Towards bounded negativity of self-intersection on general blown-up projective planes

Claudio Fontanari

Abstract

We address the problem of bounding from below the self-intersection of integral curves on the projective plane blown-up at general points. In particular, by applying classical deformation theory we obtain the expected bound in the case of either high ramification or low multiplicity.

1 Introduction

Let S be the blow-up of the complex projective plane $\mathbb{P}^2$ at $n \geq 1$ general points $p_1, \ldots, p_n$. Denote by H the hyperplane class and by E_i the exceptional divisor for $i = 1, \ldots, n$.

The following natural problem seems to be still widely open:

Question 1. *Is there a constant c_n depending only on n such that the self-intersection number C^2 satisfies $C^2 \geq c_n$ for every integral curve $C \subset S$?*

Renewed interest in this question has been recently witnessed by both Joe Harris and Brian Harbourne (see [9], Question on p. 24, and [8], Conjectures I.2.1 and I.2.7).

According to [7], the celebrated Segre-Harbourne-Gimigliano-Hirschowitz (SHGH) Conjecture (see for instance [7], Conjecture 3.1) turns out to be equivalent to the sharp inequality $C^2 \geq g - 1$, where g is the arithmetic genus of C, hence the expected lower bound is precisely $C^2 \geq -1$.

It is indeed well-known that $C^2 \geq -1$ if C is rational (see for instance [4], Proposition 2.4). The main result of [4] shows in particular that $C^2 \geq -1$ if $C \in |dH - \sum_{i=1}^{n} m_i E_i|$ with $m_k = 2$ for some k. The Mori-theoretic point of view introduced in [4] has been further developed in [11] and [5].

Here we generalize [4], Theorem 2.5, in two different directions:

Theorem 1. *Let Γ be an integral curve in $\mathbb{P}^2$ and let C be its strict transform. If Γ has at most two tangent directions at p_k for some k, then $C^2 \geq -1$.*

Theorem 2. *Let Γ be an integral curve in $\mathbb{P}^2$ and let $C \in |dH - \sum_{i=1}^n m_i E_i|$ be its strict transform. If $m_k \leq 3$ for some k, then $C^2 \geq -1$.*

Our main tool is classical deformation theory. In particular, we follow the well-established approach of Xu (see [12], [13]). We also exploit some recent refinements which appeared in [1], Lemma 3, and [10], Theorem A. For further applications of deformation theory to linear systems of divisors we refer to [3] and [2], §2.

We work over the complex field $\mathbb{C}$.

We are grateful to Edoardo Ballico, Ciro Ciliberto and Edoardo Sernesi for stimulating discussions on these topics and to the anonymous referee for her/his valuable remarks.

This research has been partially supported by GNSAGA of INdAM and MIUR Cofin 2008 - Geometria delle varietà algebriche e dei loro spazi di moduli (Italy).

2 The proofs

Proof of Theorem 1. Let $\Gamma \in |dH - (\sum_{i \neq k} m_i p_i - (m_k - 1) p_k)|$ for appropriate choices of d and m_j. The proof of [12], Lemma 1 (see also [6], Lemma 1.1) shows that the linear system $|dH - (\sum_{i \neq k} m_i p_i - (m_k - 1) p_k)|$ contains a curve $\Gamma' \neq \Gamma$. More explicitly, if $\Gamma = \{F(X, Y, Z) = 0\}$, $p_k(t) := [a(t), b(t), 1]$ with $p_k(0) = p_k$ and $p_i(t) := p_i$ for every $i \neq k$, then there is a deformation $\Gamma_t = \{F_t(X, Y, Z) = 0\}$ of Γ such that $F_0(X, Y, Z) = F(X, Y, Z)$ and Γ_t passes through $p_i(t)$ with multiplicity m_i for every $i = 1, \ldots, n$ and every t in a neighbourhood of 0. It follows that the curve Γ' defined as

$$\Gamma' = \left\{ \left. \frac{\partial F_t}{\partial t} \right|_{t=0} (X, Y, Z) = 0 \right\}$$

passes through p_i with multiplicity m_i for every $i \neq k$ and through p_k with multiplicity $m_k - 1$. Indeed, if

$$\Gamma = \{f_{m_k}(x, y) + \mathit{higher} = 0\}$$

in local affine coordinates (x, y) centered at p_k, then

$$\Gamma' = \left\{ \dot{a}(0) \frac{\partial f_{m_k}}{\partial x}(x, y) + \dot{b}(0) \frac{\partial f_{m_k}}{\partial y}(x, y) + \mathit{higher} = 0 \right\}. \tag{1}$$

Now, if Γ has at most two tangent directions at p_k, then we may assume that coordinates have been chosen so that

$$f_{m_k} = x^\alpha y^\beta$$

with $\alpha, \beta \geq 0$ and $\alpha + \beta = m_k$. It follows that

$$\Gamma' = \{\dot{a}(0)\alpha x^{\alpha-1} y^\beta + \dot{b}(0)\beta x^\alpha y^{\beta-1} + \mathit{higher} = 0\}.$$

In particular, by choosing $p_k(t) = [a(t), b(t), 1]$ such that one of $\dot{a}(0)$ and $\dot{b}(0)$ is 0 and the other is not 0 we obtain a curve $\Gamma' \neq \Gamma$ (since their multiplicity at p_k is different) of degree d (see [12], proof of Lemma 1) such that Γ' and Γ have exactly $m_k - 1$ tangents in common at p_k (counted with multiplicity).

