MULTIDIMENSIONAL TAUBERIAN THEOREMS FOR WAVELET AND NON-WAVELET TRANSFORMS

STEVAN PILIPOVIĆ AND JASSON VINDAS

Abstract. We provide Abelian and Tauberian theorems for regularization transforms of tempered distributions with values in Banach spaces, that is, transforms of the form $M^{\mathbf{f}}_{\varphi}(x,y)=(\mathbf{f}*\varphi_y)(x)$, where φ is a test function and $\varphi_y(\cdot)=y^{-n}\varphi(\cdot/y)$. If the first moment of φ vanishes, it is a wavelet type transform; otherwise, we say it is a non-wavelet type transform. It is shown that the scaling asymptotic properties of distributions can be completely characterized by boundary asymptotics of the wavelet and non-wavelet transforms plus natural Tauberian hypotheses. We apply our Tauberian results to the analysis of pointwise and local regularity of Banach space valued distributions. We also give applications to regularity theory within generalized function algebras, the stabilizations of solutions for a class of Cauchy's problems, for example $u_t=\Delta^{2k}u$, and Tauberian theorems for the Laplace transform; in addition, we find a necessary and sufficient condition for the existence of $f(t_0,\xi)\in\mathcal{S}'(\mathbb{R}^n_\xi)$, where $f(t,\xi)\in\mathcal{S}'(\mathbb{R}^n_t\times\mathbb{R}^n_\xi)$.

1. Introduction

The aim of this paper is to characterize scaling asymptotic properties of vector valued tempered distributions in terms of Tauberian type theorems for integral transforms arising from regularizations.

Fix $\varphi\in\mathcal{S}(\mathbb{R}^n)$ and set $\varphi_y(\cdot)=y^{-n}\varphi(\cdot/y)$, for $y>0$. To a tempered distribution f, we associate the integral transform given by

$$M^{f}_{\varphi}(x,y):=(f*\varphi_y)(x)\ ,\quad (x,y)\in\mathbb{R}^n\times\mathbb{R}_+\ ; \tag{1.1}$$

it is a C^∞-function on the upper half-plane. Such a transform has been studied by Drozhzhinov and Zavialov in [5, 6]; they named it the *standard average* of f with respect to φ. Depending whether the integral of φ vanishes or not, the transform is of wavelet or non-wavelet type. The common name we use in this article for both transforms is the regularizing transform.

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In this article we provide Abelian and Tauberian theorems related to regularizing transforms of Banach space-valued distributions in $\mathcal{S}'(\mathbb{R}^n, E)$ and $\mathcal{S}'_0(\mathbb{R}^n, E)$, (cf. [48] for vector-valued distributions). Moreover, in Section 7, we show that a locally convex space-valued distribution $\mathbf{f}$ (X-valued one) satisfying a Tauberian boundedness condition for $M^{\mathbf{f}}_{\varphi}(x, y)$, $(x, y) \in \mathbb{R}^n \times (0, \infty)$ or $(x, y) \in \mathbb{R}^n \times (0, 1)$ in the Banach space $E \subset X$, is actually an E-valued distribution up to an X valued distribution with support at zero or with a controled compact support around zero.

We follow the approach of [7] and consider quasiasymptotics as a mesuarement of scaling asymptotic properties within the quoted spaces of vector valued generalized functions: the distributional limit of $\mathbf{f}(x_0 + \varepsilon x)/c(\varepsilon) \to \mathbf{g}, \varepsilon \to 0$, or $\mathbf{f}(\lambda x)/c(\lambda) \to \mathbf{g}, \lambda \to \infty$, where $\mathbf{g}$ must be homogeneous with degree of homogeneity α as a distribution with values in E, i.e., $\mathbf{g}(at) = a^{\alpha}\mathbf{g}(t)$, for all $a \in \mathbb{R}_+$ while $c(t), t > 0$ must be of the form $t^{\alpha}L(t)$, with L slowly varying at zero or at infinity.

The precise relation between quasiasymptotics in $\mathcal{S}'_0(\mathbb{R}^n, E)$ and $\mathcal{S}'(\mathbb{R}^n, E)$ is important for the Abelian and Tauberian type results since our basic tool is actually the wavelet analysis on the space of E-valued tempered distributions $\mathcal{S}'_0(\mathbb{R}^n, E)$. Because of that, we consider also asymptotically homogeneous and associate homogeneous functions. They are intrinsically involved in the study of asymptotic properties of distributions at a finite point and at infinity. We refer to [51, 52, 53, 54, 60] (see also de Haan theory in [1]) for the properties of such functions as well as of slowly varying functions.

Let us remark that all our results are formulated for the quasiasymptotic boundedness and the quasiasymptotic behaviour of a Banach-valued distribution at a finite point and at infinity. They are related to those of Drozhzhinov and Zavialov [5, 6], and also to those of the authors and D. Rakić about the wavelet transform.

The paper is organized as follows. After introducing basic notions in Section 2, we extend in Section 3 the wavelet analysis given in [19, 20] to E-valued distributions, formulate a Tauberian boundedness condition and present examples of wavelet and non-wavelet transforms which correspond to specially chosen non-degenerate test functions φ [61]. We show that the non-wavelet type transform is involved in the notion of summability of divergent series [1, 18, 27, 70] and in a solution to certain Cauchy problems (for example $\Delta^k u = f$). Our class of non-degenerate functions φ contains Drozhzhinov-Zavialov wavelets [4, 8] as a proper subclass. We show how the Laplace transform of a distribution supported by a cone can be expressed as a ϕ-transform. Moreover, our ϕ-transform is used in embedding and characterization of distributions within certain generalized function algebras [2, 35]. Then we extend the scalar distribution wavelet analysis given in [19, 20] to E-valued distributions and prepare the ground for the vector-valued theory given in the next four sections.

Our Abelian result in Section 4 is essentially due to Drozhzhinov and Zavialov [5, 6], but we refine their results by adding some information through uniformity conditions included in the asymptotic behavior of the transform.

In Section 5 we present wavelet Tauberian results concerning quasiasymptotics in $\mathcal{S}_0(\mathbb{R}^n, E)$ through the form of a Tauberian boundedness condition on the sphere $|x|^2 + y^2 = 1, y > 0, x \in \mathbb{R}^n$. With this condition we formulate our Tauberian results for the ϕ-transform and the quasiasymptotic boundedness and quasiasymptotic behavior in the sense of the topology of $\mathcal{S}'_0(\mathbb{R}^n, E)$.

Our main results in Section 6 are the Tauberian theorems for the $\phi-$ and wavelet transforms in $\mathcal{S}'(\mathbb{R}^n, E)$. It involves associate asymptotically homogeneous and homogeneously bounded functions implying the transfer of the Tauberian theorems for the wavelet transform and the extension theorems for distributions defined on $\mathbb{R}^n \setminus \{0\}$ to distributions on $\mathbb{R}^n$.

Section 7 is devoted to *global and local class estimate* related to $\mathbf{f}$ taking values in a "broad" locally convex space which contains the narrower Banach space E, with $M^{\mathbf{f}}_{\phi}(x, y), (x, y) \in \mathbb{R}^n \times (0, \infty)$ satisfying Tauberian boundedness condition in E. In this case there exists a distribution $\mathbf{G}$ with values in the broad space such that $\operatorname{supp} \hat{\mathbf{G}} \subseteq \{0\}$ and $\mathbf{f} - \mathbf{G} \in \mathcal{S}'(\mathbb{R}^n, E)$. In case when the broad space is a normed one, $\mathbf{G}$ reduces simply to a polynomial. If the Tauberian boundedness condition holds locally, i.e. for $(x, y) \in \mathbb{R}^n \times (0, 1]$, we call it a local class estimate, then $\mathbf{f} - \mathbf{G} \in \mathcal{S}'(\mathbb{R}^n, E)$, where $\hat{\mathbf{G}}$ has compact support but its support may not be any longer the origin.

Applications of our theory are given in Section 8. Estrada's distributionally small distributions, element of the dual space for $\mathcal{K}$ (the space of symbols studied in [17]) are characterized through our Tauberian type theorems. This enables us to give, as our first application, the complete distributional asymptotic expansion of Riemann's "nondifferentiable function" at $t = 0$ and $t = 1$. Regularity properties of distributions embedded into generalized function algebra with prescribed boundedness condition are studied as our second application. We show that under certain growth condition with respect to ε for $(f * \varphi_\varepsilon)_\varepsilon, f \in \mathcal{S}'(\mathbb{R}^n)$, f is a smooth function with all derivatives being polinomially bounded. The third application is related to the asymptotic stabilization in time of solutions for Cauchy problems. Representing solutions by the use of kernels with appropriate properties, we find necessary conditions for the time stabilization for a solution of an evolution equation $u_t - P(\partial/\partial x)u = 0, u_{|t=0} = f$, with a homogeneous polynomial P. As a fourth application we present Tauberian theorems for Laplace transforms of $\mathbf{f}$, $\operatorname{supp} \mathbf{f} \subset \Gamma$ with the less restrictive conditions than those obtained in [3, 64] concerning the cone Γ. Then we apply our result for the Laplace transform and obtain a new proof of Littlewood's Tauberian theorem [18, 27, 29]. As the last application we provide a new and very short proof of the fact that the quasiasymptotics (boundedness or behaviour) of $f \in \mathcal{S}'(\mathbb{R}^n; E)$ over $\mathcal{D}(\mathbb{R}^n)$ is equivalent to the quasiasymptotics over $\mathcal{S}(\mathbb{R}^n)$.

