mathcal{B}$ be such that $p_1 = q_2 = r$ for some p_1 and q_2 (from the comultiplication $\sum p_1 \otimes p_2$ of p and, respectively, the comultiplication $\sum q_1 \otimes q_2$ of q). Then $p = rp'$ and $q = q'r$ for some $p', q' \in \mathcal{B}$. Let $d = q'rp'$. If $d \in \mathcal{F}$, then $\beta(p', q) = \beta(p, q') = \alpha_d$, while if $d \notin \mathcal{F}$ we have that $\beta(p', q) = \beta(p, q') = 0$ by definition. Thus (2) holds. Now let $p, q \in \mathcal{B}$ and fix some p_2 (from the comultiplication $\sum p_1 \otimes p_2$ of p) such that the corresponding

p_1 is not equal to any q_2. If $s(p_2) \neq t(q)$, then clearly $\beta(p_2, q_1) = 0$ by the definition of β. If $s(p_2) = t(q)$, then $d = qp_2 \notin \mathcal{F}$. Indeed, let r be a maximal path such that $p_1 = er$ for some path e and q ends with r, say $q = q'r$. Note that e has length at least 1, since p_1 is not equal to any of the q_2's. Then the terminal arrow of e cannot be the terminal arrow of q', and this shows that $d = p_2 q = (p_2 r)q' \notin \mathcal{F}$. Then $\beta(p_2, q) = 0$ and (3) is satisfied. Similarly, (4) is satisfied. □

3.2. **Subcoalgebras of incidence coalgebras.** In this subsection we assume that C is a subcoalgebra of the incidence coalgebra KX. Let $\mathcal{D}$ be the set of all pairs (x, y) of elements in X such that $x \leq y$ and there exists x' with $x \leq x' \leq y$ and $e_{x,x'}, e_{x',y} \in \mathcal{B}$. Fix $(x, y) \in \mathcal{D}$. Let

$$U_{x,y} = \{u \mid x \leq u \leq y \text{ and } e_{x,u}, e_{u,y} \in \mathcal{B}\}$$

and define the relation $\sim$ on $U_{x,y}$ by $u \sim v$ if and only if there exist a positive integer n, and $u_0 = u, u_1, \ldots, u_n = v$ and $z_1, \ldots, z_n$ in $U_{x,y}$, such that $z_i \leq u_{i-1}$ and $z_i \leq u_i$ for any $1 \leq i \leq n$. It is easy to see that $\sim$ is an equivalence relation on $U_{x,y}$. Let $U_{x,y}/\sim$ be the associated set of equivalence classes, and denote by $(U_{x,y}/\sim)_0$ the set of all equivalence classes $\mathcal{C}$ satisfying the following two conditions.

• If $u \in \mathcal{C}$, and $v \in X$ satisfies $v \leq u$ and $e_{v,y} \in \mathcal{B}$, then $x \leq v$.

• If $u \in \mathcal{C}$, and $v \in X$ satisfies $u \leq v$ and $e_{x,v} \in \mathcal{B}$, then $v \leq y$.

Now we can describe the balanced bilinear forms on C.

Theorem 3.2. *A bilinear form $\beta : C \times C \to K$ is C^*-balanced if and only if there is a family of scalars $(\alpha_{\mathcal{C}})_{\mathcal{C} \in \bigsqcup_{(x,y) \in \mathcal{D}} (U_{x,y}/\sim)_0}$ such that for any $e_{t,y}, e_{x,z} \in \mathcal{B}$*

$$\beta(e_{t,y}, e_{x,z}) = \begin{cases} \alpha_{\mathcal{C}}, & \text{if } (x,y) \in \mathcal{D}, z = t \in U_{x,y} \text{ and the class } \mathcal{C} \text{ of } z \text{ in } U_{x,y}/\sim \text{ is in } (U_{x,y}/\sim)_0 \\ 0, & \text{otherwise} \end{cases}$$

In particular the set of all C^*-balanced bilinear forms on C is in bijective correspondence to $K^{\bigsqcup_{(x,y) \in \mathcal{D}} (U_{x,y}/\sim)_0}$.

Proof. Assume that β is C^*-balanced. Fix some $x \leq y$ such that $U_{x,y} \neq \emptyset$. We first note that if $x \leq z \leq t \leq y$ and $z, t \in U_{x,y}$, then by applying (2) for $p = e_{z,y}, q = e_{x,t}$ and $p_1 = q_2 = e_{z,t}$, we find that $\beta(e_{t,y}, e_{x,t}) = \beta(e_{z,y}, e_{x,z})$. Now let $u, v \in U_{x,y}$ such that $u \sim v$. Let $u_0 = u, u_1, \ldots, u_n = v$ and $z_1, \ldots, z_n$ in $U_{x,y}$, such that $z_i \leq u_{i-1}$ and $z_i \leq u_i$ for any $1 \leq i \leq n$. By the above we have that $\beta(e_{u_{i-1},y}, e_{x,u_{i-1}}) = \beta(e_{u_i,y}, e_{x,u_i}) = \beta(e_{z_i,y}, e_{x,z_i})$, for any i, and this implies that $\beta(e_{u,y}, e_{x,u}) = \beta(e_{v,y}, e_{x,v})$. This shows that $\beta(e_{u,y}, e_{x,u})$ takes the same value for any u in the same equivalence class in $U_{x,y}/\sim$.

Now assume that for some $u \in U_{x,y}$ there is $v \in X$, such that $v \leq u$, $x \nleq v$ and $e_{v,y} \in \mathcal{B}$. Use (3) for $p = e_{v,y}, q = e_{x,u}$ and $p_1 = e_{v,u}$. Note that $p_1 \neq q_2$ for any q_2. We get that $\beta(e_{u,y}, e_{x,u}) = 0$. Similarly, if $u \in U_{x,y}$, and there is $v \in X$ such that $u \leq v$, $v \nleq y$ and $e_{x,v} \in \mathcal{B}$, then using (4) for

$p = e_{u,y}, q = e_{x,v}$ and $q_2 = e_{u,v}$, we find that $\beta(e_{u,y}, e_{x,u}) = 0$. We have thus showed that β has the desired form.

Conversely, assume that β has the indicated form. We show that it satisfies (2), (3) and (4). Let $p, q \in \mathcal{B}$ such that $p_1 = q_2$ for some p_1 and q_2. Then $p = e_{z,y}, q = e_{x,t}$ and $p_1 = q_2 = e_{z,t}$ for some $x \leq z \leq t \leq y$. Clearly $t \sim z$, and let $\mathcal{C}$ be the equivalence class of t in $U_{x,y}/\sim$. Then $\beta(p_2, q) = \beta(e_{t,y}, e_{x,t})$ and $\beta(p, q_1) = \beta(e_{z,y}, e_{x,z})$, and they are both equal to $\alpha_{\mathcal{C}}$ if $\mathcal{C} \in (U_{x,y}/\sim)_0$, and to 0 if $\mathcal{C} \notin (U_{x,y}/\sim)_0$. Thus (2) is satisfied.

Let now $p = e_{z,y}, p_1 = e_{z,t}, p_2 = e_{t,y}$ and $q = e_{x,u}$ such that $p_1 \neq q_2$ for any q_2. We have that $\beta(p_2, q) = \beta(e_{t,y}, e_{x,u})$. If $u \neq t$, this is clearly 0. Let $u = t$. Then $x \not\leq z$, otherwise $p_1 = q_2$ for some q_2. We have that $t \in U_{x,y}$, but the equivalence class of t in $U_{x,y}/\sim$ is not in $(U_{x,y}/\sim)_0$, since $e_{z,y} \in \mathcal{B}$, $z \leq t$, but $x \not\leq z$. It follows that $\beta(p_2, q) = 0$, and (3) holds. Similarly we can show that (4) holds. $\qquad\square$

4. Left quasi-co-Frobenius path subcoalgebras and subcoalgebras of incidence coalgebras

In this section we investigate when a path subcoalgebra of a path coalgebra or a subcoalgebra of an incidence coalgebra is left co-Frobenius. We keep the notation of Section 3. Thus C will be either a path subcoalgebra of a path coalgebra $K\Gamma$, or a subcoalgebra of an incidence coalgebra KX. The distinguished basis of C consisting of paths or segments will be denoted by $\mathcal{B}$. We note that in each of the two cases $\mathcal{B}$ is a coradical basis of C in the sense of Definition 2.1. It is easy to see that Proposition 2.3 extends to the following.

Proposition 4.1. *(i) If C is a path subcoalgebra, then for each vertex v of Γ such that $v \in C$, the injective envelope of the left (right) C-comodule Kv is (the K-span) $E_l(Kv) =< p \in \mathcal{B}|t(p) = v >$ (and $E_r(Kv) =< p \in \mathcal{B}|s(p) = v >$ respectively).*
(ii) If C is a subcoalgebra of the incidence coalgebra KX, then for any $a \in X$ such that $e_{a,a} \in C$, the injective envelope of the left (right) C-comodule $Ke_{a,a}$ is (the K-span) $E_l(Ke_{a,a}) =< e_{x,a}|x \in X, e_{x,a} \in C >$ (and $E_r(Ke_{a,a}) =< e_{a,x}|x \in X, e_{a,x} \in C >$).

The following shows that we have a good left-right duality for comodules generated by elements of the basis $\mathcal{B}$.

Lemma 4.2. *(i) Let C be a subcoalgebra of the incidence coalgebra KX, and let $e_{a,b} \in C$. Then $(C^*e_{a,b})^* \cong e_{a,b}C^*$ as right C^*-modules (or left C-comodules).*
(ii) Let C be a path subcoalgebra of $K\Gamma$, and let p be a path in C. Then $(C^*p)^* \cong pC^*$ as right C^*-modules (or left C-comodules).

Proof. (i) Clearly the set of all segments $e_{a,x}$ with $a \leq x \leq b$ is a basis of $C^*e_{a,b}$. Denote by $e_{a,x}^*$ the corresponding elements of the dual basis of $(C^*e_{a,b})^*$. Since for $c^* \in C^*$ and $a \leq x, y \leq b$ we

have

$$(e_{a,x}^* c^*)(e_{a,y}) \;=\; \sum_{a \le z \le y} c^*(e_{z,y}) e_{a,x}^*(e_{a,z})$$

$$= \begin{cases} 0, \text{ if } x \nleq y \\ c^*(e_{x,y}), \text{ if } x \le y \end{cases}$$

we get that

$$(5) \qquad\qquad e_{a,x}^* c^* = \sum_{x \le y \le b} c^*(e_{x,y}) e_{a,y}^*$$

On the other hand $e_{a,b}C^*$ has a basis consisting of all segments $e_{x,b}$ with $a \le x \le b$, and

$$(6) \qquad\qquad e_{x,b} c^* = \sum_{x \le y \le b} c^*(e_{x,y}) e_{y,b}$$

Equations (5) and (6) show that the linear map $\phi : (C^* e_{a,b})^* \to e_{a,b}C^*$ defined by $\phi(e_{a,x}^*) = e_{x,b}$, is an isomorphism of right C^*-modules.

(ii) Let $p = a_1 \dots a_n$ and $v = s(p)$. Denote $p_i = a_1 \dots a_i$ for any $1 \le i \le n$, and $p_0 = v$. Then $\{p_0, p_1, \dots, p_n\}$ is a basis of C^*p, and let $(p_i^*)_{0 \le i \le n}$ be the dual basis of $(C^*p)^*$. For any $0 \le t \le j \le n$ denote by $\overline{p_{t,j}}$ the path such that $p_j = p_t \overline{p_{t,j}}$. Then a simple computation shows that $p_i^* c^* = \sum_{i \le j \le n} c^*(\overline{p_{i,j}}) p_j^*$ for any i and any $c^* \in C^*$.

On the other hand, $\{\overline{p_{i,n}} \mid 0 \le i \le n\}$ is a basis of pC^*, and it is easy to see that $\overline{p_{i,n}} c^* = \sum_{i \le r \le n} c^*(\overline{p_{i,r}}) \overline{p_{r,n}}$ for any i and any $c^* \in C^*$. Then the linear map $\phi : (C^*p)^* \to pC^*$ defined by $\phi(p_i^*) = \overline{p_{i,n}}$ for any $0 \le i \le n$, is an isomorphism of right C^*-modules. $\square$

For a path subcoalgebra C let us denote by $R(C)$ the set of vertices v in C such that the set $\{p \in C \mid p \text{ path and } s(p) = v\}$ is finite (i.e. $E_r(Kv)$ is finite dimensional) and contains a unique maximal path. Note that this is equivalent to $E_r(Kv)$ being finite dimensional and local. Indeed, if $E_r(Kv)$ is finite dimensional and contains a unique maximal path $p = a_1 \dots a_n$, then keeping the notation from the proof of Lemma 4.2, we have that $E_r(Kv) = C^*p$ and $C^*p_{n-1} = <p_0, \dots, p_{n-1}>$ is the unique maximal C^*-submodule of C^*p. Conversely, if $E_r(Kv)$ is finite dimensional and local, then by Proposition 2.2, $E_r(Kv) = C^*p$ for some path p, and then clearly p is the unique maximal path in $\{q \in C \mid q \text{ path and } s(q) = v\}$.

