

# TRIANGULAR DECOMPOSITION OF RIGHT COIDEAL SUBALGEBRAS

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*To Susan Montgomery — famous mathematician and beautiful person*

ABSTRACT. Let  $\mathfrak{g}$  be a Kac-Moody algebra. We show that every homogeneous right coideal subalgebra  $U$  of the multiparameter version of the quantized universal enveloping algebra  $U_q(\mathfrak{g})$ ,  $q^m \neq 1$  containing all group-like elements has a triangular decomposition  $U = U^- \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} U^+$ , where  $U^-$  and  $U^+$  are right coideal subalgebras of negative and positive quantum Borel subalgebras. However if  $U_1$  and  $U_2$  are arbitrary right coideal subalgebras of respectively positive and negative quantum Borel subalgebras, then the triangular composition  $U_2 \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} U_1$  is a right coideal but not necessary a subalgebra. Using a recent combinatorial classification of right coideal subalgebras of the quantum Borel algebra  $U_q^+(\mathfrak{so}_{2n+1})$ , we find a necessary condition for the triangular composition to be a right coideal subalgebra of  $U_q(\mathfrak{so}_{2n+1})$ .

If  $q$  has a finite multiplicative order  $t > 4$ , similar results remain valid for homogeneous right coideal subalgebras of the multiparameter version of the small Lusztig quantum groups  $u_q(\mathfrak{g})$ ,  $u_q(\mathfrak{so}_{2n+1})$ .

## 1. INTRODUCTION

It is well-known that the quantized universal enveloping algebras  $U_q(\mathfrak{g})$  of the Kac-Moody algebras have so called triangular decomposition. In this paper we are studying when a right coideal subalgebra of  $U_q(\mathfrak{g})$  also has the triangular decomposition. In fact the triangular decomposition holds not only for  $U_q(\mathfrak{g})$ , but also for a large class of character Hopf algebras  $\mathfrak{A}$  having positive and negative skew-primitive generators connected by relations of the type  $x_i x_j^- - p_{ji} x_j^- x_i = \delta_i^j (1 - g_i f_i)$ , see [7, Proposition 3.4]. In Theorem 3.2 we show that a right coideal subalgebra  $U$  of  $\mathfrak{A}$  containing all group-like elements has a required triangular decomposition provided that  $U$  is homogeneous with respect to the degree function  $D$  under the identification  $D(x_i^-) = -D(x_i)$ . Interestingly, if  $\mathfrak{A} = U_q(\mathfrak{g})$ ,  $q^t \neq 1$  with  $\mathfrak{g}$  defined by a Cartan matrix of finite type then each subalgebra containing all group-like elements is homogeneous with respect to the above degree function, [7, Corollary 3.3]. Hence in Corollary 3.3, applying a recent Heckenberger—Schneider theorem, [1, Theorem 7.3], we see that for a semisimple complex Lie algebra  $\mathfrak{g}$  the quantized universal enveloping algebra  $U_q(\mathfrak{g})$ ,  $q^t \neq 1$  has not more then  $|W|^2$  different right coideal subalgebras containing the coradical. Here  $W$  is the Weyl group of  $\mathfrak{g}$ .

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We should stress that when  $U^\pm$  run through the sets of right coideal subalgebras of the quantum Borel subalgebras, the triangular composition  $U^- \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} U^+$  is a right coideal but not always a subalgebra. For example, in [7] there are given the numbers  $C_n$  of pairs that define right coideal subalgebras of  $U_q(\mathfrak{g})$  when  $\mathfrak{g} = \mathfrak{sl}_{n+1}$  is the simple Lie algebra of type  $A_n$ . Using these numbers we can find the probabilities  $p_n$  for a pair  $U^-, U^+$  to define a right coideal subalgebra of  $U_q(\mathfrak{g})$ ,  $\mathfrak{g} = \mathfrak{sl}_{n+1}$ :

$$p_2 = 72.3\%; p_3 = 43.8\%; p_4 = 23.4\%; p_5 = 11.4\%; p_6 = 5.1\%; p_7 = 2.2\%.$$

If  $\mathfrak{g}$  is the simple Lie algebra of type  $G_2$  then the probability equals  $60/144 = 41.7\%$ , see B. Pogorelsky [9, 10].

The next goal of the paper is to prove a necessary condition for two right coideal subalgebras of the quantum Borel subalgebras to define by means of the triangular composition a right coideal subalgebra of  $U_q(\mathfrak{g})$  (respectively of  $u_q(\mathfrak{g})$ ) when  $\mathfrak{g} = \mathfrak{so}_{2n+1}$  is the simple Lie algebra of type  $B_n$ . In the fourth and fifth sections we follow the classification given in [6] to recall the basic properties of right coideal subalgebras of quantum Borel algebras  $U_q^\pm(\mathfrak{so}_{2n+1})$ . In particular we lead out the following ‘‘integrability’’ condition: if all partial derivatives of a homogeneous polynomial  $f$  in positive generators of an admissible degree belong to a right coideal subalgebra  $U \supseteq G$  of  $U_q^+(\mathfrak{so}_{2n+1})$  then  $f$  itself belongs to  $U$ , see Corollary 5.3.

In Section 6 we introduce the elements  $\Phi^S(k, m)$  defined by the sets  $S \subseteq [1, 2n]$  and the ordered pairs of indices  $1 \leq k \leq m \leq 2n$ , see (6.1). We display the element  $\Phi^S(k, m)$  schematically as a sequence of black and white points labeled by the numbers  $k-1, k, k+1, \dots, m-1, m$ , where the first point is always white, and the last one is always black, while an intermediate point labeled by  $i$  is black if and only if  $i \in S$ :

$$(1.1) \quad \overset{k-1}{\circ} \quad \overset{k}{\circ} \quad \overset{k+1}{\circ} \quad \overset{k+2}{\bullet} \quad \overset{k+3}{\circ} \quad \dots \quad \overset{m-2}{\bullet} \quad \overset{m-1}{\circ} \quad \overset{m}{\bullet} .$$

These elements are very important since every right coideal subalgebra  $U \supseteq G$  of the quantum Borel algebra is generated as an algebra by  $G$  and the elements of that form, see [6, Corollary 5.7]. Moreover  $U$  is uniquely defined by its *root sequence*  $\theta = (\theta_1, \theta_2, \dots, \theta_n)$ . The root sequence satisfies  $0 \leq \theta_i \leq 2n - 2i + 1$ , and each sequence satisfying these conditions is a root sequence for some  $U$ . There exists a constructive algorithm that allows one to find the generators  $\Phi^S(k, m)$  if the sequence  $\theta$  is given, see [6, Definition 10.1 and Eq. (10.6)]. In particular one may construct all schemes (1.1) for the generators.

The minimal generators  $\Phi^S(k, m)$  (the generators that do not belong to the subalgebra generated by the other generators of that form) satisfy important duality relation  $\Phi^S(k, m) = \alpha \Phi^R(\psi(m), \psi(k))$ ,  $\alpha \neq 0$ , where by definition  $\psi(i) = 2n - i + 1$ , while  $R$  is the complement of  $\{\psi(s) - 1 \mid s \in S\}$  with respect to the interval  $[\psi(m), \psi(k)]$ , see Proposition 6.5. In particular to every minimal generator  $\Phi^S(k, m)$  correspond two essentially different schemes (1.1). Respectively, if  $\Phi^S(k, m)$  and  $\Phi^T(i, j)$  are minimal generators for given right coideal subalgebras  $U_1 \subseteq U_q^+(\mathfrak{so}_{2n+1})$  and  $U_2 \subseteq U_q^-(\mathfrak{so}_{2n+1})$  then we have four different diagrams of the form

$$(1.2) \quad \begin{array}{cccccccc} S : & \overset{k-1}{\circ} & \dots & \overset{i-1}{\bullet} & \overset{i}{\bullet} & \overset{i+1}{\circ} & \dots & \overset{m}{\bullet} & \overset{j}{\bullet} \\ T : & & & \circ & \circ & \bullet & \dots & \bullet & \dots & \bullet \end{array} .$$

In Theorem 7.2 we prove the main result of the paper: If the triangular composition  $U_2 \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} U_1$  is a subalgebra then for each pair of minimal generators one of the following two options is fulfilled:

a) no one of the possible four diagrams (1.2) has fragments of the form

$$\begin{array}{ccc} \circ & \cdots & \bullet \\ \circ & \cdots & \bullet \end{array};$$

b) one of the possible four diagrams (1.2) has the form

$$\begin{array}{ccccccc} \overset{k-1}{\circ} & \cdots & \circ & \cdots & \bullet & \cdots & \overset{m}{\bullet} \\ \circ & \cdots & \bullet & \cdots & \circ & \cdots & \bullet \end{array},$$

where no one of the intermediate columns has points of the same color.

Certainly  $U_q(\mathfrak{sl}_n)$  is a Hopf subalgebra of  $U_q(\mathfrak{so}_{2n+1})$ . If we apply the found condition to right coideal subalgebras of  $U_q(\mathfrak{sl}_n)$ , we get precisely the necessary and sufficient condition given in [7, Theorem 11.1]. Hence we have a reason to believe that the found necessary condition is also sufficient for the triangular composition to define a right coideal subalgebra of  $U_q(\mathfrak{so}_{2n+1})$ .

Finally we would like to stress that right coideal subalgebras that do not admit the triangular decomposition (inhomogeneous or not including the coradical) are also of interest due to their relations with quantum symmetric pairs, quantum Harish-Chandra modules, and quantum symmetric spaces. Many of the (left) coideal subalgebras studied by M. Noumi and G. Letzter, see the survey [8], do not admit a triangular decomposition.

## 2. BRACKET TECHNIQUE

Let  $X = \{x_1, x_2, \dots, x_n\}$  be *quantum variables*; that is, associated with each letter  $x_i$  are an element  $g_i$  of a fixed Abelian group  $G$  and a character  $\chi^i : G \rightarrow \mathbf{k}^*$ . For every word  $w$  in  $X$  let  $g_w$  or  $\text{gr}(w)$  denote an element of  $G$  that appears from  $w$  by replacing each  $x_i$  with  $g_i$ . In the same way  $\chi^w$  denotes a character that appears from  $w$  by replacing each  $x_i$  with  $\chi^i$ .

Let  $G\langle X \rangle$  denote the skew group algebra generated by  $G$  and  $\mathbf{k}\langle X \rangle$  with the commutation rules  $x_i g = \chi^i(g) g x_i$ , or equivalently  $w g = \chi^w(g) g w$ , where  $w$  is an arbitrary word in  $X$ . Certainly  $G\langle X \rangle$  is spanned by the products  $g w$ , where  $g \in G$  and  $w$  runs through the set of words in  $X$ .

The algebra  $G\langle X \rangle$  has natural gradings by the group  $G$  and by the group  $G^*$  of characters. More precisely the basis element  $g w$  belongs to the  $g \text{gr}(w)$ -homogeneous component with respect to the grading by  $G$  and it belongs to the  $\chi^w$ -homogeneous component with respect to the grading by  $G^*$ .

Let  $u$  be a homogeneous element with respect to the grading by  $G^*$ , and  $v$  be a homogeneous element with respect to the grading by  $G$ . We define a skew commutator by the formula

$$(2.1) \quad [u, v] = uv - \chi^u(g_v)vu,$$

where  $u$  belongs to the  $\chi^u$ -homogeneous component, while  $v$  belongs to the  $g_v$ -homogeneous component. Sometimes for short we use the notation  $\chi^u(g_v) = p_{uv} = p(u, v)$ . Of course  $p(u, v)$  is a bimultiplicative map:

$$(2.2) \quad p(u, v)p(u, t) = p(u, vt), \quad p(u, v)p(t, v) = p(ut, v).$$

In particular the form  $p(-, -)$  is completely defined by the *quantification matrix*  $\|p_{ij}\|$ , where  $p_{ij} = \chi^i(g_j)$ .

The brackets satisfy the following Jacobi identities for homogeneous (with respect to the both gradings) elements:

$$(2.3) \quad [[u, v], w] = [u, [v, w]] + p_{wv}^{-1}[[u, w], v] + (p_{vw} - p_{wv}^{-1})[u, w] \cdot v.$$

$$(2.4) \quad [[u, v], w] = [u, [v, w]] - p_{vu}^{-1}[v, [u, w]] + (p_{vu}^{-1} - p_{uv})v \cdot [u, w].$$

These identities can be easily verified by direct computations using (2.1), (2.2). In particular the following conditional identities are valid (both in  $G\langle X \rangle$  and in all of its homomorphic images)

$$(2.5) \quad [[u, v], w] = [u, [v, w]], \text{ provided that } [u, w] = 0.$$

$$(2.6) \quad [u, [v, w]] = p_{uv}[v, [u, w]], \text{ provided that } [u, v] = 0 \text{ and } p_{uv}p_{vu} = 1.$$

By an evident induction on the length these conditional identities admit the following generalization, see [7, Lemma 2.2].

**Lemma 2.1.** *Let  $y_1, y_2, \dots, y_m$  be linear combinations of words homogeneous in each  $x_k \in X$ . If  $[y_i, y_j] = 0$ ,  $1 \leq i < j - 1 < m$ , then the bracketed polynomial  $[y_1 y_2 \dots y_m]$  is independent of the precise alignment of brackets:*

$$(2.7) \quad [y_1 y_2 \dots y_m] = [[y_1 y_2 \dots y_s], [y_{s+1} y_{s+2} \dots y_m]], \quad 1 \leq s < m.$$

The brackets are related to the product by the following ad-identities

$$(2.8) \quad [u \cdot v, w] = p_{vw}[u, w] \cdot v + u \cdot [v, w],$$

$$(2.9) \quad [u, v \cdot w] = [u, v] \cdot w + p_{uv}v \cdot [u, w].$$

In particular, if  $[u, w] = 0$ , we have

$$(2.10) \quad [u \cdot v, w] = u \cdot [v, w].$$

The antisymmetry identity takes the form

$$(2.11) \quad [u, v] = -p_{uv}[v, u] \quad \text{provided that } p_{uv}p_{vu} = 1.$$

Further we have

$$(2.12) \quad [u, gv] = u \cdot gv - \chi^u(gg_v)gv \cdot u = \chi^u(g)g[u, v], \quad g \in G;$$

$$(2.13) \quad [gu, v] = gu \cdot v - \chi^u(g_v)v \cdot gu = g(uv - p_{uv}\chi^v(g)vu),$$

or in a bracket form

$$(2.14) \quad [gu, v] = g[u, v] + p_{uv}(1 - \chi^v(g))gv \cdot u, \quad g \in G.$$

$$(2.15) \quad [gu, v] = \chi^v(g)g[u, v] + (1 - \chi^v(g))gu \cdot v, \quad g \in G.$$

**Quantization of variables.** Let  $p_{ij}$ ,  $1 \leq i, j \leq n$  be a set of parameters,  $0 \neq p_{ij} \in \mathbf{k}$ . Let  $g_j$  be the linear transformation  $g_j : x_i \rightarrow p_{ij}x_i$  of the linear space spanned by a set of variables  $X = \{x_1, x_2, \dots, x_n\}$ . Let  $\chi^i$  denote a character  $\chi^i : g_j \rightarrow p_{ij}$  of the group  $G$  generated by  $g_i$ ,  $1 \leq i \leq n$ . We may consider each  $x_i$  as a quantum variable with parameters  $g_i, \chi^i$ .

**Algebra  $\mathfrak{F}_n$ .** Let  $X^- = \{x_1^-, x_2^-, \dots, x_n^-\}$  be a new set of variables. We consider  $X^-$  as a set of quantum variables quantized by the parameters  $p_{ji}^{-1}$ ,  $1 \leq i, j \leq n$ . More precisely we have an Abelian group  $F$  generated by elements  $f_1, f_2, \dots, f_n$  acting on the linear space spanned by  $X^-$  so that  $(x_i^-)^{f_j} = p_{ji}^{-1} x_i^-$ , where  $p_{ij}$  are the same parameters that define the quantization of the variables  $X$ . In this case  $\text{gr}(x_i^-) = f_i$ ,  $\chi^{x_i^-}(f_j) = p_{ji}^{-1}$ .

We may extend the characters  $\chi^i$  on  $G \times F$  in the following way

$$(2.16) \quad \chi^i(f_j) \stackrel{\text{df}}{=} p_{ji} = \chi^j(g_i).$$

Indeed, if  $\prod_k f_k^{m_k} = 1$  in  $F$ , then application to  $x_i^-$  implies  $\prod_k p_{ki}^{-m_k} = 1$ , hence  $\chi^i(\prod_k f_k^{m_k}) = \prod_k p_{ki}^{m_k} = 1$ .

In the same way we may extend the characters  $\chi^{x_i^-}$  on  $G \times F$  so that

$$(2.17) \quad \chi^{x_i^-} = (\chi^i)^{-1} \text{ as characters of } G \times F.$$

In what follows  $H$  denotes a quotient group  $(G \times F)/N$ , where  $N$  is an arbitrary subgroup with  $\chi^i(N) = 1$ ,  $1 \leq i \leq n$ . For example, if the quantification parameters satisfy additional symmetry conditions  $p_{ij} = p_{ji}$ ,  $1 \leq i, j \leq n$ , (as this is a case for the original Drinfeld-Jimbo and Lusztig quantifications) then  $\chi^i(g_k^{-1} f_k) = p_{ik}^{-1} p_{ki} = 1$ , and we may take  $N$  to be the subgroup generated by  $g_k^{-1} f_k$ ,  $1 \leq k \leq n$ . In this particular case the groups  $H, G, F$  may be identified.

In the general case without loss of generality we may suppose that  $G, F \subseteq H$ . Certainly  $\chi^i$ ,  $1 \leq i \leq n$  are characters of  $H$  and  $H$  still acts on the space spanned by  $X \cup X^-$  by means of these characters and their inverses.

We define the algebra  $\mathfrak{F}_n$  as a quotient of  $H\langle X \cup X^- \rangle$  by the following relations

$$(2.18) \quad [x_i, x_j^-] = \delta_i^j (1 - g_i f_i), \quad 1 \leq i, j \leq n,$$

where the brackets are defined on  $H\langle X \cup X^- \rangle$  by the above quantization of the variables  $X \cup X^-$ ; that is,  $[x_i, x_j^-] = x_i x_j^- - p_{ji} x_j^- x_i$ , for  $\chi^i(f_j) = p_{ji}$ .

We go ahead with a number of useful notes for calculation of the skew commutators in  $\mathfrak{F}_n$ . If  $u$  is a word in  $X$ , then  $u^-$  denotes a word in  $X^-$  that appears from  $u$  under the substitution  $x_i \leftarrow x_i^-$ . We have  $p(v, w^-) = \chi^v(f_w) = p(w, v)$ , while  $p(w^-, v) = (\chi^w)^{-1}(g_v) = p(w, v)^{-1}$ . Thus  $p(v, w^-)p(w^-, v) = 1$ . Therefore the Jacobi and antisymmetry identities (see, (2.3), (2.11)) take up their original "colored" form:

$$(2.19) \quad [[u, v], w^-] = [u, [v, w^-]] + p_{uv} [[u, w^-], v];$$

$$(2.20) \quad [u^-, w] = -p_{uw}^{-1} [w, u^-].$$

In the same way

$$(2.21) \quad [[u^-, v^-], w] = [u^-, [v^-, w]] + p_{v^-}^{-1} [[u^-, w], v^-].$$

Using (2.4) we have also

$$(2.22) \quad [u, [v^-, w^-]] = [[u, v^-], w^-] + p_{vu} [v^-, [u, w^-]].$$

If we put  $w^- \leftarrow [w^-, t^-]$  in (2.19) we have

$$[[u, v], [w^-, t^-]] = [u, [v, [w^-, t^-]]] + p_{wt, v} [[u, [w^-, t^-]], v].$$

Using (2.22) we get

$$[[u, v], [w^-, t^-]] = [u, [[v, w^-], t^-]] + p_{w, v} [u, [w^-, [v, t^-]]]$$

$$(2.23) \quad + p_{wt,v}[[u, [w^-, t^-]], v].$$

Using once more (2.22) we get

$$(2.24) \quad \begin{aligned} [[u, v], [w^-, t^-]] &= [u, [[v, w^-], t^-]] + p_{w,v}[u, [w^-, [v, t^-]]] \\ &+ p_{wt,v}[[[u, w^-], t^-], v] + p_{wt,v}p_{w,u}[[w^-, [u, t^-]], v]. \end{aligned}$$

We must stress that relations (2.18) are homogeneous with respect to the grading by the character group  $H^*$ , but they are not homogeneous with respect to the grading by  $H$ . Therefore once we apply relations (2.18), or other ‘‘inhomogeneous in  $H$ ’’ relations, we have to develop the bracket to its explicit form as soon as the inhomogeneous substitution applies to the right factor of the bracket. For example we have

$$(2.25) \quad [u, [x_i, x_i^-]] = u(1 - g_i f_i) - \chi^u(g_i f_i)(1 - g_i f_i)u = (1 - \chi^u(g_i f_i))u,$$

but not  $[u, [x_i, x_i^-]] = [u, 1 - g_i f_i] = [u, 1] - [u, g_i f_i] = 0$ . In fact here the bracket  $[u, 1 - g_i f_i]$  is undefined since the right factor  $1 - g_i f_i$  is inhomogeneous in  $H$  (unless  $g_i f_i = 1$ ). At the same time

$$(2.26) \quad [[x_i, x_i^-], u] = (1 - g_i f_i)u - u(1 - g_i f_i) = (\chi^u(g_i f_i) - 1)g_i f_i \cdot u,$$

and  $[[x_i, x_i^-], u] = [1 - g_i f_i, u] = [1, u] - [g_i f_i, u]$  is valid since the inhomogeneous substitution has been applied to the left factor in the brackets.

**Lemma 2.2.** *Let  $X_1, X_2$  be subsets of  $X$ . Suppose that  $u$  is a word in  $X_1$  and  $v$  is a word in  $X_2$ . If  $X_1 \cap X_2 = \emptyset$ , then in the algebra  $\mathfrak{F}_n$  we have  $[u, v^-] = 0$ .*

*Proof.* Defining relations (2.18) imply  $[x_i, x_j^-] = 0$ ,  $x_i \in X_1$ ,  $x_j \in X_2$ . Ad-identities (2.8) and (2.9) with evident induction prove the statement.  $\square$

**Lemma 2.3.** *In the algebra  $\mathfrak{F}_n$  for any pair  $(i, j)$  with  $1 \leq i, j \leq n$ ,  $i \neq j$  we have*

$$[[x_i, x_j], [x_j^-, x_i^-]] = (1 - p_{ij}p_{ji})(1 - g_i g_j f_i f_j).$$

*Proof.* Without loss of generality we may assume  $i = 1$ ,  $j = 2$ . Since  $[x_1, x_2^-] = [x_2, x_1^-] = 0$ , identity (2.24) implies

$$(2.27) \quad [[x_1, x_2], [x_2^-, x_1^-]] = [x_1, [[x_2, x_2^-], x_1^-]] + p(x_2 x_1, x_2)p(x_2, x_1)[[x_2^-, [x_1, x_1^-]], x_2].$$

Using (2.26) and then (2.12) we get

$$[x_1, [[x_2, x_2^-], x_1^-]] = ((\chi^1)^{-1}(g_2 f_2) - 1)\chi^1(g_2 f_2)g_2 f_2[x_1, x_1^-] = (1 - p_{12}p_{21})g_2 f_2(1 - g_1 f_1).$$

Taking into account (2.25), we have

$$[x_2^-, [x_1, x_1^-]] = (1 - (\chi^2)^{-1}(g_1 f_1))x_2^- = (1 - p_{21}^{-1}p_{12}^{-1})x_2^-.$$

Antisymmetry relation (2.20) implies  $[x_2^-, x_2] = -p_{22}^{-1}[x_2, x_2^-]$ . Hence

$$[[x_2^-, [x_1, x_1^-]], x_2] = (1 - p_{21}^{-1}p_{12}^{-1})(-p_{22}^{-1})(1 - g_2 f_2).$$

In (2.27) we have  $p(x_2 x_1, x_2)p(x_2, x_1) = p_{22}p_{12}p_{21}$ , hence

$$[[x_1, x_2], [x_2^-, x_1^-]] = (1 - p_{12}p_{21})(g_2 f_2 - g_1 g_2 f_1 f_2 + 1 - g_2 f_2),$$

which is required.  $\square$

**Lemma 2.4.** *In the algebra  $\mathfrak{F}_n$  for any pair  $(i, j)$  with  $1 \leq i, j \leq n$ ,  $i \neq j$  we have*

$$[[[x_i, x_j], x_j], [x_j^-, [x_j^-, x_i^-]]] = \varepsilon(1 - g_i g_j^2 f_i f_j^2),$$

where  $\varepsilon = (1 + p_{jj})(1 - p_{ij}p_{ji})(1 - p_{ij}p_{ji}p_{jj})$ .