In Section 9 we extend our result to a more general setting; all the results from Sections 3–7 hold if we replace the Banach space E by a regular injective inductive limit of an increasing limit of Banach spaces $E_n, n \in \mathbb{N}$ ([47], [28]), which have the property that a bounded set in the $\liminf_{n\to\infty}$ is actually bounded in some E_{n_0}. As an application we give a necessary and sufficient condition for a tempered distribution f on $\mathbb{R}^n_t \times \mathbb{R}^m_\xi$ to have trace at at $t = t_0$, i.e. for the existence of $f(t_0, \cdot)$ in $\mathcal{S}'(\mathbb{R}^m)$.

Applications of our results to several topics in local and microlocal analysis of Meyer, Jaffard, Holschneider and Boni [20, 26, 34] will be given in our next article.

The purpose of the Appendix is to show precise connections between quasiasymptotics in the spaces $\mathcal{S}'_0(\mathbb{R}^n, E)$ and $\mathcal{S}'(\mathbb{R}^n, E)$.

2. Notation and Preliminaries

We use the notation $\mathbb{H}^{n+1} = \mathbb{R}^n \times \mathbb{R}_+$; $\mathbb{S}^{n-1}$ is the unit sphere; $|x|$ is the Euclidean norm, $x \in \mathbb{R}^n$; $|m| = m_1 + m_2 + \cdots + m_n$, for $m \in \mathbb{N}^n$, where $\mathbb{N}$ includes 0; $\varphi^{(m)} = (\partial^{|m|}/\partial x^m)\varphi$, $m \in \mathbb{N}^n$. The space E always denotes a fixed, but arbitrary, Banach space with norm $\|\cdot\|$. If $\mathbf{a} : I \mapsto E$ and $T : I \mapsto \mathbb{R}_+$, where $I = (0, A)$ (resp. $I = (A, \infty)$) we write $\mathbf{a}(y) = o(T(y))$ as $y \to 0^+$ (resp. $y \to \infty$) if $\|\mathbf{a}(y)\| = o(T(y))$, and similarly for the big O Landau symbol; ; let $\mathbf{v} \in E$, we write $\mathbf{a}(y) \sim T(y)\mathbf{v}$ if $\mathbf{a}(y) = T(y)\mathbf{v} + o(T(y))$.

2.1. **Spaces of Distributions.** The Schwartz spaces [43] of smooth compactly supported and rapidly decreasing test functions are denoted by $\mathcal{D}(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$; their dual spaces, the scalar spaces of distributions and tempered distributions, are $\mathcal{D}'(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$. We denote by $\mathcal{E}(\mathbb{R}^n)$ the space of C^∞-functions, then $\mathcal{E}'(\mathbb{R}^n)$ is the space of compactly supported distributions. We will use the Fourier transform $\hat{\varphi}(u) = \int_{\mathbb{R}^n} \varphi(t) e^{-iu\cdot t} \mathrm{d}t$, $u \in \mathbb{R}^n$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Following [19], we define the space $\mathcal{S}_0(\mathbb{R}^n)$ of highly time-frequency localized functions over $\mathbb{R}^n$ as those elements of $\mathcal{S}(\mathbb{R}^n)$ for which all the moments vanish, i.e., $\eta \in \mathcal{S}_0(\mathbb{R}^n)$ if and only if $\int_{\mathbb{R}^n} t^m \eta(t) \mathrm{d}t = 0$, for all $m \in \mathbb{N}^n$. It is provided with the relative topology inhered from $\mathcal{S}(\mathbb{R}^n)$. This is space is also known as the Lizorkin space [30, 40], and it is invariant under Riesz potential operators. We must emphasize that $\mathcal{S}_0(\mathbb{R}^n)$ is different from the one used in [7]. The corresponding space of highly localized function over $\mathbb{H}^{n+1}$ is denoted by $\mathcal{S}(\mathbb{H}^{n+1})$. It consists of those $\Phi \in C^\infty(\mathbb{H}^{n+1})$ for which

$$\sup_{(x,y)\in\mathbb{H}^{n+1}} \left(y + \frac{1}{y}\right)^{k_1} (1 + |x|)^{k_2} \left|\frac{\partial^l}{\partial y^l} \frac{\partial^m}{\partial x^m} \Phi(x, y)\right| < \infty,$$

for all $k_1, k_2, l \in \mathbb{N}$ and $m \in \mathbb{N}^n$. The canonical topology of this space is defined in the standard way [19].

Let $\mathcal{A}(\Omega)$ be a topological vector space of test function over an open subset $\Omega \subseteq \mathbb{R}^n$. We denote by $\mathcal{A}'(\Omega, E) = L_b(\mathcal{A}(\Omega), E)$, the space of continuous linear mappings from $\mathcal{A}(\Omega)$ to E with the topology of uniform convergence over bounded subsets [50] of $\mathcal{A}(\Omega)$. We are mainly concerned with the spaces

$\mathcal{D}'(\mathbb{R}^n, E)$, $\mathcal{S}'(\mathbb{R}^n, E)$, $\mathcal{S}_0'(\mathbb{R}^n, E)$, and $\mathcal{S}'(\mathbb{H}^{n+1}, E)$; see [48] for vector-valued distributions. Let $\mathbf{f}$ be in one of these spaces of E-valued generalized functions and let φ be in the corresponding space of test functions; the value of $\mathbf{f}$ at φ will be denoted by $\langle \mathbf{f}, \varphi \rangle = \langle \mathbf{f}(t), \varphi(t) \rangle \in E$. If f is a scalar generalized function and $\mathbf{v} \in E$, we denote by $f\mathbf{v} = \mathbf{v}f$ the E-valued generalized function given by $\langle f(t)\mathbf{v}, \varphi(t) \rangle = \langle f, \varphi \rangle \mathbf{v}$. The Fourier transform of $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, E)$ is defined in the usual way, i.e., $\left\langle \hat{\mathbf{f}}(u), \varphi(u) \right\rangle = \langle \mathbf{f}(t), \hat{\varphi}(t) \rangle$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

Observe that we have a well defined continuous linear projector from $\mathcal{S}'(\mathbb{R}^n, E)$ onto $\mathcal{S}_0'(\mathbb{R}^n, E)$ as the restriction of E-valued tempered distributions to the closed subspace $\mathcal{S}_0(\mathbb{R}^n)$. It is clear that this map is surjective; however, it has no continuous right inverse [12]. We do not want to introduce a notation for this map, so if $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, E)$, we will keep calling by $\mathbf{f}$ its projection onto $\mathcal{S}_0'(\mathbb{R}^n, E)$. Note also that the kernel of this projection is the space of polynomials over $\mathbb{R}^n$ with coefficients in E (*E-valued polynomials*); therefore, $\mathcal{S}_0'(\mathbb{R}^n, E)$ can be regarded as the quotient space of $\mathcal{S}'(\mathbb{R}^n, E)$ by the space of E-valued polynomials.

If $\mathbf{f}$ is a continuous E-valued function of tempered growth on $\mathbb{R}^n$, we make the usual identification with the element $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, E)$, that is, $\langle \mathbf{f}(t), \varphi(t) \rangle := \int_{\mathbb{R}^n} \mathbf{f}(t)\varphi(t)\mathrm{d}t$. On the other hand, our convention is different for the space $\mathcal{S}'(\mathbb{H}^{n+1}, E)$. Let $\mathbf{K} \in C(\mathbb{H}^{n+1}, E)$, we say that it is of *slow growth* on $\mathbb{H}^{n+1}$ if there exist $C > 0$ and $k, l \in \mathbb{N}$ such that

$$\|\mathbf{K}(x, y)\| \leq C \left(\frac{1}{y} + y \right)^k (1 + |x|)^l, \quad (x, y) \in \mathbb{H}^{n+1};$$

we shall identify $\mathbf{K} \in \mathcal{S}'(\mathbb{H}^{n+1}, E)$ by

$$\langle \mathbf{K}(x, y), \Phi(x, y) \rangle := \int_0^\infty \int_{\mathbb{R}^n} \mathbf{K}(x, y)\Phi(x, y) \frac{\mathrm{d}x\mathrm{d}y}{y}, \quad \Phi \in \mathcal{S}(\mathbb{H}^{n+1}).$$

The choice of $y^{-1}\mathrm{d}x\mathrm{d}y$ instead of $\mathrm{d}x\mathrm{d}y$ will be clear in Subsection 3.3 below.

2.2. **Quasiasymptotics.** The quasiasymptotics [6, 7, 15, 37, 51, 52, 60, 64] measure the scaling asymptotic properties of a distribution by asymptotic comparison with Karamata regularly varying functions. Recall a measurable real valued function, defined and positive on an interval $(0, A]$ (resp. $[A, \infty)$), $A > 0$, is called *slowly varying* at the origin (resp. at infinity) [1, 44] if

$$\lim_{\varepsilon \to 0^+} \frac{L(a\varepsilon)}{L(\varepsilon)} = 1 \quad \left(\text{resp. } \lim_{\lambda \to \infty} \frac{L(a\lambda)}{L(\lambda)} = 1 \right).$$

Observe that slowly varying functions are very convenient objects to be employed in wavelet analysis since they are asymptotic invariant under rescaling at small scale (resp. large scale).

In the next definition $\mathcal{A}(\mathbb{R}^n)$ is assumed to be a space of functions for which the dilations and translations are continuous operators; consequently, these two operations can be canonically defined on $\mathcal{A}'(\mathbb{R}^n, E)$. Our interest is in $\mathcal{A} = \mathcal{D}, \mathcal{S}, \mathcal{S}_0$.