Similarly, denote by $L(C)$ the set of vertices v of C such that $E_l(Kv)$ is a finite dimensional local left C-comodule. Also, for each vertex $v \in R(C)$ let $r(v)$ denote the endpoint of the maximal path starting at v, and for $v \in L(C)$ let $l(v)$ be the starting point of the maximal path ending at v.

Similarly, for a subcoalgebra C of the incidence coalgebra KX, let $R(C)$ be the set of all $a \in X$ for which $e_{a,a} \in C$ and the set $\{x \in X \mid a \le x, e_{a,x} \in C\}$ is finite and has a unique maximal element, and $L(C)$ be the set of all $a \in X$ for which $e_{a,a} \in C$ and the set $\{x \in X \mid x \le a, e_{x,a} \in C\}$

is finite and has a unique minimal element. As before, $R(C)$ (respectively $L(C)$) consists of those $a \in X$ for which $E_r(Ke_{a,a})$ (respectively, $E_l(Ke_{a,a})$) are local, hence generated by a segment. Here $r(a) = r(e_{a,a})$ for $a \in R(C)$ denotes the maximum element in the set $\{x \mid x \geq a, e_{a,x} \in C\}$ and $l(a)$ for $a \in L(C)$ means the minimum of $\{x \mid x \leq a, e_{x,a} \in C\}$.

Proposition 4.3. *(I) Let C be a path subcoalgebra of the path coalgebra $K\Gamma$. Then the following are equivalent.*

(a) C is left co-Frobenius.

(b) C is left quasi-co-Frobenius.

(c) $R(C)$ consists of all vertices belonging to C, $r(R(C)) \subseteq L(C)$ and $lr(v) = v$, for any vertex v in C.

(d) For any path $q \in \mathcal{B}$ there exists a path $p \in \mathcal{B}$ such that $qp \in \mathcal{F}$ (for $\mathcal{F}$ defined in the previous section).

(II) Let C be a subcoalgebra of the incidence coalgebra KX. Then the following are equivalent.

(a) C is left co-Frobenius.

(b) C is left quasi-co-Frobenius.

(c) $R(C)$ consists of all $a \in X$ such that $e_{a,a} \in C$, $r(R(C)) \subseteq L(C)$ and $lr(a) = a$, $\forall a \in X$ with $e_{a,a} \in C$.

(d) For any segment $e_{x,z} \in C$ there exists $y \geq z$ such that $e_{z,y} \in C$ and the class of z in $U_{x,y}/\sim$ lies in $(U_{x,y}/\sim)_0$.

Proof. (I) (a)$\Rightarrow$(b) is clear.

(b)$\Rightarrow$(c) We apply the QcF characterization of [20] and [21]. If C is left QcF then for any vertex $v \in C$, there is a vertex $u \in C$ such that $E_r(Kv) \cong E_l(Ku)^*$. Hence $E_r(Kv)$ is local (by [19, Lemma 1.4]), and by Proposition 2.2 it is generated by a path p. As in the proof of Proposition 2.4, this means that $v \in R(C)$, and $u = r(v)$. By Lemma 4.2, $E_l(Ku) \cong E_r(Kv)^* = (C^*p)^* \cong pC^*$, so $E_l(Ku)$ is generated by p, and this shows that p is the unique maximal path ending at u. Hence, $u = r(v) \in L(X)$, and $l(u) = v$. Thus $l(r(v)) = v$.

(c)$\Rightarrow$ (d) Let $q \in \mathcal{B}$, and let $v = s(q)$. Since $v \in R(C)$, there exists a unique maximal path d starting at v, and $d = qp$ for some path p. We show that $d \in \mathcal{F}$. Denote $t(d) = v'$, and let $d = q'p'$ for some paths q', p' in $\mathcal{B}$. Let $u = t(q') = s(p')$. If there is an arrow b (in Γ_1) starting at u, such that $q'b \in \mathcal{B}$, then $q'b$ is a subpath of d, since d is the unique maximal path starting at v. It follows that p' starts with b. On the other hand, $v' = r(v) \in L(C)$ and $l(v') = lr(v) = v$, so d is the unique maximal path in $\mathcal{B}$ ending at v'. This shows that if an arrow a (in Γ_1) ends at u, and $ap' \in \mathcal{B}$, then ap' is a subpath of d, so the last arrow of q' is a. We conclude that $d \in \mathcal{F}$.

(d)$\Rightarrow$(a) Choose a family $(\alpha_d)_{d \in \mathcal{F}}$ of scalars, such that $\alpha_d \neq 0$ for any d. Associate a C^*-balanced bilinear form B on C to this family of scalars as in Theorem 3.1. Then B is right non-degenerate, so C is left co-Frobenius.

(II) (a)$\Rightarrow$(b) is clear; (b)$\Rightarrow$(c) is proved as the similar implication in (I), with paths replaced by segments.

(c)$\Rightarrow$(d) Let $e_{x,z} \in C$. If $r(x) = y$, then clearly $z \leq y$ and $e_{x,y} \in \mathcal{B}$, so $U_{x,y} = [x,y]$. Then any two elements in $U_{x,y}$ are equivalent with respect to $\sim$ (since they are both $\geq x$), so there is precisely one equivalence class in $U_{x,y}/\sim$, the whole of $U_{x,y}$. We show that this class lies in $(U_{x,y}/\sim)_0$. Indeed, if $u \in U_{x,y}$, $v \in X$, $v \leq u$ and $e_{v,y} \in \mathcal{B}$, then $v \in \{a | e_{a,y} \in \mathcal{B}\}$, and since $l(y) = l(r(x)) = x$, we must have $x \leq v$. Also, if $u \in U_{x,y}$, $v \in X$, $u \leq v$ and $e_{x,v} \in \mathcal{B}$, then $v \in \{a | e_{x,a} \in \mathcal{B}\}$. Then since $r(x) = y$, we must have $v \leq y$.

(d)$\Rightarrow$(a) follows as the similar implication in (I) if we take into account Theorem 3.2. $\qquad\square$

5. Classification of left co-Frobenius path subcoalgebras

Proposition 4.3 gives information about the structure of left co-Frobenius path subcoalgebras. The aim of this section is to classify these coalgebras. We first use Proposition 4.3 to give some examples of left co-Frobenius path subcoalgebras. These examples will be the building blocks for the classification.

Example 5.1. *Let $\Gamma = \mathbb{A}_\infty$ be the quiver such that $\Gamma_0 = \mathbb{Z}$ and there is precisely one arrow from i to $i+1$ for any $i \in \mathbb{Z}$.*

$$\mathbb{A}_\infty : \quad \ldots \longrightarrow \circ^{-1} \longrightarrow \circ^0 \longrightarrow \circ^1 \longrightarrow \circ^2 \longrightarrow \ldots$$

For any $k < l$, let $p_{k,l}$ be the (unique) path from the vertex k to the vertex l. Also denote by $p_{k,k}$ the vertex k. Let $r : \mathbb{Z} \to \mathbb{Z}$ be a strictly increasing function such that $r(n) > n$ for any $n \in \mathbb{Z}$. We consider the path subcoalgebra $K[\mathbb{A}_\infty, r]$ of $K\mathbb{A}_\infty$ with the basis

$$
\begin{aligned}
\mathcal{B} &= \bigcup_{n \in \mathbb{Z}} \{p \mid p \text{ is a path in } \mathbb{A}_\infty, s(p) = n \text{ and } \mathrm{length}(p) \leq r(n) - n\} \\
&= \{p_{k,l} \mid k, l \in \mathbb{Z} \text{ and } k \leq l \leq r(k)\}
\end{aligned}
$$

Note that $K[\mathbb{A}_\infty, r]$ is indeed a subcoalgebra, since

$$\Delta(p_{k,l}) = \sum_{i=k}^{l} p_{k,i} \otimes p_{i,l}, \quad k \leq l$$

The counit is given by

$$\varepsilon(p_{k,l}) = \delta_{k,l}$$

Note that this can also be seen as a subcoalgebra of the incidence coalgebra of $(\mathbb{N}, \leq)$, consisting of the segments $e_{k,l}$ for $k \leq l \leq r(k)$.

The construction immediately shows that the maximal path starting from n is $p_{n,r(n)}$. Note that for each $n \in \mathbb{Z}$, $p_{n,r(n)}$ is the unique maximal path into $r(n)$. If there would be another longer path $p_{l,r(n)}$ into $r(n)$ in $K[\mathbb{A}_\infty, r]$, then $l < n$. Then, since $p_{l,r(n)}$ is among the paths in $K[\mathbb{A}_\infty, r]$

which start at l we must have that it is a subpath of $p_{l,r(l)}$, and so $r(l) \geq r(n)$. But since $l < n$, this contradicts the assumption that r is strictly increasing. Therefore, we see that the conditions of Proposition 4.3 are satisfied: $p_{n,r(n)}$ is the unique maximal path in the (finite) set of all paths starting from a vertex n, and it is simultaneously the unique maximal path in the (finite) set of all paths ending at $r(n)$. Therefore if $l : L(C) = \mathrm{Im}(r) \to R(C)$ is the function used in Proposition 4.3 for $C = K[\mathbb{A}_\infty, r]$ satisfies $l(r(n)) = n$. This means that $K[\mathbb{A}_\infty, r]$ is a left co-Frobenius coalgebra.

We have that $K[\mathbb{A}_\infty, r]$ is also right co-Frobenius if and only if there is a positive integer s such that $r(n) = n + s$ for any $n \in \mathbb{Z}$. Indeed, if r is of such a form, then $K[\mathbb{A}_\infty, r]$ is right co-Frobenius by the right-hand version of Proposition 4.3. Conversely, assume that $K[\mathbb{A}_\infty, r]$ is right co-Frobenius. If r would not be surjective, let $m \in \mathbb{Z}$ which is not in the image of r. Then there is $n \in \mathbb{Z}$ such that $r(n) < m < r(n+1)$. The maximal path ending at m is $p_{n+1,m}$. Indeed, this maximal path can not start before n (since then $p_{n,r(n)}$ would be a subpath of $p_{n,m}$ different from $p_{n,m}$), and $p_{n+1,m}$ is a path in $K[\mathbb{A}_\infty, r]$, as a subpath of $p_{n+1,r(n+1)}$. Hence $r(l(m)) = r(n+1) \neq m$, and then $K[\mathbb{A}_\infty, r]$ could not be right co-Frobenius by the right-hand version of Proposition 4.3, a contradiction. Thus r must be surjective, and then it must be of the form $r(n) = n + s$ for any n, where s is an integer. Since $n < r(n)$ for any n, we must have $s > 0$. For simplicity we will denote $K[\mathbb{A}_\infty, r]$ by $K[\mathbb{A}_\infty | s]$ in the case where $r(n) = n + s$ for any $n \in \mathbb{Z}$.

Example 5.2. Let $\Gamma = \mathbb{A}_{0,\infty}$ be the subquiver of $\mathbb{A}_\infty$ obtained by deleting all the negative vertices and the arrows involving them. Thus $\Gamma_0 = \mathbb{N}$, the natural numbers (including 0).

$$\mathbb{A}_{0,\infty} : \quad \circ^0 \longrightarrow \circ^1 \longrightarrow \circ^2 \longrightarrow \circ^3 \longrightarrow \ldots$$

We keep the same notation for $p_{k,l}$ for $0 \leq k \leq l$. Let $r : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function with $r(0) > 0$ (so then $r(n) > n$ for any $n \in \mathbb{N}$), and define $K[\mathbb{A}_{0,\infty}, r]$ to be the path subcoalgebra of $K\mathbb{A}_{0,\infty}$ with basis $\{p_{k,l} \mid k, l \in \mathbb{N}, k \leq l \leq r(k)\}$. With the same arguments as in Example 5.1 we see that $K[\mathbb{A}_{0,\infty}, r]$ is a left co-Frobenius coalgebra. We note that $l(0) = 0$, and then $r(l(0)) = r(0) > 0$. By a right-hand version of Proposition 4.3, this shows that $K[\mathbb{A}_{0,\infty}, r]$ is never right co-Frobenius.

Example 5.3. For any $n \geq 2$ we consider the quiver $\mathbb{C}_n$, whose vertices are the elements of $\mathbb{Z}_n = \{\overline{0}, \ldots, \overline{n-1}\}$, the integers modulo n, and there is one arrow from $\overline{i}$ to $\overline{i+1}$ for each i.

$$\mathbb{C}_n :$$
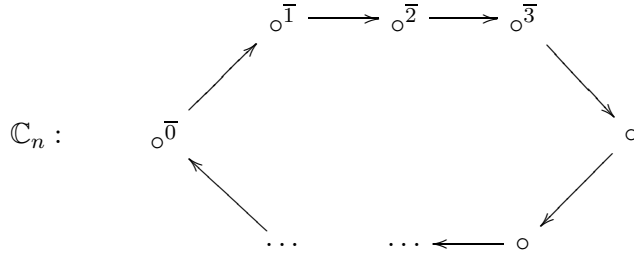
We also denote by $\mathbb{C}_1$ the quiver with one vertex, denoted by $\overline{0}$, and one arrow $\circ$, and by $\mathbb{C}_0$ the quiver with one vertex and no arrows.