*Proof.* Again, without loss of generality we may assume  $i = 1$ ,  $j = 2$ . Let us put  $u \leftarrow [x_1, x_2]$ ,  $v \leftarrow x_2$ ,  $w^- \leftarrow x_2^-$ ,  $t^- \leftarrow [x_2^-, x_1^-]$  in (2.24). We have  $[v, w^-] = [x_2, x_2^-] = 1 - g_2 f_2$ . By means of (2.26) we get  $[[v, w^-], t^-] = (\chi^{t^-}(g_2 f_2) - 1)g_2 f_2 \cdot t^-$ . Here  $\chi^{t^-}(g_2 f_2) = p_{22}^{-2} p_{12}^{-1} p_{21}^{-1}$ . Using first (2.12) and then Lemma 2.3 we get

$$(2.28) \quad [u, [[v, w^-], t^-]] = \varepsilon_1 g_2 f_2 (1 - g_1 g_2 f_1 f_2),$$

where

$$\varepsilon_1 = (p_{22}^{-2} p_{12}^{-1} p_{21}^{-1} - 1) \chi^u(g_2 f_2) (1 - p_{12} p_{21}) = (1 - p_{12} p_{21} p_{22}^2) (1 - p_{12} p_{21}).$$

Further,  $[v, t^-] = [x_2, [x_2^-, x_1^-]]$ . By (2.5) we have  $[x_2, [x_2^-, x_1^-]] = [[x_2, x_2^-], x_1^-]$ . Hence (2.26) implies  $[v, t^-] = ((\chi^1)^{-1}(g_2 f_2) - 1)g_2 f_2 \cdot x_1^-$ . By (2.12) we get  $[w^-, [v, t^-]] = p_{22}^{-2} (p_{12}^{-1} p_{21}^{-1} - 1)g_2 f_2 \cdot [x_2^-, x_1^-]$ . Using first (2.12) and then Lemma 2.3 we get

$$(2.29) \quad p_{w^-, v} [u, [w^-, [v, t^-]]] = \varepsilon_2 g_2 f_2 (1 - g_1 g_2 f_1 f_2),$$

where

$$\varepsilon_2 = p_{22} \cdot p_{22}^{-2} (p_{12}^{-1} p_{21}^{-1} - 1) \cdot \chi^u(g_2 f_2) \cdot (1 - p_{12} p_{21}) = p_{22} (1 - p_{12} p_{21})^2.$$

In the same way  $[u, w^-] = [[x_1, x_2], x_2^-] = (1 - \chi^1(g_2 f_2)) \cdot x_1$  due to (2.5) and (2.25). Further  $[[u, w^-], t^-] = (1 - p_{12} p_{21}) [x_1, [x_2^-, x_1^-]]$ . Using (2.22) we have  $[x_1, [x_2^-, x_1^-]] = p_{21} [x_2^-, [x_1, x_1^-]]$ . Hence (2.25) allows us to find  $[[u, w^-], t^-] = (1 - p_{12} p_{21}) p_{21} (1 - p_{21}^{-1} p_{12}^{-1}) \cdot x_2^-$ . This implies

$$[[[u, w^-], t^-], v] = (1 - p_{12} p_{21}) (p_{21} - p_{12}^{-1}) [x_2^-, x_2].$$

Since  $[x_2^-, x_2] = -p_{22}^{-1} [x_2, x_2^-]$ , and  $[x_2, x_2^-] = 1 - g_2 f_2$ , we get

$$(2.30) \quad p_{wt^-, v} [[[u, w^-], t^-], v] = \varepsilon_3 (1 - g_2 f_2),$$

where

$$\varepsilon_3 = p_{22}^2 p_{12} \cdot (1 - p_{12} p_{21}) (p_{21} - p_{12}^{-1}) \cdot (-p_{22}^{-1}) = \varepsilon_2.$$

Finally, by Lemma 2.3 we have  $[u, t^-] = (1 - p_{12} p_{21}) (1 - g_1 g_2 f_1 f_2)$ . If we apply (2.25) with  $x_i \leftarrow u$ ,  $x_i^- \leftarrow t^-$ , then  $[w^-, [u, t^-]] = (1 - p_{12} p_{21}) (1 - (\chi^2)^{-1}(g_1 g_2 f_1 f_2)) \cdot x_2^-$ . Hence

$$[[w^-, [u, t^-]], v^-] = (1 - p_{12} p_{21}) (1 - p_{22}^{-2} p_{12}^{-1} p_{21}^{-1}) [x_2^-, x_2].$$

Here  $[x_2^-, x_2] = -p_{22}^{-1} (1 - g_2 f_2)$ . Since  $p_{wt^-, v} p_{w^-, u} = p_{22}^2 p_{12} p_{21} p_{22}$ , we may write

$$(2.31) \quad p_{wt^-, v} p_{w^-, u} [[w^-, [u, t^-]], v^-] = \varepsilon_4 (1 - g_2 f_2),$$

where

$$\varepsilon_4 = p_{22}^2 p_{12} p_{21} p_{22} \cdot (1 - p_{12} p_{21}) (1 - p_{22}^{-2} p_{12}^{-1} p_{21}^{-1}) \cdot (-p_{22}^{-1}) = \varepsilon_1.$$

Now we see that the sum of (2.28) and (2.31) equals  $\varepsilon_1 (1 - g_1 g_2^2 f_1 f_2^2)$ , while the sum of (2.29) and (2.30) equals  $\varepsilon_2 (1 - g_1 g_2^2 f_1 f_2^2)$ . It remains to check that  $\varepsilon_1 + \varepsilon_2 = \varepsilon$ .  $\square$

The algebra  $\mathfrak{F}_n$  has a structure of Hopf algebra with the following coproduct:

$$(2.32) \quad \Delta(x_i) = x_i \otimes 1 + g_i \otimes x_i, \quad \Delta(x_i^-) = x_i^- \otimes 1 + f_i \otimes x_i^-.$$

$$(2.33) \quad \Delta(g_i) = g_i \otimes g_i, \quad \Delta(f_i) = f_i \otimes f_i.$$

In this case  $G\langle X \rangle$  and  $F\langle X^- \rangle$  are Hopf subalgebras of  $\mathfrak{F}_n$ .

The free algebra  $\mathbf{k}\langle X \rangle$  has a coordinate differential calculus

$$(2.34) \quad \partial_i(x_j) = \delta_i^j, \quad \partial_i(uv) = \partial_i(u) \cdot v + \chi^u(g_i)u \cdot \partial_i(v).$$

The partial derivatives connect the calculus with the coproduct on  $G\langle X \rangle$  via

$$(2.35) \quad \Delta(u) \equiv u \otimes 1 + \sum_i g_i \partial_i(u) \otimes x_i \pmod{G\langle X \rangle \otimes \mathbf{k}\langle X \rangle^{(2)}},$$

for all  $u \in \mathbf{k}\langle X \rangle$ . Here  $\mathbf{k}\langle X \rangle^{(2)}$  is the ideal of  $\mathbf{k}\langle X \rangle$  generated by  $x_i x_j$ ,  $1 \leq i, j \leq n$ . Symmetrically the equation

$$(2.36) \quad \Delta(u) \equiv g_u \otimes u + \sum_i g_u g_i^{-1} x_i \otimes \partial_i^*(u) \pmod{G\langle X \rangle^{(2)} \otimes \mathbf{k}\langle X \rangle}$$

defines a dual differential calculus on  $\mathbf{k}\langle X \rangle$  where the partial derivatives satisfy

$$(2.37) \quad \partial_j^*(x_i) = \delta_i^j, \quad \partial_i^*(uv) = \chi^i(g_v) \partial_i^*(u) \cdot v + u \cdot \partial_i^*(v).$$

Here  $G\langle X \rangle^{(2)}$  is the ideal of  $G\langle X \rangle$  generated by  $x_i x_j$ ,  $1 \leq i, j \leq n$ .

Similarly the algebra  $\mathbf{k}\langle X^- \rangle$  has a pair of differential calculi:

$$(2.38) \quad \partial_{-i}(x_j^-) = \delta_i^j, \quad \partial_{-i}(u^- v^-) = \partial_{-i}(u^-) \cdot v^- + \chi^{u^-}(f_i) u^- \cdot \partial_{-i}(v^-),$$

$$(2.39) \quad \partial_{-j}^*(x_i^-) = \delta_i^j, \quad \partial_{-i}^*(u^- v^-) = (\chi^i(f_v))^{-1} \partial_{-i}^*(u^-) \cdot v^- + u^- \cdot \partial_{-i}^*(v^-).$$

These calculi are related to the coproduct by the similar formulae

$$(2.40) \quad \Delta(u^-) \equiv u^- \otimes 1 + \sum_i f_i \partial_{-i}(u^-) \otimes x_i^- \pmod{F\langle X^- \rangle \otimes \mathbf{k}\langle X^- \rangle^{(2)}},$$

$$(2.41) \quad \Delta(u^-) \equiv f_u \otimes u^- + \sum_i f_u f_i^{-1} x_i^- \otimes \partial_{-i}^*(u^-) \pmod{F\langle X^- \rangle^{(2)} \otimes \mathbf{k}\langle X^- \rangle}.$$

It will be important for us that operators  $[x_i, -]$  and  $[-, x_i^-]$  defined respectively on  $\mathbf{k}\langle X^- \rangle$  and  $\mathbf{k}\langle X \rangle$  have a nice differential form (see [7, Remark, page 2586]):

$$(2.42) \quad [x_i, u^-] = \partial_{-i}^*(u^-) p(x_i, u^-) p_{ii}^{-1} - g_i f_i \partial_{-i}(u^-), \quad u^- \in \mathbf{k}\langle X^- \rangle,$$

$$(2.43) \quad [u, x_i^-] = \partial_i^*(u) - p_{ii}^{-1} p(u, x_i) \partial_i(u) g_i f_i, \quad u \in \mathbf{k}\langle X \rangle.$$

These relations are clear if  $u = x_j$ , or  $u^- = x_j^-$  while ad-identities (2.8) and (2.9) with Leibniz rules (2.34, 2.37, 2.38, 2.39) allow one to perform evident induction.

**Quantification of Kac-Moody algebras.** Let  $C = \|a_{ij}\|$  be a symmetrizable by  $D = \text{diag}(d_1, \dots, d_n)$  generalized Cartan matrix,  $d_i a_{ij} = d_j a_{ji}$ . Let  $\mathfrak{g}$  be a Kac-Moody algebra defined by  $C$ , see [2]. Suppose that the quantification parameters  $p_{ij} = p(x_i, x_j) = \chi^i(g_j)$  are related by

$$(2.44) \quad p_{ii} = q^{d_i}, \quad p_{ij} p_{ji} = q^{d_i a_{ij}}, \quad 1 \leq i, j \leq n.$$

As above  $g_j$  denotes a linear transformation  $g_j : x_i \rightarrow p_{ij} x_i$  of the linear space spanned by a set of variables  $X = \{x_1, x_2, \dots, x_n\}$ . Let  $\chi^i$  denote a character  $\chi^i : g_j \rightarrow p_{ij}$  of the group  $G$  generated by  $g_i$ ,  $1 \leq i \leq n$ . We consider each  $x_i$  as



a quantum variable with parameters  $g_i, \chi^i$ . Respectively  $\mathfrak{F}_n$  is the above defined algebra related to quantum variables  $X$ , and  $X^- = \{x_1^-, x_2^-, \dots, x_n^-\}$ , where by definition  $\text{gr}(x_i^-) = f_i, \chi^{x_i^-} = (\chi^i)^{-1}$ , see (2.17), (2.18).

In this case the multiparameter quantization  $U_q(\mathfrak{g})$  of  $\mathfrak{g}$  is a quotient of  $H\langle X \cup X^- \rangle$  defined by Serre relations with the skew brackets in place of the Lie operation:

$$(2.45) \quad [\dots \underbrace{[[x_i, x_j], x_j], \dots, x_j}_{1-a_{ji} \text{ times}}] = 0, \quad 1 \leq i \neq j \leq n;$$

$$(2.46) \quad [\dots \underbrace{[[x_i^-, x_j^-], x_j^-], \dots, x_j^-}_{1-a_{ji} \text{ times}}] = 0, \quad 1 \leq i \neq j \leq n;$$

$$(2.47) \quad [x_i, x_j^-] = \delta_i^j (1 - g_i f_i), \quad 1 \leq i, j \leq n,$$

where the brackets are defined on  $H\langle X \cup X^- \rangle$  by (2.1). Certainly relations (2.47) coincide with (2.18). Hence  $U_q(\mathfrak{g})$  is a homomorphic image of  $\mathfrak{F}_n$ . The algebra  $U_q(\mathfrak{g})$  has a structure of Hopf algebra with the coproduct (2.32), (2.33); that is, the above homomorphism is a homomorphism of Hopf algebras.

If the multiplicative order  $t$  of  $q$  is finite then the multiparameter version of the small Lusztig quantum group is defined as the homomorphic image of  $U_q(\mathfrak{g})$  subject to additional relations  $u = 0, u \in \mathbf{\Lambda}, u^- = 0, u^- \in \mathbf{\Lambda}^-$ , where  $\mathbf{\Lambda}$  is the biggest Hopf ideal of  $G\langle X \rangle$  that is contained in the ideal  $G\langle X \rangle^{(2)}$  generated by  $x_i x_j, 1 \leq i, j \leq n$ . Respectively  $\mathbf{\Lambda}^-$  is the biggest Hopf ideal of  $F\langle X^- \rangle$  that is contained in the ideal  $F\langle X^- \rangle^{(2)}$  generated by  $x_i^- x_j^-, 1 \leq i, j \leq n$ .

**Mirror generators.** Of course there is no essential difference between positive and negative quantum Borel subalgebras. More precisely, let us put  $y_i = p_{ii}^{-1} x_i^-$ ,  $y_i^- = -x_i$ . Consider  $y_i$  as a quantum variable with parameters  $f_i, (\chi^i)^{-1}$ , while  $y_i^-$  as a quantum variable with parameters  $g_i, \chi^i$ . Relations (2.45 – 2.47) are invariant under the substitution  $x_i \leftarrow p_{ii}^{-1} x_i^-, x_i^- \leftarrow -x_i$ . Hence  $y_i, y_i^-$  with  $H$  generate a subalgebra which can be identified with the quantification  $U_{q^{-1}}(\mathfrak{g})$ . At the same time this subalgebra coincides with  $U_q(\mathfrak{g})$ . In this way one may replace positive and negative quantum Borel subalgebras. We shall call the generators  $y_i = p_{ii}^{-1} x_i^-$ ,  $y_i^- = -x_i$  as *mirror generators*.

**Antipode.** Recall that the antipode  $\sigma$  by definition satisfies  $\sum a^{(1)} \cdot \sigma(a^{(2)}) = \sum \sigma(a^{(1)}) \cdot a^{(2)} = \varepsilon(a)$ . Hence (2.32) implies  $\sigma(x_i) = -g_i^{-1} x_i, \sigma(x_i^-) = -f_i^{-1} x_i^-$ . In particular if  $u$  is a word in  $X \cup X^-$ , then  $g_u \sigma(u)$  is proportional to a word in  $X \cup X^-$ , for  $\sigma$  is an antiautomorphism:  $\sigma(ab) = \sigma(b) \sigma(a)$ . Moreover, if  $u, v$  are linear combinations of words homogeneous in each  $y \in X \cup X^-$ , then we have

$$(2.48) \quad g_u g_v \sigma([u, v]) = p_{vu}^{-1} [g_v \sigma(v), g_u \sigma(u)].$$

Indeed, the left hand side equals  $g_u g_v (\sigma(v) \sigma(u) - p_{uv} \sigma(u) \sigma(v))$ , while the right hand side is  $p_{vu}^{-1} g_v \sigma(v) g_u \sigma(u) - g_u \sigma(u) g_v \sigma(v)$ . We have  $g_v \sigma(v) \cdot g_u = p_{vu} g_u \cdot g_v \sigma(v)$ , and  $g_u \sigma(u) \cdot g_v = p_{uv} g_v \cdot g_u \sigma(u)$ . This implies (2.48).

**$\Gamma$ -Grading and  $\Gamma^+ \oplus \Gamma^-$ -filtration.** We are reminded that *constitution* of a word  $u$  in  $H \cup X \cup X^-$  is a family of nonnegative integers  $\{m_y, y \in X \cup X^-\}$  such that  $u$  has  $m_y$  occurrences of  $y$ . Let  $\Gamma^+$  denote the free additive (commutative) monoid generated by  $X$ , while by  $\Gamma^-$  the free additive monoid generated by  $X^-$ . Respectively  $\Gamma^+ \oplus \Gamma^-$  is the free additive monoid generated by  $X \cup X^-$ , while  $\Gamma$  by

definition is the free commutative group generated by  $X \cup X^-$  with identification  $x_i^- = -x_i$ ,  $1 \leq i \leq n$ . We fix the following order on  $X \cup X^-$ :

$$(2.49) \quad x_1 > x_2 > \dots > x_n > x_1^- > x_2^- > \dots > x_n^-.$$

The monoid  $\Gamma^+ \oplus \Gamma^-$  is a completely ordered monoid with respect to the order

$$(2.50) \quad m_1 y_{i_1} + m_2 y_{i_2} + \dots + m_k y_{i_k} > m'_1 y_{i_1} + m'_2 y_{i_2} + \dots + m'_k y_{i_k}$$

if the first from the left nonzero number in  $(m_1 - m'_1, m_2 - m'_2, \dots, m_k - m'_k)$  is positive, where  $y_{i_1} > y_{i_2} > \dots > y_{i_k}$  in  $X \cup X^-$ .

We associate a formal degree  $D(u) = \sum_{y \in X \cup X^-} m_y y \in \Gamma^+ \oplus \Gamma^-$  to a word  $u$  in  $X \cup X^-$ , where  $\{m_y, y \in X \cup X^-\}$  is the constitution of  $u$ . Respectively, if  $f = \sum \alpha_i u_i \in H\langle X \cup X^- \rangle$ ,  $0 \neq \alpha_i \in \mathbf{k}[H]$  is a linear combination of different words, then  $D(f) = \max_i \{D(u_i)\}$ . This degree function defines a grading by  $\Gamma^+ \oplus \Gamma^-$  on  $H\langle X \cup X^- \rangle$ . However relations (2.47), (2.18) are not homogeneous with respect to this grading. Hence neither  $\mathfrak{F}_n$  nor  $U_q(\mathfrak{g})$ ,  $u_q(\mathfrak{g})$ , are graded by  $\Gamma^+ \oplus \Gamma^-$ , but certainly they have a filtration defined by the induced degree function.

Relations (2.47), (2.18) became homogeneous if we consider the degree  $D(u)$  as an element of the group  $\Gamma$  with identifications  $x_i^- = -x_i$ . Hence  $\mathfrak{F}_n$ ,  $U_q(\mathfrak{g})$ , and  $u_q(\mathfrak{g})$  have grading by  $\Gamma$  (are  $\Gamma$ -homogeneous).

### 3. TRIANGULAR DECOMPOSITION

It is well-known that there is so called triangular decomposition

$$(3.1) \quad U_q(\mathfrak{g}) = U_q^-(\mathfrak{g}) \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} U_q^+(\mathfrak{g}),$$

where  $U_q^+(\mathfrak{g})$  is the positive quantum Borel subalgebra, the subalgebra generated by  $G$  and values of  $x_i$ ,  $1 \leq i \leq n$ , while  $U_q^-(\mathfrak{g})$  is the negative quantum Borel subalgebra, the subalgebra generated by  $F$  and values of  $x_i^-$ ,  $1 \leq i \leq n$ .

The small Lusztig quantum group has the triangular decomposition also

$$(3.2) \quad u_q(\mathfrak{g}) = u_q^-(\mathfrak{g}) \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} u_q^+(\mathfrak{g}).$$

In fact the triangular decomposition holds not only for the quantizations defined by the quantum Serre relations but also for arbitrary Hopf homomorphic images of  $\mathfrak{F}_n$ . More precisely we have the following statement.

**Theorem 3.1.** ([7, Proposition 3.4]). *The algebra  $\mathfrak{A} = \langle \mathfrak{F}_n \mid u_l = 0, w_t^- = 0 \rangle$  has the triangular decomposition*

$$(3.3) \quad \mathfrak{A} = \langle \mathfrak{F}_n^- \mid w_t^- = 0 \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle \mathfrak{F}_n^+ \mid u_l = 0 \rangle$$

*provided that  $\langle \mathfrak{F}_n^- \mid w_t^- = 0 \rangle$  and  $\langle \mathfrak{F}_n^+ \mid u_l = 0 \rangle$  are Hopf algebras, and  $u_l$ ,  $l \in L$ ,  $w_t^-$ ,  $t \in T$  are homogeneous polynomials respectively in  $x_i$ ,  $1 \leq i \leq n$  and  $x_i^-$ ,  $1 \leq i \leq n$  of total degree  $> 1$ .*

Our goal in this section is to find conditions when a right coideal subalgebra of  $\mathfrak{A}$  has a triangular decomposition.

**Theorem 3.2.** *Let  $\mathfrak{A}$  be the Hopf algebra defined in the above theorem. Every  $\Gamma$ -homogeneous right coideal subalgebra  $U \supset H$  of  $\mathfrak{A}$  has a decomposition*

$$(3.4) \quad U = U^- \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} U^+,$$

*where  $U^- \supset F$  and  $U^+ \supset G$  are homogeneous right coideal subalgebras respectively of  $\langle \mathfrak{F}_n^- \mid w_t^- = 0 \rangle$  and  $\langle \mathfrak{F}_n^+ \mid u_l = 0 \rangle$ .*

*Proof.* By [5, Theorem 1.1] the algebra  $U$  has a PBW-basis over the coradical  $\mathbf{k}[H]$ . We shall prove that the PBW-basis can be constructed in such a way that each PBW-generator for  $U$  belongs to either positive or negative component of (3.3). By definition of PBW-basis (see, for example [7, Section 2]) this implies the required decomposition of  $U$ .

Recall that the PBW-basis of  $U$  is constructed in the following way, see [5, Section 4]. First, we fix a PBW-basis of  $\mathfrak{A}$  defined by the *hard super-letters* [3]. Due to the triangular decomposition (3.3) the PBW-generators for  $\mathfrak{A}$  belong to either  $\mathfrak{A}^+ = \langle \mathfrak{F}_n^+ \mid u_i = 0 \rangle$  or  $\mathfrak{A}^- = \langle \mathfrak{F}_n^- \mid w_i^- = 0 \rangle$ . Then, for each PBW-generator (hard super-letter)  $[u]$  we fix an arbitrary element  $c_u \in U$  with minimal possible  $s$ , if any, such that

$$(3.5) \quad c_u = [u]^s + \sum \alpha_i W_i R_i + \sum_j \beta_j V_j \in U, \quad \alpha_i \in \mathbf{k}, \quad \beta_j \in \mathbf{k}[H],$$

where  $W_i$  are basis words in less than  $[u]$  super-letters,  $R_i$  are basis words in greater than or equal to  $[u]$  super-letters,  $D(W_i R_i) = sD(u)$ ,  $D(V_j) < sD(u)$ . Next, Proposition 4.4 [5] implies that the set of all chosen  $c_u$  form a set of PBW-generators for  $U$ . Since  $U$  is  $\Gamma$ -homogeneous, we may choose  $c_u$  to be  $\Gamma$ -homogeneous as well.

We stress that the leading terms here are defined by the degree function with values in the additive monoid  $\Gamma^+ \oplus \Gamma^-$  freely generated by  $X \cup X^-$ , but not in the group  $\Gamma$ , see the last subsection of Section 2. Equality  $D(W_i R_i) = sD(u)$  implies that all  $W_i R_i$  in (3.5) have the same constitution in  $X \cup X^-$  as the leading term  $[u]^s$  does. Thus all  $W_i R_i$ 's and the leading term  $[u]^s$  belong to the same component of the triangular decomposition. Hence it remains to show that if  $c_u$  is  $\Gamma$ -homogeneous then there are no terms  $V_j$ . In this case all terms  $V_j$  have the same  $\Gamma$ -degree and smaller  $\Gamma^+ \oplus \Gamma^-$ -degree. We shall prove that this is impossible.

If  $[u] \in \mathfrak{A}^-$  then  $sD(u) = m_1 x_1^- + m_2 x_2^- + \dots + m_n x_n^-$ , while the  $\Gamma^+ \oplus \Gamma^-$ -degree of  $V_j$  should be less than  $m_1 x_1^- + m_2 x_2^- + \dots + m_n x_n^-$ . Hence due to definitions (2.49) and (2.50) we have  $V_j \in \mathfrak{A}^-$ . In particular the  $\Gamma$ -degree of  $V_j$  coincides with the  $\Gamma^+ \oplus \Gamma^-$ -degree, a contradiction.

Suppose that  $[u] \in \mathfrak{A}^+$ . In this case  $sD(u) = m_1 x_1 + m_2 x_2 + \dots + m_n x_n$ . Let  $d = \sum_{i \leq n} s_i x_i + \sum_{i \leq n} r_i x_i^-$  be the  $\Gamma^+ \oplus \Gamma^-$ -degree of  $V_j$ . Since  $\Gamma$ -degree of  $V_j$  coincides with  $\Gamma$ -degree of  $[u]^s$ , we have  $s_i - r_i = m_i$ ,  $1 \leq i \leq n$ . This implies  $s_i = m_i + r_i \geq m_i$ . At the same time definition (2.50) and the condition  $d < sD(u)$  imply  $s_k < m_k$ , where  $k$  is the smallest index such that  $s_k \neq m_k$ . Thus  $s_k = m_k$  for all  $1 \leq k \leq n$ . This yields  $r_k = 0$ ,  $1 \leq k \leq n$ . In particular  $\Gamma$ -degree of  $V_j$  coincides with the  $\Gamma^+ \oplus \Gamma^-$ -degree, again a contradiction.  $\square$

**Corollary 3.3.** *Let  $\mathfrak{g}$  be a semisimple complex Lie algebra. If  $q$  is not a root of 1, then  $U_q(\mathfrak{g})$  has at most  $|W|^2$  different right coideal subalgebras containing the coradical, where  $W$  is the Weyl group of  $\mathfrak{g}$ .*

*Proof.* Due to Heckenberger—Schneider theorem, [1, Theorem 7.3], each of the quantum Borel subalgebras  $U_q^\pm(\mathfrak{g})$  has exactly  $|W|$  different right coideal subalgebras containing the coradical. At the same time by [7, Corollary 3.3] every subalgebra of  $U_q(\mathfrak{g})$  containing  $H$  is  $\Gamma$ -homogeneous. Hence by Theorem 3.2 we have a decomposition (3.4). We see that there are just  $|W|^2$  options to form the right hand side of (3.4).  $\square$

We should stress that when  $U^\pm$  run through the sets of right coideal subalgebras of the quantum Borel subalgebras the tensor product in the right hand side of (3.4) is a right coideal but not always a subalgebra.