Definition 2.1. *Let $\mathbf{f} \in \mathcal{A}'(\mathbb{R}^n, E)$ and let L be slowly varying at the origin (resp. at infinity). We say that:*

(i) $\mathbf{f}$ *is quasiasymptotically bounded of degree $\alpha \in \mathbb{R}$ at the point $x_0 \in \mathbb{R}^n$ (resp. at infinity) with respect to L in $\mathcal{A}'(\mathbb{R}^n, E)$ if*

$$\sup_{0<\varepsilon\leq 1} \frac{1}{\varepsilon^{\alpha} L(\varepsilon)} \left\| \langle \mathbf{f}\left(x_0+\varepsilon t\right), \varphi(t) \rangle \right\| < \infty, \quad \textit{for each } \varphi \in \mathcal{A}(\mathbb{R}^n),$$

$$\left(\textit{resp.} \ \sup_{1\leq \lambda} \frac{1}{\lambda^{\alpha} L(\lambda)} \left\| \langle \mathbf{f}\left(\lambda t\right), \varphi(t) \rangle \right\| < \infty \right).$$

In such a case we write,

$$\mathbf{f}\left(x_0+\varepsilon t\right) = O\left(\varepsilon^{\alpha} L(\varepsilon)\right) \quad \textit{as } \varepsilon \to 0^{+} \quad \textit{in } \mathcal{A}'(\mathbb{R}^n, E)$$
$$\big(\textit{resp.} \ \mathbf{f}\left(\lambda t\right) = O\left(\lambda^{\alpha} L(\lambda)\right) \quad \textit{as } \lambda \to \infty \quad \textit{in } \mathcal{A}'(\mathbb{R}^n, E)\big).$$

(ii) $\mathbf{f}$ *has quasiasymptotic behavior of degree $\alpha \in \mathbb{R}$ at the point $x_0 \in \mathbb{R}^n$ (resp. at infinity) with respect to L in $\mathcal{A}'(\mathbb{R}^n, E)$ if there exists $\mathbf{g} \in \mathcal{A}'(\mathbb{R}^n, E)$ such that for each $\varphi \in \mathcal{A}(\mathbb{R}^n)$ the following limit holds with respect to the norm of E*

$$\lim_{\varepsilon\to 0^{+}} \frac{1}{\varepsilon^{\alpha} L(\varepsilon)} \langle \mathbf{f}\left(x_0+\varepsilon t\right), \varphi(t) \rangle = \langle \mathbf{g}(t), \varphi(t) \rangle \in E,$$

$$\left(\textit{resp.} \ \lim_{\lambda\to\infty} \frac{1}{\lambda^{\alpha} L(\lambda)} \langle \mathbf{f}\left(\lambda t\right), \varphi(t) \rangle \right).$$

We write

$$\mathbf{f}\left(x_0+\varepsilon t\right) \sim \varepsilon^{\alpha} L(\varepsilon) \mathbf{g}(t) \quad \textit{as } \varepsilon \to 0^{+} \quad \textit{in } \mathcal{A}'(\mathbb{R}^n, E) \tag{2.1}$$
$$\big(\textit{resp.} \ \mathbf{f}\left(\lambda t\right) \sim \lambda^{\alpha} L(\lambda) \mathbf{g}(t) \quad \textit{as } \lambda \to \infty \quad \textit{in } \mathcal{A}'(\mathbb{R}^n, E)\big).$$

We shall also employ the following notation for denoting the quasiasymptotic behavior (2.1)

$$\mathbf{f}\left(x_0+\varepsilon t\right) = \varepsilon^{\alpha} L(\varepsilon) \mathbf{g}(t) + o\left(\varepsilon^{\alpha} L(\varepsilon)\right) \quad \text{as } \varepsilon \to 0^{+} \quad \text{in } \mathcal{A}'(\mathbb{R}^n, E)$$
$$\big(\text{resp.} \ \mathbf{f}\left(\lambda t\right) = \lambda^{\alpha} L(\lambda) \mathbf{g}(t) + o\left(\lambda^{\alpha} L(\lambda)\right) \quad \text{as } \lambda \to \infty \quad \text{in } \mathcal{A}'(\mathbb{R}^n, E)\big),$$

which has certain advantage when considering (quasi)asymptotic expansions.

It is easy to show [15, 37, 64] that $\mathbf{g}$ in (2.1) must be homogeneous with degree of homogeneity α as a generalized function in $\mathcal{A}'(\mathbb{R}^n, E)$, i.e., $\mathbf{g}(at) = a^{\alpha}\mathbf{g}(t)$, for all $a \in \mathbb{R}_{+}$. We refer to [7] for an excellent presentation of the theory of multidimensional homogeneous distributions; such results are valid for E-valued distributions too.

Let $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, E)$ have quasiasymptotic behavior (resp. be quasiasymptotically bounded) in $\mathcal{S}'(\mathbb{R}^n, E)$, it is trivial to see that $\mathbf{f}$ has the same quasiasymptotic properties when it is seen as an element of $\mathcal{S}_0'(\mathbb{R}^n, E)$; however, the converse is in general not true. The precise relation between quasi-asymptotics in $\mathcal{S}'_0(\mathbb{R}^n, E)$ and $\mathcal{S}'(\mathbb{R}^n, E)$ will be of vital importance for our further investigations, it will be studied in detail in the Appendix A (see Propositions A.1 and A.2).

Example 2.2. *Lojasiewicz point values.* If $\mathcal{A}'(\mathbb{R}^n, E) = \mathcal{D}'(\mathbb{R}^n, E)$, $\alpha = 0$ and $\mathbf{g}(t) = \mathbf{v} \in E$, a constant E-valued distribution, in (ii) of Definition 2.1 (as $\varepsilon \to 0^+$), then we say that the *distributional point value* of $\mathbf{f}$ at x_0 is $\mathbf{v}$. We denote this by $\mathbf{f}(x_0) = \mathbf{v}$, *distributionally.* The notion of point values for distributions is due to Łojasiewicz [31, 32] (see also [16, 55, 56]).

Example 2.3. *Moment asymptotic expansions.* Let $\mathbf{f} \in \mathcal{E}'(\mathbb{R}^n, E)$, a compactly supported E-valued distribution. Then $\mathbf{f}$ satisfies the Estrada-Kanwal moment asymptotic expansion [13, 15], i.e.,

$$\mathbf{f}(\lambda t) \sim \sum_{|m|=0}^{\infty} \frac{(-1)^{|m|}}{m! \lambda^{|m|+n}} \delta^{(m)}(t) \mu_m(\mathbf{f}) \quad \text{as } \lambda \to \infty \ \text{ in } \mathcal{S}'(\mathbb{R}^n, E), \tag{2.2}$$

where $\mu_m(\mathbf{f}) = \langle \mathbf{f}(t), t^m \rangle \in E$ are the moments of $\mathbf{f}$, in the sense that if $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then, for each $N \in \mathbb{N}$,

$$\langle \mathbf{f}(\lambda t), \varphi(t) \rangle = \sum_{|m| \leq N} \frac{\varphi^{(m)}(0)}{m! \lambda^{|m|+n}} \mu_m(\mathbf{f}) + O\left(\frac{1}{\lambda^{N+n+1}}\right) \quad \text{as } \lambda \to \infty,$$

Consequently, this shows that the quasiasymptotics of distributions is not a local notion at infinity; in contrast with the the case at finite points where the notion is actually local. The moment asymptotic expansion is valid in many other important distribution spaces [15]. Distributions having an expansion of the type (2.2) are said to be *distributionally small at infinity*, we shall provide a wavelet characterization of such distributions in Subsection 8.1. We refer to [15] for the numerous and interesting applications of the moment asymptotic expansions.

3. Wavelet and Non-Wavelet Transforms of E-valued Distributions

We shall present in Subsection 3.1 some basic properties of wavelet and non-wavelet type transforms of E-valued tempered distributions. We then discuss examples in Subsection 3.2. Section 3.3 deals with wavelet analysis on the space $\mathcal{S}'_0(\mathbb{R}^n, E)$. For test functions we set $\check{\varphi}(\cdot) = \varphi(-\cdot)$ and $\varphi_y(\cdot) = y^{-n}\varphi(\cdot/y)$. The moments of φ are denoted by $\mu_m(\varphi) = \int_{\mathbb{R}^n} t^m \varphi(t) \mathrm{d}t$, $m \in \mathbb{N}^n$.

3.1. Wavelet and Non-Wavelet Transforms.

Let $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, E)$. We set, as in the introduction,

$$M^{\mathbf{f}}_{\varphi}(x, y) := (\mathbf{f} * \varphi_y)(x) \in E, \quad (x, y) \in \mathbb{H}^{n+1}, \tag{3.1}$$

the regularazing transform of $\mathbf{f}$ with respect to the test function $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Notice that $M^{\mathbf{f}}_{\varphi} \in C^{\infty}(\mathbb{H}^{n+1}, E)$.

We shall distinguish two cases of the regularazing transform.

If $\mu_0(\varphi) \neq 0$, we say that (3.1) is a *non-wavelet* type transform. Furthermore, let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\mu_0(\phi) = \int_{\mathbb{R}^n} \phi(t)\mathrm{d}t = 1$. *The $\phi-$transform* of $\mathbf{f}$ is

$$F_{\phi}\mathbf{f}(x,y) := M^{\mathbf{f}}_{\check{\phi}}(x,y) = \langle \mathbf{f}(x+yt), \phi(t)\rangle \in E, \quad (x,y) \in \mathbb{H}^{n+1}. \tag{3.2}$$

It should be observed that the $\phi-$transform essentially encloses all non-wavelet cases of (3.1) after a normalization. The terminology of $\phi-$transforms is from [16, 55, 58, 59].

The second case of (3.1) is the wavelet one, i.e., $\mu_0(\varphi) = 0$. Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ satisfy $\mu_0(\psi) = \int_{\mathbb{R}^n} \psi(t)\mathrm{d}t = 0$, we then call ψ a *wavelet. The wavelet transform* of $\mathbf{f}$ with respect to ψ is defined by

$$\mathcal{W}_{\psi}\mathbf{f}(x,y) := M^{\mathbf{f}}_{\check{\bar{\psi}}}(x,y) = \left\langle \mathbf{f}(x+yt), \bar{\psi}(t)\right\rangle \in E, \quad (x,y) \in \mathbb{H}^{n+1}. \tag{3.3}$$

In the sequel we shall restrict our attention to those wavelets which posses nice reconstruction properties (cf. Subsection 3.3 below).