Let $n \geq 1$ and $s > 0$ be integers. Let $K[\mathbb{C}_n, s]$ be the path subcoalgebra of the path coalgebra $K\mathbb{C}_n$, spanned by all paths of length at most s. Denote by $q_{\overline{k}|l}$ the path (in $\mathbb{C}_n$) of length l starting at the $\overline{k}$, for any $k \in \mathbb{Z}_n$ and $0 < l \leq s$. Also denote by $q_{\overline{k}|0}$ the vertex $\overline{k}$. Since the comultiplication and counit in $K\mathbb{C}_n$ are given by

$$\Delta(q_{\overline{k}|l}) = \sum_{i=0}^{l} q_{\overline{k},i} \otimes q_{\overline{k+i},l-i},$$

$$\varepsilon(q_{\overline{k}|l}) = \delta_{0,l}$$

we see that indeed $K[\mathbb{C}_n, s] = < q_{\overline{k}|l} \mid \overline{k} \in \mathbb{Z}, 0 \leq l \leq s >$ is a subcoalgebra of $K\mathbb{C}_n$. Clearly $q_{\overline{k}|s}$ is the unique maximal path in $K[\mathbb{C}_n, s]$ starting at $\overline{k}$, so $\overline{k} \in R(K[\mathbb{C}_n, s])$ and $r(\overline{k}) = \overline{k+s}$. Also $\overline{k+s} \in L(K[\mathbb{C}_n, s])$ and the maximal path ending at $\overline{k+s}$ is also $q_{\overline{k}|s}$, thus $lr(\overline{k}) = \overline{k}$, and by Proposition 4.3 we get that $K[\mathbb{C}_n, s]$ is a left co-Frobenius coalgebra. Since it has finite dimension $n(s+1)$, it is right co-Frobenius, too.

For a path subcoalgebra $C \subseteq K\Gamma$, denote by $C \cap \Gamma$ the subgraph of Γ consisting of arrows and vertices of Γ belonging to C.

Lemma 5.4. If $C \subseteq K\Gamma$ is a left co-Frobenius path subcoalgebra, then $C \cap \Gamma = \bigsqcup_i \Gamma_i$, a disjoint union of subquivers of Γ, where each Γ_i is of one of types $\mathbb{A}_\infty$, $\mathbb{A}_{0,\infty}$ or $\mathbb{C}_n$, $n \geq 0$, and $C = \bigoplus_i C_i$, where C_i, a path subcoalgebra of $K\Gamma_i$, is the subcoalgebra of C spanned by the paths of $\mathcal{B}$ contained in Γ_i.

Proof. Let v be a vertex in $C \cap \Gamma$. By Proposition 4.3 there is a unique maximal path $p \in \mathcal{B}$ starting at v, and any path in $\mathcal{B}$ starting at v is a subpath of p. This shows that at most one arrow in $\mathcal{B}$ starts at v (the first arrow of p, if p has length > 0). We show that at most one arrow in $\mathcal{B}$ ends at v, too. Otherwise, if we assume that two different arrows a and a' in $\mathcal{B}$ end at v, let $s(a) = u$ and $s(a') = u'$ (clearly $u \neq u'$, since at most one arrow starts at u), and let q and q' be the maximal paths in $\mathcal{B}$ starting at u and u', respectively. Then $q = az$ and $q' = a'z'$ for some paths z and z' starting at v. But then z and z' are subpaths of p, so one of them, say z, is a subpath of the other one. If $w = t(z)$, then $w = r(u)$, so $w \in L(C)$ and any path in $\mathcal{B}$ ending at w is a subpath of $q = az$. This provides a contradiction, since $a'z$ is in $\mathcal{B}$ (as a subpath of q')

and ends at w, but it is not a subpath of q.

We also have that if there is no arrow in $\mathcal{B}$ starting at a vertex v, then there is no arrow in $\mathcal{B}$ ending at v either. Indeed, we have that the maximal path in $\mathcal{B}$ starting at v has length zero, so $r(v) = v$, and then $v \in L(C)$ and $l(v) = v$, which shows that no arrow in $\mathcal{B}$ ends at v.

Now taking the connected components of $C \cap \Gamma$ (regarded just as an undirected graph), and then considering the (directed) arrows, we find that $C \cap \Gamma = \bigsqcup_i \Gamma_i$ for some subquivers Γ_i which can be of the types $\mathbb{A}_\infty$, $\mathbb{A}_{0,\infty}$ or $\mathbb{C}_n$, and this ends the proof. $\qquad\square$

Lemma 5.5. *Let $C \subseteq K\Gamma$ be a left co-Frobenius path subcoalgebra. Let $u, v \in C \cap \Gamma$ be different vertices, and denote by p_u and p_v the maximal paths starting at u and v, respectively. Then p_u is not a subpath of p_v.*

Proof. Assume otherwise, so p_u is a subpath of p_v. We know that p_u and p_v end at $r(u)$ and $r(v)$, respectively. Let q be the subpath of p_v which starts at v and ends at $r(u)$. Since p_u is a subpath of p_v, then q contains p_u, too. Then both q and p_u end at $r(u)$, and since by Proposition 4.3 p_u is maximal with this property, we get that $q = p_u$. This means that $u = v$ (as starting points of p_a and q), a contradiction. $\qquad\square$

Now we are in the position to give the classification result for left co-Frobenius path subcoalgebras.

Theorem 5.6. *Let C be a path subcoalgebra of the path coalgebra $K\Gamma$, and let $\mathcal{B}$ be a basis of paths of C. Then C is left co-Frobenius if and only if $C \cap \Gamma = \bigsqcup_i \Gamma_i$, a disjoint union of subquivers of Γ of one of types $\mathbb{A}_\infty$, $\mathbb{A}_{0,\infty}$ or $\mathbb{C}_n$, $n \geq 1$, and the path subcoalgebra C_i of $K\Gamma_i$ spanned by the paths of $\mathcal{B}$ contained in Γ_i is of type $K[\mathbb{A}_\infty, r]$ if $\Gamma_i = \mathbb{A}_\infty$, of type $K[\mathbb{A}_{0,\infty}, r]$ if $\Gamma_i = \mathbb{A}_{0,\infty}$, of type $K[\mathbb{C}_n, s]$ with $s \geq 1$ if $\Gamma_i = \mathbb{C}_n$, $n \geq 1$, and of type K if $\Gamma_i = \mathbb{C}_0$. In this case $C = \bigoplus_i C_i$, in particular left co-Frobenius path subcoalgebras are direct sums of coalgebras of types $K[\mathbb{A}_\infty, r]$, $K[\mathbb{A}_{0,\infty}, r]$, $K[\mathbb{C}_n, s]$ or K.*

Proof. By Lemma 5.4 we have that $C \cap \Gamma = \bigsqcup_i \Gamma_i$, and any Γ_i is of one of the types $\mathbb{A}_\infty$, $\mathbb{A}_{0,\infty}$ or $\mathbb{C}_n$, $n \geq 0$. Moreover, $C = \bigoplus_i C_i$, so C is left co-Frobenius if and only if all C_i's are left co-Frobenius (see for example [15, Chapter 3]). If all C_i's are of the indicated form, then they are left co-Frobenius by Examples 5.1, 5.2 and 5.3, and then so is C. Assume now that C is left co-Frobenius. Then each C_i is left co-Frobenius, so we can reduce to the case where Γ is one of $\mathbb{A}_\infty$, $\mathbb{A}_{0,\infty}$ or $\mathbb{C}_n$, and $C \cap \Gamma = \Gamma$. As before, for each vertex v we denote by $r(v)$ the end-point of the unique maximal path in C starting at v, and by p_v this maximal path. Also denote by $m(v)$ the length of p_v.

Case I. Let $\Gamma = \mathbb{C}_n$. If $n = 0$, then $C \cong K$. If $n = 1$, then $C \cong K[\mathbb{C}_1, s]$, since $m(\overline{0}) = s > 0$ because $\Gamma_1 \subset C$, so there must be at least some nontrivial path in C. If $n \geq 2$, we have that

$m(\overline{k}) \leq m(\overline{k+1})$ for any $\overline{k} \in \mathbb{Z}_n$, since otherwise $p_{\overline{k+1}}$ would be a subpath of $p_{\overline{k}}$, a contradiction by Lemma 5.5. Thus $m(\overline{0}) \leq m(\overline{1}) \leq \ldots m(\overline{n-1}) \leq m(\overline{0})$, so $m(\overline{0}) = m(\overline{1}) = \ldots m(\overline{n-1}) = m(\overline{0}) = s$ for some $s \geq 0$. Since $C \cap \Gamma = \Gamma$, there are non-trivial paths in C, so $s > 0$, and then clearly $C \cong K[\mathbb{C}_n, s]$.

Case II. If $\Gamma = \mathbb{A}_\infty$ or $\Gamma = \mathbb{A}_{0,\infty}$, then for any n (in $\mathbb{Z}$ if $\Gamma = \mathbb{A}_\infty$, or in $\mathbb{N}$ if $\Gamma = \mathbb{A}_{0,\infty}$) we have that $m(n) \leq m(n+1)$, otherwise p_{n+1} would be a subpath of p_n, again a contradiction. Then if we take $r(n) = n + m(n)$ for any n, we have that r is a strictly increasing function. Clearly $r(n) > n$, since $m(n) = 0$ would contradict $C \cap \Gamma = \Gamma$. Now it is obvious that $C \cong K[\Gamma, r]$.   □

Corollary 5.7. *Let $C \subseteq K\Gamma$ be a left and right co-Frobenius path subcoalgebra. Then C is a direct sum of coalgebras of the type $K[\mathbb{A}_\infty|s]$, $K[\mathbb{C}_n, s]$ or K.*

Proof. It follows directly from Theorem 5.6 and the discussion at the end of each of Examples 5.1, 5.2 and 5.3, concerned to the property of being left and right co-Frobenius.   □

Remark 5.8. (1) We have a uniqueness result for the representation of a left co-Frobenius path subcoalgebras as a direct sum of coalgebras of the form $K[\mathbb{A}_\infty, r]$, $K[\mathbb{A}_{0,\infty}, r]$, $K[\mathbb{C}_n, s]$ or K. To see this, we first note that an easy computation shows that the dual algebra of a coalgebra of any of these four types does not have non-trivial central idempotents, so it is indecomposable as an algebra. Now if $(C_i)_{i \in I}$ and $(D_j)_{j \in J}$ are two families of coalgebras with indecomposable dual algebras such that $\oplus_{i \in I} C_i \simeq \oplus_{j \in J} D_j$ as coalgebras, then there is a bijection $\phi : J \to I$ such that $D_j \simeq C_{\phi(j)}$ for any $j \in J$. Indeed, if $f : \oplus_{i \in I} C_i \to \oplus_{j \in J} D_j$ is a coalgebra isomorphism, then the dual map $f^* : \prod_{j \in J} D_j^* \to \prod_{i \in I} C_i^*$ is an algebra isomorphism. Since all C_i^*'s and D_j^*'s are indecomposable, there is a bijection $\phi : J \to I$ and some algebra isomorphisms $\gamma_j : D_j^* \to C_{\phi(j)}^*$ for any $j \in J$, such that for any $(d_j^*)_{j \in J} \in \prod_{j \in J} D_j^*$, the map f^* takes $(d_j^*)_{j \in J}$ to the element of $\prod_{i \in I} C_i^*$ having $\gamma_j(d_j^*)$ on the $\phi(j)$-th slot. Regarding $C = (C_i)_{i \in I}$ as a left C^*-module, and $D = \oplus_{j \in J} D_j$ as a left D^*-module in the usual way, with actions denoted by $\cdot$, we have that $f(f^*(d^*) \cdot c) = d^* \cdot f(c)$ for any $c \in C$ and $d^* \in D^*$. This shows that f induces coalgebra isomorphisms $C_{\phi(j)} \simeq D_j$ for any $j \in J$.

(2) The coalgebras of types $K[\mathbb{A}_\infty, r]$, $K[\mathbb{A}_{0,\infty}, r]$, $K[\mathbb{C}_n, s]$ or K can be easily classified if we take into account that the sets of grouplike elements are just the vertices and the non-trivial skew-primitives are scalar multiples of the arrows. There are no isomorphic coalgebras of two different types among these four types. Moreover: (i) $K[\mathbb{A}_\infty, r] \simeq K[\mathbb{A}_\infty, r']$ if and only if there is an integer h such that $r'(n) = r(n+h)$ for any integer n; (ii) $K[\mathbb{A}_{0,\infty}, r] \simeq K[\mathbb{A}_{0,\infty}, r']$ if and only if $r = r'$; (iii) $K[\mathbb{C}_n, s] \simeq K[\mathbb{C}_m, s']$ if and only $n = m$ and $s = s'$.