Our next goal is to state and prove a necessary condition for two right coideal subalgebras  $U^+, U^-$  of the quantum Borel algebras to define in (3.4) a right coideal subalgebra of  $U_q(\mathfrak{so}_{2n+1})$  (respectively of  $u_q(\mathfrak{so}_{2n+1})$ ).

#### 4. STRUCTURE OF QUANTUM BOREL SUBALGEBRAS OF $U_q(\mathfrak{so}_{2n+1})$

In this section we follow [6] to recall the basic properties of quantum Borel subalgebras  $U_q^\pm(\mathfrak{so}_{2n+1})$ . In what follows we fix a parameter  $q$  such that  $q^2 \neq \pm 1$ ,  $q^3 \neq 1$ . Let  $\sim$  denote the projective equality:  $a \sim b$  if and only if  $a = \alpha b$ , where  $0 \neq \alpha \in \mathbf{k}$ .

If  $C$  is a Cartan matrix of type  $B_n$ , relations (2.44) take up the form

$$(4.1) \quad p_{nn} = q, p_{ii} = q^2, \quad p_{i+1}p_{i+1} = q^{-2}, \quad 1 \leq i < n;$$

$$(4.2) \quad p_{ij}p_{ji} = 1, \quad j > i + 1.$$

Starting with parameters  $p_{ij}$  satisfying these relations, we define the group  $G$  and the character Hopf algebra  $G\langle X \rangle$  as in the above section. In this case the quantum Borel algebra  $U_q^+(\mathfrak{so}_{2n+1})$  is defined as a quotient of  $G\langle X \rangle$  by the following relations

$$(4.3) \quad [x_i, [x_i, x_{i+1}]] = 0, \quad 1 \leq i < n; \quad [x_i, x_j] = 0, \quad j > i + 1;$$

$$(4.4) \quad [[x_i, x_{i+1}], x_{i+1}] = [[[x_{n-1}, x_n], x_n], x_n] = 0, \quad 1 \leq i < n - 1.$$

Here we slightly modify Serre relations (2.45) so that the left hand side of each relation is a bracketed Lyndon-Shirshov word. It is possible to do due to the following general relation in  $\mathbf{k}\langle X \rangle$ , see [4, Corollary 4.10]:

$$(4.5) \quad [\dots \underbrace{[[x_i, x_j], x_j], \dots, x_j}_a \sim \underbrace{[x_j, [x_j, \dots [x_j, x_i] \dots]}_a],$$

provided that  $p_{ij}p_{ji} = p_{jj}^{1-a}$ .

**Definition 4.1.** The elements  $u, v$  are said to be *separated* if there exists an index  $j$ ,  $1 \leq j \leq n$ , such that either  $u \in \mathbf{k}\langle x_i \mid i < j \rangle$ ,  $v \in \mathbf{k}\langle x_i \mid i > j \rangle$  or vice versa  $u \in \mathbf{k}\langle x_i \mid i > j \rangle$ ,  $v \in \mathbf{k}\langle x_i \mid i < j \rangle$ .

**Lemma 4.2.** *In the algebra  $U_q^+(\mathfrak{so}_{2n+1})$  every two separated homogeneous in each  $x_i \in X$  elements  $u, v$  (skew) commute,  $[u, v] = 0$ , in particular  $u \cdot v \sim v \cdot u$ .*

*Proof.* The statement follows from the second group of defining relations (4.3) due to (2.8), (2.9).  $\square$

**Definition 4.3.** In what follows  $x_i$ ,  $n < i \leq 2n$  denotes the generator  $x_{2n-i+1}$ . Respectively,  $u(k, m)$ ,  $1 \leq k \leq m \leq 2n$  is the word  $x_k x_{k+1} \cdots x_{m-1} x_m$ . If  $1 \leq i \leq 2n$ , then  $\psi(i)$  is the number  $2n - i + 1$ , so that  $x_i = x_{\psi(i)}$ . We shall frequently use the following properties of  $\psi$ : if  $i < j$ , then  $\psi(i) > \psi(j)$ ;  $\psi(\psi(i)) = i$ ;  $\psi(i+1) = \psi(i) - 1$ ,  $\psi(i-1) = \psi(i) + 1$ .

**Definition 4.4.** If  $k \leq i < m \leq 2n$ , then we define

$$(4.6) \quad \sigma_k^m \stackrel{\text{df}}{=} p(u(k, m), u(k, m)),$$

$$(4.7) \quad \mu_k^{m,i} \stackrel{\text{df}}{=} p(u(k, i), u(i+1, m)) \cdot p(u(i+1, m), u(k, i)).$$

Of course, one can easily find the  $\sigma$ 's and the  $\mu$ 's by means of (4.1), (4.2). More precisely, by [6, Eq. (3.10)] we have

$$(4.8) \quad \sigma_k^m = \begin{cases} q, & \text{if } m = n, \text{ or } k = n + 1; \\ q^4, & \text{if } m = \psi(k); \\ q^2, & \text{otherwise.} \end{cases}$$

If  $m < \psi(k)$ , then by [6, Eq. (3.13)] we have

$$(4.9) \quad \mu_k^{m,i} = \begin{cases} q^{-4}, & \text{if } m > n, i = \psi(m) - 1; \\ 1, & \text{if } i = n; \\ q^{-2}, & \text{otherwise.} \end{cases}$$

If  $m = \psi(k)$ , then by [6, Eq. (3.14)] we have

$$(4.10) \quad \mu_k^{m,i} = \begin{cases} q^2, & \text{if } i = n; \\ 1, & \text{otherwise.} \end{cases}$$

If  $m > \psi(k)$ , then then by [6, Eq. (3.15)] we have

$$(4.11) \quad \mu_k^{m,i} = \begin{cases} q^{-4}, & \text{if } k \leq n, i = \psi(k); \\ 1, & \text{if } i = n; \\ q^{-2}, & \text{otherwise.} \end{cases}$$

We define the bracketing of  $u(k, m)$ ,  $k \leq m$  as follows.

$$(4.12) \quad u[k, m] = \begin{cases} [[[\dots [x_k, x_{k+1}], \dots], x_{m-1}], x_m], & \text{if } m < \psi(k); \\ [x_k, [x_{k+1}, [\dots, [x_{m-1}, x_m] \dots]]], & \text{if } m > \psi(k); \\ \beta[u[n+1, m], u[k, n]], & \text{if } m = \psi(k), \end{cases}$$

where  $\beta = -p(u(n+1, m), u(k, n))^{-1}$  normalizes the coefficient at  $u(k, m)$ . Conditional identity (2.7) and the second group of defining relations (4.3) show that the value of  $u[k, m]$  in  $U_q^+(\mathfrak{so}_{2n+1})$  is independent of the precise alignment of brackets provided that  $m \leq n$  or  $k > n$ . Formula (2.48) and evident induction show that

$$(4.13) \quad g_k g_{k+1} \cdots g_m \sigma(u[k, m]) \sim u[\psi(m), \psi(k)],$$

where  $\sigma$  is the antipode.

**Lemma 4.5.** ([6, Corollary 3.13]). *If  $m \neq \psi(k)$ ,  $k \leq n < m$ , then in  $U_q^+(\mathfrak{so}_{2n+1})$  we have*

$$(4.14) \quad u[k, m] = [u[k, n], u[n+1, m]] = \beta[u[n+1, m], u[k, n]],$$

where  $\beta = -p(u(n+1, m), u(k, n))^{-1}$ .

**Proposition 4.6.** ([6, Proposition 3.14]). *If  $m \neq \psi(k)$ ,  $k \leq i < m$ , then in  $U_q^+(\mathfrak{so}_{2n+1})$  for each  $i$ ,  $k \leq i < m$  we have*

$$[u[k, i], u[i+1, m]] = u[k, m]$$

with only two possible exceptions being  $i = \psi(m) - 1$ , and  $i = \psi(k)$ . In particular this decomposition holds for arbitrary  $i$  if  $m \leq n$  or  $k > n$ .

**Proposition 4.7.** *Let  $k \leq i < j < m$ . If  $m \neq \psi(i) - 1$ ,  $j \neq \psi(k)$ , and  $m \neq \psi(k)$  then  $[u[k, i], u[j+1, m]] = 0$ . If  $m \neq \psi(i) - 1$ ,  $j \neq \psi(k)$ , and  $i \neq \psi(j) - 1$ , then  $[u[j+1, m], u[k, i]] = 0$ .*

*Proof.* The former statement follows from [6, Proposition 3.15]. Let  $m \neq \psi(i) - 1$ ,  $j \neq \psi(k)$ , and  $i \neq \psi(j) - 1$ . If additionally  $m \neq \psi(k)$  then still [6, Proposition 3.15] applies. Assume  $m = \psi(k)$ . We shall use the following two relations

$$(4.15) \quad [x_\lambda, [x_{\lambda-1}x_\lambda x_{\lambda+1}]] = [[x_{\lambda-1}x_\lambda x_{\lambda+1}], x_\lambda] = 0, l * too$$

where  $1 < \lambda < 2n$ ,  $\lambda \neq n, n+1$ . The latter one is precisely [6, Eq. (3.7)] with  $k \leftarrow \lambda$  if  $\lambda < n$ , and with  $k \leftarrow \psi(\lambda)$  if  $\lambda > n+1$ . The former one follows from antisymmetry identity (2.11), for

$$p(x_\lambda, x_{\lambda-1}x_\lambda x_{\lambda+1})p(x_{\lambda-1}x_\lambda x_{\lambda+1}, x_\lambda) = q^{-2}q^4q^{-2} = 1.$$

That equalities imply the following two ones

$$(4.16) \quad [x_\lambda, u[k, a]] = 0, \quad k \leq \lambda < a \leq n;$$

$$(4.17) \quad [u[k, a], x_\lambda] = 0, \quad n < k < \lambda \leq a.$$

Indeed, if in (4.16) we have  $\lambda = k$  then  $[x_k, u[k, a]] = [[x_k, [x_k, x_{k+1}]], u[k+2, a]] = 0$ , for in this case  $u[k, a]$  is independent of the precise alignment of brackets, see Lemma 2.1, and of course  $[x_k, u[k+2, a]] = 0$  due to Lemma 4.2. If  $\lambda > k$  then

$$[x_\lambda, u[k, a]] \sim [u[k, \lambda-2], [x_\lambda, [x_{\lambda-1}x_\lambda x_{\lambda+1}]], u[\lambda+2, a]] = 0,$$

for  $[x_\lambda, u[k, \lambda-2]] = [x_\lambda, u[\lambda+2, a]] = 0$ . The proof of (4.17) is quite similar.

Let  $i \leq n < j$ . In this case the equality  $[u[1+j, m], u[k, i]] = 0$  follows from (4.16) with  $a \leftarrow i$  if  $1+j > \psi(i)$ . If  $1+j < \psi(i)$  this follows from (4.17) with  $k \leftarrow 1+j$ ,  $a \leftarrow m$ . We have  $1+j \neq \psi(i)$ , for  $i \neq \psi(j) - 1$ .

Let  $i < j \leq n$ . By Lemma 4.5 we have  $u[1+j, m] = [u[1+j, n], u[n+1, m]]$ . At the same time  $[u[n+1, m], u[k, i]] = 0$  due to (4.17) with  $k \leftarrow n+1$ ,  $a \leftarrow m$ , while  $[u[1+j, n], u[k, i]] = 0$  since  $u[k, i]$  and  $u[1+j, n]$  are separated, see Lemma 4.2.

Similarly, if  $n < i < j$  then by Lemma 4.5 we have  $u[k, i] = [u[k, n], u[n+1, i]]$ . At the same time  $[u[1+j, m], u[k, n]] = 0$  due to (4.16) with  $a \leftarrow n$ , while  $[u[1+j, m], u[n+1, i]] = 0$  since  $u[1+j, m]$  and  $u[n+1, i]$  are separated.  $\square$

The elements  $u[k, m]$  are important due to the following statements.

**Proposition 4.8.** [6, Proposition 4.1]. *If  $q^3 \neq 1$ ,  $q^4 \neq 1$ , then values of the elements  $u[k, m]$ ,  $k \leq m < \psi(k)$  form a set of PBW-generators for the algebra  $U_q^+(\mathfrak{so}_{2n+1})$  over  $\mathbf{k}[G]$ . All heights are infinite.*

**Proposition 4.9.** ([6, Proposition 4.5]). *If the multiplicative order  $t$  of  $q$  is finite,  $t > 4$ , then the values of  $u[k, m]$ ,  $k \leq m < \psi(k)$  form a set of PBW-generators for  $u_q^+(\mathfrak{so}_{2n+1})$  over  $\mathbf{k}[G]$ . The height  $h$  of  $u[k, m]$  equals  $t$  if  $m = n$  or  $t$  is odd. If  $m \neq n$  and  $t$  is even, then  $h = t/2$ . In all cases  $u[k, m]^h = 0$  in  $u_q^+(\mathfrak{so}_{2n+1})$ .*

We stress that due to (4.8) the height  $h$  here equals the multiplicative order of  $p_{uu}$ , where  $u = u[k, m]$ . The coproduct on  $u[k, m]$ ,  $k \leq m \leq 2n$  is given by the following elegant formula, see [6, Theorem 4.3]:

$$(4.18) \quad \begin{aligned} \Delta(u[k, m]) &= u[k, m] \otimes 1 + g_{k \rightarrow m} \otimes u[k, m] \\ &+ \sum_{i=k}^{m-1} \tau_i (1 - q^{-2}) g_{k \rightarrow i} u[i+1, m] \otimes u[k, i], \end{aligned}$$

where by definition  $g_{k \rightarrow i} = g_k g_{k+1} \cdots g_i = g(u[k, i])$ , and

$$(4.19) \quad \tau_i = q^{\delta_i^n} = \begin{cases} q, & \text{if } i = n; \\ 1, & \text{otherwise.} \end{cases}$$

Formula (4.18) with (2.35) and (2.36) allows one to find the differentiation formulae

$$(4.20) \quad \partial_i(u[k, m]) = \begin{cases} (1 - q^{-2})\tau_k u[k+1, m], & \text{if } i \in \{k, \psi(k)\}, k < m; \\ 0, & \text{if } i \notin \{k, \psi(k)\}; \\ 1, & \text{if } i \in \{k, \psi(k)\}, k = m. \end{cases}$$

$$(4.21) \quad \partial_i^*(u[k, m]) = \begin{cases} (1 - q^{-2})\tau_{m-1} u[k, m-1], & \text{if } i \in \{m, \psi(m)\}, m > k; \\ 0, & \text{if } i \notin \{m, \psi(m)\}; \\ 1, & \text{if } i \in \{m, \psi(m)\}, m = k. \end{cases}$$

These differentiation formulae with differential representation of the simplest adjoint operators (2.42), (2.43) allows one to find the (skew) bracket of basis elements  $u[k, m]^{\mp}$  with the main generators  $x_i^{\pm}$ .

**Lemma 4.10.** *If  $k < m$ , then in  $U_q(\mathfrak{so}_{2n+1})$  we have*

$$(4.22) \quad [u[k, m], x_i^-] \sim \begin{cases} 0, & \text{if } i \notin \{k, m, \psi(k), \psi(m)\}; \\ g_k f_k u[k+1, m], & \text{if } i \in \{k, \psi(k)\}, m \neq \psi(k); \\ u[k, m-1], & \text{if } i \in \{m, \psi(m)\}, m \neq \psi(k). \end{cases}$$

*Proof.* The statement follows from (2.43), (4.21), and (4.20).  $\square$

**Lemma 4.11.** *If  $i < j$ , then in  $U_q(\mathfrak{so}_{2n+1})$  we have*

$$(4.23) \quad [x_k, u[i, j]^-] \sim \begin{cases} 0, & \text{if } k \notin \{i, j, \psi(i), \psi(j)\}; \\ g_i f_i u[i+1, j], & \text{if } k \in \{i, \psi(i)\}, j \neq \psi(i); \\ u[i, j-1], & \text{if } k \in \{j, \psi(j)\}, j \neq \psi(i). \end{cases}$$

*Proof.* The statement follows from (2.42), (4.21), and (4.20).  $\square$

**Corollary 4.12.** *If either  $k, m, \psi(k), \psi(m) \notin [i, j]$  or  $i, j, \psi(i), \psi(j) \notin [k, m]$ , then*

$$[u[k, m], u[i, j]^-] = 0.$$

*Proof.* If  $k, m, \psi(k), \psi(m) \notin [i, j]$ , then due to Lemma 4.11 we have  $[u[k, m], x_t^-] = 0$  for every  $t \in [i, j]$ . Hence ad-identity (2.9) and evident induction imply the required equality, for  $u[i, j]^-$  belongs to the subalgebra generated by  $x_t^-, t \in [i, j]$ . If  $i, j, \psi(i), \psi(j) \notin [k, m]$ , then in perfect analogy we use ad-identity (2.8) and Lemma 4.10.  $\square$

## 5. ROOTS AND RELATED PROPERTIES OF QUANTUM BOREL SUBALGEBRAS

Recall that a *root* of a homogeneous right coideal subalgebra  $U$  is degree of a PBW-generator of  $U$ , see [7, Definition 2.9]. Due to [6, Corollary 5.7] all roots of a homogeneous right coideal subalgebra  $U \supset G$  of positive quantum Borel subalgebra have the form  $[k : m] \stackrel{\text{df}}{=} x_k + x_{k+1} + \cdots + x_{m-1} + x_m = D(u[k, m])$ , where  $1 \leq k \leq m \leq 2n$ . Here  $x_{2n-i+1} = x_i$ , see Definition 4.3. An  $U$ -root is *simple* if it is not a sum of two or more other  $U$ -roots.

In what follows  $\Sigma(U)$  denotes the submonoid of  $\Gamma^+$  generated by all  $U$ -roots. Certainly degree of any nonzero homogeneous element from  $U$  belongs to  $\Sigma(U)$ . Moreover if  $q$  is not a root of 1, then all PBW-generators have infinite heights. Hence in this case  $\Sigma(U)$  is precisely the set of all degrees of nonzero homogeneous elements from  $U$ . Simple  $U$ -roots are nothing more than indecomposable elements from  $\Sigma(U)$ . In particular [6, Lemma 8.9] shows that  $U$  is uniquely defined by  $\Sigma(U)$  : if  $\Sigma(U) = \Sigma(U_1)$ , then  $U = U_1$ . The following statement shows that the lattice of right coideal subalgebras that contain the coradical is isomorphic to some lattice of submonoids of  $\Gamma^+$ .

**Proposition 5.1.** *Let  $U, U_1 \supseteq G$  be (homogeneous) right coideal subalgebras of  $U_q^+(\mathfrak{so}_{2n+1})$ ,  $q^t \neq 1$  (respectively of  $u_q^+(\mathfrak{so}_{2n+1})$ , if  $q^t = 1$ ,  $t > 4$ ). Then  $U \subseteq U_1$  if and only if  $\Sigma(U) \subseteq \Sigma(U_1)$ .*

*Proof.* If  $U \subseteq U_1$ , then every PBW-generator  $a$  of  $U$  belongs to  $U_1$ . In particular  $a$  is a (noncommutative) polynomial in  $G$  and PBW-generators of  $U_1$ . Hence every  $U$ -root, being a degree of some  $a$ , is a sum of  $U_1$ -roots (degrees of PBW-generators of  $U_1$ ); that is,  $\Sigma(U) \subseteq \Sigma(U_1)$ .

Let  $\Sigma(U) \subseteq \Sigma(U_1)$ . Consider the subalgebra  $U_2$  generated by  $U$  and  $U_1$ . Certainly this is a right coideal subalgebra. At the same time

$$\Sigma(U_1) \subseteq \Sigma(U_2) \subseteq \Sigma(U) + \Sigma(U_1) = \Sigma(U_1),$$

which implies  $\Sigma(U_1) = \Sigma(U_2)$ , and  $U_1 = U_2 \supseteq U$ .  $\square$

The proved statement implies the following nice characterization of elements from  $U$  in terms of degrees of its partial derivatives. Recall that the subalgebra  $A$  of  $U_q^+(\mathfrak{so}_{2n+1})$  or  $u_q^+(\mathfrak{so}_{2n+1})$  generated over  $\mathbf{k}$  by  $x_1, x_2, \dots, x_n$  has a noncommutative differential calculus (2.34). Due to (2.35) the subalgebra  $U_A \stackrel{\text{df}}{=} U \cap A$  is differential:  $\partial_i(U_A) \subseteq U_A$ ,  $1 \leq i \leq n$ . Conversely, if  $U_A$  is any differential subalgebra of  $A$  homogeneous in each  $x_i$ , then the subalgebra  $U$  generated by  $U_A$  and  $G$  is a right coideal subalgebra of  $U_q^+(\mathfrak{so}_{2n+1})$  or  $u_q^+(\mathfrak{so}_{2n+1})$ , see [7, Lemma 2.10]. Let  $\partial_u$ ,  $u = x_{i_1} x_{i_2} \cdots x_{i_m}$  denote the differential operator  $\partial_{i_1} \partial_{i_2} \cdots \partial_{i_m}$ . Certainly if  $f \in U_A$ ,  $\partial_u(f) \neq 0$ , then degree of  $\partial_u(f)$  belongs to  $\Sigma(U)$ , for  $\partial_u(f) \in U_A \subset U$ . Interestingly the converse statement is true as well.

**Proposition 5.2.** *Let  $U \supseteq G$  be a (homogeneous) right coideal subalgebra of  $U_q^+(\mathfrak{so}_{2n+1})$ ,  $q^t \neq 1$  (respectively of  $u_q^+(\mathfrak{so}_{2n+1})$ , if  $q^t = 1$ ,  $t > 4$ ). If  $f \in A$  is a homogeneous element such that for each differential operator  $\partial_u$  we have  $D(\partial_u(f)) \in \Sigma(U)$  or  $\partial_u(f) = 0$ , then  $f \in U$ .*

*Proof.* Consider the differential subalgebra  $B$  generated by  $U_A$  and  $f$ . As an algebra  $B$  is generated by  $U_A$  and all  $\partial_u(f)$ . Hence degrees of all nonzero homogeneous elements from  $B$  belong to  $\Sigma(U)$  (in particular  $D(f) = D(\partial_\emptyset(f)) \in \Sigma(U)$ ). Proposition 5.1 applied to the pair  $U, BG$  implies  $BG \subseteq U$ , and  $f \in U$ .  $\square$

We stress that the condition  $D(\partial_u(f)) \in \Sigma(U)$  is equivalent to  $D(f) \in \Sigma(U) + D(u)$ . Hence we may restate the proved statement:  $f \in U$  if and only if  $\partial_u(f) = 0$  for all words  $u$  such that  $D(f) \notin \Sigma(U) + D(u)$ . To put it another way, we have a representation of homogeneous components  $U_A^{(\gamma)}$ ,  $\gamma \in \Gamma^+$  of  $U_A$  in the form of



kernel of a set of differential operators:

$$(5.1) \quad U_A^{(\gamma)} = \bigcap_{\gamma \notin \Sigma(U) + D(u)} \text{Ker } \partial_u.$$

Moreover Proposition 5.2 shows that right coideal subalgebras are differentially closed in the following sense.

**Corollary 5.3.** *If under the conditions of the above proposition  $D(f) \in \Sigma(U)$  and  $\partial_i(f) \in U$ ,  $1 \leq i \leq n$ , then  $f \in U$ .*

*Proof.* Indeed, if  $\partial_i(f) \in U$ ,  $1 \leq i \leq n$ , then of course  $\partial_u(f) \in U$  for all nonempty words  $u$ . In particular either  $D(\partial_u(f)) \in \Sigma(U)$  or  $\partial_u(f) = 0$ . Proposition 5.2 applies.  $\square$

Needless to say that all statements of this and the above sections remain valid for negative quantum Borel subalgebra too. In particular all roots of a homogeneous right coideal subalgebra  $U^- \supset F$  of negative quantum Borel subalgebra have the form  $[i : j]^- \stackrel{df}{=} x_i^- + x_{i+1}^- + \dots + x_{j-1}^- + x_j^-$ , where  $1 \leq i \leq j \leq 2n$ .

## 6. MINIMAL GENERATORS FOR RIGHT COIDEAL SUBALGEBRAS OF THE QUANTUM BOREL ALGEBRA

Let  $S$  be a set of integer numbers from the interval  $[1, 2n]$ . A (noncommutative) polynomial  $\Phi^S(k, m)$ ,  $1 \leq k \leq m \leq 2n$  is defined by induction on the number  $r$  of elements in the set  $S \cap [k, m] = \{s_1, s_2, \dots, s_r\}$ ,  $k \leq s_1 < s_2 < \dots < s_r < m$  as follows:

$$(6.1) \quad \Phi^S(k, m) = u[k, m] - (1 - q^{-2}) \sum_{i=1}^r \alpha_{km}^{s_i} \Phi^S(1 + s_i, m) u[k, s_i],$$

where  $\alpha_{km}^s = \tau_s p(u(1 + s, m), u(k, s))^{-1}$ , while the  $\tau$ 's was defined in (4.19).