Definition 3.1. *We say that the test function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ is non-degenerate if for any $\omega \in \mathbb{S}^{n-1}$ the function of one variable $R_{\omega}(r) = \hat{\varphi}(r\omega) \in C^{\infty}[0,\infty)$ is not identically zero, that is,*

$$\operatorname{supp} R_{\omega} \neq \emptyset, \quad \textit{for each } \omega \in \mathbb{S}^{n-1}.$$

We say that $\psi \in \mathcal{S}(\mathbb{R}^n)$ is a non-degenerate wavelet if it is a non-degenerate test function and additionally $\mu_0(\psi) = 0$.

Obviously, test functions for which $\mu_0(\varphi) \neq 0$ are always non-degenerate. We mention particular important cases of non-degenerate wavelets in Example 3.7.

There is a remarkable difference between the wavelet and non-wavelet transforms. Indeed, the following proposition shows such a difference. We give a quick proof of it by using Łojasiewicz point values (cf. Example 2.2), the argument is essentially the same as in [58].

Proposition 3.2. *Let $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, E)$ and let $\phi \in \mathcal{S}(\mathbb{R}^n)$, then*

$$\lim_{y \to 0^+} M^{\mathbf{f}}_{\varphi}(\cdot, y) = \mu_0(\varphi)\mathbf{f}, \quad \textit{in } \mathcal{S}'(\mathbb{R}^n, E). \tag{3.4}$$

Proof. Since $\mathcal{S}(\mathbb{R}^n)$ is a Montel space [50], the Banach-Steinhaus theorem implies that it is enough to show the convergence of (3.4) for the topology of pointwise convergence [50]. Let $\rho \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\left\langle M^{\mathbf{f}}_{\varphi}(x,y), \rho(x) \right\rangle = \left\langle \mathbf{h}(yt), \varphi(t) \right\rangle, \quad 0 < y < 1,$$

where $\mathbf{h}(u) = \langle \mathbf{f}(x), \rho(x+u)\rangle$, $u \in \mathbb{R}^n$, is a smooth E-valued function of slow growth. The Łojasewicz point value $\mathbf{h}(0)$ exists and equals the ordinary value and thus

$$\lim_{y\to 0^+}\left\langle \mathbf{h}\left(yt\right),\varphi\left(t\right)\right\rangle = \mathbf{h}\left(0\right)\int_{\mathbb{R}^n}\varphi(t)\mathrm{d}t = \mu_0(\varphi)\left\langle \mathbf{f}\left(x\right),\rho\left(x\right)\right\rangle,$$

as required. □

Since for the $\phi-$transform

$$\lim_{y\to 0^+} F_\phi f(\cdot,y) = f, \quad \text{in } \mathcal{S}'(\mathbb{R}^n),$$

the Hahn-Banach theorem implies the ensuing important corollary.

Corollary 3.3. *Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and let $\sigma > 0$. Then, the linear span of the set of the dilates (at scale less than σ) and translates of φ, that is, $\{\varphi((\,\cdot - x)/y) : (x,y) \in \mathbb{R}^n \times (0,\sigma)\}$, is dense in $\mathcal{S}(\mathbb{R}^n)$ if and only if $\mu_0(\varphi) \neq 0$.*

A property shared by the wavelet and non-wavelet transforms is the following one: They map continuously tempered distributions to smooth functions of slow growth on $\mathbb{H}^{n+1}$.

Proposition 3.4. *Let $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n,E)$ and let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then, $M^{\mathbf{f}}_{\varphi} \in C^\infty(\mathbb{H}^{n+1},E)$ is a function of slow growth on $\mathbb{H}^{n+1}$. In addition, the linear map $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n,E) \mapsto M^{\mathbf{f}}_{\varphi} \in \mathcal{S}'(\mathbb{H}^{n+1},E)$ is continuous for the topologies of uniform convergence over bounded sets. Furthermore, if $\mathfrak{B} \subset \mathcal{S}'(\mathbb{R}^n,E)$ is bounded for the topology of pointwise convergence, then there exist k,l and $C > 0$ such that*

$$\left\|M^{\mathbf{f}}_{\varphi}(x,y)\right\| \leq C\left(\frac{1}{y}+y\right)^k \left(1+|x|\right)^l, \quad \textit{for all } \mathbf{f} \in \mathfrak{B}. \tag{3.5}$$

Proof. Since $\mathcal{S}'(\mathbb{R}^n,E)$ is the inductive limit of an (strictly) increasing sequence of Banach spaces [50, 63], it is bornological. Therefore, we should show that this map takes bounded sets to bounded ones. Let $\mathfrak{B} \subset \mathcal{S}'(\mathbb{R}^n,E)$ be a bounded set. The Banach-Steinhaus theorem implies that $\mathfrak{B}$ is bounded for the topology of bounded convergence if and only if it is bounded for the topology of pointwise convergence; it is also an equicontinuous set, from where we obtain the existence of $k_1 \in \mathbb{N}$ and $C_1 > 0$ such that

$$\left\|\left\langle \mathbf{f},\rho\right\rangle\right\| \leq C_1 \sup_{t\in\mathbb{R}^n,|m|\leq k_1}\left(1+|t|\right)^{k_1}\left|\rho^{(m)}(t)\right|, \quad \text{for all } \rho \in \mathcal{S}(\mathbb{R}^n) \text{ and } \mathbf{f} \in \mathfrak{B}.$$

Consequently,

$$\begin{aligned}\left\|M^{\mathbf{f}}_{\varphi}(x,y)\right\| &= \frac{1}{y^n}\left\|\left\langle \mathbf{f}(t),\varphi\left(\frac{x-t}{y}\right)\right\rangle\right\| \\ &\leq C_1\left(\frac{1}{y}+y\right)^{n+k_1}\sup_{u\in\mathbb{R}^n,|m|\leq k_1}\left(1+|x|+y\left|u\right|\right)^{k_1}\left|\varphi^{(m)}\left(u\right)\right| \\ &\leq C\left(\frac{1}{y}+y\right)^{n+2k_1}\left(1+|x|\right)^{k_1}, \quad \text{for all } \mathbf{f} \in \mathfrak{B},\end{aligned}$$

where $C = C_1 \sup_{u\in\mathbb{R}^n, |m|\leq k_1} (1+|u|)^{k_1} \left|\varphi^{(m)}(u)\right|$. So, we obtain (3.5) with $k = n + 2k_1$ and $l = k_1$. If $\mathfrak{C} \subset \mathcal{S}(\mathbb{H}^{n+1})$ is a bounded set of test functions, we have

$$\left\|\left\langle M_{\varphi}^{\mathbf{f}}(x,y), \Phi(x,y)\right\rangle\right\| = \left\|\int_0^\infty \int_{\mathbb{R}^n} M_{\varphi}^{\mathbf{f}}(x,y)\Phi(x,y)\frac{\mathrm{d}x\mathrm{d}y}{y}\right\|$$
$$\leq C\int_0^\infty \int_{\mathbb{R}^n} \left(\frac{1}{y}+y\right)^k (1+|x|)^l \left|\Phi(x,y)\right| \frac{\mathrm{d}x\mathrm{d}y}{y},$$

which stays bounded as $\mathbf{f} \in \mathfrak{B}$ and $\Phi \in \mathfrak{C}$. Therefore, the set

$$\left\{M_{\varphi}^{\mathbf{f}} :\ \mathbf{f} \in \mathfrak{B}\right\} \subset \mathcal{S}'(\mathbb{H}^{n+1}, E)$$

is bounded, and hence the map is continuous. □

3.2. **Examples of Wavelet and Non-wavelet Transforms.** Let us discuss some examples of regularizing transforms. We shall return to these examples in Section 8 where we will provide applications of the Tauberian theorems from Section 6 and Section 7.

Our first example is one-dimensional and shows how the $\phi-$transform is related to summability of nummerical series.

Example 3.5. *The $\phi-$transform and summability of series.* Let $\{c_n\}_{n=0}^\infty$ be a sequence of complex numbers and let $\rho \in \mathcal{S}(\mathbb{R})$ with $\rho(0) = 1$. We say that the (possible divergent) series $\sum_{n=0}^\infty c_n$ is (ρ) summable to β if

$$\sum_{n=0}^{\infty} c_n\rho(yn) \ \text{ converges for all } y > 0, \tag{3.6}$$

and

$$\lim_{y\to 0^+} \sum_{n=0}^{\infty} c_n\rho(yn) = \beta. \tag{3.7}$$

One readily verifies that this summability method is regular [18], in the sense that it sums convergent series to their actual values of convergence. Furthermore, different choices of the kernel ρ lead to many familiar methods of summability. For example, if $\rho(u) = e^{-u}$ for $u > 0$, one then recovers the well known Abel method [18, 27], in such a case one writes for Abel summable series

$$\sum_{n=0}^{\infty} c_n = \beta \quad (\mathrm{A}).$$

Another instance is provided by $\rho(u) = u/(e^u - 1)$, $u > 0$, the kernel of Lambert summability which is so important in number theory [27, 70].

Assume further that $\{c_n\}_{n=0}^\infty$ is of slow growth, i.e., there is $k \in \mathbb{N}$ such that $c_n = O\left(n^k\right)$. Obviously, (3.6) is always fulfilled under this assumption. Define $f(t) = \sum_{n=0}^\infty c_n e^{itn}$, a periodic distribution over the real line.