6. EXAMPLES

It is known (see [24], [11]) that any pointed coalgebra can be embedded in a path coalgebra. Thus it is expected that there is a large variety of co-Frobenius subcoalgebras of path coalgebras

if we do not restrict only to the class of path subcoalgebras. The aim of this section is to provide several such examples. We first explain a simple construction connecting incidence coalgebras and path coalgebras, and producing examples as we wish.

As a pointed coalgebra, any incidence coalgebra can be embedded in a path coalgebra. However, there is a more simple way to define such an embedding for incidence coalgebras than for arbitrary pointed coalgebras. Indeed, let X be a locally finite partially ordered set. Consider the quiver Γ with vertices the elements of X, and such that there is an arrow from x to y if and only if $x < y$ and there is no element z with $x < z < y$. With this notation, it is an easy computation to check the following.

Proposition 6.1. *The linear map* $\phi : KX \to K\Gamma$, *defined by*

$$\phi(e_{x,y}) = \sum_{\substack{p \text{ path} \\ s(p)=x, t(p)=y}} p$$

for any $x, y \in X, x \leq y$, *is an injective coalgebra morphism.*

Note that in the previous proposition $\phi(KX)$ is in general a subcoalgebra of $K\Gamma$ which is not a path subcoalgebra. This suggests that when we deal with left co-Frobenius subcoalgebras of incidence coalgebras, which of course embed themselves in $K\Gamma$ (usually not as path subcoalgebras), structures that are more complicated than those of left co-Frobenius path subcoalgebras can appear. Thus the classification of left co-Frobenius subcoalgebras of incidence coalgebras is probably more difficult. The next example is evidence in this direction.

Example 6.2. *Let* $s \geq 2$ *and* $X = \{a_n | n \in \mathbb{Z}\} \cup (\cup_{n\in\mathbb{Z}}\{b_{n,i} | 1 \leq i \leq s\})$ *with the ordering* $\leq$ *such that* $a_n < b_{n,i} < a_{n+1}$ *for any integer* n *and any* $1 \leq i \leq s$, *and* $b_{n,i}$ *and* $b_{n,j}$ *are not comparable for any* n *and* $i \neq j$.
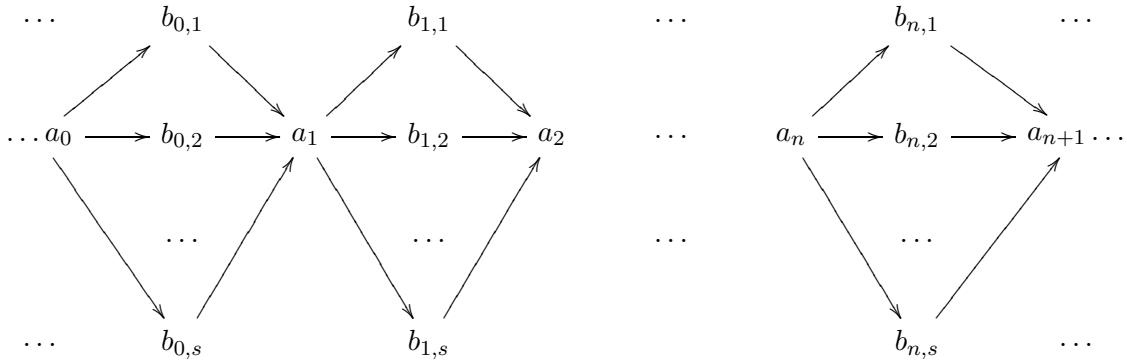
Let C *be the subcoalgebra of* KX *spanned by the following elements*

• *the elements* $e_{x,x}$, $x \in X$.

• *all segments* $e_{x,y}$ *of length 1.*

• *the segments* $e_{a_n,a_{n+1}}$, $n \in \mathbb{Z}$.

• *the segments* $e_{b_{n,i},b_{n+1,i}}$, *with* $n \in \mathbb{Z}$ *and* $1 \leq i \leq s$.

Then by applying Proposition 4.3, we see that C *is co-Frobenius.*

If we take the subcoalgebra D *of* C *obtained by restricting to the non-negative part of* X, *i.e.* D *is spanned by the elements* $e_{x,y}$ *in the indicated basis of* C *with both* x *and* y *among* $\{a_n | n \geq 0\} \cup (\cup_{n\geq 0}\{b_{n,i} | 1 \leq i \leq s\})$, *we see that* D *is left co-Frobenius, but not right co-Frobenius.*

Now let Γ be the quiver associated to the ordered set X as in the discussion above.

If $\phi : KX \to K\Gamma$ is the embedding described in Proposition 6.1, then $\phi(C)$ is a co-Frobenius subcoalgebra of $K\Gamma$. We see that $\phi(C)$ is the subspace of $K\Gamma$ spanned by the vertices of Γ, the paths of length 1, the paths $[b_{n,i}a_{n+1}b_{n+1,i}]$ with $n \in \mathbb{Z}$ and $1 \leq i \leq s$, and the elements $\sum_{1 \leq i \leq s}[a_n b_{n,i} a_{n+1}]$ with $n \in \mathbb{Z}$, thus $\phi(C)$ is not a path subcoalgebra. Here we denoted by $[b_{n,i}a_{n+1}b_{n+1,i}]$ and $[a_n b_{n,i} a_{n+1}]$ the paths following the indicated vertices and the arrows between them. By restricting to the non-negative part of X, a similar description can be given for $\phi(D)$, a subcoalgebra of $K\Gamma$ which is left co-Frobenius, but not right co-Frobenius.

It is possible to embed some of the co-Frobenius path subcoalgebras in other path coalgebras as subcoalgebras which are not path subcoalgebras.

Example 6.3. *Consider the quiver $\mathbb{A}_\infty$ with vertices indexed by the integers, with the path from i to j denoted by $p_{i,j}$. Consider the path subcoalgebra $D = K[\mathbb{A}_\infty|2]$, with basis $\{p_{i,i}, p_{i,i+1}, p_{i,i+2} | i \in \mathbb{Z}\}$. We also consider the quiver Γ below*
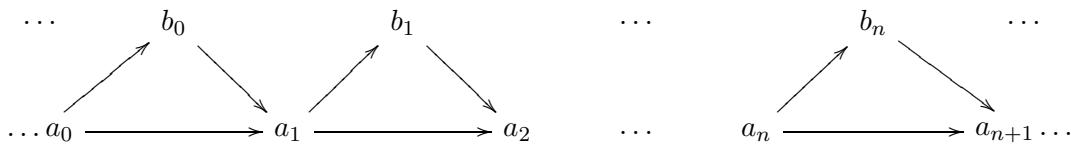
Then $\mathbb{A}_\infty$ is a subquiver of Γ if we identify a_i with $2i$ and b_i with $2i+1$ for any integer i. Thus $K\mathbb{A}_\infty$ is a subcoalgebra of $K\Gamma$ in the obvious way, and then so is D. However, there is another way to embed D in $K\Gamma$. Indeed, the linear map $\phi : D \to K\Gamma$, defined such that

$$\phi(p_{2i,2i}) = a_i, \phi(p_{2i+1,2i+1}) = b_i,$$

$$\phi(p_{2i,2i+1}) = [a_i b_i], \phi(p_{2i+1,2i+2}) = [b_i a_{i+1}],$$

$$\phi(p_{2i,2i+2}) = [a_i a_{i+1}] + [a_i b_i a_{i+1}],$$

$$\phi(p_{2i+1,2i+3}) = [b_i a_{i+1} b_{i+1}]$$

for any $i \in \mathbb{Z}$, is an injective morphism of coalgebras. Here we denoted by $[a_i b_i]$, $[a_i b_i a_{i+1}]$, etc, the paths following the respective vertices and arrows. We conclude that the subcoalgebra

$C = \phi(D)$ of $K\Gamma$, spanned by all vertices a_n, b_n, all arrows $[a_n a_{n+1}], [a_n b_n], [b_n a_{n+1}]$ and the elements $[a_n b_n a_{n+1}] + [a_n a_{n+1}]$ and $[b_n a_{n+1} b_{n+1}]$, is co-Frobenius. Note that D is not a path subcoalgebra of $K\Gamma$. This can be also seen as the subcoalgebra of the incidence coalgebra of $\mathbb{Z}$ with basis consisting of segments of length at most 2.

Note that in the above example, we can also consider a similar situation but with all segments $e_{n,n+i}$ of the incidence coalgebra of $\mathbb{Z}$ which have length less or equal to a certain positive integer s $(i \leq s)$; the same properties as above would then hold for this situation.

Example 6.4. *We consider the same situation as above, but we restrict the graph Γ to only the non-negative part:*
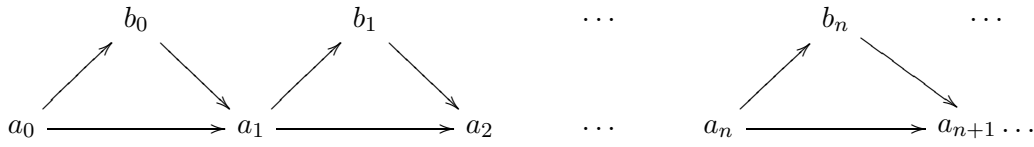
Equivalently, we consider the subcoalgebra of the incidence coalgebra of $\mathbb{N}$ with a basis of all segments of length less or equal to 2 (or $\leq s$ for more generality). This coalgebra is only going to be left co-Frobenius and not right co-Frobenius, it is a subcoalgebra of an incidence coalgebra, and it can also be regarded as a subcoalgebra of a path coalgebra, but without a basis of paths.

Now we prove a simple, but useful result, which shows that the category of incidence coalgebras is closed under tensor product of coalgebras.

Proposition 6.5. *Let X, Y be locally finite partially ordered sets. Consider on $X \times Y$ the order $(x, y) \leq (x', y')$ if and only if $x \leq y$ and $x' \leq y'$. Then there is an isomorphism of coalgebras $K(X \times Y) \cong KX \otimes KY$.*

Proof. It is clear that $X \times Y$ is locally finite. We show that the natural isomorphism of vector spaces $\varphi : K(X \times Y) \to KX \otimes KY$, $\varphi(e_{(x,y),(x',y')}) = e_{x,x'} \otimes e_{y,y'}$ is a morphism of coalgebras. This is well defined by the definition of the order relation on $X \times Y$. For comultiplication we have

$$
\begin{aligned}
\sum \varphi(e_{(x,y),(x',y')})_1 \otimes (e_{(x,y),(x',y')})_2 &= \sum_{\substack{x \leq a \leq x' \\ y \leq b \leq y'}} e_{x,a} \otimes e_{y,b} \otimes e_{a,x'} \otimes e_{x',b} \\
&= \sum_{(x,y) \leq (a,b) \leq (x',y')} \varphi(e_{(x,y),(a,b)}) \otimes \varphi(e_{(a,b),(x',y')}) \\
&= \varphi((e_{(x,y),(x',y')})_1) \otimes \varphi((e_{(x,y),(x',y')})_2)
\end{aligned}
$$

and it is also easy to see that $\varepsilon_{KX \otimes KY} \circ \varphi = \varepsilon_{K(X \times Y)}$. $\square$

Example 6.6. *Consider the ordered set $(\mathbb{Z} \times \mathbb{Z}, \leq)$, with order given by the direct product of the orders of $(\mathbb{Z}, \leq)$ and $(\mathbb{Z}, \leq)$. Thus $(i, j) \leq (p, q)$ if and only if $i \leq p$ and $j \leq q$. We know from*

Proposition 6.5 that $\psi : K\mathbb{Z} \otimes K\mathbb{Z} \to K(\mathbb{Z} \times \mathbb{Z})$, $\psi(e_{i,p} \otimes e_{j,q}) = e_{(i,j),(p,q)}$, is an isomorphism of coalgebras.