We display the element  $\Phi^S(k, m)$  schematically as a sequence of black and white points labeled by the numbers  $k-1, k, k+1, \dots, m-1, m$ , where the first point is always white, and the last one is always black, while an intermediate point labeled by  $i$  is black if and only if  $i \in S$ :

$$(6.2) \quad \begin{array}{ccccccccccc} k-1 & k & k+1 & k+2 & k+3 & \dots & m-2 & m-1 & m \\ \circ & \circ & \circ & \bullet & \circ & \dots & \bullet & \circ & \bullet \end{array}$$

Sometimes, if  $k \leq n < m$ , it is more convenient to display the element  $\Phi^S(k, m)$  in two lines putting the points labeled by indices  $i, \psi(i)$  that define the same variable  $x_i = x_{\psi(i)}$  in one column:

$$(6.3) \quad \begin{array}{ccccccccccc} & & & m & \dots & \bullet & \psi(i) & \dots & n+1 \\ & & & \bullet & \dots & \bullet & \circ & \dots & \bullet \\ k-1 & \circ & \dots & \psi(m) & \dots & \bullet & i & \dots & n \\ & & & \circ & \dots & \bullet & \bullet & \dots & \circ \end{array}$$

The elements  $\Phi^S(k, m)$  are very important since every right coideal subalgebra  $U \supseteq G$  of the quantum Borel subalgebra is generated as an algebra by  $G$  and the elements of this form, see [6, Corollary 5.7]. Moreover  $U$  is uniquely defined by its *root sequence*  $\theta = (\theta_1, \theta_2, \dots, \theta_n)$ . The root sequence satisfies  $0 \leq \theta_i \leq 2n - 2i + 1$ , and each sequence satisfying these conditions is a root sequence for some  $U$ . There exists a constructive algorithm that allows one to find the generators  $\Phi^S(k, m)$  if the sequence  $\theta$  is given, see [6, Definition 10.1 and Eq. (10.6)]. More precisely the algorithm allows one to find all possible values of the numbers  $k, m$

and the sets  $S$ . In particular one may construct all schemes (6.2) for the generators. However the explicit form of  $\Phi^S(k, m)$  needs complicated inductive procedure (6.1). These generators satisfy two additional important properties. First, their degrees,  $D(\Phi^S(k, m)) = x_k + x_{k+1} + \dots + x_m$ , are simple  $U$ -roots; that is,  $D(\Phi^S(k, m))$  is not a sum of nonzero degrees of other elements from  $U$ , see [6, Claims 7,8]. Next, the set  $S$  is always  $(k, m)$ -regular in the sense of the following definition, see [6, Claim 5].

**Definition 6.1.** Let  $1 \leq k \leq n < m \leq 2n$ . A set  $S$  is said to be *white*  $(k, m)$ -regular if for every  $i$ ,  $k - 1 \leq i < m$ , such that  $k \leq \psi(i) \leq m + 1$  either  $i$  or  $\psi(i) - 1$  does not belong to  $S \cup \{k - 1, m\}$ .

A set  $S$  is said to be *black*  $(k, m)$ -regular if for every  $i$ ,  $k \leq i \leq m$ , such that  $k \leq \psi(i) \leq m + 1$  either  $i$  or  $\psi(i) - 1$  belongs to  $S \setminus \{k - 1, m\}$ .

A set  $S$  is said to be  $(k, m)$ -regular if it is either black or white  $(k, m)$ -regular.

If  $m \leq n$ , or  $k > n$  (or, equivalently, if  $u[k, m]$  is of degree  $\leq 1$  in  $x_n$ ), then by definition each set  $S$  is both white and black  $(k, m)$ -regular.

To illustrate the notion of a regular set, we shall need a *shifted representation* that appears from (6.3) by shifting the upper line to the left by one step and putting the colored point labeled by  $n$ , if any, to the vacant position (so that this point appears twice in the shifted scheme):

$$(6.4) \quad \begin{array}{cccccccccccc} & & & & m & \dots & \circ & n+i & \dots & n+1 & n & \circ \leftarrow \\ & & & & \bullet & \dots & & \circ & \dots & \bullet & & \\ k-1 & \circ & \dots & \psi(m)-1 & \dots & \bullet & \bullet & \dots & n-1 & n & \circ & \\ & & & & \bullet & \dots & \bullet & \bullet & \dots & \circ & \circ & \end{array}$$

If  $k \leq n < m$  and  $S$  is white  $(k, m)$ -regular, then  $n \notin S$ , for  $\psi(n) - 1 = n$ . If additionally  $m < \psi(k)$ , then taking  $i = \psi(m) - 1$  we get  $\psi(i) - 1 = m$ , hence the definition implies  $\psi(m) - 1 \notin S$ . We see that if  $m < \psi(k)$ ,  $k \leq n < m$ , then  $S$  is white  $(k, m)$ -regular if and only if the shifted scheme of  $\Phi^S(k, m)$  given in (6.4) has no black columns:

$$(6.5) \quad \begin{array}{cccccccccccc} & & & & m & \dots & \bullet & n+i & \circ & \dots & n & \circ \leftarrow \\ & & & & \bullet & \dots & \bullet & \circ & \dots & \bullet & & \\ k-1 & \dots & \psi(m)-1 & \dots & \circ & \dots & \circ & \bullet & \circ & \dots & n & \circ \\ & & & & \circ & \dots & \circ & \bullet & \circ & \dots & \circ & \end{array}$$

In the same way, if  $m > \psi(k)$ , then for  $i = \psi(k)$  we get  $\psi(i) - 1 = k - 1$ , hence  $\psi(k) \notin S$ . That is, if  $m > \psi(k)$ ,  $k \leq n < m$ , then  $S$  is white  $(k, m)$ -regular if and only if the shifted scheme (6.4) has no black columns and the first from the left complete column is a white one.

$$(6.6) \quad \begin{array}{cccccccccccc} & & & & m & \dots & \psi(k) & \dots & \bullet & n+i & \circ & \dots & n & \circ \leftarrow \\ & & & & \bullet & \dots & \circ & \dots & \bullet & \circ & \dots & \bullet & & \\ k-1 & \dots & \psi(k) & \dots & \circ & \dots & \circ & \bullet & \circ & \dots & n & \circ & \circ & \\ & & & & \circ & \dots & \circ & \bullet & \circ & \dots & \circ & \dots & \circ & \end{array}$$

*All in all, a set  $S$  is white  $(k, m)$ -regular, where  $1 \leq k \leq n < m \leq 2n$ , if the shifted scheme obtained by painting  $k - 1$  black does not contain columns with two black points.*

Similarly, if  $k \leq n < m$  and  $S$  is black  $(k, m)$ -regular, then  $n \in S$ . If additionally  $m < \psi(k)$ , then taking  $i = \psi(m) - 1$  we get  $\psi(i) - 1 = m$ , hence  $\psi(m) - 1 \in S$ . We see that if  $m < \psi(k)$ ,  $k \leq n < m$ , then  $S$  is black  $(k, m)$ -regular if and only if the shifted scheme (6.4) has no white columns and the first from the left complete

column is a black one.

$$(6.7) \quad \begin{array}{cccccccc} & & & \bullet & \dots & \bullet & \overset{n+i}{\circ} & \bullet & \dots & \bullet & \leftarrow \leftarrow \\ & & & \psi(m)-1 & & \dots & \bullet & \circ & & \dots & \bullet \\ k-1 & \dots & \psi(m)-1 & \dots & \dots & \dots & \bullet & \circ & \dots & \dots & \bullet \\ \circ & & \bullet & & & & \bullet & & & & \bullet \end{array}$$

If  $m > \psi(k)$ , then for  $i = \psi(k)$  we get  $\psi(i) - 1 = k - 1$ , hence  $\psi(k) \in S$ . That is, if  $m > \psi(k)$ ,  $k \leq n < m$ , then  $S$  is black  $(k, m)$ -regular if and only if the shifted scheme (6.4) has no white columns:

$$(6.8) \quad \begin{array}{cccccccc} & & & \bullet & \dots & \bullet & \overset{n+i}{\circ} & \bullet & \dots & \bullet & \leftarrow \leftarrow \\ & & & \psi(k) & & \dots & \bullet & \circ & & \dots & \bullet \\ m & \dots & \psi(k) & \dots & \dots & \dots & \bullet & \circ & & \dots & \bullet \\ & & & k-1 & & \dots & \circ & \bullet & & \dots & \bullet \\ \circ & & \bullet & & & & \bullet & & & & \bullet \end{array}$$

All in all, a set  $S$  is black  $(k, m)$ -regular, where  $1 \leq k \leq n < m \leq 2n$ , if the shifted scheme obtained by painting  $m$  white does not contain columns with two white points.

At the same time we should stress that if  $m = \psi(k)$ , then no one set is  $(k, m)$ -regular. Indeed, for  $i = k - 1$  we have  $\psi(i) - 1 = m$ . Hence both of the elements  $i, \psi(i) - 1$  belong to  $S \cup \{k - 1, m\}$ , and therefore  $S$  is not white  $(k, \psi(k))$ -regular. If we take  $i = m$ , then  $\psi(i) - 1 = k - 1$ , and no one of the elements  $i, \psi(i) - 1$  belongs to  $S \setminus \{k - 1, m\}$ . Thus  $S$  is neither black  $(k, \psi(k))$ -regular.

**Lemma 6.2.** *A set  $S$  is white (black)  $(k, m)$ -regular if and only if its complement  $\overline{S}$  with respect to  $[k, m]$  is black (white)  $(k, m)$ -regular.*

*Proof.* The shifted scheme for  $\Phi^{\overline{S}}(k, m)$  appears from that for  $\Phi^S(k, m)$  by changing the color of all points except the first one,  $k - 1$ , and the last one,  $m$ . Under this re-coloring a scheme of type (6.4) is transformed to (6.7), while a scheme of type (6.5) is transformed to (6.8) and vice versa.  $\square$

**Lemma 6.3.** *A set  $S$  is white (black)  $(k, m)$ -regular if and only if  $\psi(S) - 1$  is white (black)  $(\psi(m), \psi(k))$ -regular. Here  $\psi(S) - 1 = \{\psi(s) - 1 \mid s \in S\}$ .*

*Proof.* The shifted scheme for  $\Phi^{\psi(S)-1}(\psi(m), \psi(k))$  appears from that for  $\Phi^S(k, m)$  by switching rows and changing the color of the first and the last points. Under that transformation a scheme of type (6.5) is transformed to (6.6), while a scheme of type (6.7) is transformed to (6.8) and vice versa.  $\square$

**Theorem 6.4.** ([6, Corollary 10.4]). *If  $q$  is not a root of 1 then every right coideal subalgebra of  $U_q^+(\mathfrak{so}_{2n+1})$  that contains  $G$  is generated as an algebra by  $G$  and a set of elements  $\Phi^S(k, m)$  with  $(k, m)$ -regular sets  $S$ . If  $q^t = 1$ ,  $t > 4$ , then this is the case for every homogeneous right coideal subalgebra of  $u_q^+(\mathfrak{so}_{2n+1})$  that contains  $G$ .*

Of course this theorem is valid for negative quantum Borel subalgebra as well. In this case the generators take up the form  $\Phi_-^S(k, m)$  with  $(k, m)$ -regular sets  $S$ , where  $\Phi_-^S(k, m)$ , is the element (6.1) under the replacement  $x_i \leftarrow x_i^-$ ,  $1 \leq i \leq n$ .

**Proposition 6.5.** *If  $S$  is a  $(k, m)$ -regular set, then*

$$\Phi^S(k, m) \sim \Phi^T(\psi(m), \psi(k)),$$

where  $T = \overline{\psi(S) - 1}$  is a  $(\psi(m), \psi(k))$ -regular set and  $\psi(S) - 1$  denotes the set  $\{\psi(s) - 1 \mid s \in S\}$ , while the complement is related to the interval  $[\psi(m), \psi(k))$ .

*Proof.* The proof follows from [6, Proposition 7.10] since due to Lemmas 6.2 and 6.3 the set  $S$  is white (black)  $(k, m)$ -regular if and only if  $T$  is black (white)  $(\psi(m), \psi(k))$ -regular.  $\square$

**Lemma 6.6.** *Let  $S$  be a white  $(k, m)$ -regular set. Assume  $s$  is a black point on the scheme (6.2), and  $k - 1 \leq t < s \leq m$ . Then  $S$  is white  $(1 + t, s)$ -regular if and only if either  $\psi(t) - 1$  is a white point or  $\psi(t) - 1 \notin [t, s]$ . In particular if either  $t$  is black or  $t = k - 1$ , then  $S$  is white  $(1 + t, s)$ -regular.*

*Proof.* The general statement follows from interpretation of regular sets given on diagrams (6.5), (6.6). The points  $t, \psi(t) - 1$  form a column on the shifted scheme. Hence if either  $t$  is black or  $t = k - 1$ , then  $\psi(t) - 1$  is white or it does not appear on the scheme at all, that is  $\psi(t) - 1 \notin [k - 1, m] \supseteq [t, s]$ .  $\square$

Similarly we have the following statement.

**Lemma 6.7.** *Let  $S$  be a black  $(k, m)$ -regular set. Assume  $t$  is a white point on the scheme (6.2), and  $k - 1 \leq t < s \leq m$ . Then  $S$  is black  $(1 + t, s)$ -regular if and only if either  $\psi(s) - 1$  is a black point or  $\psi(s) - 1 \notin [t, s]$ . In particular if either  $s$  is white or  $s = m$ , then  $S$  is black  $(1 + t, s)$ -regular.*

**Lemma 6.8.** ([6, Corollaries 7.7, 7.13]) *Let  $k \leq t < m$ . The decomposition*

$$(6.9) \quad \Phi^S(k, m) \sim [\Phi^S(k, t), \Phi^S(1 + t, m)]$$

*is valid if either  $S \cup \{t\}$  is white  $(k, m)$ -regular and  $t \notin S$ , or  $S$  is black  $(k, m)$ -regular and  $t \notin S \setminus \{n\}$ .*

**Lemma 6.9.** ([6, Corollaries 7.5, 7.14]) *Let  $k \leq s < m$ . The decomposition*

$$(6.10) \quad \Phi^S(k, m) \sim [\Phi^S(1 + s, m), \Phi^S(k, s)]$$

*is valid if either  $S$  is white  $(k, m)$ -regular and  $s \in S \cup \{n\}$ , or  $S \setminus \{s\}$  is black  $(k, m)$ -regular and  $s \in S$ .*

We stress that due to Lemmas 6.6, 6.7 in these lemmas the set  $S$  appears to be both  $(k, t)$ -regular and  $(1 + t, m)$ -regular; that is, the multiple use of the lemmas is admissible.

**Lemma 6.10.** *If  $S$  is  $(k, m)$ -regular set, then we have*

$$(6.11) \quad g_{k \rightarrow m} \sigma(\Phi^S(k, m)) \sim \Phi^{\psi(S)-1}(\psi(m), \psi(k)) \sim \Phi^{\bar{S}}(k, m),$$

*where  $\bar{S}$  is the complement of  $S$  with respect to  $[k, m]$ , and  $\sigma$  is the antipode.*

*Proof.* Assume  $S$  is white  $(k, m)$ -regular. We use induction on the number  $r$  of elements in the intersection  $S \cap [k, m]$ . If  $r = 0$ , then the left hand side equals  $g_{k \rightarrow m} \sigma(u[k, m]) \sim u[\psi(m), \psi(k)]$  due to (4.13). Proposition 6.5 with  $S \leftarrow [k, m]$  implies  $u[\psi(m), \psi(k)] \sim \Phi^{[k, m]}(k, m)$ , which is required. If  $r > 0$  then we choose  $s \in S, k \leq s < m$ . By Lemma 6.9 we have decomposition (6.10). Using (2.48) and the inductive supposition, we have

$$(6.12) \quad g_{k \rightarrow m} \sigma(\Phi^S(k, m)) \sim [\Phi^{\psi(S)-1}(\psi(s), \psi(k)), \Phi^{\psi(S)-1}(\psi(m), \psi(1 + s))].$$

At the same time Lemma 6.3 implies that  $\psi(S) - 1$  is a white  $(\psi(m), \psi(k))$ -regular set, and  $\psi(1 + s) = \psi(s) - 1 \in \psi(S) - 1$ . Hence we may apply Lemma 6.9, that shows that the right hand side of (6.12) is proportional to  $\Phi^{\psi(S)-1}(\psi(m), \psi(k))$ .

This proves the first proportion in (6.11). The second one follows from Proposition 6.5.

If  $S$  is black  $(k, m)$ -regular, then Lemma 6.5 reduces the consideration to white regular case.  $\square$

**Lemma 6.11.** *Let  $U^S(k, m)$  be the right coideal subalgebra generated by  $G$  and by an element  $\Phi^S(k, m)$  with a  $(k, m)$ -regular set  $S$ . In this case the monoid  $\Sigma(U^S(k, m))$  defined in the above section coincides with the monoid  $\Sigma$  generated by all  $[1 + t : s]$  with  $t$  being a white point and  $s$  being a black point on the scheme (6.2).*

*Proof.* Proposition 9.3 [6] implies that degrees of all homogeneous elements from  $U^S(k, m)$  belong to  $\Sigma$ . Hence  $\Sigma(U^S(k, m)) \subseteq \Sigma$ . At the same time Lemma 9.7 [6] says that every indecomposable in  $\Sigma$  element  $[1 + t : s]$  is a simple  $U^S(k, m)$ -root. Since certainly  $\Sigma$  is generated by its indecomposable elements, we have  $\Sigma \subseteq \Sigma(U^S(k, m))$ .  $\square$

**Lemma 6.12.** *Let  $S$  be a white  $(k, m)$ -regular set,  $t < s$  be respectively white and black points on the scheme (6.2). If  $\psi(1 + t)$  is not a black point (it is white or does not appear on the scheme at all) then  $[1 + t : s]$  is a simple  $U^S(k, m)$ -root, and  $\Phi^S(1 + t, s) \in U^S(k, m)$ .*

*Proof.* By [6, Lemma 9.5] the element  $[1 + t : s]$  is indecomposable in  $\Sigma$ . Hence by Lemma 6.11 it is a simple  $U^S(k, m)$ -root. At the same time [6, Theorem 9.8] implies  $\Phi^S(1 + t, s) \in U^S(k, m)$ .  $\square$

**Lemma 6.13.** *Let  $S$  be a black  $(k, m)$ -regular set,  $t < s$  be respectively white and black points on the scheme (6.2). If  $\psi(1 + s)$  is not a white point then  $[1 + t : s]$  is a simple  $U^S(k, m)$ -root, and  $\Phi^S(1 + t, s) \in U^S(k, m)$ .*

*Proof.* Similarly by [6, Lemma 9.6] the element  $[1 + t : s]$  is indecomposable in  $\Sigma$ . Hence by Lemma 6.11 it is a simple  $U^S(k, m)$ -root, while [6, Theorem 9.8] implies  $\Phi^S(1 + t, s) \in U^S(k, m)$ .  $\square$

**Lemma 6.14.** *Let  $S$  be a  $(k, m)$ -regular set. If  $t < s$  are respectively white and black points on the scheme (6.2), then  $\Phi^S(1 + t, s) \in U^S(k, m)$  unless  $t < n < s$ .*

*Proof.* Let  $S$  be white  $(k, m)$ -regular. Assume  $s \leq n$ . The point  $\psi(k)$  is not black on the schemes (6.5), (6.6). Hence Lemma 6.12 with  $t \leftarrow k - 1$ ,  $s \leftarrow s$  implies  $\Phi^S(k, s) \in U^S(k, m)$ . Again by Lemma 6.12 applied to  $U^S(k, s)$  we get  $\Phi^S(1 + t, s) \in U^S(k, s) \subseteq U^S(k, m)$ .

Assume  $t \geq n$ . The point  $n = \psi(n + 1)$  is white on the schemes (6.5), (6.6). Therefore Lemma 6.12 with  $t \leftarrow n$ ,  $s \leftarrow m$  implies  $\Phi^S(1 + n, m) \in U^S(k, m)$ . Again by Lemma 6.12 applied to  $U^S(1 + n, m)$  we get  $\Phi^S(1 + t, s) \in U^S(1 + n, m) \subseteq U^S(k, m)$ .

If  $S$  is black  $(k, m)$ -regular, then we may apply Lemma 6.13 in a similar way or just use the duality given in Proposition 6.5.  $\square$

## 7. NECESSARY CONDITION

Let  $U^- \supseteq F$  and  $U^+ \supseteq G$  be right coideal subalgebras of respectively negative and positive quantum Borel subalgebras. As we mentioned in the above section  $U^+$  is generated as algebra by  $G$  and elements of the form  $\Phi^S(k, m)$  with  $(k, m)$ -regular sets  $S$ . Respectively  $U^-$  is generated as algebra by  $F$  and elements of the

form  $\Phi_-^T(i, j)$  with  $(i, j)$ -regular sets  $T$ . Here  $\Phi_-^T(i, j)$  appears from  $\Phi^T(i, j)$  given in (6.1) under the substitutions  $x_t \leftarrow x_t^-$ ,  $1 \leq t \leq 2n$ .

To state a necessary condition for tensor product (3.4) to be a subalgebra we display the regular generators  $\Phi^S(k, m)$  and  $\Phi_-^T(i, j)$  graphically as defined in (6.2):

$$(7.1) \quad \begin{array}{cccccccc} S & k-1 & \dots & i-1 & i & i+1 & \dots & m \\ & \circ & & \bullet & \bullet & \circ & & \bullet \\ T & & & \circ & \circ & \bullet & \dots & \bullet \dots \bullet \\ & & & & & & & j \end{array} \cdot$$

We shall call this scheme a  $S_k^m T_i^j$ -scheme. Sometimes in this notation we omit those of the indices that are fixed in the context. For example if  $k, m, i, j$  are fixed, this is a  $ST$ -scheme. Lemma 6.5 shows that the element  $\Phi^S(k, m)$  up to a scalar factor equals the element  $\Phi^{\overline{\psi(S)-1}}(\psi(m), \psi(k))$  that has essentially different representation (6.2). By this reason to the pair  $\Phi^S(k, m), \Phi_-^T(i, j)$  we may associate three more schemes:

$$(7.2) \quad \begin{array}{cccccccc} S & k-1 & \dots & \psi(j)-1 & \psi(j) & \psi(j)+1 & \dots & m \\ & \circ & & \bullet & \bullet & \circ & & \bullet \\ T^* & & & \circ & \bullet & \bullet & \dots & \circ \dots \psi(i) \\ & & & & & & & \bullet \end{array} \cdot$$

Here  $T^*$  is the set  $\overline{\psi(T)-1}$ , the complement of  $\{\psi(t)-1 \mid t \in T\}$  with respect to  $[\psi(j), \psi(i)]$ . By definition this is the  $S_k^m T_{\psi(j)}^{*\psi(i)}$ -scheme, or shortly the  $ST^*$ -scheme.

$$(7.3) \quad \begin{array}{cccccccc} S^* & & & \psi(m)-1 & \dots & j-2 & j-1 & j & \dots & \psi(k) \\ & & & \circ & & \bullet & \circ & \circ & & \bullet \\ T & i-1 & \dots & \bullet & \dots & \circ & \circ & \bullet & & \end{array} \cdot$$

Here  $S^*$  is the set  $\overline{\psi(S)-1}$ , the complement of  $\{\psi(s)-1 \mid s \in S\}$  with respect to  $[\psi(j), \psi(i)]$ . By definition this is the  $S_{\psi(m)}^{*\psi(k)} T_i^j$ -scheme, or shortly the  $S^*T$ -scheme.

$$(7.4) \quad \begin{array}{cccccccc} S^* & & & \psi(m)-1 & \dots & \psi(i)-1 & \psi(i) & \dots & \psi(k) \\ & & & \circ & & \circ & \circ & & \bullet \\ T^* & \psi(j)-1 & \dots & \circ & \dots & \bullet & \bullet & & \end{array} \cdot$$

Again by definition this is the  $S_{\psi(m)}^{*\psi(k)} T_{\psi(j)}^{*\psi(i)}$ -scheme, or shortly the  $S^*T^*$ -scheme.

**Definition 7.1.** A scheme is said to be *balanced* if it has no fragments of the form

$$(7.5) \quad \begin{array}{ccc} \circ & \dots & \bullet \\ \circ & \dots & \bullet \end{array} \cdot$$

**Theorem 7.2.** Consider the triangular decomposition of a right coideal subalgebra given in Theorem 3.2

$$(7.6) \quad U = U^- \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} U^+.$$

If  $\Phi^S(k, m), \Phi_-^T(i, j)$  are the regular generators respectively of  $U^+$  and  $U^-$  defined by simple roots  $[k : m]$  and  $[i : j]^-$ , then either all four schemes (7.1 – 7.4) defined by this pair are balanced, or one of them has the form

$$(7.7) \quad \begin{array}{cccccc} \circ & \dots & \circ & \dots & \bullet & \dots & \bullet \\ \circ & \dots & \bullet & \dots & \circ & \dots & \bullet \end{array},$$

where no one intermediate column has points of the same color.

The next lemma shows that to see that a given pair satisfies the conclusion of the theorem it is sufficient to check just two first schemes (7.1), (7.2).