Moreover, set $\phi = (1/2\pi)\hat{\rho}$; thus, the $\phi-$transform of f is precisely

$$F_\phi f(x,y) = \frac{1}{2\pi}\left\langle e^{ixu}\hat{f}(u), \rho(yu)\right\rangle = \sum_{n=0}^{\infty} c_n e^{ixn}\rho(yn).$$

Consequently, (3.7) becomes equivalent to a statement on the (radial) boundary behavior of the $\phi-$transform, namely,

$$\lim_{y\to 0^+} F_\phi f(0,y) = \beta.$$

In Section 8 we shall use these ideas to produce a new proof of Littlewood's Tauberian theorem for power series (Example 8.10).

Example 3.6. *Embedding of distributions into generalized function algebras.* The second important example of $\phi-$transforms points out its relation with the theory of algebras of generalized functions [2, 35]. If $\phi \in \mathcal{S}(\mathbb{R}^n)$ is a *mollifier* with all higher order vanishing moments, i.e., a test function such that

$$\mu_0(\phi) = 1 \text{ and } \mu_m(\phi) = 0, \quad \text{for all } |m| \geq 1, \tag{3.8}$$

then, for scalar distributions, the $\phi-$transform is nothing but the standard embedding of $f \in \mathcal{S}'(\mathbb{R}^n)$ into the special Colombeau algebra $\mathcal{G}(\mathbb{R}^n)$ of generalized functions (cf. Subsection 8.2), namely, the net

$$f_\varepsilon(x) = F_\phi f(x,\varepsilon), \quad 0<\varepsilon<1,\ x\in\mathbb{R}^n,$$

which determines the class $[f_\varepsilon] \in \mathcal{G}(\mathbb{R}^n)$. Likewise, the $\phi-$transform also induces the embedding of $f \in \mathcal{S}'(\mathbb{R}^n)$ into the algebra $\mathcal{G}_{\mathcal{T}}(\mathbb{R}^n)$ of tempered generalized functions [2, 35]. We will use this interpretation of the $\phi-$transform in Subsection 8.2 to give applications to regularity theory within the framework of algebras of generalized functions.

We now give examples of non-degenerate wavelets.

Example 3.7. *Drozhzhinov-Zavialov wavelets.* We say that a polynomial P is non-degenerate (at the origin) if for each $\omega \in \mathbb{S}^{n-1}$ one has that

$$P(r\omega) \not\equiv 0, \quad r \in \mathbb{R}_+.$$

Drozhzhinov and Zavialov have considered the class of wavelets $\psi \in \mathcal{S}(\mathbb{R}^n)$, $\mu_0(\psi) = 0$, for which there exists $N \in \mathbb{N}$ such that

$$T^N_{\hat{\psi}}(u) = \sum_{|m|\leq N} \frac{\hat{\psi}^{(m)}(0)u^m}{m!},$$

the Taylor polynomial of order N at the origin, is non-degenerate; these wavelets were used in [6] to obtain Tauberian theorems for distributions. It should be noticed that this type of wavelets are included in Definition 3.1; naturally, Definition 3.1 gives much more wavelets. For instance, any non-degenerate wavelet from $\mathcal{S}_0(\mathbb{R}^n)$ obviously fails to be of this kind. An explicit example of a non-degenerate wavelet $\psi \in \mathcal{S}_0(\mathbb{R}^n)$ is given in the

Fourier side by $\hat{\psi}(u) = e^{-|u|-(1/|u|)}$, $u \in \mathbb{R}^n$. Furthermore, if $\psi_1 \in \mathcal{S}(\mathbb{R}^n)$ satisfies $\hat{\psi}_1(u) = e^{-|u|-(1/|u|)} + u_1^2$ for $|u| < 1$, where $u = (u_1, u_2, \dots, u_n)$, then $\psi_1 \in \mathcal{S}(\mathbb{R}^n) \setminus \mathcal{S}_0(\mathbb{R}^n)$ is a non-degenerate wavelet but all its Taylor polynomials vanish on the axis $u_1 = 0$.

Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be a mollifier that satisfies (3.8) (cf. Example 3.6) and let P be a non-degenerate polynomial of degree k, then $\psi = P(-i\partial/\partial t)\phi$ is a wavelet of the type considered by Drozhzhinov and Zavialov; indeed, $T^k_{\hat{\psi}}(u) = P(u)$. Wavelets of the form $\psi = \Delta^d \phi$ were used in [24] to study Hölder-Zygmund regularity in algebras of generalized functions.

Example 3.8. *The $\phi-$transform as solution to Cauchy problems.* When the test function is of certain special form, the $\phi-$transform can become the solution to a PDE. We discuss a particular case in this example. Let the set $\Gamma \subseteq \mathbb{R}^n$ be a closed convex cone with vertex at the origin. In particular, we may have $\Gamma = \mathbb{R}^n$. Let P be a homogeneous polynomial of degree d such that $\Re e\, P(iu) < 0$ for all $u \in \Gamma \setminus \{0\}$. We denote [63, 64] by $\mathcal{S}'_\Gamma \subseteq \mathcal{S}'(\mathbb{R}^n)$ the subspace of distributions supported by Γ.

Consider the Cauchy problem

$$\frac{\partial}{\partial t} U(x,t) = P\left(\frac{\partial}{\partial x}\right) U(x,t), \quad \lim_{t\to 0^+} U(x,t) = f(x) \quad \text{in } \mathcal{S}'(\mathbb{R}^n), \tag{3.9}$$

$$\operatorname{supp} \hat{f} \subseteq \Gamma, \quad (x,t) \in \mathbb{H}^{n+1},$$

within the class of functions of slow growth over $\mathbb{H}^{n+1}$, that is,

$$\sup_{(x,t)\in\mathbb{H}^{n+1}} |U(x,t)| \left(t + \frac{1}{t}\right)^{-k_1} (1+|x|)^{-k_1} < \infty, \quad \text{for some } k_1, k_2 \in \mathbb{N}.$$

One readily verifies that (3.9) has a unique solution. Indeed,

$$U(x,t) = \frac{1}{(2\pi)^n} \left\langle \hat{f}(u), e^{ix\cdot u} e^{tP(iu)} \right\rangle = \frac{1}{(2\pi)^n} \left\langle \hat{f}(u), e^{ix\cdot u} e^{P\left(it^{1/d}u\right)} \right\rangle,$$

is the sought solution. We can find [64] a test function $\eta \in \mathcal{S}(\mathbb{R}^n)$ with the property $\eta(u) = e^{P(iu)}$, $u \in \Gamma$; setting $\phi = (2\pi)^{-n}\hat{\eta}$, we express U as a $\phi-$transform,

$$U(x,t) = \left\langle f(\xi), \frac{1}{t^{n/d}} \phi\left(\frac{\xi - x}{t^{1/d}}\right) \right\rangle = F_\phi f(x,y), \quad \text{with } y = t^{1/d}. \tag{3.10}$$

If $d = 2k$ is a positive even integer and $P(\xi) = (-1)^{k-1} |\xi|^d$, then we may take $\Gamma = \mathbb{R}^n$, the differential operator becomes $P(\partial/\partial x) = (-1)^{k-1}\Delta^d$, and ϕ is the Fourier inverse transform of $\eta(u) = e^{-|u|^d}$. In particular, when $d = 2$, (3.9) is the Cauchy problem for the heat equation and $\phi(\xi) = (2\sqrt{\pi})^{-n} e^{-\xi^2/4}$.

We will study in Subsection 8.3 necessary conditions for the asymptotic stabilization in time of the solution U to (3.9).

Example 3.9. *Laplace transforms as $\phi-$transforms.* Let Γ be a closed convex acute cone [63, 64] with vertex at the origin. Its conjugate cone is denoted by Γ^*. The definition of an acute cone tells us that Γ^* has non-empty interior, set $C_\Gamma = \operatorname{int}\Gamma^*$ and $T^{C_\Gamma} = \mathbb{R}^n + iC_\Gamma$. We denote by $\mathcal{S}'_\Gamma(E)$ the subspace of E-valued tempered distributions supported by Γ. Given $\mathbf{h} \in \mathcal{S}'_\Gamma(E)$, its *Laplace transform* [63] is

$$\mathcal{L}\left\{\mathbf{h}; z\right\} = \left\langle \mathbf{h}(u), e^{iz\cdot u}\right\rangle, \quad z \in T^{C_\Gamma};$$

it is a holomorphic E-valued function on the tube domain T^{C_Γ}. Fix $\omega \in C_\Gamma$. We may write $\mathcal{L}\left\{\mathbf{h}; x + i\sigma\omega\right\}$, $x \in \mathbb{R}^n$, $\sigma > 0$, as a $\phi-$transform. In fact, choose $\eta_\omega \in \mathcal{S}(\mathbb{R}^n)$ such that $\eta_\omega(u) = e^{-\omega\cdot u}$, $u \in \Gamma$; then, with $\phi_\omega = (2\pi)^{-n}\hat{\eta}_\omega$ and $\hat{\mathbf{f}} = (2\pi)^n\mathbf{h}$,

$$\mathcal{L}\left\{\mathbf{h}; x + i\sigma\omega\right\} = F_{\phi_\omega}\mathbf{f}(x,\sigma). \tag{3.11}$$

Notice that this is a particular case of Example 3.8 with $P_\omega(\xi) = i\omega\cdot\xi$.

3.3. **Wavelet Analysis on $\mathcal{S}'_0(\mathbb{R}^n, E)$.** In this subsection we briefly sketch how to extend the scalar distribution wavelet analysis given in [19] to E-valued generalized functions. We complement the theory with some new results.