With the notation preceding Proposition 6.1, the quiver Γ associated to the locally finite ordered set $(\mathbb{Z} \times \mathbb{Z}, \leq)$ is

$$
\begin{array}{ccccccc}
\cdots & & \cdots & & \cdots & & \\
\uparrow & & \uparrow & & \uparrow & & \\
\cdots \longrightarrow a_{n-1,k+1} & \longrightarrow & a_{n,k+1} & \longrightarrow & a_{n+1,k+1} & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
\cdots \longrightarrow a_{n-1,k} & \longrightarrow & a_{n,k} & \longrightarrow & a_{n+1,k} & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
\cdots \longrightarrow a_{n-1,k-1} & \longrightarrow & a_{n,k-1} & \longrightarrow & a_{n+1,k-1} & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
\cdots & & \cdots & & \cdots & &
\end{array}
$$

where we just denoted the vertices by $a_{n,k}$ instead of just (n,k). Let $\phi : K(\mathbb{Z} \times \mathbb{Z}) \to K\Gamma$ be the embedding from Proposition 6.1. If we consider the subcoalgebra $K[\mathbb{A}_\infty|1]$ of $K\mathbb{Z}$, we have that $K[\mathbb{A}_\infty|1] \otimes K[\mathbb{A}_\infty|1]$ is a subcoalgebra of $K\mathbb{Z} \otimes K\mathbb{Z}$, so then $C = \phi\psi(K[\mathbb{A}_\infty|1] \otimes K[\mathbb{A}_\infty|1])$, which is the subspace spanned by the vertices of Γ, the arrows of Γ, and the elements $[a_{n,k}a_{n+1,k}a_{n+1,k+1}] + [a_{n,k}a_{n,k+1}, a_{n+1,k+1}]$, is a subcoalgebra of $K\Gamma$. Since $K[\mathbb{A}_\infty|1]$ is co-Frobenius, and the tensor product of co-Frobenius coalgebras is co-Frobenius (see [21, Proposition 4.15]), we obtain that C is a co-Frobenius coalgebra. Alternatively, it can be seen that $\psi(K[\mathbb{A}_\infty|1] \otimes K[\mathbb{A}_\infty|1])$, which is the subspace spanned by the elements $e_{(n,k),(n,k)}, e_{(n,k),(n+1,k)}, e_{(n,k),(n,k+1)}, e_{(n,k),(n+1,k+1)}$ with arbitrary $n, k \in \mathbb{Z}$, is co-Frobenius by applying Proposition 4.3. C can be seen as both a subcoalgebra of an incidence coalgebra and of a path coalgebra, but not with a basis of paths. We note that C is not even isomorphic to a path subcoalgebra. Indeed, if it were so, it should be isomorphic to some $K[\mathbb{A}_\infty|s]$, since it is infinite dimensional and indecomposable. But in C, for any grouplike element g there are precisely two other grouplike elements h with the property that the set of non-trivial (h,g)-skew-primitive elements is nonempty, while for any grouplike element g of $K[\mathbb{A}_\infty|s]$ there is only one such h.

With similar arguments, we can give a more general version of the previous example, by considering finite tensor products of coalgebras of type $K[\mathbb{A}_\infty|s]$, as follows.

Example 6.7. *Let $D = K[\mathbb{A}_\infty|s_1] \otimes K[\mathbb{A}_\infty|s_2] \otimes \ldots \otimes K[\mathbb{A}_\infty|s_m]$, where $m \geq 2$ and $s_1, \ldots, s_m$ are positive integers. Then D is co-Frobenius as a tensor product of co-Frobenius coalgebras, and D embeds in the m-fold tensor product $K\mathbb{Z} \otimes K\mathbb{Z} \otimes \ldots \otimes K\mathbb{Z}$. But this last tensor product is isomorphic to the incidence coalgebra of the ordered set $\mathbb{Z}^m = \mathbb{Z} \times \mathbb{Z} \times \ldots \times \mathbb{Z}$, with the direct product order. The image of D via this embedding is the subcoalgebra E of $K(\mathbb{Z} \times \mathbb{Z} \times \ldots \times \mathbb{Z})$*

spanned by all the segments $e_{(n_1,\ldots,n_m),(k_1,\ldots,k_m)}$ with $n_1 \leq k_1 \leq n_1+s_1, \ldots, n_m \leq k_m \leq n_m+s_m$.
Now if we consider the quiver Γ associated to the ordered set $\mathbb{Z} \times \mathbb{Z} \times \ldots \times \mathbb{Z}$ as in the beginning
of this section, we have an embedding of $K(\mathbb{Z} \times \mathbb{Z} \times \ldots \times \mathbb{Z})$ in $K\Gamma$. Denote the vertices of Γ
by $a_{n_1,\ldots,n_m}$. The image of E through this embedding is the subcoalgebra C of $K\Gamma$ spanned by
all the elements of the form $S(\Gamma, (n_1,\ldots,n_m),(k_1,\ldots,k_m))$, with $n_1,\ldots,n_m,k_1,\ldots,k_m$ integers
such that $n_1 \leq k_1 \leq n_1 + s_1,\ldots,n_m \leq k_m \leq n_m + s_m$, where by $S(\Gamma, (n_1,\ldots,n_m),(k_1,\ldots,k_m))$
we denote the sum of all paths in Γ starting at $a_{n_1,\ldots,n_m}$ and ending at $a_{k_1,\ldots,k_m}$. Thus C is
a co-Frobenius subcoalgebra of $K\Gamma$, which is also isomorphic to a subcoalgebra of an incidence
coalgebra. However, C is not a path subcoalgebra, and not even isomorphic to a path subcoalgebra.
Indeed, for any grouplike element g of E there are precisely m grouplike elements h for which
there are non-trivial (h,g)-skew-primitive elements, while in a co-Frobenius path subcoalgebra
for any grouplike element g there is at most one such h.

Remark 6.8. *We note that the co-Frobenius coalgebra C constructed in Example 6.2 is not*
isomorphic to a coalgebra of the form $K[\mathbb{A}_\infty|s_1] \otimes K[\mathbb{A}_\infty|s_2] \otimes \ldots \otimes K[\mathbb{A}_\infty|s_m]$. Indeed, if
$g = b_{n,i}$ there exists exactly one grouplike element h of C such that there are non-trivial (h,g)-
skew-primitive elements (this is $h = a_{n+1}$), and if $g = a_n$ there exist s such grouplike elements
h (these are $b_{n,1},\ldots,b_{n,s}$). On the other hand, in $K[\mathbb{A}_\infty|s_1] \otimes K[\mathbb{A}_\infty|s_2] \otimes \ldots \otimes K[\mathbb{A}_\infty|s_m]$ for
any grouplike element g there exist precisely m such elements h.

We end with another explicit example, which shows that there are co-Frobenius subcoalgebras
of path coalgebras that are isomorphic neither to a path subcoalgebra nor to a subcoalgebra of
an incidence coalgebra.

Example 6.9. *Let Γ be the graph:*

and let C be the subcoalgebra of the path coalgebra of Γ having a basis the elements a_n, x_n, y_n
and $y_n + x_ny_n$. This is, in fact, isomorphic to $K[\mathbb{C}_1|1] \otimes K[\mathbb{A}_\infty|1]$, so it is co-Frobenius. By the
classification theorem for co-Frobenius path subcoalgebras and the structure of the skew-primitive
elements of C, we see that C is not isomorphic to a path subcoalgebra. We note that it is not
isomorphic either to a subcoalgebra of an incidence coalgebra, because in an incidence coalgebra,
if g is any grouplike element, there is no (g,g)- skew-primitive element, while in C for each
grouplike $g = a_n$, x_n is a (g,g)- skew-primitive.

7. Coreflexivity for path subcoalgebras and subcoalgebras of incidence

coalgebras

We recall from [26, 29] that a coalgebra C is called coreflexive if any finite dimensional left (or equivalently, any right) C^*-module is rational. This is also equivalent to asking that the natural embedding of C into the finite dual of C^*, $C \to (C^*)^0$ is surjective (so an isomorphism), or that any left (equivalently, any right) cofinite ideal is closed in the finite topology. See [25, 26, 29, 30] for further equivalent characterizations.

Theorem 7.1. *Let C be a coalgebra with the property that for any finite dimensional subcoalgebra V there exists a finite dimensional subcoalgebra W such that $V \subseteq W$ and $W^\perp W^\perp = W^\perp$. Then C is coreflexive if and only if its coradical C_0 is coreflexive.*

Proof. If C is coreflexive, then so s C_0, since a subcoalgebra of a coreflexive coalgebra is coreflexive (see [18, Proposition 3.1.4]). Conversely, let C_0 be coreflexive. We prove that any finite dimensional left C^*-module M is rational, by induction on the length $l(M)$ of M. If $l(M) = 1$, i.e. M is simple, then M is also a left $C^*/J(C^*)$-module. Since $C^*/J(C^*) \simeq C_0^*$ and C_0 is coreflexive, we have that M is a rational C_0^*-module, so then it is a rational C^*-module, too. Assume now that the statement is true for length $< n$, where $n > 1$, and let M be a left C^*-module of length n. Let M' be a simple submodule of M, and consider the associated exact sequence

$$0 \to M' \to M \to M'' \to 0$$

By the induction hypothesis M' and M'' are rational. By [15, Theorem 2.2.14] we have that $ann_{C^*}(M')$ and $ann_{C^*}(M'')$ are co-finite dimensional closed two-sided ideals in C^*. Using [15, Corollary 1.2.8 and Proposition 1.5.23], we have that $ann_{C^*}(M') = U_1^\perp$ and $ann_{C^*}(M'') = U_2^\perp$ for some finite dimensional subcoalgebras of C. Using the hypothesis for $V = U_1 + U_2$, we have that there is a finite dimensional subcoalgebra W of C such that $U_1 \subseteq W$, $U_2 \subseteq W$ and $W^\perp W^\perp = W^\perp$. Then $W^\perp = W^\perp W^\perp \subseteq U_1^\perp U_2^\perp = ann_{C^*}(M')ann_{C^*}(M'') \subseteq ann_{C^*}(M)$, and M is a rational C^*-module by using again [15, Theorem 2.2.14]. $\square$

Proposition 7.2. *Let C be the path coalgebra $K\Gamma$, where Γ is a quiver such that there are finitely many paths between any two vertices. Then for any finite dimensional subcoalgebra V of C there exists a finite dimensional subcoalgebra W such that $V \subseteq W$ and $W^\perp W^\perp = W^\perp$. As a consequence, C is coreflexive if and only if the coradical C_0 (which is a grouplike coalgebra over the set of vertices of Γ) is coreflexive.*

Proof. Let V be a finite dimensional subcoalgebra of $C = K\Gamma$. An element $c \in V$ is of the form $c = \sum_{i=1,n} \alpha_i p_i, \alpha_i \neq 0$, a linear combination of paths $p_1, \ldots, p_n$. Consider the set of all vertices one of these paths pass through, and let S_0 be the union of all these sets of vertices when c runs through the elements of V. Since V is finite dimensional, we have that S_0 is finite (in fact, one

can see that S_0 consists of all vertices in Γ which belong to V, so that KS_0 is the socle of V).
Let P be the set of all paths p such that $s(p), t(p) \in S_0$. We consider the set S of all vertices at
least one path of P passes through. It is clear that P is finite, and then so is S. We note that
if $v_1, v_2 \in S$ and p is a path from v_1 to v_2, then any vertex on p lies in S. Indeed, v_1 is on a
path from u_1 to u_1' (vertices in S_0), and let p_1 be its subpath from u_1 to v_1. Similarly, v_2 is on
a path from u_2 to u_2' (in S_0), and let p_2 be the subpath from v_2 to u_2'. Then $p_1pp_2 \in P$, so any
vertex of p is in S. Let W be the subspace spanned by all paths starting and ending at vertices
in S. It is clear that any subpath of a path in W is also in W, so then W is a finite dimensional
subcoalgebra containing V (sine S_0 is contained in S).

We show that $W^\perp W^\perp = W^\perp$. For this, let $\eta \in W^\perp$, and we construct $f_1, f_2, g_1, g_2 \in W^\perp$
such that $\eta = f_1 g_1 + f_2 g_2$. We define $f_i(p)$ and $g_i(p)$, $i = 1, 2$, on all paths p by induction on
the length of p. For paths p of length zero, i.e. p is a vertex v, we define $f_i(v) = g_i(v) = 0$,
$i = 1, 2$, for any $v \in S$, while for $v \notin S$, we set $f_1(v) = g_2(v) = 1$, and $f_2(v)$ and $g_2(v)$ are
such that $g_1(v) + f_2(v) = \eta(v)$. Then clearly $\eta = f_1 g_1 + f_2 g_2$ on paths of length zero. For
the induction step, assume that we have defined f_i and g_i, $i = 1, 2$, on all paths of length
$< l$, and that $\eta = f_1 g_1 + f_2 g_2$ on any such path. Let now p be a path of length l, starting
at u and ending at v. If $u, v \in S$, then we define $f_i(p) = g_i(p) = 0$, $i = 1, 2$, and clearly
$\eta(p) = \sum_{1 \leq i \leq 2} \sum_{qr=p} f_i(q) g_i(r)$, since both sides are zero. If either $u \notin S$ or $v \notin S$, we need the
following equality to hold.

$$(7) \qquad f_1(u)g_1(p) + f_1(p)g_1(v) + f_2(u)g_2(p) + f_2(p)g_2(v) = \eta(p) - \sum_{\substack{i=1,2}} \sum_{\substack{qr=p \\ q \neq p, r \neq p}} f_i(q)g_i(r)$$

We note that the terms of the right-hand side of the equality (7) have already been defined,
because when $p = qr$ and $q \neq p$, $r \neq p$, the length of the paths q and r is strictly less than the
length of p. We define $f_1(p)$ and $g_2(p)$ to be any elements of K, and then since either $f_1(u) = 1$
or $g_2(v) = 1$ (since either $u \notin S$ or $v \notin S$), we can choose suitable $g_1(p)$ and $f_2(p)$ such that (7)
holds true.

The fact that C is coreflexive if and only if so is C_0 follows now directly from Theorem 7.1. $\qquad \square$

Moreover, we can extend the result in the previous proposition to subcoalgebras of path coalge-
bras.