**Lemma 7.3.** *ST-Scheme (7.1) is balanced if and only if so is  $S^*T^*$ -scheme (7.4). Similarly  $ST^*$ -scheme (7.2) is balanced if and only if so is  $S^*T$ -scheme (7.3).  $ST$ -Scheme (7.1) has the form (7.7) if and only if so does  $S^*T^*$ -scheme (7.4). Respectively  $ST^*$ -scheme (7.2) has the form (7.7) if and only if so does  $S^*T$ -scheme (7.3).*

*Proof.* Consider a transformation  $\rho$  of schemes that moves a point  $a$  to  $\psi(a) - 1$  and changes the color. This transformation maps  $ST$ -scheme to  $S^*T^*$ -scheme and  $ST^*$ -scheme to  $S^*T$ -scheme. At the same time it changes the order of columns. In particular the fragment of the form (7.5) transforms to a fragment of the same form with  $t \leftarrow \psi(s) - 1$ ,  $s \leftarrow \psi(t) - 1$ .  $\square$

## 8. ADDITIONAL RELATIONS

In this and the next technical sections we are going to describe two important cases when  $[\Phi^S(k, m), \Phi_{-}^T(i, j)]$  belongs to  $\mathbf{k}[H]$ . The first one (Theorem 8.1) is the case when  $ST$ -scheme has the form (7.7), while the second one (Theorem 9.5) provides conditions when this bracket equals zero.

We fix the following notations. Let  $h_i$  denote  $g_i f_i \in H$ , while  $g_{k \rightarrow m}$  is the product  $g_k g_{k+1} \dots g_m$ , respectively  $f_{k \rightarrow m} = f_k f_{k+1} \dots f_m$ , and  $h_{k \rightarrow m} = g_{k \rightarrow m} f_{k \rightarrow m}$ . In the same way  $\chi^{k \rightarrow m} = \chi^k \chi^{k+1} \dots \chi^m$ . Similarly  $P_{k \rightarrow m, i \rightarrow j}$  is  $\chi^{k \rightarrow m} (g_{i \rightarrow j}) = \chi^{i \rightarrow j} (f_{k \rightarrow m})$ . Of course we have  $P_{k \rightarrow m, i \rightarrow j} = P_{\psi(m) \rightarrow \psi(k), \psi(j) \rightarrow \psi(i)}$ . In these notations Definition 4.4 takes the form  $\sigma_k^m = P_{k \rightarrow m, k \rightarrow m}$ ;  $\mu_k^{m, i} = P_{k \rightarrow i, i+1 \rightarrow m} \cdot P_{i+1 \rightarrow m, k \rightarrow i}$ .

**Theorem 8.1.** *If  $S$  is a  $(k, m)$ -regular set then*

$$[\Phi^S(k, m), \Phi_{-}^{\bar{S}}(k, m)] \sim 1 - h_{k \rightarrow m},$$

where  $\bar{S}$  is a complement of  $S$  with respect to the interval  $[k, m]$ .

*Proof.* We use induction on  $m - k$ . If  $m = k$ , the statement is clear.

Suppose firstly that  $n \notin [k, m]$ . In this case each set is both black and white  $(k, m)$ -regular. Hence by Lemma 6.8 and Lemma 6.9 with  $t = m - 1$  we have

$$\Phi^S(k, m) \sim \begin{cases} [\Phi^S(k, m-1), x_m] & \text{if } m-1 \notin S; \\ [x_m, \Phi^S(k, m-1)] & \text{if } m-1 \in S, \end{cases}$$

and

$$\Phi_{-}^{\bar{S}}(k, m) \sim \begin{cases} [x_m^-, \Phi_{-}^{\bar{S}}(k, m-1)] & \text{if } m-1 \notin S; \\ [\Phi_{-}^{\bar{S}}(k, m-1), x_m^-] & \text{if } m-1 \in S. \end{cases}$$

Let us fix for short the following designations:  $u = \Phi^S(k, m-1)$ ,  $v^- = \Phi_{-}^{\bar{S}}(k, m-1)$ . By the inductive supposition we have  $[u, v^-] = \alpha(1 - h_{k \rightarrow m-1})$ ,  $\alpha \neq 0$ . Consider the algebra  $\mathfrak{F}_2$  defined by the quantum variables  $z_1, z_2$  with  $g_{z_1} = \text{gr}(u) = g_{k \rightarrow m-1}$ ,  $\chi^{z_1} = \chi^u$ ,  $g_{z_2} = g_m$ ,  $\chi^{z_2} = \chi^m$ , and respectively  $g_{z_1^-} = \text{gr}(v^-) = f_{k \rightarrow m-1}$ ,  $\chi^{z_1^-} = (\chi^u)^{-1}$ ,  $g_{z_2^-} = f_m$ ,  $\chi^{z_2^-} = (\chi^m)^{-1}$ . Since due to Lemma 2.2 we have  $[u, x_m^-] = [x_m, v^-] = 0$ , the map  $z_1 \rightarrow u$ ,  $z_2 \rightarrow x_m$ ,  $z_1^- \rightarrow \alpha^{-1}v^-$ ,  $z_2^- \rightarrow x_m^-$  has an extension up to a homomorphism of algebras. Hence by Lemma 2.3 we have  $[[u, x_m], [x_m^-, v^-]] = \varepsilon(1 - h_{k \rightarrow m})$ , where the coefficient  $\varepsilon = (1 - p(z_1, z_2)p(z_2, z_1))$  equals  $1 - q^{-2}$ , for  $p(z_1, z_2) = p(u, x_m) = p_{km}p_{k+1m} \dots p_{m-1m}$  and  $p(z_2, z_1) =$

$p(x_m, u) = p_{mk}p_{m k+1} \dots p_{m m-1}$ . Since conditions of Lemma 2.3 are invariant under the substitution  $i \leftrightarrow j$ , we have also  $[[x_m, u], [v^-, x_m^-]] = \varepsilon(1 - h_{k \rightarrow m})$ , which is required.

Now consider the case  $n \in [k, m)$ . Suppose that  $S$  is white  $(k, m)$ -regular and  $m < \psi(k)$ . In this case  $\bar{S}$  is black  $(k, m)$ -regular. Let  $t$  denote the first white point next in order to  $\psi(m) - 1$ . Since  $n$  is a white point, we have  $t \leq n$ .

$$(8.1) \quad \begin{array}{ccccccccccc} & & & m & & \circ & \circ & \psi(t) & \psi(t)-1 & \dots & n \\ & & & \bullet & & & & \circ & * & & \circ \leftarrow \\ k-1 & \dots & \psi(m)-1 & & \bullet & \bullet & t-1 & t & & \dots & n \\ \circ & & \circ & & \circ & & \circ & \circ & & \dots & \circ \end{array}$$

The set  $S \cup \{\psi(t) - 1\}$  is white  $(k, m)$ -regular, unless  $\psi(t) - 1 = n$ . Hence by Lemma 6.8 and Lemma 6.9 we have

$$\Phi^S(k, m) \sim \begin{cases} [\Phi^S(k, \psi(t) - 1), \Phi^S(\psi(t), m)] & \text{if } \psi(t) - 1 \notin S \cup \{n\}; \\ [\Phi^S(\psi(t), m), \Phi^S(k, \psi(t) - 1)] & \text{if } \psi(t) - 1 \in S \cup \{n\}. \end{cases}$$

Similarly  $\bar{S} \setminus \{\psi(t) - 1\}$  is black  $(k, m)$ -regular, unless  $\psi(t) - 1 = n$ . The condition  $\psi(t) - 1 \notin \bar{S} \setminus \{n\}$  is equivalent to  $\psi(t) - 1 \in S \cup \{n\}$ . Hence these lemmas imply also

$$\Phi_{-}^{\bar{S}}(k, m) \sim \begin{cases} [\Phi_{-}^{\bar{S}}(\psi(t), m), \Phi_{-}^{\bar{S}}(k, \psi(t) - 1)] & \text{if } \psi(t) - 1 \notin S \cup \{n\}; \\ [\Phi_{-}^{\bar{S}}(k, \psi(t) - 1), \Phi_{-}^{\bar{S}}(\psi(t), m)] & \text{if } \psi(t) - 1 \in S \cup \{n\}. \end{cases}$$

Let us fix for short the following designations:  $u = \Phi^S(k, \psi(t) - 1)$ ,  $v = \Phi^S(\psi(t), m)$ ,  $w^- = \Phi_{-}^{\bar{S}}(k, \psi(t) - 1)$ ,  $y^- = \Phi_{-}^{\bar{S}}(\psi(t), m)$ . By the inductive supposition we have

$$(8.2) \quad [u, w^-] = \alpha(1 - h_{k \rightarrow \psi(t)-1}), \quad [v, y^-] = \beta(1 - h_{\psi(t) \rightarrow m}),$$

where  $\alpha \neq 0$ ,  $\beta \neq 0$ .

**Assume**  $t \neq n$  (equivalently,  $\psi(t) - 1 \neq n$ ). In this case  $u$  and  $w^-$  have further decompositions according to Lemmas 6.8, 6.9:

$$(8.3) \quad u = [\Phi^S(n+1, \psi(t) - 1), \Phi^S(k, n)], \quad w^- = [\Phi_{-}^{\bar{S}}(k, n), \Phi_{-}^{\bar{S}}(n+1, \psi(t) - 1)].$$

Moreover,  $S$  and  $\bar{S}$  are both black and white  $(k, n)$ -regular. Since  $\psi(m) - 1$ ,  $t$  are white points for  $S$  and black points for  $\bar{S}$ , we have

$$\Phi^S(k, n) = [[a_1, a_2], a_3], \quad \Phi_{-}^{\bar{S}}(k, n) = [b_3^-, [b_2^-, b_1^-]],$$

where  $a_1 = \Phi^S(k, \psi(m) - 1)$ ,  $a_2 = \Phi^S(\psi(m), t)$ ,  $a_3 = \Phi^S(t+1, n)$ , and similarly  $b_1^- = \Phi_{-}^{\bar{S}}(k, \psi(m) - 1)$ ,  $b_2^- = \Phi_{-}^{\bar{S}}(\psi(m), t)$ ,  $b_3^- = \Phi_{-}^{\bar{S}}(t+1, n)$ . All points of the interval  $[\psi(m), t)$  are black for  $S$  (of course if  $t = \psi(m)$ , then this interval is empty). Hence all points of the interval  $[\psi(t), m)$  are white (otherwise  $S$  is not white  $(k, m)$ -regular). In particular

$$(8.4) \quad v = \Phi^S(\psi(t), m) = \Phi^{\emptyset}(\psi(t), m) = u[\psi(t), m].$$

At the same time, using Lemma 6.5, we have

$$(8.5) \quad a_2 = \Phi^S(\psi(m), t) = \Phi^{[\psi(m), t]}(\psi(m), t) \sim \Phi^{\emptyset}(\psi(t), m) = u[\psi(t), m].$$

Hence by (8.2) we have  $[a_2, y^-] \sim [v, y^-] \sim 1 - h_{\psi(t) \rightarrow m}$ . Lemma 2.2 implies

$$0 = [a_1, y^-] = [a_3, y^-] = [\Phi^S(n+1, \psi(t) - 1), y^-].$$

Therefore

$$[\Phi^S(k, n), y^-] = [[[a_1, a_2], a_3], y^-] \stackrel{(2.19)}{\sim} [[[a_1, a_2], y^-], a_3]$$



$$\stackrel{(2.5)}{=} [[a_1, [a_2, y^-]], a_3] \stackrel{(2.25)}{\sim} [a_1, a_3] = 0,$$

for  $a_1, a_3$  are separated in  $U_q^+(\mathfrak{so}_{2n+1})$ . Thus, (8.3) implies  $[u, y^-] = 0$ .

In perfect analogy we have  $[v, w^-] = 0$ . Consider the algebra  $\mathfrak{F}_2$  defined by quantum variables  $z_1, z_2$  with  $g_{z_1} = \text{gr}(u) = g_{k \rightarrow \psi(t)-1}$ ,  $\chi^{z_1} = \chi^u$ ,  $g_{z_2} = \text{gr}(v) = g_{\psi(t) \rightarrow m}$ ,  $\chi^{z_2} = \chi^v$ , and respectively  $g_{z_1^-} = \text{gr}(w^-) = f_{k \rightarrow \psi(t)-1}$ ,  $\chi^{z_1^-} = (\chi^u)^{-1}$ ,  $g_{z_2^-} = \text{gr}(y^-) = f_{\psi(t) \rightarrow m}$ ,  $\chi^{z_2^-} = (\chi^v)^{-1}$ . Due to (8.2) and  $[u, y^-] = [v, w^-] = 0$ , the map  $z_1 \rightarrow u, z_2 \rightarrow v, z_1^- \rightarrow \alpha^{-1}w^-, z_2^- \rightarrow \beta^{-1}y^-$  has an extension up to a homomorphism of algebras. Hence by Lemma 2.3 we have  $[[u, v], [y^-, w^-]] = \varepsilon(1 - h_{k \rightarrow m})$ , where the coefficient  $\varepsilon$  equals  $1 - q^{-2}$ , for  $p(z_1, z_2)p(z_2, z_1) = p(u, v)p(v, u) = \mu_k^{m, \psi(t)-1} = q^{-2}$  due to (4.9). Conditions of Lemma 2.3 are invariant under the substitution  $i \leftrightarrow j$ . Hence we have also  $[[v, u], [w^-, y^-]] = \varepsilon(1 - h_{k \rightarrow m})$ , which proves the required relation for  $t \neq n$ .

**Assume**  $t = n$ . In this case  $\Phi^S(k, m) = [v, u]$ ,  $\Phi_{-}^{\bar{S}}(k, m) = [w^-, y^-]$  and we have

$$u = \Phi^S(k, n) = [a_1, b_1], \quad w^- = \Phi_{-}^{\bar{S}}(k, n) = [b_2^-, b_1^-],$$

where  $a_1 = \Phi^S(k, \psi(m) - 1)$ ,  $a_2 = \Phi^S(\psi(m), n)$ , and  $b_1^- = \Phi_{-}^{\bar{S}}(k, \psi(m) - 1)$ ,  $b_2^- = \Phi_{-}^{\bar{S}}(\psi(m), n)$ . Equalities (8.4) and (8.5) with  $t \leftarrow n$  show that  $a_2 \sim v$ . Hence  $\Phi^S(k, m) = [v, u] \sim [v, [a_1, v]]$ , while  $[v, [a_1, v]] \sim [[a_1, v], v]$  due to conditional identity (2.11), for  $p(a_1 v, v)p(v, a_1 v) = \mu_k^{m, n} = 1$ , see (4.9). Similarly

$$\Phi_{-}^{\bar{S}}(k, m) = [w^-, y^-] \sim [[b_2^-, b_1^-], y^-] \sim [[y^-, b_1], y^-] \sim [y^-, [y^-, b_1^-]].$$

Consider the algebra  $\mathfrak{F}_2$  defined by quantum variables  $z_1, z_2$  with  $g_{z_1} = \text{gr}(a_1) = g_{k \rightarrow \psi(m)-1}$ ,  $\chi^{z_1} = \chi^{a_1}$ ,  $g_{z_2} = \text{gr}(v) = g_{n+1 \rightarrow m}$ ,  $\chi^{z_2} = \chi^v$ , and respectively  $g_{z_1^-} = \text{gr}(b_1^-) = f_{k \rightarrow \psi(m)-1}$ ,  $\chi^{z_1^-} = \chi^{b_1^-} = (\chi^{a_1})^{-1}$ ,  $g_{z_2^-} = \text{gr}(y^-) = f_{n+1 \rightarrow m}$ ,  $\chi^{z_2^-} = \chi^{y^-} = (\chi^v)^{-1}$ . By the considered above case “ $n \notin [k, m]$ ” we have  $[a_1, b_1^-] = \gamma(1 - h_{k, \psi(m)-1})$ . Since Lemma 2.2 implies  $[a_1, y^-] = [v, b_1^-] = 0$ , the map  $z_1 \rightarrow a_1, z_2 \rightarrow v, z_1^- \rightarrow \gamma^{-1}b_1^-, z_2^- \rightarrow \beta^{-1}y^-$  has an extension up to a homomorphism of algebras. Hence by Lemma 2.4 we have  $[[[a_1, v], v], [y^-, [y^-, b_1^-]]] = \varepsilon(1 - h_{k \rightarrow m})$ .

It remains to note that  $\varepsilon \neq 0$ . Definition (4.6) implies  $p(z_2, z_2) = p(v, v) = \sigma_{n+1}^m$ , while (4.8) shows that  $\sigma_{n+1}^m = q$ . Further,  $p(z_1, z_2)p(z_2, z_1) = p(a_1, v)p(v, a_1) = \mu_k^{n, \psi(m)-1} = q^{-2}$ , see (4.7), (4.9). Hence  $\varepsilon = (1 + q)(1 - q^{-2})(1 - q^{-1}) \neq 0$ . This completes the proof of the case “ $m < \psi(k)$ ,  $S$  is white  $(k, m)$ -regular”.

If  $S$  is black  $(k, m)$ -regular and still  $m < \psi(k)$ , then by Lemma 6.2 the set  $\bar{S}$  is white  $(k, m)$ -regular. Hence  $[\Phi_{-}^{\bar{S}}(k, m), \Phi^S(k, m)] \sim 1 - h_{k \rightarrow m}$ . Let us apply  $h_{k \rightarrow m}\sigma$ , where  $\sigma$  is the antipode, to this equality. By (2.48) and (6.11) we have  $[\Phi_{-}^{\bar{S}}(k, m), \Phi^S(k, m)] \sim 1 - h_{k \rightarrow m}$ . It remains to apply antisymmetry (2.20).

If  $m > \psi(k)$  then Lemma 6.5 reduces consideration to the case “ $m < \psi(k)$ .”  $\square$

**Corollary 8.2.** *If  $k \leq m \neq \psi(k)$ , then in the algebra  $U_q(\mathfrak{so}_{2n+1})$  we have*

$$(8.6) \quad [u[k, m], u[\psi(m), \psi(k)]^-] \sim 1 - h_{k \rightarrow m}.$$

*Proof.* Proposition 6.5 with  $S = \emptyset$  applied to the mirror generators implies  $u[\psi(m), \psi(k)]^- \sim \Phi_{-}^{[k, m]}(k, m)$ . Hence Theorem 8.1 works.  $\square$

## 9. PAIRS WITH STRONG SCHEMES

In this section we determine when  $[\Phi^S(k, m), \Phi_-^T(i, j)]$  equals zero. Let us consider firstly the case  $S = T = \emptyset$ .

**Proposition 9.1.** *Let  $i \neq k$ ,  $j \neq m$ ,  $k \leq m$ ,  $i \leq j$ . If  $\psi(m), \psi(k) \notin [i, j]$  or, equivalently,  $k, m \notin [\psi(j), \psi(i)]$ , then in  $U_q(\mathfrak{so}_{2n+1})$  we have*

$$[u[k, m], u[i, j]^-] = 0.$$

*Proof.* If  $m = \psi(k)$ , then conditions  $\psi(m), \psi(k) \notin [i, j]$  certainly imply  $k, m, \psi(m), \psi(k) \notin [i, j]$ , and one may use Corollary 4.12. If  $j = \psi(i)$ , then  $\psi(t) \notin [i, j]$  if and only if  $t \notin [i, j]$ . Hence again  $\psi(m), \psi(k) \notin [i, j]$  implies  $k, m, \psi(m), \psi(k) \notin [i, j]$ , and Corollary 4.12 applies. Thus, further we may assume  $m \neq \psi(k)$ ,  $j \neq \psi(i)$ .

We shall use induction on the parameter  $m - k + j - i$ . If either  $m = k$  or  $j = i$ , then the statement follows from (4.22) and (4.23). Assume  $k < m$ ,  $i < j$ . Condition  $\psi(m), \psi(k) \notin [i, j]$  holds if and only if one of the following two options is fulfilled:

- A.  $\psi(m) < i < j < \psi(k)$ ;
- B.  $\psi(m) < \psi(k) < i < j$ , or  $i < j < \psi(m) < \psi(k)$ ;

Let us consider these options separately.

**A.** Since  $\psi(k), \psi(m) \notin [i, j]$ , by Corollary (4.12) we may suppose that either  $k \in [i, j]$  or  $m \in [i, j]$ . The option **A** is equivalent to  $k < \psi(j) < \psi(i) < m$ , for  $\psi$  changes the order.

By Proposition 4.6 with  $i \leftarrow \psi(j) - 1$  we have  $u[k, m] = [u[k, \psi(j) - 1], u[\psi(j), m]]$ . Indeed, the exceptional equality  $\psi(j) - 1 = \psi(m) - 1$  implies a contradiction  $j = m$ . The exceptional equality  $\psi(j) - 1 = \psi(k)$  implies  $j = k - 1$ , hence  $j < k < m$ , in particular  $k, m \notin [i, j]$ .

Similarly, Proposition 4.6 with  $k \leftarrow \psi(j)$ ,  $i \leftarrow \psi(i)$  shows that  $u[\psi(j), m] = [u[\psi(j), \psi(i)], u[\psi(i) + 1, m]]$ . Indeed, we have  $m \neq \psi(\psi(j)) = j$ , and  $\psi(i) \neq \psi(\psi(j)) = j$ . The remaining condition,  $\psi(i) \neq \psi(m) - 1$ , is also valid since otherwise  $i = m + 1$ , and again  $k, m \notin [i, j]$ , and again Corollary 4.12 applies.

Let us fix the for short following designations:  $u = u[k, \psi(j) - 1]$ ,  $v = u[\psi(j), \psi(i)]$ ,  $w = u[\psi(i) + 1, m]$ ,  $z^- = u[i, j]^-$ . Corollary 8.2 implies  $[v, z^-] \sim 1 - h_v$ . Proposition 4.7 with  $i \leftarrow \psi(j) - 1$ ,  $j \leftarrow \psi(i)$  shows that  $[u, w] = 0$ , for  $m = \psi(\psi(j) - 1) - 1$  is equivalent to  $m = j$ , while  $\psi(i) = \psi(k)$  is equivalent to  $i = k$ . Using (2.19) we have

$$(9.1) \quad [u[k, m], u[i, j]^-] = [[u, [v, w]], z^-] = [u, [[v, w], z^-]] + p_{z^-, v} [[u, z^-], [v, w]].$$

If  $j \neq n$ , then  $\psi(j) - 1 \neq j$ , and still  $\psi(k) \neq \psi(i) - 1$ . Hence by the inductive supposition with  $m \leftarrow \psi(j) - 1$  we have  $[u, z^-] = 0$ . If  $i \neq n + 1$ , then  $\psi(i) + 1 \neq i$ , and still  $m \neq \psi(\psi(i) + 1) = i - 1$ . Hence by the inductive supposition with  $k \leftarrow \psi(i) + 1$  we have  $[w, z^-] = 0$ . Thus for  $i \neq n + 1$ ,  $j \neq n$  we may continue (9.1):

$$(9.2) \quad \sim [u, [[v, z^-], w]] \stackrel{(2.26)}{\sim} [u, h_v \cdot w] \stackrel{(2.12)}{\sim} h_v [u, w] = 0.$$

Suppose that  $j = n$ . In this case we have  $k \in [i, j] = [i, n]$ , for  $m > \psi(j) = n + 1 > j$ . Moreover  $i \neq n + 1$ , for  $i < j = n$ . Hence still  $[w, z^-] = 0$ , and the first addend in (9.1) is zero (see arguments in (9.2)).

By additional induction on  $t - k$  we shall prove the following equation:

$$(9.3) \quad [u[k, t], u[i, t]^-] = \sum_{b=k-1}^{t-1} \alpha_b u[i, b]^- \cdot u[k, b],$$

where  $i < k \leq t \leq n$ ,  $0 \neq \alpha_b \in \mathbf{k}$ , and by definition  $u[k, k-1] = 1$ .

If  $t = k$ , the formula follows from (4.23). In the general case by the main inductive supposition we have  $[u[k, t-1], u[i, t]^-] = 0$ , for  $k \neq i$ ,  $t-1 \neq t$ ;  $\psi(k)$ ,  $\psi(t-1) \notin [i, t]$ ; and  $t-1 \neq \psi(k)$ ,  $t \neq \psi(i)$  due to  $i < k \leq t \leq n$ . Therefore

$$[u[k, t], u[i, t]^-] = [[u[k, t-1], x_t], u[i, t]^-] \stackrel{(2.5)}{=} [u[k, t-1], [x_t, u[i, t]^-]].$$

Here we would like to apply inhomogeneous substitution (4.23) to the right factor of brackets. To do this we must fix the coefficient:

$$\begin{aligned} &\sim u[k, t-1] \cdot u[i, t-1]^- - \chi^{k \rightarrow t-1}(g_t f_{i \rightarrow t}) u[i, t-1]^- \cdot u[k, t-1] \\ &= [u[k, t-1], u[i, t-1]^-] + \alpha_{t-1} u[i, t-1]^- \cdot u[k, t-1], \end{aligned}$$

where  $\alpha_{t-1} = \chi^{k \rightarrow t-1}(f_{i \rightarrow t-1})(1 - \chi^{k \rightarrow t-1}(h_t)) \neq 0$ . Thus by induction on  $t$  we get (9.3).