Although Proposition 3.2 makes impossible to recover an E-valued tempered distribution as the boundary value of its wavelet transform, the non-degenerate wavelets from $\mathcal{S}_0(\mathbb{R}^n)$ enjoy excellent reconstruction properties as long as we are interested in the projection of the tempered distribution onto $\mathcal{S}'_0(\mathbb{R}^n, E)$. Observe that if the wavelet belongs to $\mathcal{S}_0(\mathbb{R}^n)$, the wavelet transform with respect to this wavelet is continuous $\mathcal{S}'(\mathbb{R}^n, E) \mapsto \mathcal{S}'(\mathbb{H}^{n+1}, E)$, as can be inferred from Proposition 3.4; however it is not injective, since it maps every E-valued polynomial to $\mathbf{0}$, as follows from the moment vanishing properties of the wavelet. This fact makes necessary to work on $\mathcal{S}'_0(\mathbb{R}^n, E)$ if one wishes to have reconstruction of distributions from their wavelet transforms.

Let $\psi \in \mathcal{S}_0(\mathbb{R}^n)$. We have that [19, Thm 19.0.1] $\mathcal{W}_\psi : \mathcal{S}_0(\mathbb{R}^n) \mapsto \mathcal{S}(\mathbb{H}^{n+1})$ is a continuous linear map. We are interested in those wavelets for which $\mathcal{W}_\psi$ admits a left inverse. For wavelet-based reconstruction, we shall use the wavelet synthesis operator [19]. Given $\Phi \in \mathcal{S}(\mathbb{H}^{n+1})$, we define the *wavelet synthesis operator* with respect to the wavelet ψ as

$$\mathcal{M}_\psi\Phi(t) = \int_0^\infty \int_{\mathbb{R}^n} \Phi(x,y)\frac{1}{y^n}\psi\left(\frac{t-x}{y}\right)\frac{\mathrm{d}x\mathrm{d}y}{y}\,, \quad t \in \mathbb{R}^n. \tag{3.12}$$

One can show that $\mathcal{M}_\psi : \mathcal{S}(\mathbb{H}^{n+1}) \to \mathcal{S}_0(\mathbb{R}^n)$ is continuous [19, p. 74].

We shall say that the wavelet $\psi \in \mathcal{S}_0(\mathbb{R}^n)$ admits a *reconstruction wavelet* if there exists $\eta \in \mathcal{S}_0(\mathbb{R}^n)$ such that

$$c_{\psi,\eta}(\omega) = \int_0^\infty \overline{\hat{\psi}}(r\omega)\hat{\eta}(r\omega)\frac{\mathrm{d}r}{r}\,, \quad \omega \in \mathbb{S}^{n-1}, \tag{3.13}$$

is independent of the direction ω; in such a case we set $c_{\psi,\eta} := c_{\psi,\eta}(\omega)$. The wavelet η is called a reconstruction wavelet for ψ.

It is easy to find explicit examples of wavelets admitting reconstruction wavelets; in fact, any non-trivial rotation invariant element of $\mathcal{S}_0(\mathbb{R}^n)$ is itself its own reconstruction wavelet.

If ψ is admits the reconstruction wavelet η, one has the reconstruction formula [19] for the wavelet transform on $\mathcal{S}_0(\mathbb{R}^n)$

$$\mathrm{Id}_{\mathcal{S}_0(\mathbb{R}^n)} = \frac{1}{c_{\psi,\eta}}\mathcal{M}_\eta \mathcal{W}_\psi. \tag{3.14}$$

We now characterize those wavelets which have a reconstruction wavelet. Actually, the class of non-degenerate wavelets from $\mathcal{S}_0(\mathbb{R}^n)$ (cf. Definition 3.1) coincides with the class of wavelets admitting reconstruction wavelets.

Proposition 3.10. *Let $\psi \in \mathcal{S}_0(\mathbb{R}^n)$. Then, ψ admits a reconstruction wavelet if and only if it is non-degenerate.*

Proof. The necessity is clear, for if $\hat{\psi}(rw_0)$ identically vanishes in the direction of $w_0 \in \mathbb{R}$, then $c_{\psi,\eta}(w_0) = 0$ (cf. (3.13)) for any $\eta \in \mathcal{S}_0(\mathbb{R}^n)$.

Suppose now that ψ is non-degenerate, we will construct a reconstruction wavelet for it. As in (3.13), we write $c_{\psi,\psi}(\omega) = \int_0^\infty |\hat{\psi}(r\omega)|^2 (\mathrm{d}r/r) > 0$, $\omega \in \mathbb{S}^{n-1}$. Set

$$\varrho(r,w) = \frac{\hat{\psi}(rw)}{c_{\psi,\psi}(w)}\ , \quad (r,w) \in [0,\infty) \times \mathbb{S}^{n-1};$$

obviously, if we prove that $\varrho(|u|, u/|u|) \in \mathcal{S}(\mathbb{R}^n)$ and all its partial derivatives vanish at the origin, then η given by $\hat{\eta}(u) = \varrho(|u|, u/|u|)$ will be a reconstruction wavelet for ψ; actually, $c_{\psi,\eta} = 1$. By the characterization theorem for polar coordinates of test functions from $\mathcal{S}(\mathbb{R}^n)$ [7, Prop. 1.1], the fact $\hat{\eta} \in \mathcal{S}(\mathbb{R}^n)$ is a consequence of the relations

$$\left(\frac{\partial}{\partial r}\right)^k \varrho(r,\omega)\bigg|_{r=0} = 0, \quad k = 0, 1, \dots\ ;$$

the same relations show that all partial derivatives of $\hat{\eta}$ vanish at the origin, and hence $\eta \in \mathcal{S}_0(\mathbb{R}^n)$. □

In [19], (3.14) was extended to $\mathcal{S}_0'(\mathbb{R}^n)$ via duality arguments, the main step being the formula

$$\int_0^\infty \int_{\mathbb{R}^n} \mathcal{W}_\psi f(x,y)\Phi(x,y)\frac{\mathrm{d}x\mathrm{d}y}{y} = \left\langle f(t), \mathcal{M}_{\bar{\psi}}\Phi\ (t)\right\rangle,$$

valid for $\Phi \in \mathcal{S}(\mathbb{H}^{n+1})$ and $f \in \mathcal{S}_0'(\mathbb{R}^n)$. It can be easily extended to the E-valued case, as the next proposition shows.

Proposition 3.11. *Let $\mathbf{f} \in \mathcal{S}_0'(\mathbb{R}^n, E)$ and $\psi \in \mathcal{S}_0(\mathbb{R}^n)$. Then*

$$\int_0^\infty \int_{\mathbb{R}^n} \mathcal{W}_\psi \mathbf{f}(x,y)\Phi(x,y)\frac{\mathrm{d}x\mathrm{d}y}{y} = \left\langle \mathbf{f}(t), \mathcal{M}_{\bar{\psi}}\Phi\ (t)\right\rangle, \quad \Phi \in \mathcal{S}(\mathbb{H}^{n+1}). \tag{3.15}$$

Proof. The same argument used in Proposition 3.4 shows that $\mathcal{W}_\psi : \mathcal{S}_0'(\mathbb{R}^n, E) \mapsto \mathcal{S}'(\mathbb{H}^{n+1}, E)$ is continuous. The linear map $T : \mathcal{S}_0'(\mathbb{R}^n, E) \mapsto \mathcal{S}'(\mathbb{H}^{n+1}, E)$ given by

$$\langle (T\mathbf{f})(x,y), \Phi(x,y)\rangle = \left\langle \mathbf{f}(t), \mathcal{M}_{\bar{\psi}}\Phi\,(t)\right\rangle,$$

is continuous as well. Thus, if we show that $\mathcal{W}_\psi$ and T coincide on a dense subset of $\mathcal{S}_0'(\mathbb{R}^n, E)$, we would have (3.15). The nuclearity [50] of $\mathcal{S}_0'(\mathbb{R}^n)$ implies that $\mathcal{S}_0'(\mathbb{R}^n) \otimes E \subset \mathcal{S}_0'(\mathbb{R}^n, E)$ is dense; thus, it is enough to verify (3.15) for $\mathbf{f} = f\mathbf{v}$, where $f \in \mathcal{S}_0'(\mathbb{R}^n)$ and $\mathbf{v} \in E$. Now, the scalar case implies

$$\begin{aligned}\langle \mathcal{W}_\psi(f\mathbf{v})(x,y), \Phi(x,y)\rangle &= \langle \mathcal{W}_\psi f(x,y), \Phi(x,y)\rangle\,\mathbf{v} = \left\langle f(t), \mathcal{M}_{\bar{\psi}}\Phi\,(t)\right\rangle \mathbf{v}\\ &= \left\langle f(t)\mathbf{v}, \mathcal{M}_{\bar{\psi}}\Phi\,(t)\right\rangle,\end{aligned}$$

as required. □

We now extend the definition of the wavelet synthesis operator (3.12) to $\mathcal{S}_0'(\mathbb{H}^{n+1}, E)$. Let $\mathbf{K} \in \mathcal{S}_0'(\mathbb{H}^{n+1}, E)$, we define $\mathcal{M}_\psi : \mathcal{S}_0'(\mathbb{H}^{n+1}, E) \mapsto \mathcal{S}_0'(\mathbb{R}^n, E)$, a continuous linear map, as

$$\langle \mathcal{M}_\psi \mathbf{K}(t), \rho(t)\rangle = \left\langle \mathbf{K}(x,y), \mathcal{W}_{\bar{\psi}}\rho(x,y)\right\rangle, \quad \rho \in \mathcal{S}_0(\mathbb{R}^n).$$

So, we have the ensuing reconstruction formula for the wavelet transform.