Hence Bezout's theorem yields

$$d^2 = \Gamma.\Gamma' \geq \sum_{i \neq k} m_i^2 + m_k(m_k - 1) + m_k - 1 = \sum_{i=1}^{n} m_i^2 - 1$$

and

$$C^2 = d^2 - \sum_{i=1}^{n} m_i^2 \geq -1.$$

□

Proof of Theorem 2. By Theorem 1, we may assume that $m_k = 3$ and Γ has an ordinary singularity at p_k. Let S_k be the blow-up of $\mathbb{P}^2$ at the $n-1$ points $\{p_1, \dots, p_n\} \setminus \{p_k\}$ and let $\sigma_k : S \to S_k$ be the blow-up of p_k. If $C_k \subset S_k$ is the strict transform of $\Gamma \subset \mathbb{P}^2$, then it is enough to show that $C_k^2 \geq m_k^2 - 1$.

In order to do so, we follow the proof of Lemma 3 in [1] (see also [10], Theorem A). Indeed, the argument of [6], Lemma 1.1, and of [12], Lemma 1, as recalled above at the beginning of the proof of Theorem 1, yields non-zero sections $s \in H^0(C, L)$, where

$$L := (\sigma_k^*(\mathcal{O}_{C_k}(C_k)) \otimes \mathcal{O}_S((1 - m_k)E_k))\,|_C = \mathcal{O}_S(C + E_k)|_C.$$

In our notation, the sections s are induced by the strict transforms on S of the curves (1). Since Γ has at least two tangent directions at p_k, then as in [13], proof of Lemma 1, Case (1) on p. 202, we have that $\frac{\partial f_{m_k}}{\partial x}(x, y)$ and $\frac{\partial f_{m_k}}{\partial y}(x, y)$ are linearly independent modulo higher degree terms. It follows that the strict transforms on S of

$$\begin{aligned} \Gamma'_1 &= \left\{ \frac{\partial f_{m_k}}{\partial x}(x, y) + \mathit{higher} = 0 \right\}, \\ \Gamma'_2 &= \left\{ \frac{\partial f_{m_k}}{\partial y}(x, y) + \mathit{higher} = 0 \right\} \end{aligned}$$

together with $C + E_k$ generate a net of curves in $\mathbb{P}H^0(S, \mathcal{O}_S(C + E_k))$ and induce two linearly independent sections $s_1, s_2 \in H^0(C, L)$. Hence we get a map $\phi : C \to \mathbb{P}^1$ of degree $\deg(\phi) \leq \deg(L) = C_k^2 - m_k(m_k - 1)$.

Now, if C is rational then it is well-known that $C^2 \geq -1$ (see for instance [4], Proposition 2.4). Otherwise, we have

$$2 \leq \deg(\phi) \leq C_k^2 - m_k(m_k - 1),$$

hence

$$C_k^2 \geq m_k^2 - m_k + 2 = m_k^2 - 1$$

since $m_k = 3$.

□

References

[1] F. Bastianelli: Remarks on the nef cone on symmetric products of curves. Manuscripta Math. 130 (2009), 113-120.

[2] L. Chiantini and C. Ciliberto: Weakly defective varieties. Trans. Amer. Math. Soc. 354 (2002), 151-178.

[3] C. Ciliberto and R. Miranda: A deformation theory approach to linear systems with general triple points. Matematiche (Catania) 55 (2000), 259-270 (2002).

[4] T. de Fernex: Negative curves on very general blow-ups of $\mathbb{P}^2$. Projective varieties with unexpected properties, 199–207, de Gruyter, Berlin, 2005.

[5] T. de Fernex: On the Mori cone of blow-ups of the plane. arXiv:1001.5243 (2010).

[6] L. Ein and R. Lazarsfeld: Seshadri constants on smooth surfaces. Journées de Géométrie Algébrique d'Orsay (Orsay, 1992). Astérisque 218 (1993), 177–186.

[7] B. Harbourne: The (unexpected) importance of knowing alpha. Projective Varieties with Unexpected Properties, 267–272, de Gruyter, Berlin, 2005.

[8] B. Harbourne: Global aspects of the geometry of surfaces. In: Summer School on Linear Systems. Pedagogical University of Cracow, Poland, March 23–27, 2009, available online at http://www.math.unl.edu/~bharbourne1/LecturesJune8-09.pdf.

[9] J. Harris: The Interpolation Problem. In: Classical Algebraic Geometry Today. MSRI, Berkeley, January 26–30, 2009, available online at http://jessica2.msri.org/attachments/13549/13549.pdf.

[10] A. L. Knutsen, W. Syzdek, and T. Szemberg: Moving curves and Seshadri constants. Math. Res. Lett. 16 (2009), 711-719.

[11] F. Monserrat: Lins Neto's examples of foliations and the Mori cone of blow-ups of $\mathbb{P}^2$. arXiv:0901.1565 (2009).

[12] G. Xu: Curves in $\mathbb{P}^2$ and symplectic packings. Math Ann. 299 (1994), 609–613.

[13] G. Xu: Ample line bundles on smooth surfaces. J. Reine Angew. Math. 469 (1995), 199-209.

Claudio Fontanari
Dipartimento di Matematica
Università di Trento
Via Sommarive 14
38123 Trento, Italy.
E-mail address: fontanar@science.unitn.it