Relation (9.3) with  $t = n$  takes the form  $[u, z^-] = \sum_{b=k-1}^{n-1} \alpha_b u[i, b]^- \cdot u[k, b]$ . Since the first addend in (9.1) is zero, we may continue (9.1):

$$\sim \left[ \sum_{b=k-1}^{n-1} \alpha_b u[i, b]^- \cdot u[k, b], [v, w] \right].$$

We have seen that  $[v, w] = u[n+1, m]$ . At the same time  $[u[k, b], u[n+1, m]] = 0$  by Proposition 4.7 with  $i \leftarrow b$ ,  $j \leftarrow n$ . Indeed,  $m \neq \psi(b) - 1$  since  $m > \psi(i)$  and  $i < k \leq b$  implies  $\psi(i) > \psi(b)$ , while  $n \neq \psi(k)$  since  $k \in [i, n]$ . It remains to note that  $[u[i, b]^- , u[n+1, m]] \stackrel{(2.20)}{\sim} [u[n+1, m], u[i, b]^-] = 0$  by the inductive supposition with  $k \leftarrow n+1$ ,  $j \leftarrow b$ , for now  $n+1 \neq i$ ,  $m \neq b$ ,  $\psi(n+1) = n \notin [i, b]$ ,  $\psi(m) \notin [i, b]$ , and of course  $m \neq \psi(n+1) = n = j$ ,  $b \neq \psi(i) > n$ .

Similarly we consider the case  $i = n+1$ . In this case  $m \in [i, j] = [n+1, j]$ , for  $k < \psi(i) = n < i$ . Moreover,  $j \neq n$ , for  $n+1 = i < j$ . Hence still  $[u, z^-] = 0$ ; that is, the second addend in (9.1) equals zero, and by means of (2.19) we continue (9.1):

$$(9.4) \quad = [u, [[v, w], z^-]] = [u, [v, [w, z^-]]] + p_{z, w} [u, [[v, z^-], w]].$$

Arguments in (9.2) show that here the second addend is zero. Since  $[u, w] = [u, z^-] = 0$ , we have  $[u, [w, z^-]] = 0$ . Hence conditional identity (2.5) implies that the first addend in (9.4) equals

$$(9.5) \quad [[u, v], [w, z^-]] = [u[k, n], [w, z^-]].$$

By downward induction on  $t$  we shall prove the following equation:

$$(9.6) \quad [u[t, m], u[t, j]^-] = \sum_{a=t+1}^{\mu+1} \alpha_a h_{t \rightarrow a-1} u[a, j]^- \cdot u[a, m],$$

where  $n < t$ ,  $\mu = \min\{m, j\}$ ,  $0 \neq \alpha_a \in \mathbf{k}$ , and by definition  $u[\mu+1, \mu] = 1$ .

If  $t = j$  or  $t = m$  the equation follows from (4.22) and (4.23). In the general case by the main inductive supposition we have  $[u[t, m], u[t+1, j]^-] = 0$ , for  $t \neq t+1$ ,  $m \neq j$ ,  $\psi(t)$ ,  $\psi(m) \notin [t+1, j]$ . Therefore

$$\begin{aligned} [u[t, m], u[t, j]^-] &= [u[t, m], [x_t^-, u[t+1, j]^-]] \\ &\stackrel{(2.5)}{=} [[u[t, m], x_t^-], u[t+1, j]^-] \stackrel{(4.22)}{\sim} [h_t \cdot u[t+1, m], u[t+1, j]^-]. \end{aligned}$$

To prove (9.6) it remains to apply (2.14) and the inductive supposition for downward induction. Here the new coefficient  $\alpha_{t+1}$  is nonzero since  $\chi^{t+1 \rightarrow j}(h_t) = q^{-2} \neq 1$ .

Equation (9.6) with  $t = n + 1$  implies

$$(9.7) \quad [u[k, n], [w, z^-]] = \left[ u[k, n], \sum_{a=n+2}^{m+1} \alpha_a h_{n+1 \rightarrow a-1} u[a, j]^- \cdot u[a, m] \right].$$

By Proposition 4.7 with  $i \leftarrow n$ ,  $j \leftarrow a - 1$  we have  $[u[k, n], u[a, m]] = 0$ , for  $m \neq \psi(n) - 1 = n$  since  $m > i = n + 1$ , and  $a - 1 \neq \psi(k)$  since  $a - 1 \leq m \leq j < \psi(k)$ .

Due to (2.12) it remains to note that  $[u[k, n], u[a, j]^-] = 0$  by the inductive supposition with  $m \leftarrow n$ ,  $i \leftarrow a$ , for now  $n \neq j$ ,  $k \neq a$ ,  $\psi(k) \notin [a, j]$ ,  $\psi(n) = n + 1 \notin [a, j]$ , and  $n \neq \psi(k) > n$ ,  $j \neq \psi(a) < n$ .

**B.** In this case  $\psi(k)$ ,  $\psi(m) \notin [i, j]$ , hence by Corollary (4.12) we may suppose that either  $k \in [i, j]$  or  $m \in [i, j]$ . Application of  $\psi$  shows that the option **B** is equivalent to

$$(9.8) \quad \psi(j) < \psi(i) < k < m, \text{ or } k < m < \psi(j) < \psi(i).$$

In particular again due to Corollary (4.12) we may suppose that either  $i \in [k, m]$  or  $j \in [k, m]$ , for (9.8) implies  $\psi(i), \psi(j) \notin [k, m]$ . Since  $i \neq k$ ,  $j \neq m$ , it remains to consider two configurations:  $k < i \leq m < j$  and  $i < k \leq j < m$ . Moreover, the substitution  $i \leftrightarrow k$ ,  $j \leftrightarrow m$  transforms the original conditions **B** to equivalent form (9.8). Therefore it suffices to consider just one of the above configurations.

Suppose that  $k < i \leq m < j$ . In this case Proposition 4.6 with  $k \leftarrow i$ ,  $m \leftarrow j$ ,  $i \leftarrow i$  shows that  $u[i, j]^- = [x_i^-, u[i + 1, j]^-]$ , unless  $i = \psi(j) - 1$ . If  $i \neq \psi(j) - 1$ , then by the inductive supposition we have  $[u[k, m], u[i + 1, j]^-] = 0$ , for now  $k < i$ ,  $m \neq j$  and  $\psi(k), \psi(m) \notin [i, j] \supset [i + 1, j]$ . Hence by (2.5) and (4.22) we have

$$[u[k, m], [x_i^-, u[i + 1, j]^-]] = [u[k, m], x_i^-], u[i + 1, j]^- = \delta_i^m \cdot [u[k, m - 1], u[i + 1, j]],$$

for  $i \neq k$ ,  $i \neq \psi(k)$ ,  $i \neq \psi(m)$ , see original conditions **B**. At the same time if  $\delta_i^m \neq 0$  (that is  $m = i$ ), then  $[u[k, m - 1], u[i + 1, j]] = 0$  by Proposition 4.7 with  $m \leftarrow j$ ,  $i \leftarrow m - 1$ ,  $j \leftarrow i = m$ , for  $j \neq \psi(k)$ ,  $j \neq \psi(m - 1) - 1 = \psi(m)$ ,  $i = m \neq \psi(k)$  due to the original conditions **B**. Thus, it remains to check the case  $i = \psi(j) - 1$ .

Equality  $i = \psi(j) - 1$  with  $k < i < m$  imply  $k < \psi(j) \leq m$ , this contradicts to (9.8). Hence in this case we have  $i = m$ . Moreover,  $k < i$  implies

$$\psi(k) > \psi(i) = \psi(\psi(j) - 1) = j + 1 > i = m.$$

In particular  $\psi(k) \neq m - 1$ . Hence by Proposition 4.6 with  $i \leftarrow m$  we have  $[u[k, m] = [u[k, m - 1], x_m]$ . Corollary 4.12 implies both  $[u[k, m - 1], u[i, j]^-] = 0$  and  $[u[k, m - 1], u[i + 1, j]^-] = 0$ , for  $m - 1 = i - 1 \notin [i, j] \supset [i + 1, j]$ ,  $\psi(m - 1) = \psi(i - 1) = \psi(\psi(j) - 2) = j + 2 \notin [i, j] \supset [i + 1, j]$ . Thus by (2.5) and (2.12) we have

$$\begin{aligned} [u[k, m], u[i, j]^-] &= [[u[k, m - 1], x_m], u[i, j]^-] = [u[k, m - 1], [x_i, u[i, j]^-]] \\ &= [u[k, m - 1], h_i \cdot u[i + 1, j]^-] \sim h_i \cdot [u[k, m - 1], u[i + 1, j]^-] = 0. \end{aligned}$$

This completes the proof.  $\square$

**Proposition 9.2.** *Let  $i \neq k$ ,  $j \neq m$ ,  $k \leq m$ ,  $i \leq j$ . If  $\psi(j), \psi(i) \notin [k, m]$  or, equivalently,  $i, j \notin [\psi(m), \psi(k)]$ , then in  $U_q(\mathfrak{so}_{2n+1})$  we have*

$$[u[k, m], u[i, j]^-] = 0.$$

*Proof.* Substitution  $i \leftrightarrow k$ ,  $j \leftrightarrow m$  transforms the conditions of Proposition 9.2 to the conditions of Proposition 9.1. Let us apply Proposition 9.1 with  $i \leftrightarrow k$ ,  $j \leftrightarrow m$  to the mirror generators  $y_i = p_{ii}^{-1}x_i^-$ ,  $y_i^- = -x_i$ . We get  $[u[i, j]_y, u[k, m]_{y^-}] = 0$ . However  $u[i, j]_y \sim u[i, j]^-$ ,  $u[k, m]_{y^-} \sim u[k, m]$ . It remains to apply (2.20).  $\square$

**Proposition 9.3.** *Let  $i \neq k$ ,  $j \neq m$ . If*

$$(9.9) \quad \psi(j) \leq k \leq \psi(i) \leq m$$

*or, equivalently,*

$$(9.10) \quad \psi(m) \leq i \leq \psi(k) \leq j$$

*then in  $U_q(\mathfrak{so}_{2n+1})$  we have*

$$(9.11) \quad [u[k, m], u[i, j]^-] \sim h_{k \rightarrow \psi(i)} u[\psi(k) + 1, j]^- \cdot u[\psi(i) + 1, m]$$

*provided that  $\psi(m) \neq i$  or  $\psi(k) \neq j$ . Here by definition we set  $u[j + 1, j]^- = u[m + 1, m] = 1$ .*

*Proof.* We note that condition (9.9) is equivalent to the condition (9.10) since  $\psi$  changes the order. Let  $u = u[k, \psi(i)]$ ,  $v = u[\psi(i) + 1, m]$ ,  $w^- = u[i, \psi(k)]^-$ ,  $t^- = u[\psi(k) + 1, j]^-$ . Of course  $v = 1$  if  $m = \psi(i)$ , while  $t^- = 1$  if  $j = \psi(k)$ . By Lemma 8.2 we have

$$(9.12) \quad [u, w^-] = [u[k, \psi(i)], u[i, \psi(k)]^-] \sim 1 - h_u,$$

while Proposition 9.1 with  $k \leftarrow \psi(i) + 1$ ,  $i \leftarrow \psi(k) + 1$  shows that

$$(9.13) \quad [v, t^-] = [u[\psi(i) + 1, m], u[\psi(k) + 1, j]^-] = 0,$$

for  $\psi(m)$ ,  $\psi(\psi(i) + 1) = i - 1 \notin [\psi(k) + 1, j]$  due to (9.10). At the same time Proposition 9.1 with  $m \leftarrow \psi(i)$ ,  $i \leftarrow \psi(k) + 1$  implies

$$(9.14) \quad [u, t^-] = [u[k, \psi(i)], u[\psi(k) + 1, j]^-] = 0,$$

where  $k \neq n + 1$ ,  $j \neq \psi(i)$ . Indeed,  $\psi(\psi(i)) = i$ ,  $\psi(k) \notin [\psi(k) + 1, j]$  due to (9.10), while  $k \neq \psi(k) + 1$  due to  $k \neq n + 1$ . Similarly Proposition 9.2 with  $k \leftarrow \psi(i) + 1$ ,  $j \leftarrow \psi(k)$  shows that

$$(9.15) \quad [v, w^-] = [u[\psi(i) + 1, m], u[i, \psi(k)]^-] = 0, \quad \text{if } i \neq n + 1, m \neq \psi(k),$$

for  $\psi(\psi(k)) = k$ ,  $\psi(i) \notin [\psi(i) + 1, m]$  due to (9.9), while  $\psi(i) + 1 \neq i$  due to  $i \neq n + 1$ .

We shall prove firstly the proposition when the parameters are in the **general position**; that is, when  $i, k \neq n + 1$ ,  $i \neq m + 1$ ,  $k \neq j + 1$ ,  $m \neq \psi(k)$ ,  $j \neq \psi(i)$ .

By Proposition 4.6 with  $i \leftarrow \psi(i)$  we have  $u[k, m] = [u, v]$  provided that  $m > \psi(i) \neq \psi(m) - 1$ , for  $\psi(i) \neq \psi(k)$ . The same proposition with  $k \leftarrow i$ ,  $m \leftarrow j$ ,  $i \leftarrow \psi(k)$  shows that  $u[i, j]^- = [w^-, t^-]$  provided that  $j > \psi(k) \neq \psi(j) - 1$ . In particular if in the general position we have additionally  $\psi(i) \neq m$ ,  $\psi(k) \neq j$ , then  $u[k, m] = [u, v]$ ,  $u[i, j]^- = [w^-, t^-]$ , and all relations (9.12 — 9.15) hold. Hence we have the required proportions

$$[[u, v], [w^-, t^-]] \stackrel{(2.24)}{\sim} [[[u, w^-], t^-], v] \stackrel{(2.26)}{\sim} [h_u \cdot t^-, v] \stackrel{(2.15)}{\sim} h_u t^- \cdot v.$$

The omitted coefficient after the application of (2.26) is  $\chi^{t^-}(h_u) - 1$ , while

$$\chi^{t^-}(h_u) = \chi_-^{\psi(k)+1 \rightarrow j}(h_{k \rightarrow \psi(j)}) = \chi_-^{\psi(k)+1 \rightarrow j}(h_{i \rightarrow \psi(k)}) = (\mu_i^{j, \psi(k)})^{-1} = q^2 \neq 1$$

due to definition (4.7) and relations (4.9) and (4.11). Similarly the omitted coefficient after the application of (2.15) is  $1 - \chi^v(h_u)$ , while

$$\chi^v(h_u) = \chi^{\psi(i)+1 \rightarrow m}(h_{k \rightarrow \psi(i)}) = \mu_k^{m, \psi(i)} = q^{-2} \neq 1.$$

If in the general position we have  $\psi(i) = m$ ,  $\psi(k) \neq j$ , then  $u[k, m] = u$ ,  $u[i, j]^- = [w^-, t^-]$ ,  $[u, t^-] = 0$ . Hence we again have the required relation

$$[u, [w^-, t^-]] \stackrel{(2.22)}{\sim} [[u, w^-], t^-] \stackrel{(2.26)}{\sim} h_u \cdot t^-$$

with the omitted coefficient  $\chi^{t^-}(h_u) - 1 = q^2 - 1$ . Similarly, if in the general position we have  $\psi(i) \neq m$ ,  $\psi(k) = j$ , then  $u[k, m] = [u, v]$ ,  $u[i, j]^- = w^-$ ,  $[v, w^-] = 0$ , and

$$(9.16) \quad [[u, v], w^-] \stackrel{(2.19)}{\sim} [[u, w^-], v] \stackrel{(2.26)}{=} (1 - q^{-2})h_u \cdot v.$$

This completes the proof if  $k, m, i, j$  are in the general position. Consider the exceptional cases.

**1.  $k = n + 1$ .** In this case  $i \neq n + 1$ , for  $i \neq k$ . In particular by (9.15) we have  $[v, w^-] = 0$ . Moreover  $i \neq m + 1$ , for  $\psi(j) \leq n + 1 \leq \psi(i) \leq m$  and  $\psi(m) \leq i \leq n \leq j$  imply  $i \leq n < m$ . Hence  $u[k, m] = [u, v]$  if  $m \neq \psi(i)$ , and  $u[k, m] = u$  otherwise.

**1.1.** If  $j = n$ , then  $u[i, j]^- = w^-$ , for  $j = \psi(k) = n$ . Moreover we may assume  $m \neq \psi(i)$  (otherwise one may apply Lemma 8.2); that is,  $u[k, m] = [u, v]$ . Now algebraic manipulations (9.16) prove the required relation

$$(9.17) \quad [u[n + 1, m], u[i, n]^-] \sim h_{n+1 \rightarrow \psi(i)} \cdot u[\psi(i) + 1, m], \quad \psi(i) < m.$$

**1.2.** Let  $j = n + 1$ . By definition (4.12) we have  $u[i, j]^- = [u[i, n]^-, x_{n+1}^-]$ , and of course  $x_{n+1}^- = x_n^-$ . Hence Jacobi identity (2.22) and (4.22) show that  $[u[k, m], u[i, j]^-]$  is a linear combination of the following two terms

$$[[u[n + 1, m], u[i, n]^-], x_{n+1}^-], \quad [h_{n+1} \cdot u[n + 2, m], u[i, n]^-].$$

We claim that the former term equals zero. Indeed, if  $\psi(i) = m$ , then by Lemma 8.2 we have  $[u[n + 1, m], u[i, n]^-] \sim 1 - h_{i \rightarrow n}$ . However

$$\chi^{n+1}(h_{i \rightarrow n}) = \chi^n(g_{n-1}f_{n-1}g_n f_n) = p_{n-1}p_{n-1}p_{nn}p_{nn} = 1.$$

Hence (2.26) shows that the former term equals zero. If  $\psi(i) < m$ , then by (9.17) we have  $[u[n + 1, m], u[i, n]^-] \sim h_{n+1 \rightarrow \psi(i)} \cdot u[\psi(i) + 1, m]$ . Since  $\psi(i) \geq k = n + 1$ , Lemma 2.2 implies  $[u[\psi(i) + 1, m], x_{n+1}^-] = 0$ . At the same time  $\chi^{n+1}(h_{n+1 \rightarrow \psi(i)}) = \chi^{n+1}(h_{i \rightarrow n}) = 1$ . Thus (2.15) reduces the former term to zero.

To find the latter term we note that

$$\chi^{i \rightarrow n}(h_{n+1}) = \chi^{n-1}(g_n f_n) \chi^n(g_n f_n) = p_{n-1}p_{n-1}p_{nn}p_{nn} = 1.$$

Hence by (2.15) the latter term is proportional to  $h_{n+1} \cdot [u[n + 2, m], u[i, n]^-]$ . Since the points  $k' = n + 2$ ,  $m' = m$ ,  $i' = i$ ,  $j' = n$  are in the general position, we may apply (9.11):

$$[u[n + 2, m], u[i, n]^-] \sim h_{n+2 \rightarrow \psi(i)} u[n, n]^- \cdot u[\psi(i) + 1, m],$$

which is required, for  $u[n, n]^- = u[n + 1, n + 1]^- = x_n^-$ , and  $h_{n+1} \cdot h_{n+2 \rightarrow \psi(i)} = h_{n+1 \rightarrow \psi(i)}$ .

**1.3.** Let  $j > n + 1$ ,  $i < n$ . By definition (4.12) we have  $u[k, m] = [x_{n+1}, u[n + 2, m]]$ . Relation (4.23) shows that  $[x_{n+1}, u[i, j]^-] = 0$ . Hence conditional identity (2.5) implies

$$(9.18) \quad [u[k, m], u[i, j]^-] = [x_{n+1}, [u[n + 2, m], u[i, j]^-]].$$

At the same time the points  $k' = n + 2$ ,  $m' = m$ ,  $i' = i$ ,  $j' = j$  are in the general position. Moreover  $i < n$  implies  $\psi(i) + 1 > n + 2$ , hence  $[x_{n+1}, u[\psi(i) + 1, m]] = 0$  by Lemma 4.2. This allows us to continue (9.18) applying (2.12), (2.9):

$\sim [x_{n+1}, h_{n+2 \rightarrow \psi(i)} u[n, j]^- \cdot u[\psi(i) + 1, m]] \sim h_{n+2 \rightarrow \psi(i)} [x_{n+1}, u[n, j]^-] \cdot u[\psi(i) + 1, m]$ , which is required due to (4.23).

**1.4.** Let  $j > n + 1$ ,  $i = n$ . In this case by definition (4.12) we have  $u[i, j]^- = [x_n^-, u[n + 1, j]^-]$ . Jacobi identity (2.22) and (4.22) show that  $[u[k, m], u[i, j]^-]$  is a linear combination of the following two terms

$$[h_{n+1} u[n + 2, m], u[n + 1, j]^-], \quad [x_n^-, [u[n + 1, m], u[n + 1, j]^-]].$$

Proposition 9.1 implies  $[u[n + 2, m], u[n + 1, j]^-] = 0$ , for both  $\psi(n + 2) = n - 1$ , and  $\psi(m)$  are less than  $n + 1$ . At the same time

$$\chi^{n+1 \rightarrow j}(h_{n+1}) = \chi^{\psi(j) \rightarrow n}(h_n) = \chi^{n-1}(g_n f_n) \chi^n(g_n f_n) = p_{n-1} n p_{n-1} p_{nn}^2 = 1.$$

Hence by (2.14) the first term equals zero. Due to (9.6) the second term takes the form

$$(9.19) \quad \left[ x_n^-, \sum_{a=n+2}^{\mu} \alpha_a h_{n+1 \rightarrow a-1} u[a, j]^- \cdot u[a, m] \right],$$

where  $\mu = \min\{j, m\}$ . By Lemma 2.2 we have  $[x_n^-, u[a, m]] = 0$  for all  $a$ . At the same time  $[x_n^-, u[a, j]^-] = 0$  for all  $a > n + 2$ , see Lemma 4.2, while  $[x_n^-, u[n + 2, j]^-] = u[n + 1, j]^-$  since  $x_n = x_{n+1}$ . Hence in (9.19) remains just one term that corresponds to  $a = n + 2$ . By (2.12) and (2.9) this term is proportional to

$$h_{n+1} u[n + 1, j]^- \cdot u[n + 2, m],$$

which coincides with the right hand side of (9.11) with  $k = n + 1$ ,  $i = n$ .

**2.  $k = j + 1$ .** In this case inequality  $\psi(j) \leq k$  reads  $\psi(j) \leq j + 1$ , or, equivalently,  $2n - j + 1 \leq j + 1$ ; that is,  $j \geq n$ . If  $j = n$ , then we turn to the considered above case  $k = n + 1$ . Thus we have to consider just the case  $j > n$ . In this case  $k = j + 1 > n + 1$ , and  $j = k - 1 < \psi(i)$  since by the conditions of the proposition we have  $k \leq \psi(i)$ .

We shall prove firstly by downward induction on  $i$  with fixed  $j, k$  the following proportion

$$(9.20) \quad [u, u[i, j]^-] \sim h_{k \rightarrow \psi(i)} u[\psi(k) + 1, j]^-.$$

If  $i = \psi(k)$  then (9.20) follows from (4.23). Let  $i < \psi(k)$ . In this case by Proposition 4.6 we have  $u = [u[k, \psi(i) - 1], x_i]$ , for  $k > n$ . At the same time Proposition 9.2 implies  $[u[k, \psi(i) - 1], u[i, j]^-] = 0$  since  $\psi(j) \leq n < j = k - 1 < k$ , and  $\psi(j), \psi(i) \notin [k, \psi(i) - 1]$ . Hence conditional identity (2.5) with (4.23) show that

$$[u, u[i, j]^-] = [u[k, \psi(i) - 1], [x_i, u[i, j]^-]] \sim [u[k, \psi(i) - 1], h_i \cdot u[i + 1, j]^-].$$

This relation, after application of (2.12), and the inductive supposition imply (9.20), for  $h_i = h_{\psi(i)}$ ,  $\psi(i) - 1 = \psi(i + 1)$ .

If  $m = \psi(i)$  then  $u[k, m] = u$ , while (9.20) coincides with the required (9.11). Let  $m > \psi(i)$ . In this case  $u[k, m] = [u, v]$ , for  $k > n$ , see Proposition 4.6. Lemma 2.2 shows that  $[v, u[i, j]^-] = 0$ ; indeed,  $v = u[\psi(i) + 1, m]$  depends only in  $x_s$  with  $s < i$ , while  $u[i, j]^-$  depends only in  $x_s^-$  with  $i \leq s \leq n$ , for  $j < \psi(i)$ . We have

$$(9.21) \quad [[u, v], u[i, j]^-] \stackrel{(2.19)}{\sim} [[u, u[i, j]^-], v] \stackrel{(9.20)}{\sim} [h_{k \rightarrow \psi(i)} u[\psi(k) + 1, j]^-, v].$$

Again by Lemma 2.2 we get  $[u[\psi(k) + 1, j]^{-}, v] = 0$ . Therefore we may continue (9.21) applying (2.14):

$$\sim (1 - \chi^v(h_{k \rightarrow \psi(i)})) h_{k \rightarrow \psi(i)} u[\psi(k) + 1, j]^{-} \cdot v$$

which is required since by definition  $v = u[\psi(i) + 1, m]$ , and

$$\chi^v(h_{k \rightarrow \psi(i)}) = \chi^{\psi(i)+1 \rightarrow m}(h_{k \rightarrow \psi(i)}) = \mu_k^{m, \psi(i)} = q^{-2} \neq 1.$$

**3.  $\mathbf{i} = \mathbf{n} + 1$  or  $\mathbf{i} = \mathbf{m} + 1$ .** Conditions of the proposition are invariant under the transformation  $i \leftrightarrow k, j \leftrightarrow m$ . At the same time this transformation reduce the condition “ $i = n + 1$  or  $i = m + 1$ ” to the considered above cases **1** or **2**. Hence for the mirror generators  $y_i = p_{ii}^{-1} x_i^{-}, y_i^{-} = -x_i$  we have

$$[u[i, j]_y, u[k, m]_y^{-}] \sim h_{k \rightarrow \psi(i)} u[\psi(i) + 1, m]_y^{-} \cdot u[\psi(k) + 1, j]_y.$$

However  $u[a, b]_y \sim u[a, b]^{-}, u[a, b]_y^{-} \sim u[a, b]$ . It remains to apply (2.20) and to note that by Proposition 9.1 the factors in the right hand side of (9.11) skew commute each other, for  $\psi(m) \leq i \leq \psi(k) \leq j$  implies  $\psi(m), \psi(\psi(i) + 1) = i - 1 \notin [\psi(k) + 1, j]$ .