Proposition 7.3. *Let C be a subcoalgebra of a path coalgebra $K\Gamma$, such that there are only
finitely many paths between any two vertices in Γ . Then C is coreflexive if and only if C_0 is
coreflexive.*

Proof. Let Γ' be the subquiver of Γ whose vertices are all the vertices v of Γ such that there is
an element $c = \sum_i \alpha_i p_i \in C$, where the α_i's are nonzero scalars and the p_i's are distinct paths,
and at least one p_i passes through v. The arrows of Γ' are all the arrows of Γ between vertices

of Γ'. Clearly, there are only finitely many paths between any two vertices in Γ'. Then we have that C is a subcoalgebra of $K\Gamma'$ and $C_0 = (K\Gamma')_0$. Obviously, $C_0 \subset (K\Gamma')_0$; for the converse, let us consider a vertex u in Γ', so there is $c \in C$ such that $c = \sum_i \alpha_i p_i$, with $\alpha_i \neq 0$ and distinct p_i's, and some p_k passes through u. Let us write then $p_k = qr$ such that q ends at u and r begins at u. Since C is a subcoalgebra of $K\Gamma'$ it is also a sub-bicomodule, so then $r^*cq^* \in C$, where $q^*, r^* \in (K\Gamma')^*$ are equal to 1 on q, r respectively and 0 on al other paths of $K\Gamma'$. Now

$$r^* p_i q^* = \sum_{p_i = stw} q^*(s) tr^*(w)$$

and the only non-zero terms can occur if $p_i = qt_i r$, so with t_i is a path starting and ending at u (loop at u). Let J be the set of these indices. In this situation $r^* p_i q^* = t_i$. Note that since the p_i's are different, the t_j's, $j \in J$ are different, too. Also, since $p_k = qr$, there is at least such a j. We have $r^* c q^* = \sum_j \alpha_j p_j$, with all p_j beginning and ending at u, and $p_k = u$. Let $l \in J$ be an index such that p_l has maximum length among the p_j's. We note then that $p_l^* p_j = 0$ if $j \neq l$, since for any decomposition $p_j = st$, we have $t \neq p_l$ because of the maximality of p_l and of the fact that $p_j \neq p_k$. However, $p_l^* p_l = u$. Therefore, $p_l^* c = \alpha_l u \in C$, so $u \in C$ since $\alpha_l \neq 0$.

Thus if C_0 is coreflexive, we have that $(K\Gamma')_0$ is coreflexive, and then by Lemma 7.2, we have that $K\Gamma'$ is coreflexive. Then C is coreflexive, as a subcoalgebra of $K\Gamma'$. Conversely, if C is coreflexive, then clearly C_0 is coreflexive. $\qquad\square$

Corollary 7.4. *Let C be a subcoalgebra of an incidence coalgebra KX. Then C is coreflexive if and only if C_0 is coreflexive.*

Proof. As showed in Section 6, KX can be embedded in a path coalgebra $K\Gamma$ for which there are finitely many paths between any two vertices. Then C is isomorphic to a subcoalgebra of a $K\Gamma$ and we apply Proposition 7.3. $\qquad\square$

We note that by the results of Radford and Heynemann [18, Theorem 3.7.3], this shows that if K is an infinite field, it is reasonable to expect that a sub-incidence coalgebra is co-reflexive. More precisely, we recall the following set-theoretical facts. Given a set X, an ultrafilter $\mathcal{F}$ on X is called an *Ulam* ultrafilter if $\mathcal{F}$ is closed under countable intersection. A set is called *non-measurable* if every Ulam ultrafilter is principal (i.e. it equals the collection of all subsets of X containing some fixed $x \in X$). The class of non-measurable sets contains the countable sets and is closed under usual set-theoretic constructions, such as the power set, products and unions. In fact, if such a set exists (which is not inconsistent with ZFC), its cardinal has to be "very large" (i.e. inaccessible in the sense of set theory). Thus it is "reasonable" to expect that KX will be co-reflexive if K is an infinite field for any set X one might usually consider.

We now take a close look at the connection between the path algebra and path coalgebra. For a quiver Γ, denote $K[\Gamma]$ the quiver algebra and $K\Gamma$ the path coalgebra as so far. We have:

Proposition 7.5. *If Γ is a quiver such that between any two vertices there are only finitely many paths, then $K[\Gamma]^0 = K\Gamma$, so the path coalgebra is the finite dual of the path algebra.*

Proof. We have $K[\Gamma]^0 \subset K[\Gamma]^* = \prod_{p\,\text{path}} Kp^*$, where $p^* \in K[\Gamma]^*$ has value 1 on path p and 0 on any other path. Recall that the finite dual $K[\Gamma]^0$ consists of $f \in K[\Gamma]^*$ such that $\ker(f)$ contains an ideal of finite codimension. Let I be an ideal of $K[\Gamma]$ of finite codimension. Note that $A = K[\Gamma]$ is an algebra with a complete set of orthogonal idempotents $\{a | a \text{ vertex in } \Gamma\}$. Note that if F is a set of idempotents (vertices in Γ) of A which don't belong to I, then F is linearly independent over I. Indeed, if $\sum_{a\in F} \lambda_a a \in I$, then $c(\sum_{a\in F} \lambda_a a) = \lambda_c c \in I$ for each $c \in F$, so $\lambda_c = 0$ since $c \notin I$. Since A/I is finite dimensional, this shows that except a finite set, all vertices of Γ are elements of I. Let $S = \{a \text{ vertex in } \Gamma | a \in I\}$. Note that any path p starting or ending at a vertex in S belongs to I, since $p = s(p)p = pt(p) \in I$ if either $s(p) \in I$ or $t(p) \in I$; let the set of these paths be P, let H be the vector space spanned by P and P' the set of the rest of the paths in Γ. Since there are only finitely many vertices outside S, we see that P' is finite (P' consists of paths which begin and end outside S) so H has finite codimension. This shows that $(A/H)^*$ is spanned by $p^* \in A^*$ for $p \in P'$. Hence if $f \in A^*$ such that $\ker(f) \supseteq I$, then $\ker(f) \supseteq H^\perp$, so we can regard f as an element of $(A/H)^*$ and we easily get that f is a linear combination of $\{p^* | p \in P'\}$. This shows that $f \in \bigoplus_{p\,\text{path in }\Gamma} Kp \subset \prod_{p\,\text{path in }\Gamma} Kp$. Also, note that for each $p^* \in K[\Gamma]^*$, the span of all paths of Γ not contained in p is an ideal contained in $\ker(p^*)$. This shows that $K[\Gamma]^0 = \bigoplus_{p\,\text{path in }\Gamma} Kp^*$ is isomorphic to the path coalgebra $K\Gamma$ as a vector space. Now, we have the the formula

$$p^*(ts) = \sum_{p=qr} q^*(t)r^*(s)$$

for any paths p, t, s such that the concatenation ts is possible. This shows that the comultiplication of $K[\Gamma]^0$ is $p^* \to \sum_{p=qr} q^* \otimes r^*$; also one can define $\varepsilon_{K[\Gamma]^0}(p^*) = \delta_{0,length(p)} = p^*(\sum_{a\,\text{vertex}} a)$ (this makes sense since only finitely many a's are outside $\ker(p)$) and this gives an isomorphism of coalgebras $K[\Gamma]^0 \cong K\Gamma$. $\qquad\square$

Note that in general one has that $K[\Gamma]^0 \supseteq K\Gamma$, i.e. the path coalgebra can be embedded in the finite dual of the path algebra. It is possible to have a strict containment: if A is the path algebra of a loop

then $A = K[X]$, the polynomial algebra in one indeterminate. The finite dual of this algebra is

$$\varinjlim_{f\,\text{irreducible};\, n\in\mathbb{N}} (K[X]/(f^n))^* = \bigoplus_{f\,\text{irreducible}} [\varinjlim_{n\in\mathbb{N}}(K[X]/(f^n))^*],$$ while the path coalgebra is precisely the divided power coalgebra, which can be written as $\varinjlim_{n\in\mathbb{N}} (K[X]/(X^n))^*$. These are obviously not isomorphic.

We give now an application of our considerations on coreflexive coalgebras.

If Γ, Γ' are quivers, then let us note that $\Gamma \times \Gamma'$ is also a quiver as follows: the vertices are all pairs (a, a') for a, a' vertices in Γ and Γ' respectively; the arrows in $\Gamma \times \Gamma'$ are the pairs (a, x') and (a', x), where a, a' are vertices and x, x' are arrows in Γ and Γ' respectively. Let $p = [x_1 x_2 \ldots x_n]$ is a path in Γ going through the vertices $[a_0, a_1, \ldots, a_n]$ and $q = y_1 y_2 \ldots y_k$ is a path in Γ' going through vertices $[b_0, b_1, \ldots, b_k]$ (some vertices may repeat). We look at the 2 dimensional lattice $L(p, q)$ obtained as $L(p, q) = \{a_0, \ldots a_n\} \times \{b_0, \ldots, b_k\}$. Recall that a lattice walk is a path starting at (a_0, b_0) and ending at (a_n, b_k) and always going either one step to the right or one step upwards (there are $\binom{n+k}{k}$ such walks). Then any lattice walk in L is a path in $\Gamma \times \Gamma'$, and in fact, conversely, any path in $\Gamma \times \Gamma'$ is a walk in the lattice associated in this way to p, q paths in Γ in Γ'. For two such paths p, q let us denote $W(p, q)$ the set of such walks in the lattice $L(p, q)$ regarded as a subset of $\Gamma \times \Gamma'$. We have

Lemma 7.6. *The linear map* $\alpha : K\Gamma \otimes K\Gamma' \hookrightarrow K[\Gamma \times \Gamma']$ *defined as* $\alpha(p \otimes q) = \sum\limits_{w \in W(p,q)} w$ *for* $p \in \Gamma$, $q \in \Gamma'$ *is an injective morphism of K-coalgebras.*

Proof. We keep the notations above. Denote δ and Δ the comultiplications of $K\Gamma \times K\Gamma'$ and $K[\Gamma \times \Gamma']$. If we write $p = p'p''$ and $q = q'q''$, then if $u' \in W(p', q')$ and $u'' \in W(p'', q'')$, then $u'u'' \in W(p, q)$ in the obvious way. We have

$$
\begin{aligned}
\Delta(\alpha(p \otimes q)) &= \Delta(\sum_{w \in W(p,q)} w) = \\
&= \sum_{u'u''=w} u' \otimes u'' \\
&= \sum_{p=p'p'', q=q'q''} \sum_{u' \in W(p',q') u'' \in W(p'',q'')} u' \otimes u'' \\
&= \sum_{p=p'p'', q=q'q''} (\sum_{u' \in W(p',q')} u') \otimes (\sum_{u'' \in W(p'',q'')} u'') \\
&= \sum_{p=p'p''q=q'q''} \alpha(p' \otimes q') \otimes \alpha(p'' \otimes q'') \\
&= (\alpha \otimes \alpha)(\delta(p \otimes q))
\end{aligned}
$$

This shows that α is a morphism of coalgebras (the compatibility with counits is easily verified). To prove injectivity, let us denote by (p^*, q^*) the linear map on $K[\Gamma \times \Gamma']$ which equals 1 on the path $[(x_1, b_0), \ldots, (x_n, b_0), (a_n, y_1), \ldots, (a_n, y_k)] = [(p, b_0); (a_n, q)]$ and 0 on the rest of the paths. Note that by multiplying $\alpha(p \otimes q)$ to the left by all the elements of $K[\Gamma \times \Gamma']^*$ of the form (a^*, r^*) with a vertex in Γ and r path in Γ', the only element of the type (s, b) with s path in Γ and b vertex in Γ' that can be obtained is (p, b_0). This allows us to obtain p out of $\alpha(p \otimes q)$; similarly, one recovers q and this proves injectivity. This ends the proof. $\qquad\square$

Combining the above, we derive a result about tensor products of certain coreflexive coalgebras. It is known that a tensor product of a coreflexive and a strongly coreflexive coalgebra is coreflexive (see [26]; see also [30]). It is not known whether the tensor product of coreflexive coalgebras is necessarily coreflexive. We have though the following corrolaries:

Proposition 7.7. *Let C, D be co-reflexive subcoalgebras of path coalgebras $K\Gamma$ and $K\Gamma'$ respectively such that between any two vertices in Γ and Γ' respectively there are only finitely many paths. Then $C \otimes D$ is co-reflexive.*

Proof. Without any loss of generality we may assume that $C_0 = (K\Gamma)_0 = K^{(\Gamma_0)}$ and $D_0 = (K\Gamma')_0 = K^{(\Gamma_0)}$ (otherwise we replace Γ and Γ' by appropriate subquivers), where $K^{(\Gamma_0)}$ denotes the grouplike coalgebra with basis the set Γ_0 of vertices of Γ. Now $C \otimes D$ is a subcoalgebra of $K\Gamma \otimes K\Gamma'$, so by Lemma 7.6, it also embeds in $K(\Gamma \times \Gamma')$. Since the coradical of $K(\Gamma \times \Gamma')$ is $K^{(\Gamma_0 \times \Gamma'_0)}$, and $K^{(\Gamma_0 \times \Gamma'_0)} \simeq K^{(\Gamma_0)} \otimes K^{(\Gamma'_0)} = C_0 \otimes D_0 \subseteq C \otimes D$, we must have that $(C \otimes D)_0 = K^{(\Gamma_0 \times \Gamma'_0)}$. We claim that $K^{(\Gamma_0 \times \Gamma'_0)}$ is coreflexive. Indeed, this is obvious if Γ_0 and Γ'_0 are both finite. Otherwise, $\mathrm{card}(\Gamma_0 \times \Gamma'_0) = \max\{\mathrm{card}(\Gamma_0), \mathrm{card}(\Gamma'_0)\}$, hence $K^{(\Gamma_0 \times \Gamma'_0)}$ is isomorphic either to $K^{(\Gamma_0)}$ or to $K^{(\Gamma'_0)}$, so it is coreflexive. Now by Proposition 7.3 we have that $C \otimes D$ is coreflexive. $\square$

Corollary 7.8. *If C, D are coreflexive subcoalgebras of incidence coalgebras, then $C \otimes D$ is coreflexive.*

Proof. It follows immediately from the embeding of C and D in path coalgebras verifying the hypothesis or Proposition 7.7. $\square$

8. Hopf algebra structures on path subcoalgebras

In this section we discuss the possibility of extending the coalgebra structure of a path subcoalgebra to a Hopf algebra structure. First of all, it is a simple application of Proposition 2.4 to see when a finite dimensional path coalgebra has a Hopf algebra structure.