Proposition 3.12. *Let $\psi \in \mathcal{S}_0(\mathbb{R}^n)$ be non-degenerate and let $\eta \in \mathcal{S}_0(\mathbb{R}^n)$ be a reconstruction wavelet for it. Then,*

$$\mathrm{Id}_{\mathcal{S}_0'(\mathbb{R}^n, E)} = \frac{1}{c_{\psi,\eta}} \mathcal{M}_\eta \mathcal{W}_\psi. \tag{3.16}$$

Furthermore, we have the desingularization formula,

$$\langle \mathbf{f}(t), \rho(t)\rangle = \frac{1}{c_{\psi,\eta}} \int_0^\infty \int_{\mathbb{R}^n} \mathcal{W}_\psi \mathbf{f}(x,y) \mathcal{W}_{\bar{\eta}}\rho(x,y) \frac{\mathrm{d}x\mathrm{d}y}{y}, \tag{3.17}$$

for all $\mathbf{f} \in \mathcal{S}_0'(\mathbb{R}^n, E)$ and $\rho \in \mathcal{S}_0(\mathbb{R}^n)$.

Proof. We apply the definition of $\mathcal{M}_\eta$, Proposition 3.11 and (3.14), and use the fact that $c_{\psi,\eta} = c_{\bar{\eta},\bar{\psi}}$,

$$\begin{aligned}\frac{1}{c_{\psi,\eta}} \langle \mathcal{M}_\eta \mathcal{W}_\psi \mathbf{f}(t), \rho(t)\rangle &= \frac{1}{c_{\psi,\eta}} \int_0^\infty \int_{\mathbb{R}^n} \mathcal{W}_\psi \mathbf{f}(x,y) \mathcal{W}_{\bar{\eta}}\rho(x,y) \frac{\mathrm{d}x\mathrm{d}y}{y}\\ &= \frac{1}{c_{\psi,\eta}} \langle \mathcal{W}_\psi \mathbf{f}(x,y), \mathcal{W}_{\bar{\eta}}\rho(x,y)\rangle\\ &= \left\langle \mathbf{f}, \frac{1}{c_{\bar{\eta},\bar{\psi}}} \mathcal{M}_{\bar{\psi}} \mathcal{W}_{\bar{\eta}}\rho \right\rangle\\ &= \langle \mathbf{f}(t), \rho(t)\rangle,\end{aligned}$$

so both (3.16) and (3.17) have been established. □

The next result provides a second characterization of non-degenerate wavelets from $\mathcal{S}_0(\mathbb{R}^n)$, a corollary of the inversion formula.

Corollary 3.13. *Let $\psi \in \mathcal{S}_0(\mathbb{R}^n)$. Then, the linear span of the set of dilates and translates of $\bar{\psi}$, $\left\{\bar{\psi}((\,\cdot - x)/y) : (x,y) \in \mathbb{H}^{n+1}\right\}$, is dense in $\mathcal{S}_0(\mathbb{R}^n)$ if and only if ψ is a non-degenerate wavelet.*

Proof. The direct implication is a consequence of the Hahn-Banach theorem and the inversion formula (Proposition 3.12). On the other hand, suppose that there is $\omega_0 \in \mathbb{S}^{n-1}$ such that $\hat{\psi}(r\omega_0) = 0$ for all $r \in \mathbb{R}_+$. Let $f \in \mathcal{S}'(\mathbb{R}^n)$ be the distribution whose Fourier transform is given by $\langle \hat{f}, \rho \rangle = \int_0^\infty \rho(r\omega_0)\mathrm{d}r$, then $\mathcal{W}_\psi f(x,y) = 0$, for all $(x,y) \in \mathbb{H}^{n+1}$, which implies that f identically vanishes on the clousure of the linear span of the dilates and translates of $\bar{\psi}$. This yields the converse. □

In analogy to [19, Thm. 28.0.1], we can characterize the bounded sets of $\mathcal{S}_0'(\mathbb{R}^n, E)$. One can also characterize some types of convergent nets. The next propositions will be very important for the subsequent sections.

Proposition 3.14. *Let $\psi \in \mathcal{S}_0(\mathbb{R}^n)$ be a non-degenerate wavelet. A necessary and sufficient condition for a set $\mathfrak{B} \subset \mathcal{S}_0'(\mathbb{R}^n, E)$ to be bounded for the topology of pointwise convergence (or bounded convergence) of $\mathcal{S}_0'(\mathbb{R}^n, E)$ is the existence of $k, l \in \mathbb{N}$ and $C > 0$ such that*

$$\|\mathcal{W}_\psi \mathbf{f}(x,y)\| \leq C \left(\frac{1}{y} + y\right)^k (1 + |x|)^l, \quad \text{for all } \mathbf{f} \in \mathfrak{B}. \tag{3.18}$$

Proof. The necessity can be established as in the proof of Proposition 3.4. For the sufficiency, we only need to show the boundedness of $\mathfrak{B}$ for the topology of pointwise convergence [50], in view of the Banach-Steinhaus theorem. Let η be a reconstruction wavelet for ψ. Let $\rho \in \mathcal{S}_0(\mathbb{R}^n)$, by the wavelet desigularization formula (cf. Proposition 3.12) and (3.18),

$$\|\langle \mathbf{f}, \rho \rangle\| \leq \frac{C}{c_{\psi,\eta}} \int_0^\infty \int_{\mathbb{R}^n} \left(\frac{1}{y} + y\right)^k (1 + |x|)^l \,|\mathcal{W}_{\bar{\eta}} \rho(x,y)|\, \frac{\mathrm{d}x\mathrm{d}y}{y}\,,$$

and the last quantity is uniformly bounded for $\mathbf{f} \in \mathfrak{B}$ since $\mathcal{W}_{\bar{\eta}}\rho \in \mathcal{S}(\mathbb{H}^{n+1})$. This completes the proof. □

Proposition 3.15. *Let $\psi \in \mathcal{S}_0(\mathbb{R}^n)$ be a non-degenerate wavelet. Necessary and sufficient conditions for the net $\{\mathbf{f}_\lambda\}_{\lambda \in \mathbb{R}_+}$ to be convergent ($\lambda \to \infty$), for the topology of pointwise convergence (or bounded convergence) of $\mathcal{S}_0'(\mathbb{R}^n, E)$, are the existence of the limits (with respect to the norm of E)*

$$\lim_{\lambda \to \infty} \mathcal{W}_\psi \mathbf{f}_\lambda(x,y), \quad \text{for each } (x,y) \in \mathbb{H}^{n+1}, \tag{3.19}$$

and the existence of $k, l \in \mathbb{N}$ and $C, \lambda_0 > 0$ such that

$$\|\mathcal{W}_\psi \mathbf{f}_\lambda(x,y)\| \leq C \left(\frac{1}{y} + y\right)^k (1 + |x|)^l, \quad \text{for all } \lambda_0 \leq \lambda. \tag{3.20}$$

In such a case, the limit generalized function $\mathbf{h} = \lim_{\lambda\to\infty} \mathbf{f}_\lambda$ *satisfies*

$$\mathcal{W}_\psi \mathbf{h}(x,y) = \lim_{\lambda\to\infty} \mathcal{W}_\psi \mathbf{f}_\lambda(x,y),$$

uniformly over compact subsets of $\mathbb{H}^{n+1}$.

Proof. By Proposition 3.14, (3.20) is itself equivalent to the boundedness of $\{\mathbf{f}_\lambda\}$ for large values of λ, which in turn is equivalent to the equicontinuity of the set for large values of λ (Banach-Steinhaus theorem). Because of the standard result [50, p. 356], the convergence of $\{\mathbf{f}_\lambda\}_{\lambda\in\mathbb{R}_+}$ is then equivalent to the pointwise convergence of the net of linear mappings over a dense subset of $\mathcal{S}_0(\mathbb{R}^n)$. But (3.19) gives precisely this convergence over the linear span of $\left\{\bar{\psi}((\,\cdot - x)/y) : (x,y) \in \mathbb{H}^{n+1}\right\}$, which is actually dense (Corollary 3.13). The last property follows by the definition of convergence in $\mathcal{S}_0'(\mathbb{R}^n, E)$, since if $K \subset \mathbb{H}^n$ is a compact set, then $\left\{y^{-n}\bar{\psi}((\,\cdot - x)/y) : (x,y) \in K\right\}$ is compact in $\mathcal{S}_0(\mathbb{R}^n)$. □

4. Abelian Results

We present in this section an Abelian proposition for the transform $M_\varphi^{\mathbf{f}}$. Its Tauberian counterparts will be the main subject of the next two sections. This Abelian result is essentially due to Drozhzhinov and Zavialov [5, 6] (cf. [59, 61]), but we refine their results by adding some information about uniformity in the asymptotics. Let $x_0 \in \mathbb{R}^n$ and $0 \le \vartheta < \pi/2$, we denote by $C_{x_0,\vartheta}$ the cone of angle ϑ in $\mathbb{H}^{n+1}$ with vertex at x_0, namely,

$$C_{x_0,\vartheta} = \left\{(x,y) \in \mathbb{H}^{n+1} : |x - x_0| \le (\tan\vartheta) y\right\}.$$

Proposition 4.1. *Let L be slowly varying at the origin (resp. at infinity) and let* $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, E)$.