**4.  $\mathbf{j} = \psi(\mathbf{i})$ .** If also  $m = \psi(k)$  then (9.9) reads  $i \leq k \leq \psi(i) \leq \psi(k)$ , where the first and the last inequalities are not consistent provided that  $i \neq k$ . Hence we assume  $m \neq \psi(k)$ . Denote for short

$$u = u[k, m], \quad v^{-} = u[n + 1, j]^{-}, \quad w^{-} = u[i, n]^{-}.$$

By definition (4.12) we have  $u[i, j]^{-} \sim [v^{-}, w^{-}]$ .

If  $k \leq n$  then  $\psi(k) \notin [i, n]$ . We have also  $\psi(m) \notin [i, n]$ , for Eq. (9.9) with  $m \neq j = \psi(i)$  imply  $\psi(m) < i$ . Hence by Proposition 9.1 with  $j \leftarrow n$  we have  $[u, w^{-}] = 0$ . At the same time  $\psi(j) \leq k \leq \psi(n + 1) \leq m, \psi(n + 1) \neq j$ . Therefore already proved case of the proposition with  $i \leftarrow n + 1$  implies

$$[u, v^{-}] \sim h_{k \rightarrow n} u[\psi(k) + 1, j]^{-} \cdot u[n + 1, m].$$

Taking into account Jacobi identity (2.22) we have

$$(9.22) \quad [u, [v^{-}, w^{-}]] = [[u, v^{-}], w^{-}] \sim [h_{k \rightarrow n} u[\psi(k) + 1, j]^{-} \cdot u[n + 1, m], w^{-}].$$

The second statement of Proposition 4.7 with  $k \leftarrow i, i \leftarrow n, j \leftarrow \psi(k), m \leftarrow j$  implies  $[u[\psi(k) + 1, j]^{-}, w^{-}] = 0$ . Indeed, the conditions of Proposition 4.7 under that replacement are:  $j \neq \psi(n) - 1, \psi(k) \neq \psi(i)$ , and  $n \neq \psi(\psi(k)) - 1$ . They are valid since  $j = \psi(i) > n, k \neq i$ , and  $k \leq n$  respectively. Further, using Definition 4.4 and representations (4.8), (4.9), we have also

$$\chi^{i \rightarrow n}(h_{k \rightarrow n}) = P_{i \rightarrow n, k \rightarrow n} P_{k \rightarrow n, i \rightarrow n} = (\sigma_k^n)^2 \mu_i^{n, k-1} = q^2 \cdot q^{-2} = 1.$$

Hence ad-identity (2.8) and identity (2.14) imply that the right hand side of (9.22) equals

$$h_{k \rightarrow n} u[\psi(k) + 1, j]^{-} \cdot [u[n + 1, m], w^{-}].$$

Here  $\psi(n) = n + 1 \leq \psi(i) \leq m$ . Hence we may again use already proved case of the proposition with  $k \leftarrow n + 1, j \leftarrow n$ . This yields  $[u[n + 1, m], w^{-}] \sim h_{n+1 \rightarrow j} u[1 + j, m]$ , which proves (9.11), for  $h_{k \rightarrow n} \cdot h_{n+1 \rightarrow j} = h_{k \rightarrow \psi(i)}$  in the case  $j = \psi(i)$ .

If  $k > n$  then in perfect analogy we have  $[v^{-}, u] \sim [u, v^{-}] = 0$ , while  $[w^{-}, u] \sim [u, w^{-}] \sim h_{k \rightarrow j} u[1 + \psi(k), n]^{-} \cdot u[1 + j, m]$ . Therefore

$$\begin{aligned} [u, [v^{-}, w^{-}]] &\sim [[v^{-}, w^{-}], u] = [v^{-}, [w^{-}, u]] \\ &\sim h_{k \rightarrow j} [v^{-}, u[1 + \psi(k), n]^{-}] \cdot u[1 + j, m], \end{aligned}$$



since  $[v^-, u[1 + j, m]] \sim [u[1 + j, m], v^-] = 0$  according to Lemma 2.2. We have  $[v^-, u[1 + \psi(k), n]^-] \sim u[1 + \psi(k), j]^-$ , see Lemma 4.5. This completes the case  $j = \psi(i)$ .

**5.  $\mathbf{m} = \psi(\mathbf{k})$ .** By means of the mirror generators one may reduce the consideration to the case  $j = \psi(i)$ . The proposition is completely proved.  $\square$

**Definition 9.4.** A scheme (7.1) is said to be *strongly white* provided that the following three conditions are met: first, it has no black-black columns; then, the first from the left column is incomplete; and next, if there are at least two complete columns, then the first from the left complete column is a white-white one.

A scheme (7.1) is said to be *strongly black* provided that the following three conditions are met: first, it has no white-white columns; then, the last column is incomplete; and next, if there are at least two complete columns, then the last complete column is a black-black one.

A scheme is said to be *strong* if it is either strongly white or strongly black.

Alternatively we may define a strong scheme as follows. Let  $S'$ -scheme be a scheme that appears from the  $S$ -scheme (6.2) by changing colors of the first and the last points. Then  $ST$ -scheme is strongly white (black) if and only if both  $ST$ -scheme and  $S'T'$ -scheme have no black-black (white-white) columns.

We stress that the map  $\rho$  defined in Lemma 7.3 transforms strongly white schemes to strongly black ones and vice versa. Therefore the  $ST$ -scheme is strong if and only if the  $S^*T^*$ -scheme is strong. Similarly, the  $ST^*$ -scheme is strong if and only if the  $S^*T$ -scheme is strong.

**Theorem 9.5.** *Suppose that  $S, T$  are respectively  $(k, m)$ - and  $(i, j)$ -regular sets. If  $ST$ - and  $ST^*$ -schemes are strong, then  $[\Phi^S(k, m), \Phi_-^T(i, j)] = 0$ .*

*Proof.* Without loss of generality we may suppose that both schemes are strongly white. Indeed, the mirror generators allow us, if necessary, to switch the roles of  $S$  and  $T$ , while Lemma 6.5 and Lemma 7.3 allow us to find a pair of strongly white schemes. Moreover, once  $ST$ - and  $ST^*$ -schemes are strongly white, Lemma 6.5 allows one to switch the roles of  $T$  and  $T^*$ . Thus, without loss of generality, we may suppose also that  $T$  is white  $(i, j)$ -regular.

1. Assume  $S$  is white  $(k, m)$ -regular. We shall use double induction on numbers of elements in  $S \cap [k, m)$  and in  $T \cap [i, j)$ . If both intersections are empty then  $i \neq k$ ,  $j \neq m$ , for  $ST$ -scheme is strongly white.

$$\begin{array}{cccccccccccccccc}
 & & \circ & \overset{k}{\circ} & \dots & \circ & \circ & \circ & \overset{m}{\bullet} & & \circ & \overset{k}{\circ} & \circ & \circ & \dots & \circ & \circ & \overset{m}{\bullet} \\
 & & \circ & \overset{i}{\circ} & \circ & \dots & \circ & \overset{j}{\bullet} & & & \circ & \overset{\psi(j)}{\bullet} & \bullet & \dots & \bullet & \bullet & \overset{\psi(i)}{\bullet} & \bullet
 \end{array}$$

Similarly  $k, m \notin [\psi(j), \psi(i)]$ , for  $ST^*$ -scheme is strongly white. Hence Proposition 9.1 applies.

If  $s \in S \cap [k, m)$ , then by Lemma 6.9 we have  $\Phi^S(k, m) \sim [\Phi^S(1+s, m), \Phi^S(k, s)]$ . It is easy to see that  $S^sT$ - and  $S^sT^*$ -schemes (the schemes for the pair  $\Phi^S(k, s), \Phi_-^T(i, j)$ ) are still strongly white, while  $S$  is still white  $(k, s)$ - and  $(1+s, m)$ -regular. Hence the inductive supposition implies  $[\Phi^S(k, s), \Phi_-^T(i, j)] = 0$ . By the same reason  $[\Phi^S(1+s, m), \Phi_-^T(i, j)] = 0$ . Now Jacobi identity (2.19) implies the required equality.

It remains to consider the case  $S \cap [k, m) = \emptyset$ ; that is,  $\Phi^S(k, m) = u[k, m]$ . If  $t \in T \cap [i, j)$ , then by Lemma 6.9 we have  $\Phi_-^T(i, j) \sim [\Phi_-^T(1+t, j), \Phi_-^T(i, t)]$ . In this case  $T$  is still  $(k, s)$ - and  $(1+s, m)$ -regular, while  $ST^t$ -scheme is strongly white. At

the same time  $ST_{\psi(t)}^*$ -scheme is not strongly white only if  $\psi(t) - 1 = k - 1$  (the first from the left column is complete).

Hence by the inductive supposition we have  $[\Phi^S(k, m), \Phi_-^T(i, t)] = 0$  with one exception being  $\psi(t) = k$ . Similarly  $[\Phi^S(k, m), \Phi_-^T(1 + t, j)] = 0$  with one exception being  $\psi(t) - 1 = m$ . Hence, if in the set  $T \cap [i, j)$  there exists a point  $t \neq \psi(k)$ ,  $t \neq \psi(m) - 1$ , then Jacobi identity (2.22) implies the required equality. Certainly if  $T \cap [i, j)$  has more than two elements then such a point does exist.

If  $T \cap [i, j)$  has two points then there is just one exceptional configuration for the main  $ST^*$ -scheme:

$$\begin{array}{cccccccccccccc} & & & & & \circ & \overset{k}{\circ} & \circ & \cdots & \circ & \circ & \overset{m}{\bullet} & & & & \\ \circ & \psi(j) & \bullet & \cdots & \bullet & \circ & \bullet & \bullet & \cdots & \bullet & \bullet & \circ & \bullet & \cdots & \bullet & \psi(i) \end{array}$$

In this case  $T \cap [i, j) = \{t_1, t_2\}$ , where  $\psi(t_2) - 1 = k - 1$ ,  $\psi(t_1) - 1 = m$ . Let  $a = \Phi^S(k, m) = u[k, m]$ ,  $b^- = \Phi_-^T(i, j)$ ,  $u_0^- = u[i, t_1]^- = u[i, \psi(m) - 1]^-$ ,  $u_1^- = u[1 + t_1, t_2]^- = [\psi(m), \psi(k)]^-$ ,  $u_2^- = u[1 + t_2, j]^- = u[1 + \psi(k), j]^-$ . Using Lemma 6.9 twice, we have  $b^- \sim [[u_2^-, u_1^-], u_0^-]$ . Lemma 8.2 implies  $[a, u_1^-] \sim 1 - h_a$ . Inequality  $\psi(m) \leq \psi(k)$  implies both  $\psi(m), \psi(k) \notin [i, \psi(m) - 1] = [i, t_1]$  and  $\psi(m), \psi(k) \notin [1 + \psi(k), j] = [1 + t_2, j]$ . Therefore by Proposition 9.1 we have  $[a, u_0^-] = 0$  unless  $m = \psi(m) - 1$ , and  $[a, u_2^-] = 0$  unless  $k = 1 + \psi(k)$ . At the same time  $k = 1 + \psi(k)$  implies  $k = n + 1$ , and hence  $n = \psi(k) = t_2 \in T \cap [i, j)$ , which is impossible, for  $T$  is white  $(i, j)$ -regular. Similarly,  $m = \psi(m) - 1$  implies  $m = n$ , and hence  $n = \psi(m) - 1 = t_1 \in T \cap [i, j)$ , which is wrong by the same reason.

Taking into account the proved relations, we may write

$$\begin{aligned} [a, b^-] &\sim [a, [[u_2^-, u_1^-], u_0^-]] \stackrel{(2.5)}{=} [[a, [u_2^-, u_1^-]], u_0^-] \\ &\stackrel{(2.22)}{\sim} [[u_2^-, [a, u_1^-]], u_0^-] \stackrel{(2.25)}{\sim} [u_2^-, u_0^-]. \end{aligned}$$

Here we have applied inhomogeneous substitution (2.25) to the left factor in the brackets. Proposition 4.7 with  $k \leftarrow i$ ,  $m \leftarrow j$ ,  $i \leftarrow t_1$ ,  $j \leftarrow t_2$  implies  $[u_2^-, u_0^-] = 0$  provided that  $j \neq \psi(i)$ ,  $j \neq \psi(t_1) - 1$ ,  $t_2 \neq \psi(i)$ , and  $t_1 \neq \psi(t_2) - 1$ . The first inequality is valid since  $T$  is  $(i, j)$ -regular. The second and third inequalities are equivalent to  $j \neq m$  and  $\psi(k) \neq \psi(i)$  respectively. However  $j \neq m$  and  $k \neq i$  are valid, for the main  $ST$ -scheme is strongly white. The equality  $t_1 = \psi(t_2) - 1$  is equivalent to  $m = t_2$ , while in this case on the  $ST$ -scheme we have a black-black column.

If  $T \cap [i, j) = \{t\}$  then there are just two exceptional configuration for the main  $ST^*$ -scheme, where  $\psi(t) = k$  in case A, and  $\psi(t) = m + 1$  in case B:

$$\begin{array}{c} \text{A :} \\ \circ \quad \psi(j) \quad \bullet \quad \bullet \quad \circ \quad \bullet \quad \bullet \quad \bullet \end{array} \begin{array}{c} \circ \quad \overset{k}{\circ} \quad \circ \quad \circ \quad \overset{m}{\bullet} \\ \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \end{array} ; \begin{array}{c} \text{B :} \\ \circ \quad \overset{k}{\circ} \quad \circ \quad \circ \quad \overset{m}{\bullet} \\ \circ \quad \psi(j) \quad \bullet \quad \bullet \quad \circ \quad \bullet \quad \bullet \end{array} \begin{array}{c} \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \end{array}$$

In case A we keep the above notations  $a = u[k, m]$ ,  $b^- = \Phi_-^T(i, j)$ ,  $u_0^- = u[i, t]^-$ ,  $u_1^- = u[1 + t, j]^-$ . Lemma 6.9 implies  $b^- = [u_1^-, u_0^-]$ . We have  $\psi(m), \psi(k) \notin [1 + t, j] = [1 + \psi(k), j]$ . Moreover  $k \neq 1 + t$ , for otherwise the first from the left complete column on the main  $ST$ -scheme is white-black which contradicts the definition of a strongly white scheme (here  $t \neq j$  and therefore the scheme has at least two complete columns). Hence Proposition 9.1 implies  $[a, u_1^-] = 0$ . Since  $\psi(t) = k \leq$

$\psi(i) \leq m$ , Proposition 9.3 shows that  $[a, u_0^-] \sim h_{k \rightarrow \psi(i)} u[\psi(i) + 1, m]$ . Thus we get

$$[a, b^-] = [a, [u_1^-, u_0^-]] \stackrel{(2.22)}{\sim} [u_1^-, [a, u_0^-]] \stackrel{(2.12)}{\sim} h_{k \rightarrow \psi(i)} [u_1^-, u[\psi(i) + 1, m]] = 0.$$

The latter equality follows from antisymmetry identity (2.20) and Proposition 9.1. Indeed,  $\psi(i) + 1 \neq 1 + t = 1 + \psi(k)$ , for  $i \neq k$ , while in configuration A we have  $\psi(m)$ ,  $\psi(\psi(i) + 1) \notin [1 + t, j]$  since  $\psi(i) + 1, m \notin [\psi(j), \psi(1 + t)] = [\psi(j), k - 1]$ . This allows one to apply Proposition 9.1.

In case B we consider the points  $k' = \psi(m)$ ,  $m' = \psi(k)$ ,  $i' = \psi(j)$ ,  $j' = \psi(i)$ , and  $t' = \psi(t) - 1 = m$ . These points are in configuration A. Therefore we have  $[u[k', m'], \Phi_-^{\{t'\}}(i', j')] = 0$ . Let us apply  $g_{k \rightarrow m} f_{i \rightarrow j} \sigma$ , where  $\sigma$  is the antipode. Using properties of the antipode given in (2.48), (4.13), (6.11) we get the required equality.

2. If  $S$  is black  $(k, m)$ -regular, but not white  $(k, m)$ -regular, then  $n \in [k, m]$ , and  $n$  is a black point on the scheme  $S$ . Lemma 6.8 implies  $\Phi^S(k, m) = [\Phi^S(k, n), \Phi^S(n + 1, m)]$ . By definition  $S$ , as well as any other set, is white  $(k, n)$ - and  $(n + 1, m)$ -regular. Since  $ST$ - and  $ST^*$ -schemes are strongly white, the point  $n$  is not black on the schemes  $T, T^*$ . At the same time  $n$  is a white point on  $T$  if and only if it is a black point on  $T^*$ . Hence  $n$  does not appear on  $T, T^*$  at all,  $n \notin [i - 1, j]$ . In particular  $S^n T^-$ , and  $S^n T^*$ -schemes (the schemes for the pair  $\Phi^S(k, n), \Phi_-^T(i, j)$ ) are still strongly white. The above considered case implies  $[\Phi^S(k, n), \Phi_-^T(i, j)] = 0$ . By the same reason  $[\Phi^S(n + 1, m), \Phi_-^T(i, j)] = 0$ . It remains to apply Jacobi identity (2.19).  $\square$

## 10. PROOF OF THE MAIN THEOREM

**Lemma 10.1.** *Let  $k \leq s < n$ . If  $s \in S$ , then*

$$(10.1) \quad \left[ \Phi^S(k, n), \Phi_{\bar{S}}(k, s) \right] \sim \Phi^S(1 + s, n),$$

where  $\bar{S}$  is the complement of  $S$  with respect to  $[k, s]$ .

*Proof.* By Lemma 6.9 we have  $\Phi^S(k, m) = [\Phi^S(1 + s, n), \Phi^S(k, s)]$ . At the same time  $[\Phi^S(1 + s, n), \Phi_{\bar{S}}(s, n)] = 0$  due to Lemma 2.2. Taking into account Theorem 8.1 we have

$$\begin{aligned} & \left[ [\Phi^S(1 + s, n), \Phi^S(k, s)], \Phi_{\bar{S}}(k, s) \right] \\ & \stackrel{(2.5)}{=} \left[ \Phi^S(1 + s, n), [\Phi^S(k, s), \Phi_{\bar{S}}(k, s)] \right] \stackrel{(2.25)}{\sim} \Phi^S(1 + s, n), \end{aligned}$$

where the coefficient of the proportion equals  $1 - \chi^{1+s \rightarrow n}(h_{k \rightarrow s}) = 1 - \mu_k^{n, s} = 1 - q^{-2}$ , see (4.9).  $\square$

**Lemma 10.2.** *Let  $i \neq k$ ,  $m \leq n$ . If the  $S_k^m T_i^m$ -scheme has only one black-black column (the last one), and the first complete column is white-white then*

$$(10.2) \quad [\Phi^S(k, m), \Phi_-^T(i, m)] = \sum_{b=\nu-1}^{m-1} \alpha_b \Phi_-^T(i, b) \cdot \Phi^S(k, b),$$

where  $\nu = \max\{i, k\}$ , while  $\alpha_b \neq 0$  if and only if the column  $b$  is white-white. Here by definition  $\Phi^S(k, k - 1) = \Phi_-^T(i, i - 1) = 1$ .

*Proof.* For the sake of definiteness, assume that  $k < i$  (if  $i < k$  then the proof is quite similar). We use induction on the number of white-white columns on the  $S_k^m T_i^m$ -scheme. If there is just one white-white column then this is the first from the left column labeled by  $i - 1$ . Moreover all intermediate complete columns are white-black or black-white. Hence Theorem 8.1 implies  $[\Phi^S(i, m), \Phi_-^T(i, m)] \sim 1 - h_{i \rightarrow m}$ . By Lemma 6.8 we have  $\Phi^S(k, m) = [\Phi^S(k, i - 1), \Phi^S(i, m)]$ . At the same time  $[\Phi^S(k, i - 1), \Phi_-^T(i, m)] = 0$  due to Lemma 2.2. Hence

$$\begin{aligned} & [[\Phi^S(k, i - 1), \Phi^S(i, m)], \Phi_-^T(i, m)] \\ & \stackrel{(2.5)}{=} [\Phi^S(k, i - 1), [\Phi^S(i, m), \Phi_-^T(i, m)]] \stackrel{(2.25)}{\sim} \Phi^S(k, i - 1), \end{aligned}$$

which is required, for the coefficient of the proportion equals  $1 - \chi^{k \rightarrow i - 1}(h_{i \rightarrow m}) = 1 - \mu_k^{m, i - 1} = 1 - q^{-2} \neq 0$  (recall that  $m \leq n$ ).

To make the inductive step, let  $a$  be the maximal white-white column. Then all columns between  $a$  and  $m$  are black-white or white-black. Hence Theorem 8.1 implies  $[\Phi^S(1 + a, m), \Phi_-^T(1 + a, m)] \sim 1 - h_{1+a \rightarrow m}$ . Let us fix for short the following designations:

$$u = \Phi^S(k, a), \quad v = \Phi^S(1 + a, m), \quad w^- = \Phi_-^T(i, a), \quad t^- = \Phi_-^T(1 + a, m).$$

Then by Lemma 6.8 we have  $\Phi^S(k, m) = [u, v]$ ,  $\Phi_-^T(i, m) = [w^-, t^-]$ . Lemma 2.2 implies  $[u, t^-] = [v, w^-] = 0$ . At the same time  $[v, t^-] \sim 1 - h_{1+a \rightarrow m}$ , while  $[u, w^-]$  equals the left hand side of (10.2) with  $m \leftarrow a$ . Applying inductive supposition to  $[u, w^-]$  we see that  $[[u, w^-], t^-] = 0$ . Indeed, we may apply inhomogeneous substitution (10.2) to the left factor of the bracket. Then for each  $b < a$  we have  $[\Phi^S(k, b), t^-] = 0$  by Lemma 2.2, while  $[\Phi_-^T(i, b), t^-] = 0$  due to Lemma 4.2. Additionally, using (2.25) with  $x_i \leftarrow v$ ,  $x_i^- \leftarrow t^-$ , we have  $[w^-, [v, t^-]] \sim w^-$ , for  $\chi^{w^-}(g_v f_t) = (\mu_i^{m, a})^{-1} = q^2 \neq 1$  according to (4.9).