Proposition 8.1. *If the path coalgebra $K\Gamma$ is finite dimensional, then it has a Hopf algebra structure if and only if it is cosemisimple, i.e. Γ has no arrows.*

Proof. If the finite dimensional $K\Gamma$ has a Hopf algebra structure, then it has non-zero integrals, so it is left (and right) co-Frobenius, and $K\Gamma$ is cosemisimple by Proposition 2.4. Conversely, if there are no arrows, then $K\Gamma$ can be endowed with the group Hopf algebra structure obtained if we consider a group structure on the set of vertices. $\square$

Next, we are interested in finding examples of Hopf algebra structures that can be defined on some path subcoalgebras. At this point we discuss only cases where the resulting Hopf algebra

has non-zero integrals, i.e. it is left (or right) co-Frobenius. Thus the path subcoalgebras that we consider are among the ones in Corollary 5.7. We ask the following general question.

PROBLEM. *Which of the left and right co-Frobenius path subcoalgebras (classified in Corollary 5.7) can be endowed with a Hopf algebra structure?*

In the rest of this section we solve the problem in the case where K is a field containing primitive roots of unity of any positive order, in particular K has characteristic zero. We will make this assumption on K from this point on. We just note that some of the constructions can be also done in positive characteristic, if we just require that K contains certain primitive roots of unity and the characteristic of K is large enough.

Proposition 8.2. *(I) Let $s > 0$ be an integer. Let q be a primitive $(s+1)th$ root of unity in K. Let G be a group such that there exist an element $g \in Z(G)$ of infinite order and a character $\chi \in G^*$ such that $\chi(g) = q$. Also let $\alpha \in K$. Consider the algebra generated by the elements of G (and preserving the group multiplication on these elements), and x, subject to relations*

$$xh = \chi(h)hx \text{ for any } h \in G$$

$$x^{s+1} = \alpha(g^{s+1} - 1)$$

(that is, the free or amalgamated product $K[x] * K[G]$, quotient out by the above relations). Then this algebra has a unique Hopf algebra structure such that the elements of G are grouplike elements, $\Delta(x) = 1 \otimes x + x \otimes g$, and $\varepsilon(x) = 0$. We denote this Hopf algebra structure by $H_\infty(s, q, G, g, \chi, \alpha)$.

(II) Let $n \geq 2$ and $s > 0$ be integers such that $s + 1$ divides n. Let q be a primitive $(s+1)th$ root of unity in K. Consider a group G such that there exist an element $g \in Z(G)$ of order n and a character $\chi \in G^*$ such that $\chi(g) = q$. Also let $\alpha \in K$. Consider the algebra generated by the elements of G (and preserving the group multiplication on these elements), and x, subject to relations

$$xh = \chi(h)hx \text{ for any } h \in G$$

$$x^{s+1} = \alpha(g^{s+1} - 1)$$

Then this algebra has a unique Hopf algebra structure such that the elements of G are grouplike elements, $\Delta(x) = 1 \otimes x + x \otimes g$, and $\varepsilon(x) = 0$. We denote this Hopf algebra structure by $H_n(s, q, G, g, \chi, \alpha)$.

Proof. We consider an approach similar to the one in [4]. For both (I) and (II) we consider the Hopf group algebra KG, and its Ore extension $KG[X, \overline{\chi}]$, where $\overline{\chi}$ is the algebra automorphism of KG such that $\overline{\chi}(h) = \chi(h)h$ for any $h \in G$. Since $g \in Z(G)$, this Ore extension has a unique Hopf algebra structure such that $\Delta(X) = 1 \otimes X + X \otimes g$ and $\varepsilon(X) = 0$, by the universal property for Ore extensions (see for example [4, Lemma 1.1]). Since $(1 \otimes X)(X \otimes g) = q(X \otimes g)(1 \otimes X)$,

the quantum binomial formula shows that $\Delta(X^{s+1}) = 1 \otimes X^{s+1} + X^{s+1} \otimes g^{s+1}$, so then the ideal $(X^{s+1} - \alpha(g^{s+1} - 1))$ is in fact a Hopf ideal of $KG[X, \overline{\chi}]$. Then we can consider the factor Hopf algebra $KG[X, \overline{\chi}]/(X^{s+1} - \alpha(g^{s+1} - 1))$, and this is just the desired Hopf algebra $H_\infty(s, q, G, g, \chi, \alpha)$ in case (I) and $H_n(s, q, G, g, \chi, \alpha)$ in case (II). $\qquad\square$

In the following example we give examples of co-Frobenius path subcoalgebras that can be endowed with Hopf algebra structures. Moreover, we don't only introduce one such structure, but a family of Hopf algebra structures on each path subcoalgebra considered in the example.

Example 8.3. *(i) $K[\mathbb{A}_\infty|s]$ can be endowed with a Hopf algebra structure for any $s \geq 1$. Indeed, let q be a primitive $(s+1)$th root of unity in K, and let $\alpha \in K$. We define a multiplication (on basis elements, then extended linearly) on $K[\mathbb{A}_\infty|s]$ by*

(8)
$$
p_{i,i+u} p_{j,j+v} = \begin{cases} q^{ju} \binom{u+v}{u}_q p_{i+j,i+j+u+v}, & \text{if } u+v \leq s \\ \alpha q^{ju} \dfrac{(u+v-s-1)_q!}{(u)_q!(v)_q!}(p_{i+j+s+1,u+v+i+j} - p_{i+j,u+v+i+j-s-1}), & \text{if } u+v \geq s+1 \end{cases}
$$

where $\binom{u+v}{u}_q$ denotes the q-binomial coefficient. Then this multiplication makes $K[\mathbb{A}_\infty|s]$ an algebra, which together the initial coalgebra structure define a Hopf algebra structure on $K[\mathbb{A}_\infty|s]$. Indeed, we can see this by considering the Hopf algebra $H_\infty(s, q, C_\infty, c, \chi, \alpha)$, where C_∞ is the (multiplicative) infinite cyclic group generated by an element c, and the character χ is defined by $\chi(c) = q$. Thus $H_\infty(s, q, C_\infty, c, \chi, \alpha)$ is generated as an algebra by the elements c and x, subject to relations $xc = qcx$ and $x^{s+1} = \alpha(c^{s+1} - 1)$, and with coalgebra structure such that $\Delta(c) = c \otimes c$, $\varepsilon(c) = 1$, and $\Delta(x) = 1 \otimes x + x \otimes c$. Since $(1 \otimes x)(x \otimes c) = q(x \otimes c)(1 \otimes x)$, we can apply the quantum binomial formula and get that

$$
\Delta(x^u) = \sum_{0 \leq h \leq u} \binom{u}{h}_q x^{u-h} \otimes c^{u-h} x^h
$$

and then

$$
\Delta\left(\frac{1}{(u)_q!} c^i x^u\right) = \sum_{0 \leq h \leq u} \frac{1}{(u-h)_q!} c^i x^{u-h} \otimes \frac{1}{(h)_q!} c^{i+u-h} x^h
$$

for any $0 \leq u \leq s$ and any integer i. Therefore if we denote $\frac{1}{(u)_q!} c^i x^u$ by $P_{i,i+u}$, this means that $\Delta(P_{i,i+u}) = \sum_{0 \leq h \leq u} P_{i,i+h} \otimes P_{i+h,i+u}$, showing that the linear isomorphism $\phi : K[\mathbb{A}_\infty|s] \to H_\infty(s, q, C_\infty, c, \chi, \alpha)$ taking $p_{i,i+u}$ to $P_{i,i+u}$ for any $0 \leq u \leq s$ and $i \in \mathbb{Z}$, is an isomorphism of coalgebras. Now we just transfer the algebra structure of $H_\infty(s, q, C_\infty, c, \chi, \alpha)$ through ϕ^{-1} and get precisely the multiplication defined by (8).

(ii) Let us consider now the coalgebra C which is a direct sum of a family of copies of (the same) $K[\mathbb{A}_\infty|s]$, indexed by a non-empty set P. Then C can be endowed with a Hopf algebra structure. To see this, we extend the above example as follows. Let G be a group such that there exist an element $g \in Z(G)$ of infinite order and a character $\chi \in G^*$ for which $q = \chi(g)$ is a primitive

$(s+1)$th root of unity, and moreover the factor group $G/ < g >$ is in bijection with the set P (note that such a triple (G, g, χ) always exists; we can take for instance a group structure on the set P, $G = C_\infty \times P$, g a generator of C_∞, and χ defined such that $\chi(g) = q$ and $\chi(p) = 1$ for any $p \in P$). For simplicity of the notation, we can assume that P is a set of representatives for the $< g >$-cosets of G. Consider the Hopf algebra $A = H_\infty(s, q, G, g, \chi, \alpha)$. Then the subalgebra B of A generated by g and x is a Hopf subalgebra isomorphic to $K[\mathbb{A}_\infty|s]$ as a coalgebra, and $A = \oplus_{p \in P} pB$ is a direct sum of subcoalgebras, all isomorphic to $K[\mathbb{A}_\infty|s]$. Thus A is isomorphic as a coalgebra to C, and we can transfer the Hopf algebra structure of A to C.

(iii) Assume that $n \geq 2$ and $s+1$ divides n. Then $K[\mathbb{C}_n, s]$ can be endowed with a Hopf algebra structure. Indeed, we proceed as for $K[\mathbb{A}_\infty|s]$, but replacing the Hopf algebra $H_\infty(s, q, C_\infty, c, \chi, \alpha)$ by $H_n(s, q, C_\infty, c, \chi, \alpha)$. Also, as in (ii), any arbitrary direct sum of copies of the same $K[\mathbb{C}_n, s]$ can be endowed with a Hopf algebra structure isomorphic to some $H_n(s, q, G, g, \chi, \alpha)$ for some q, G, g, χ, α, where q is a primitive $(s+1)$th root of unity, G is a group, $g \in Z(G)$ is an element of order n, $\chi \in G^*$ is a character such that $\chi(g) = q$, and $\alpha \in K$ is a scalar.

Now we can prove the main result of this section.

Theorem 8.4. *Assume that K is a field containing primitive roots of unity of any positive order (in particular, K has characteristic 0). Then a co-Frobenius path subcoalgebra C can be endowed with a Hopf algebra structure if and only if it is of one of the following three types:*

(I) A direct sum of copies of the same $K[\mathbb{A}_\infty|s]$ for some $s \geq 1$. In this case, any Hopf algebra structure on C is isomorphic to a Hopf algebra of the form $H_\infty(s, q, G, g, \chi, \alpha)$ for some q, G, g, χ, α, where q is a primitive $(s+1)$th root of unity, G is a group, $g \in Z(G)$ is an element of infinite order, $\chi \in G^*$ is a character such that $\chi(g) = q$, and $\alpha \in K$ is a scalar.

(II) A direct sum of copies of the same $K[\mathbb{C}_n, s]$ for some $n \geq 2$ and $s \geq 1$ such that $s + 1$ divides n. In this case, any Hopf algebra structure on C is isomorphic to a Hopf algebra of the form $H_n(s, q, G, g, \chi, \alpha)$ for some q, G, g, χ, α, where q is a primitive $(s+1)$th root of unity, G is a group, $g \in Z(G)$ is an element of order n, $\chi \in G^*$ is a character such that $\chi(g) = q$, and $\alpha \in K$ is a scalar.

(III) A direct sum of copies of K. In this case, any Hopf algebra structure on C is isomorphic to a group Hopf algebra KG for some group G.

Proof. By Example 8.3 we see that a coalgebra of type (I) or (II) has a Hopf algebra structure. Obviously, a coalgebra of type (III) is a grouplike coalgebra KX for some set X, so then it has a Hopf algebra structure, obtained if we endow X with a group structure.