All that relations allow us to simplify (2.24):

$$\begin{aligned} & [[u, v], [w^-, t^-]] \sim [u, [w^-, [v, t^-]]] = u \cdot w^- - p(u, w^- vt^-) w^- \cdot u \\ (10.3) \quad & = [u, w^-] + p(u, w^-)(1 - p(u, vt^-)) w^- \cdot u. \end{aligned}$$

Here we apply inhomogeneous substitution to the right factor of the bracket. By this reason we have to develop the bracket to its explicit form. We have  $p(u, vt^-) = p(u, v)p(t, u) = \mu_k^{m, a} = q^{-2} \neq 1$ . Thus inductive supposition applied to  $[u, w^-]$  shows that (10.3) is the required sum.  $\square$

**Lemma 10.3.** *Let  $S$  be a black  $(k, m)$ -regular set,  $k \leq n < m$ . We have*

$$(10.4) \quad \varepsilon^- \otimes \varepsilon^0 \otimes \text{id} \left( \left[ \Phi^S(k, m), \Phi_{-}^{\overline{S}}(k, n) \right] \right) \neq 0.$$

*This nonzero element has degree  $[\psi(m) : n] = [n + 1 : m]$ . Here  $\varepsilon^-$ ,  $\varepsilon^0$  are the counits of  $U_q^-$  and  $\mathbf{k}[H]$  respectively, the tensor product of maps is related to the triangular decomposition (3.1), (3.2); while  $\overline{S}$  is a complement of  $S$  with respect to  $[k, n]$ .*

*Proof.* Let us fix for short the following designations:

$$u = \Phi^S(k, n), \quad v = \Phi^S(1 + n, m), \quad w^- = \Phi_{-}^{\overline{S}}(k, n).$$

By Lemma 6.8 we have  $\Phi^S(k, m) = [u, v]$ , while Jacobi identity (2.19) implies that  $[\Phi^S(k, m), \Phi_{-}^{\bar{S}}(k, n)]$  is a linear combination of two elements,  $[u, [v, w^{-}]]$  and  $[[u, w^{-}], v]$ . The latter one equals zero since due to Theorem 8.1 we have  $[u, w^{-}] \sim 1 - h_v$ , and coefficient of (2.26) with  $x_i \leftarrow u, x_i^{-} \leftarrow w^{-}, u \leftarrow v$  is  $\chi^v(g_u f_w) - 1 = \mu_k^{m, n} - 1 = 0$ , see (4.9), (4.11). Further, due to Proposition 6.5 we have

$$(10.5) \quad [v, w^{-}] \sim \left[ \overline{\Phi^{\psi(S)-1}}(\psi(m), n), \Phi_{-}^{\bar{S}}(k, n) \right].$$

Let us show that we may apply Lemma 10.2 to this bracket. If  $a \in [k, n]$  is a black point on  $\bar{S}$  then  $a$  is a white point on  $S$ . If additionally  $a \in [\psi(m), n]$  then  $\psi(a) - 1 \in [n, m]$ . Moreover since  $S$  is black  $(k, m)$ -regular, the point  $\psi(a) - 1$  is black on  $S$ . Hence  $a = \psi(\psi(a) - 1) - 1$  is a white point on  $\overline{\psi(S) - 1}$ . Thus the  $\overline{\psi(S) - 1}_{\psi(m)}^n \bar{S}_k^n$ -scheme has no black-black columns except the last one. The first from the left complete column is labeled by  $\nu - 1$ , where  $\nu = \max\{\psi(m), k\}$ . If  $\psi(m) < k$  then  $\psi(k)$  is black on  $S$ , see (6.8), hence  $k - 1 = \psi(\psi(k)) - 1$  is white on  $\overline{\psi(S) - 1}$ . If  $k < \psi(m)$  then  $\psi(m) - 1$  is black on  $S$ , see (6.7), hence  $\psi(m) - 1$  is white on  $\bar{S}$ . Thus in both cases the first from the left complete column is white-white. By Lemma 10.2 we may continue (10.5):

$$(10.6) \quad = \sum_{b=\nu-1}^{n-1} \alpha_b \Phi_{-}^{\bar{S}}(k, b) \cdot \overline{\Phi^{\psi(S)-1}}(\psi(m), b) \stackrel{\text{df}}{=} \sum_{b=\nu-1}^{n-1} \alpha_b w_b^{-} \cdot v_b.$$

In order to find  $[u, [v, w^{-}]]$  we would like to substitute the found value of  $[v, w^{-}]$ . However this is inhomogeneous substitution to the right factor of the bracket. Therefore we have to develop the brackets to their explicit form and analyze the coefficients. We have

$$\begin{aligned} p(u, vw^{-})p(u, w_b^{-}v_b)^{-1} &= p(u, v)p(u, v_b)^{-1}p(w, u)p(w_b, u)^{-1} \\ &= P_{k \rightarrow n, 1+n \rightarrow \psi(b)-1} P_{1+b \rightarrow n, k \rightarrow n} \\ &= P_{k \rightarrow n, 1+n \rightarrow \psi(b)-1} P_{1+n \rightarrow \psi(b)-1, k \rightarrow n} = \mu_k^{\psi(b)-1, n}, \end{aligned}$$

see definition (4.7). Relations (4.9–4.11) show that  $\mu_k^{\psi(b)-1, n} = 1$  unless  $b = k - 1$ . If  $b = k - 1$  then  $\mu_k^{\psi(b)-1, n} = \mu_k^{\psi(k), n} = q^2$ , see (4.10). Thus all brackets  $[u, w_b^{-} \cdot v_b]$  have the same coefficient as  $[u, vw^{-}]$  does with only one exception being  $b = k - 1$ .

If  $k < \psi(m)$  then of course  $b \neq k - 1$ , for  $b \geq \nu - 1 = \psi(m) - 1 \geq k$ . Hence in this case by ad-identity (2.9) the element  $[u, [v, w^{-}]]$  splits in linear combination of two sums:

$$(10.7) \quad \sum_{b=\psi(m)-1}^{n-1} \alpha_b \left[ \Phi^S(k, n), \Phi_{-}^{\bar{S}}(k, b) \right] \cdot \overline{\Phi^{\psi(S)-1}}(\psi(m), b).$$

and

$$(10.8) \quad \sum_{b=\psi(m)-1}^{n-1} \alpha_b \Phi_{-}^{\bar{S}}(k, b) \cdot \left[ \Phi^S(k, n), \overline{\Phi^{\psi(S)-1}}(\psi(m), b) \right].$$

By Lemma 10.1 we have  $[\Phi^S(k, n), \Phi_{-}^{\bar{S}}(k, b)] \sim \Phi^S(1 + b, n)$ , for  $b$  is a white point on  $\bar{S}$  (otherwise  $\alpha_b = 0$ ). In particular all terms in (10.7) belong to the positive quantum Borel subalgebra, and hence application of  $\varepsilon^{-} \otimes \varepsilon^0 \otimes \text{id}$  does not change

this sum. Application of  $\varepsilon^- \otimes \varepsilon^0 \otimes \text{id}$  to (10.8) kill all terms, for  $\varepsilon^-(\overline{\Phi^S}(k, b)) = 0$ ,  $b \geq k$ . Thus the left hand side of (10.4) takes up the form

$$(10.9) \quad \alpha \Phi^S(\psi(m), n) + \sum_{b=\psi(m)}^{n-1} \alpha_b \Phi^S(1+b, n) \cdot \overline{\Phi^{\psi(S)-1}}(\psi(m), b),$$

where  $\alpha = \alpha_{\psi(m)-1} \neq 0$ . We may decompose all terms in this expression using definition (6.1). As a result we will get a polynomial, say  $F$ , in  $u[i, j]$ ,  $1 \leq i \leq j \leq n$ . It is very important to note that all first from the left factors  $u[i, j]$  in all monomials of  $F$  satisfy  $i > \psi(m)$  with only one exception,  $\alpha u[\psi(m), n]$ , coming from the first term of (10.9). In particular  $u[i, j] < u[\psi(m), n]$  (recall that  $x_1 > x_2 > \dots > x_n$ , while words in  $X$  are ordered lexicographically). Hence further diminishing process of decomposition in PBW-basis (see [3, Lemma 7]) produces words in  $u[i, j]$ ,  $j < \psi(i)$  that start with lesser than  $u[\psi(m), n]$  elements. This means that  $\alpha u[\psi(m), n]$  is still the leading term of (10.9) after the PBW-decomposition. In particular (10.9) is not zero.

If  $\psi(m) < k$  then again by ad-identity (2.9) the element  $[u, [v, w^-]]$  splits in sums (10.7), (10.8) with  $\sum_{b=\psi(m)-1}^{n-1} \leftarrow \sum_{b=k}^{n-1}$  and an additional term that corresponds to the value  $b = k - 1$ . Since  $\alpha_{k-1} \neq 0$ , this term is proportional to

$$(10.10) \quad u \cdot v_{k-1} - p(u, vw^-) v_{k-1} \cdot u,$$

where  $v_{k-1} = \overline{\Phi^{\psi(S)-1}}(\psi(m), k-1)$  was defined in (10.6). We have already seen that  $p(u, vw^-) = p(u, v_{k-1}) q^2$ . At the same time  $p(v_{k-1}, u) p(u, v_{k-1}) = p_{k-1, k} p_{k, k-1} = q^{-2}$ , for  $p_{ij} p_{ji} = 1$ ,  $j > i+1$ , see (4.2). Hence  $p(u, v_{k-1}) q^2 = p(v_{k-1}, u)^{-1}$ . Therefore the term (10.10) is proportional to  $[v_{k-1}, u]$  with coefficient  $-p(v_{k-1}, u)^{-1}$ . Taking into account formula (6.9), we have

$$(10.11) \quad [v_{k-1}, u] = [\overline{\Phi^{\psi(S)-1}}(\psi(m), k-1), \Phi^S(k, n)] = \Phi^R(\psi(m), n),$$

where  $R = \left( \overline{\psi(S)} - 1 \cap [\psi(m), k-1] \right) \cup (S \cap [k, n])$ .

Certainly the map  $\varepsilon^- \otimes \varepsilon^0 \otimes \text{id}$  kills all terms of (10.8) with  $b \geq k$ , while Lemma 10.1 implies  $[\Phi^S(k, n), \overline{\Phi^S}(k, b)] \sim \Phi^S(1+b, n)$ ,  $b \geq k$ . Thus the left hand side of (10.4) is proportional to the sum

$$(10.12) \quad \Phi^R(\psi(m), n) + \sum_{b=k}^{n-1} \alpha'_b \Phi^S(1+b, n) \cdot \overline{\Phi^{\psi(S)-1}}(\psi(m), b).$$

This is a nonzero element precisely by the same reasons as (10.9) is.  $\square$

*Proof of Theorem 7.2.* Suppose that there exists a pair of simple roots such that one of schemes (7.1-7.4) has fragment (7.1) and no one of these schemes has form (7.7). Among all that pairs we choose a pair  $[k : m]$ ,  $[i : j]^-$  that has fragment (7.1) with minimal possible  $s - t$  on one of the schemes. Actually, due to Lemma 7.3, there are at least two of the schemes that have fragments with that minimal value of  $s - t$ . Without loss of generality, changing if necessary notations  $S \leftrightarrow S^*$  or  $T \leftrightarrow T^*$  or both, we may assume that the  $ST$ -scheme has that fragment. Since  $s - t$  is minimal, there are no white-white or black-black columns between  $t$  and  $s$ . Hence due to Theorem 8.1 we have

$$(10.13) \quad [\Phi^S(1+t, s), \Phi^T_-(1+t, s)] \sim 1 - h_{1+t \rightarrow s},$$

provided that  $S$  or, equivalently,  $T$  is  $(1+t, s)$ -regular.

Let, first,  $s \leq n$ . In this case by definition  $S$  is  $(1+t, s)$ -regular, while due to Lemma 6.14 we have  $\Phi^S(k, s) \in U^S(k, m) \subseteq U^+$ ,  $\Phi_-^T(1+t, s) \in U_-^S(i, j) \subseteq U^-$ . Hence we get

$$(10.14) \quad [\Phi^S(k, s), \Phi_-^T(1+t, s)] \in [U^+, U^-] \subseteq U.$$

At the same time by Lemma 6.8 we have a decomposition

$$(10.15) \quad \Phi^S(k, s) \sim [\Phi^S(k, t), \Phi^S(1+t, s)].$$

Lemma 2.2 implies

$$(10.16) \quad [\Phi^S(k, t), \Phi_-^T(1+t, s)] = 0.$$

Applying first (2.5), and then (2.25) with  $x_i \leftarrow \Phi^S(1+t, s)$ ,  $x_i^- \leftarrow \Phi_-^T(1+t, s)$  due to (10.13) we see that the left hand side of (10.14) is proportional to  $\Phi^S(k, t)$ , in which case the coefficient equals  $1 - \chi^{k \rightarrow t}(h_{1+t \rightarrow s}) = 1 - \mu_k^{s, t} = 1 - q^{-2}$ , see (4.7), (4.9). Thus  $\Phi^S(k, t) \in U \cap U_q^+(\mathfrak{so}_{2n+1}) = U^+$ ; that is,  $[k : t]$  is an  $U^+$ -root. According to Lemma 6.11 we have  $[1+t : m] \in \Sigma(U^S(k, m)) \subseteq \Sigma(U^+)$ . This implies that  $t = k - 1$ , for otherwise we have a contradiction:  $[k : m] = [k : t] + [1+t : m]$  is a decomposition of a simple  $U^+$ -root in  $\Sigma(U^+)$ . Similarly, due to the mirror symmetry, we have  $t = i - 1$ ; that is,  $k = i = 1 + t$ .

Now we are going to show that  $m = s$ . Equality  $t = k - 1$  implies

$$(10.17) \quad [\Phi^S(k, m), \Phi_-^T(k, s)] \in [U^+, U^-] \subseteq U.$$

Let  $s = n$ . In this case  $n$  is black on  $S$ ; that is,  $S$  is black  $(k, m)$ -regular. We have  $\varepsilon^- \otimes \varepsilon^0 \otimes \text{id}(U) \subseteq U^+$ . Hence if  $m \neq s = n$ , Lemma 10.3 allows us to find in  $U^+$  a nonzero element of degree  $[n+1 : m]$ . In particular  $[n+1 : m] \in \Sigma(U^+)$ . Hence  $[k : m] = [k : n] + [1+n : m]$  is a decomposition of a simple  $U^+$ -root in  $\Sigma(U^+)$ . A contradiction that implies  $m = n = s$ .

Let, further,  $s < n$ . If  $S$  is white  $(k, m)$ -regular, or if  $S$  is black  $(k, m)$ -regular and  $\psi(s) - 1$  is not white, then  $S$  is still  $(1+s, m)$ -regular, while Lemma 6.9 provides a decomposition

$$(10.18) \quad \Phi^S(k, m) \sim [\Phi^S(1+s, m), \Phi^S(k, s)].$$

Let us show that Theorem 9.5 implies

$$(10.19) \quad [\Phi^S(1+s, m), \Phi_-^T(k, s)] = 0.$$

To see this we have to check that  $S_{1+s}^m T_k^s$ - and  $S_{1+s}^m T_{\psi(s)}^{*\psi(k)}$ -schemes are strong. The first one has just one complete column, hence it is both strongly white and strongly black. Suppose firstly that  $S$  is white  $(k, m)$ -regular. Let us show that if  $s \neq n$  (even if  $s > n$ ), then the  $S_{1+s}^m T_{\psi(s)}^{*\psi(k)}$ -scheme is strongly white.

If  $a$  is a black point on  $T_{\psi(s)}^{*\psi(k)}$ ,  $\psi(s) \leq a < \psi(k)$ , then by definition  $\rho(a) = \psi(a) - 1$  is white on  $T_k^s$ . The inequalities  $k \leq \rho(a) < s$  imply that the point  $\rho(a)$  is intermediate for the minimal fragment (7.1), recall that now  $t = k - 1$ . Therefore  $\rho(a)$  is black on  $S$ . Since  $S$  is white  $(k, m)$ -regular, the point  $a = \rho(\rho(a))$  is not black on  $S$ . If  $a = \psi(k)$ , then still  $a$  is not black on  $S$ , see (6.6). Thus the  $S_{1+s}^m T_{\psi(s)}^{*\psi(k)}$ -scheme has no intermediate complete black-black columns.

Since  $s \neq n$ , we have  $\psi(s) \neq 1 + s$ . Hence the first from the left column is incomplete.

If there are at least two complete columns, then  $m \geq \psi(s)$ . In this case the first from the left complete column has the label  $a = \psi(s) - 1 = \rho(s)$ . Since  $s$  is black on  $S$ , and  $S$  is white  $(k, m)$ -regular, the point  $\rho(s)$  is white on  $S$ . Thus the first from the left complete column is white-white one, and  $S_{1+s}^m T_{\psi(s)}^{*\psi(k)}$ -scheme is strongly white.

Similarly we shall show that if  $S$  is black  $(k, m)$ -regular and  $\psi(s) - 1$  is not white,  $s < n$ , then the  $S_{1+s}^m T_{\psi(s)}^{*\psi(k)}$ -scheme is strongly black. If  $a$  is a white point on  $T_{\psi(s)}^{*\psi(k)}$ ,  $\psi(s) \leq a < \psi(k)$ , then by definition  $\rho(a) = \psi(a) - 1$  is black on  $T_k^s$ , and hence it is white on  $S$ . Since  $S$  is black  $(k, m)$ -regular, the point  $a = \rho(\rho(a))$  is not white on  $S$ . Thus the  $S_{1+s}^m T_{\psi(s)}^{*\psi(k)}$ -scheme has no complete white-white columns (recall that now  $\psi(s) - 1$  is not white on  $S$ , hence the column  $\psi(s) - 1$  is not white-white).

The last column is incomplete, for  $\psi(k) \neq m$ .

If there are at least two complete columns; that is,  $m \geq \psi(s)$ , then the last complete column is labeled by  $m$  or by  $\psi(k)$ . In the former case  $\psi(m) - 1$  is black on  $S$ , see (6.7). Hence, as an intermediate point for (7.1), it is white on  $T$ . Therefore  $m = \psi(\psi(m) - 1) - 1$  is black on  $T^*$ . It is still black on  $T_{\psi(s)}^{*\psi(k)}$ , for  $m \neq \psi(s) - 1$ . In the latter case  $\psi(k)$  is black on  $S$ , see (6.8). Hence it is black on  $S_{1+s}^m$  too, for  $\psi(k) \neq s$  (recall that now  $k \leq s < n$ ). Thus the  $S_{1+s}^m T_{\psi(s)}^{*\psi(k)}$ -scheme is strongly black. This completes the proof of (10.19).

Now we show how (10.13) with  $t = k - 1$  and (10.17–10.19) imply  $s = m$ . Applying first (2.5), and then (2.25) due to (10.13) we see that the left hand side of (10.17) is proportional to  $\Phi^S(1 + s, m)$ , in which case  $\chi^{1+s \rightarrow m}(h_{k \rightarrow s}) = \mu_k^{m, s} \neq 1$ , with only one exception being  $s = n$ , see (4.9–4.11). Thus  $\Phi^S(1 + s, m) \in U \cap U_q^+(\mathfrak{so}_{2n+1}) = U^+$ ; that is,  $[1 + s : m]$  is an  $U^+$ -root. According to Lemma 6.11 we have  $[k : s] \in \Sigma(U^S(k, m)) \subseteq \Sigma(U^+)$ . This implies  $s = m$ , for otherwise we have a forbidden decomposition of a simple  $U^+$ -root,  $[k : m] = [k : s] + [1 + s : m]$ .

If  $S$  is black  $(k, m)$ -regular and  $\psi(s) - 1$  is white on  $S$ , then we may not use (10.18). However in this case  $\Phi^S(k, n) \in U^S(k, m) \subseteq U^+$ , and certainly  $S$  is white  $(k, n)$ -regular. Hence instead of (10.18) we may consider the decomposition  $\Phi^S(k, n) \sim [\Phi^S(1 + s, n), \Phi^S(k, s)]$ , while instead of (10.19) use  $[\Phi^S(1 + s, n), \Phi_-^T(k, s)] = 0$ , which is valid due to Lemma 2.2. Hence we get  $[1 + s : n] \in \Sigma(U^+)$ , for now  $s < n$ . This also provides a forbidden decomposition,  $[k : m] = [k : s] + [1 + s : n] + [n + 1 : \psi(s) - 1] + [\psi(s) : m]$ , unless  $s = m$ . Here  $[n + 1 : \psi(s) - 1] = [1 + s : n]$ , while  $[\psi(s) : m] \in \Sigma(U^+)$  due to Lemma 6.11.

Thus in all cases  $s = m$ . Due to the mirror symmetry we have also  $s = j$ ; that is, the  $ST$ -scheme has the form (7.7). This contradiction completes the case “ $s \leq n$ .”

Let, then,  $n \leq t$ . By Lemma 7.3 the  $S^*T^*$ -scheme also contains a fragment (7.1) with  $t \leftarrow \psi(s) - 1$ ,  $s \leftarrow \psi(t) - 1$ . Since  $n \leq t$  implies  $\psi(t) - 1 \leq n$ , one may apply already considered case to the  $S^*T^*$ -scheme.

Let, next,  $t < n < s$ . In this case the  $n$ th column, as an intermediate one, is either white-black or black-white. Since the color of the point  $n$  defines the color of regularity,  $S$  and  $T$  have different color of regularity. For the sake of definiteness, we assume that  $S$  is white  $(k, m)$ -regular, while  $T$  is black  $(i, j)$ -regular (otherwise one may change the roles of  $S$  and  $T$  considering the mirror generators).

If  $\psi(t) - 1$  is a black point on the scheme  $S$ , then on the  $ST^*$ -scheme we have a new fragment of the form (7.1) with  $t \leftarrow n$ ,  $s \leftarrow \psi(t) - 1$ , for the color of  $\rho(t) = \psi(t) - 1$



on the scheme  $T^*$  is also black. Certainly  $\psi(t) - 1 - n = n - t < s - t$ , for  $n < s$ ; that is, we have found a lesser fragment. Hence  $\psi(t) - 1$  is not black on the scheme  $S$ . Lemma 6.12 implies  $\Phi^S(1 + t, s) \in U^+$ , while Lemma 6.6 shows that  $S$  is white  $(1 + t, s)$ -regular. In particular (10.13) is valid. Moreover  $S \cup \{t\}$  is still white  $(k, m)$ -regular, hence we have decomposition (6.9). In perfect analogy  $\psi(s) - 1$  is not white on the scheme  $T$ . Hence Lemma 6.13 implies  $\Phi_-^T(1 + t, s) \in U^-$ , and we have decomposition (6.10) of  $\Phi_-^T(i, j)$ .

By definition of a white regular set the point  $\psi(k - 1) - 1 = \psi(k)$  is not black on the scheme  $S$ , see (6.5), (6.6). Hence Lemma 6.12 implies  $\Phi^S(k, s) \in U^+$ . Therefore (10.14) is still valid. Lemma 6.8 implies decomposition (10.15), for  $\psi(t) - 1$  is not black on the scheme  $S$ . Let us show that Theorem 9.5 implies (10.16).

Indeed, the  $S_k^t T_{1+t}^s$ -scheme has just one complete column, hence it is strongly white (and of course it is strongly black too). Let us check the  $S_k^t T_{\psi(s)}^{*\psi(t)-1}$ -scheme. If  $a$  is a black point on  $S_k^t$ ,  $k \leq a < t$ , then  $\psi(a) - 1$  is not black on  $S$ , for  $(a, \psi(a) - 1)$  is a column of the shifted scheme of white  $(k, m)$ -regular set  $S$ . At the same time if  $\psi(s) \leq a < \psi(t) - 1$ , then  $s > \psi(a) - 1 > t$ . In particular  $\psi(a) - 1$  appears on the scheme  $S$ , and it is a white point on  $S$ . Further,  $\psi(a) - 1$  is an intermediate point on the minimal fragment (7.1), hence it is black on the scheme  $T$ . Therefore  $\psi(\psi(a) - 1) - 1 = a$  is a white point on  $T^*$ . Since  $a \neq \psi(t) - 1$  yet, it is a white point on  $T_{\psi(s)}^{*\psi(t)-1}$  as well. Thus the  $S_k^t T_{\psi(s)}^{*\psi(t)-1}$ -scheme has no intermediate complete black-black columns.

Consider the last column,  $a = t$ . Since  $T$  is black  $(i, j)$ -regular, and  $(t, \psi(t) - 1)$  is a column of the shifted  $T$ -scheme, the point  $\psi(t) - 1$  is not white on  $T$ . Therefore  $t = \psi(\psi(t) - 1) - 1$  is not black on  $T^*$ . It is neither black on  $T_{\psi(s)}^{*\psi(t)-1}$ , for  $t = \psi(t) - 1$  implies  $t = n$ , while now  $t < n$ .

Let  $b$  be a label of the first from the left complete column of the  $S_k^t T_{\psi(s)}^{*\psi(t)-1}$ -scheme,  $b = \max\{k - 1, \psi(s) - 1\}$ . In this case  $k - 1 \neq \psi(s) - 1$ , for  $\psi(k)$  is not black on  $S$ , see (6.6). In particular the first from the left column is incomplete.

If  $k < \psi(s)$ ,  $b = \psi(s) - 1$ , then  $(b, s)$  is a column of the shifted  $S$ -scheme. Hence  $b$  is white on  $S$ . It is still white on  $S_k^t$ , for  $\psi(s) - 1$  is not white on  $T$  in particular  $b \neq t$ . Thus, the first from the left complete column on the  $S_k^t T_{\psi(s)}^{*\psi(t)-1}$ -scheme is white-white one.

If  $k > \psi(s)$ ,  $b = k - 1$ , then due to (6.7) the point  $\psi(k)$  is white on  $S$ . We have  $t < n \leq \psi(k) < s$ ; that is,  $\psi(k)$  is an intermediate point of the fragment (7.1). Hence  $\psi(k)$  is black on  $T$ , while  $k - 1 = \psi(\psi(k)) - 1$  is white on  $T^*$ . Thus, the first from the left complete column on the  $S_k^t T_{\psi(s)}^{*\psi(t)-1}$ -scheme is still white-white one.

This proves that  $S_k^t T_{\psi(s)}^{*\psi(t)-1}$ -scheme is strongly white, and one may apply Theorem 9.5 to see that (10.16) is valid.

While considering the case “ $s \leq n$ ”, we have seen how relations (10.13–10.16) with  $\mu_k^{s,t} \neq 1$  imply  $t = k - 1$ . Here  $\mu_k^{s,t} \neq 1$  according to (4.9–4.11), for  $t \neq n$ , and  $s$ , being a black point on  $S$ , is not equal to  $\psi(k)$ . Thus  $t = k - 1$ .

Consider the  $T^* S^*$ -scheme that corresponds to the mirror generators. This scheme contains a fragment (7.1) with  $t \leftarrow \psi(s) - 1$ ,  $s \leftarrow \psi(t) - 1$ . In this case  $T^*$  is white  $(\psi(j), \psi(j))$ -regular. Therefore we may apply already proved “ $t = k - 1$ ” to that situation. We get  $\psi(s) - 1 = \psi(j) - 1$ ; that is,  $j = s$ .

Further, relations (10.17) and (10.18) are valid. While considering the case “ $s \leq n$ ”, we have seen that if  $t = k - 1$ , then the  $S_{1+s}^m T_{\psi(s)}^{*\psi(k)}$ -scheme is strongly white even if  $s > n$ . Hence Theorem 9.5 implies (10.19). At the same time we know that relations (10.17–10.19) imply  $s = m$ .

Applying this result to the  $T^*S^*$ -scheme that corresponds to the mirror generators we have  $\psi(k) = \psi(i)$ ; that is  $k = i = t - 1$ ,  $m = j = s$ . Thus,  $ST$ -scheme has the form (7.7). This contradiction completes the proof.

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