Conversely, let C be a co-Frobenius path subcoalgebra which can be endowed with a Hopf algebra structure. By Corollary 5.7, C is isomorphic to a direct sum of coalgebras of types $K[\mathbb{A}_\infty|s]$, $K[\mathbb{C}_n, s]$ or K. We have that $G = G(C)$, the set of all vertices of C, is a group with the induced

multiplication. We look at the identity element 1 of this group and distinguish three cases.

Case 1. If 1 is a vertex in a connected component of type $K[\mathbb{A}_\infty|s]$, denote the vertices of this connected component by $(v_n)_{n \in \mathbb{Z}}$ such that $v_0 = 1$. Also denote by a_n the arrow from v_n to v_{n+1} for any $n \in \mathbb{Z}$. If $g = v_1$, then $\Delta(a_1) = 1 \otimes a_1 + a_1 \otimes g$, and $a_1 \notin C_0$. Then $\Delta(ga_1) = g \otimes ga_1 + ga_1 \otimes g^2$, and $ga_1 \notin C_0$, so $P_{g^2,g}(C) \nsubseteq C_0$. Since the only $h \in G$ such that $P_{h,g}(C)$ is not trivial (i.e. $\neq K(h-g)$, or equivalently, not contained in C_0) is $h = v_2$, we obtain that $v_2 = g^2$. Recurrently we see that $v_n = g^n$ for any positive integer n, and also for any negative integer n.

Let us take some $h \in G$. Then $\Delta(ha_1) = h \otimes ha_1 + ha_1 \otimes hg$ and $ha_1 \notin C_0$, so $P_{hg,h}(C) \neq K(hg - h)$. Hence there is an arrow starting at h and ending at hg in C; as before, inductively we get that there are in C arrows as follows

$$\ldots \longrightarrow \underset{hg^{-1}}{\circ} \longrightarrow \underset{h}{\circ} \longrightarrow \underset{hg}{\circ} \longrightarrow \underset{hg^2}{\circ} \longrightarrow \ldots$$

which shows that the vertex h belongs to a connected component D of type $K[\mathbb{A}_\infty|s']$ for some $s' \geq 1$. Moreover, $\Delta(a_1 h) = h \otimes a_1 h + a_1 h \otimes gh$, we also have $P_{gh,h}(C) \neq K(gh - h)$, so there is an arrow from h to gh in C. This shows that $hg = gh$, so then g must lie in $Z(G)$.

If we denote by $p_{h,g^i h}$ the unique path from h to $g^i h$, for any $h \in G$ and $i \geq 0$, we have that

$$\Delta(p_{1,g^s}) - 1 \otimes p_{1,g^s} - p_{1,g^s} \otimes g^s \in C_{s-1} \otimes C_{s-1}$$

and $p_{1,g^s} \notin C_{s-1}$. Then

$$\Delta(hp_{1,g^s}) - h \otimes hp_{1,g^s} - hp_{1,g^s} \otimes hg^s \in C_{s-1} \otimes C_{s-1}$$

and $hp_{1,g^s} \notin C_{s-1}$. But it is easy to check that in the path coalgebra $K[\Gamma]$ (whose subcoalgebra is C) we have that $\Delta(c) - h \otimes c - c \otimes hg^s \in (K\Gamma)_{s-1} \otimes (K\Gamma)_{s-1}$ if and only if $c \in (K\Gamma)_{s-1} + Kp_{h,hg^s}$. Applying this for $c = hp_{1,g^s} \notin C_{s-1}$, we obtain that $hp_{1,g^s} = c' + \gamma p_{h,hg^s}$ for some $c' \in (K\Gamma)_{s-1}$ and $\gamma \in K^*$. This shows that p_{h,hg^s} must be in C, so it lies in D, too, which implies that $s' \geq s$ (otherwise D cannot have paths of length s).

Similarly, since

$$\Delta(h^{-1}p_{h,hg^{s'}}) - 1 \otimes h^{-1}p_{h,hg^{s'}} - h^{-1}p_{h,hg^{s'}} \otimes g^{s'} \in C_{s'-1} \otimes C_{s'-1}$$

and $h^{-1}p_{h,hg^{s'}} \notin C_{s'-1}$, we obtain that $s \geq s'$. In conclusion $s' = s$, and C is a direct sum of coalgebras isomorphic to $K[\mathbb{A}_\infty|s]$.

In order to uncover the Hopf algebra structure of C, we use the Lifting Method proposed in [2]. Since $C_0 = K\Gamma$ is a Hopf subalgebra of C, the coradical filtration $C_0 \subseteq C_1 \subseteq \ldots$ of C is a Hopf algebra filtration, and we can consider the associated graded space $\operatorname{gr} C = C_0 \oplus \frac{C_1}{C_0} \oplus \ldots$, which has a graded Hopf algebra structure. Denote $H = K\Gamma$, the degree 0 component of $\operatorname{gr} C$, and by $\gamma : H \to \operatorname{gr} C$ the inclusion morphism. The natural projection $\pi : \operatorname{gr} C \to H$ is a Hopf algebra

morphism. Then the coinvariants $R = (\mathrm{gr}\, C)^{co\,H}$ with respect to the right H-coaction induced via π, i.e.

$$R = \{z \in \mathrm{gr}\, C \mid (I \otimes \pi)\Delta(z) = z \otimes 1\}$$

is a left Yetter-Drinfeld module over H, with left H-action defined by $h \cdot r = \sum \gamma(h_1)rS(\gamma(h_2))$ for any $h \in H, r \in R$, and left H-coaction $\delta(r) = \sum r_{(-1)} \otimes r_{(0)} = (\pi \otimes I)\Delta(r)$. Moreover, R is a graded subalgebra of $\mathrm{gr}\, C$, with grading denoted by $R = \oplus_{n \geq 0} R(n)$, and it also has a coalgebra structure with comultiplication $\Delta_R(r) = \sum r^{(1)} \otimes r^{(2)} = \sum r_1 \gamma\pi(S(r_2)) \otimes r_3$, and these make R a braided Hopf algebra in the category ${}^H_H YD$ of Yetter-Drinfeld modules over H. The Hopf algebra $\mathrm{gr}\, C$ can be reconstructed from R by bosonization, i.e. $\mathrm{gr}\, C \simeq R\#H$, the biproduct of R and H. The multiplication of this biproduct is the smash product given by $(r\#h)(p\#v) = \sum r(h_1 \cdot p)\#h_2 v$, while the comultiplication is the smash coproduct $\Delta(r\#h) = \sum(r^{(1)}\#(r^{(2)})_{(-1)}h_1) \otimes (r^{(2)})_{(0)}\#h_2$.

Since in our case C_i is the span of all paths of length at most i in C, if $z = \hat{c} \in R(n)$, then $c = \sum_i \alpha_i p_i$, a linear combination of paths p_i of length i, and $\sum_i \alpha_i \hat{p_i} \otimes t(p_i) = \sum_i \alpha_i \hat{p_i} \otimes 1$. Then $\alpha_i = 0$ for any i such that $t(p_i) \neq 1$, showing that $R(i)$ is spanned by the classes of the paths of length i which end at 1. We conclude that $R(i)$ has dimension 1 for any $0 \leq i \leq s$, and $\dim(R) = s+1$. By [2, Theorem 3.2] (see also [9, Proposition 3.4]) we have that R is isomorphic to a quantum line, i.e. $R \simeq R_q(H, v, \chi)$ for some primitive $(s+1)$'th root of unity q, an element $v \in G$ and a character $\chi \in G^*$ such that $\chi(v) = q$, and $\chi(h)hv = \chi(h)vh$ for any $h \in G$, i.e. $v \in Z(G)$ (we use the notation of [9, Section 2]). As an algebra we have $R_q(H, v, \chi) = K[y]/(y^{s+1})$, and the coalgebra structure is such that the elements $d_0 = 1, d_1 = y, d_2 = \frac{y^2}{(2)_q!}, \ldots, \frac{y^s}{(s)_q!}$ form a divided power sequence, i.e. $\Delta(d_i) = \sum_{0 \leq j \leq i} d_j \otimes d_{i-j}$ for any $0 \leq i \leq s$. The H-action on $R_q(H, v, \chi)$ is such that $h \cdot y = \chi(h)y$ for any $h \in G$, and the H-coaction is such that $y \mapsto v \otimes y$. By [9, Proposition 3.1], there exists a $(1, v)$-skew-primitive z in C, which is not in C_0, such that $vz = qzv$, C is generated as an algebra by z and G, and the class $\hat{z}$ in $\frac{C_1}{C_0}$ corresponds to the element $y\#1$ in $R_q(H, v, \chi)\#H$ via the isomorphism $\mathrm{gr}\, C \simeq R_q(H, v, \chi)\#H$. It follows that v must be g^{-1}. Since for $h \in G$ both zh and hz are $(h, g^{-1}h)$-skew-primitives, we must have $zh = \lambda hz + \beta(g^{-1}h - h)$ for some scalars λ and β. But $zhg = (\lambda hz + \beta(g^{-1}h - h))g = q\lambda hgz + \beta(h - hg)$, and on the other hand $zgh = qgzh = q\lambda ghz + q\beta(h - hg)$, showing that $\beta = 0$. Thus $zh = \lambda hz$, and passing to $\mathrm{gr}\, C$, this gives $\hat{z}h = \lambda h\hat{z}$. But in $R_q(H, v, \chi)\#H$ we have that $(1\#h)(y\#1) = \chi(h)(y\#1)(1\#h)$, so $\lambda = \chi(h)$. Therefore $zh = \chi(h)hz$. Replace the generator z by $x = gz$, which is a $(g, 1)$-skew-primitive. By the quantum binomial formula we see that $\Delta(x^{s+1}) = 1 \otimes x^{s+1} + x^{s+1} \otimes g^{s+1}$, so then $x^{s+1} = \alpha(g^{s+1} - 1)$ for some scalar α. Now it is clear that $C \simeq H_\infty(s, q^{-1}, G, g, \chi, \alpha)$.

Case 2. If 1 is a vertex in a connected component D of type $K[\mathbb{C}_1, s]$, with $s \geq 1$, then let x be the arrow from 1 to 1, which is a primitive element, i.e. $\Delta(x) = x \otimes 1 + 1 \otimes x$. Then for any $g \in G$

we have that $\Delta(gx) = gx \otimes g + g \otimes gx$, and $gx \notin C_0$, so there is an arrow from gx to gx. This shows that C must be a direct sum of coalgebras of type $K[\mathbb{C}_1, s']$ (for possible different values of s'). Then looking at $\Delta(x^i) - x^i \otimes 1 - 1 \otimes x^i$, it is easy to show by induction that x^i lies in D for any $i \geq 1$. Since x is a non-zero primitive element, the set $(x^i)_{i \geq 1}$ is linearly independent, a contradiction to the finite dimensionality of D. Thus this situation cannot occur.

If 1 is a vertex in a connected component of type $K[\mathbb{C}_n, s]$, with $n \geq 2$, the proof goes as in Case 1, and leads us to the conclusion that C is a direct sum of coalgebras isomorphic to $K[\mathbb{C}_n, s]$, and that C is isomorphic as a Hopf algebra to one of the form $H_n(s, q, G, g, \chi, \alpha)$. The only difference is that instead of using the paths $p_{h, g^i h}$, we deal with paths denoted by $p_{h|l}$, and meaning the path of length l starting at the vertex h. Also, since $\chi(g) = q$, a $(s+1)$'th root of unity, and $g^n = 1$, we must have that $s + 1$ divides n.

Case 3. If 1 is a vertex in a connected component of type K, then proceeding as in Case 1, we can see that there are no arrows in C, so C is a direct sum of copies of K. Thus C is a grouplike coalgebra, and Hopf algebra structures on C are just group Hopf algebras. $\qquad\square$

We note that the above theorem completely classifies finite dimensional (co)path Hopf algebras, i.e. finite dimensional Hopf algebras which are isomorphic to a path algebra, or to a path coalgebra, or more generally, a Hopf algebra which is a quotient of a finite dimensional path algebra by an ideal generated by paths, or a finite dimensional subcoalgebra of a path coalgebra, with a basis of paths. These are the algebras KG, $H_n(s, q, G, g, \chi, \alpha)$ and their duals, because a finite dimensional Hopf algebra is Frobenius as an algebra and co-Frobenius as a coalgebra. Moreover, we can also apply the classification for the situation when the finite dimensional Hopf algebra is a quotient of a path algebra $K[\Gamma]$ of a locally finite quiver Γ (i.e. there are only finitely many paths between any two vertices in Γ) by an ideal I generated by paths: in this case, by Proposition 7.5, the orthogonal $C = I^\perp \subset K\Gamma$ of I is a finite dimensional coalgebra and has as a basis of paths $\{p^* | p \notin I\}$ (with the identification of Proposition 7.5), and $C \cong (A/I)^*$. Thus Hopf algebra structures on A/I come from dualizing Hopf algebra structures on C, and the same result holds, i.e. such Hopf algebra structures are isomorphic to KG^*, or $H_n(s, q, G, g, \chi, \alpha)^*$.

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