(i) *Assume that* $\mathbf{f}$ *is quasiasymptotically bounded of degree α at the point x_0 (resp. at infinity) with respect to L in $\mathcal{S}'(\mathbb{R}^n, E)$. Then, there exist $k, l \in \mathbb{N}$, $C > 0$ and $\varepsilon_0 > 0$ (resp. $\lambda_0 > 1$) such that for all $(x,y) \in \mathbb{H}^{n+1}$*

$$\left\|M_\varphi^{\mathbf{f}}(x_0 + \varepsilon x, \varepsilon y)\right\| \le C\varepsilon^\alpha L(\varepsilon)\left(\frac{1}{y} + y\right)^k (1 + |x|)^l, \quad 0 < \varepsilon \le \varepsilon_0, \tag{4.1}$$

$$\left(\textit{resp.} \ \left\|M_\varphi^{\mathbf{f}}(\lambda x, \lambda y)\right\| \le C\lambda^\alpha L(\lambda)\left(\frac{1}{y} + y\right)^k (1 + |x|)^l, \quad \lambda_0 \le \lambda\right).$$

(ii) *If* $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, E)$ *has the quasiasymptotic behavior* $\mathbf{f}(x_0 + \varepsilon t) \sim \varepsilon^\alpha L(\varepsilon)\mathbf{g}(t)$ *as* $\varepsilon \to 0^+$ *(resp.* $\mathbf{f}(\lambda t) \sim \lambda^\alpha L(\lambda)\mathbf{g}(t)$ *as* $\lambda \to \infty$*) in* $\mathcal{S}'(\mathbb{R}^n, E)$, *and if* $0 \le \vartheta < \pi/2$, *then*

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in C_{0,\vartheta}}} |(x,y)|^{-\alpha}\left\|\frac{1}{L(|(x,y)|)} M_\varphi^{\mathbf{f}}(x_0 + x, y) - M_\varphi^{\mathbf{g}}(x,y)\right\| = 0 \tag{4.2}$$

$$\left(resp. \lim_{\substack{|(x,y)|\to\infty \\ (x,y)\in C_{0,\vartheta}}} |(x,y)|^{-\alpha} \left\| \frac{1}{L\left(|(x,y)|\right)} M_{\varphi}^{\mathbf{f}}(x,y) - M_{\varphi}^{\mathbf{g}}\left(x,y\right) \right\| = 0 \right);$$

in particular, for each fixed $(x,y)\in\mathbb{H}^{n+1}$,

$$\lim_{\varepsilon\to 0^{+}} \frac{1}{\varepsilon^{\alpha} L\left(\varepsilon\right)} M_{\varphi}^{\mathbf{f}}(x_0+\varepsilon x,\varepsilon y) = M_{\varphi}^{\mathbf{g}}\left(x,y\right) \quad in\ E \tag{4.3}$$

$$\left(resp. \lim_{\lambda\to\infty} \frac{1}{\lambda^{\alpha} L\left(\lambda\right)} M_{\varphi}^{\mathbf{f}}(\lambda x,\lambda y) = M_{\varphi}^{\mathbf{g}}(x,y) \right).$$

Proof. The estimate (4.1) from Part (i) follows immediately from Proposition 3.4 by considering the bounded set

$$\left\{ \frac{1}{\varepsilon^{\alpha}L(\varepsilon)}\mathbf{f}(x_0+\varepsilon\,\cdot\,) : 0<\varepsilon\le 1 \right\} \quad \left(\text{resp. } \left\{ \frac{1}{\lambda^{\alpha}/L(\lambda)}\,\mathbf{f}(\lambda\,\cdot\,) : 1\le\lambda \right\} \right).$$

For (ii), we may assume that $x_0=0$. Next, observe that $(x,y)\in C_{0,\vartheta}$ can be written as $x=r\xi$ and $y=r\cos\theta$, where $r>0$, $\xi\in\mathbb{R}^n$, $|\xi|=\sin\theta$ and $0\le\theta\le\vartheta$. So,

$$M_{\varphi}^{\mathbf{f}}(r\xi, r\cos\theta) = \left\langle \mathbf{f}(rt), \frac{1}{(\cos\theta)^n}\varphi\left(\frac{\xi-t}{\cos\theta}\right) \right\rangle.$$

Since the subset $\mathfrak{C}=\{(1/\cos\theta)\varphi\left((\xi-\ \cdot\ )/\cos\theta\right) : 0\le\theta\le\vartheta\}$ is a compact set in $\mathcal{S}(\mathbb{R}^n)$, the Banach-Steinhaus theorem implies that the quasiasymptotic behavior of $\mathbf{f}$ holds uniformly when evaluated at test functions of $\mathfrak{C}$. Then, as $r\to 0^{+}$ (resp. $r\to\infty$),

$$\frac{1}{r^{\alpha}L(r)}M_{\varphi}^{\mathbf{f}}(r\xi, r\cos\theta) \to \left\langle \mathbf{g}(t), \frac{1}{(\cos\theta)^n}\varphi\left(\frac{\xi-t}{\cos\theta}\right) \right\rangle = M_{\varphi}^{\mathbf{g}}(\xi,\cos\theta),$$

uniformly in $|\xi|=\sin\theta$ and $0\le\theta\le\vartheta$. Thus, we have shown (4.2). On the other hand, if again $x=r\xi$ and $y=r\cos\theta$, where r,ξ and θ are fixed, we have that, as $h\to 0^{+}$ (resp. $h\to\infty$),

$$\begin{aligned} M_{\varphi}^{\mathbf{f}}(hx,hy) &\sim (rh)^{\alpha}L(hr)\left\langle \mathbf{g}(t), \frac{1}{(\cos\theta)^n}\varphi\left(\frac{\xi-t}{\cos\theta}\right) \right\rangle \\ &= h^{\alpha}L(hr)\left\langle \mathbf{g}(rt), \frac{1}{(\cos\theta)^n}\varphi\left(\frac{\xi-t}{\cos\theta}\right) \right\rangle \sim h^{\alpha}L(h)M_{\varphi}^{\mathbf{g}}(x,y), \quad \text{in } E, \end{aligned}$$

because of the homogeneity of $\mathbf{g}$ and the fact that L is slowly varying. Hence, (4.3) has been proved. □

5. Wavelet Tauberian Characterization of Quasiasymptotics in $\mathcal{S}_0'(\mathbb{R}^n, E)$

The purpose of this section is to characterize the quasiasymptotic behavior in the space $\mathcal{S}_0'(\mathbb{R}^n,E)$ in terms of the asymptotic behavior of the wavelet transform with respect to a non-degenerate wavelet from $\mathcal{S}_0(\mathbb{R}^n)$. Our characterization is of Tauberian character and it is related to (4.1) and

(4.3) for the wavelet transform. We begin with a preliminary proposition which shows that conditions (4.1) and (4.3) are equivalent to (apparently) weaker ones.

Proposition 5.1. *Let* $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, E)$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$, *and let* L *be slowly varying at the origin (resp. at infinity). Then,*

(i) *The estimate (4.1) is equivalent to one of the form (k may be a different exponent)*

$$\limsup_{\varepsilon \to 0^+} \sup_{|x|^2+y^2=1,\, y>0} \frac{y^k}{\varepsilon^\alpha L(\varepsilon)} \left\| M^{\mathbf{f}}_\varphi (x_0 + \varepsilon x, \varepsilon y) \right\| < \infty \tag{5.1}$$

$$\left(\textit{resp. } \limsup_{\lambda \to \infty} \sup_{|x|^2+y^2=1,\, y>0} \frac{y^k}{\lambda^\alpha L(\lambda)} \left\| M^{\mathbf{f}}_\varphi (\lambda x, \lambda y) \right\| < \infty \right).$$

(ii) *If*

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^\alpha L(\varepsilon)} M^{\mathbf{f}}_\varphi (x_0 + \varepsilon x, \varepsilon y) = M_{x,y} \in E \tag{5.2}$$

$$\left(\textit{resp. } \lim_{\lambda \to \infty} \frac{1}{\lambda^\alpha L(\lambda)} M^{\mathbf{f}}_\varphi (\lambda x, \lambda y) = M_{x,y} \in E \right)$$

exists for each $(x, y) \in \mathbb{H}^{n+1} \cap \mathbb{S}^n$, *then it exists for all* $(x, y) \in \mathbb{H}^{n+1}$.

Proof. By translating, we may assume that $x_0 = 0$.

Part (i). We only need to showthat (5.1) implies (4.1). Our assumption is that there are constants $C_1, h_0 > 0$ such that

$$\left\| M^{\mathbf{f}}_\varphi (h\xi, h\cos\vartheta) \right\| < \frac{C_1}{(\cos\vartheta)^k} h^\alpha L(h),$$

$|\xi|^2 + (\cos\vartheta)^2 = 1$ and $0 < h \le h_0$ (resp. $h_0 \le h$). We can assume that $1 + |\alpha| \le k$ and $h_0 < 1$ (resp. $1 < h_0$). Potter's estimate [1, p. 25] implies that we may assume that

$$\frac{L(hr)}{L(h)} < C_2 \frac{(1+r)^2}{r}, \quad \text{for } h, hr \in (0, h_0] \ \ (\text{resp. } h, hr \in [h_0, \infty)\,). \tag{5.3}$$

In addition, since $1/L(h) = o(h^{-1})$ as $h \to 0^+$ (resp. $1/L(h) = o(h)$, as $h \to \infty$) [1, 44], we can assume

$$\frac{1}{L(h)} < \frac{C_3}{h}, \quad \text{for } 0 < h \le h_0 \quad \left(\text{resp. } \frac{1}{L(h)} < C_3 h, \quad \text{for } h_0 \le h \right). \tag{5.4}$$

After this preparation, we are ready to give the proof. For $(x, y) \in \mathbb{H}^{n+1}$ write $x = r\xi$ and $y = r\cos\vartheta$, with $r = |(x, y)|$. We always keep $h \le h_0$

(resp. $h_0 \leq h$). If $rh \leq h_0$ (resp. $h_0 \leq rh$), we have that

$$\begin{aligned}\left\|M_{\varphi}^{\mathbf{f}}\left(hr\xi, hr\cos\vartheta\right)\right\| &< \frac{C_1}{y^k} h^{\alpha} L(hr) r^{\alpha+k} < C_1 C_2 h^{\alpha} L(h) \frac{(1+r)^{\alpha+k+1}}{y^k} \\ &< C_4 h^{\alpha} L(h) \left(\frac{1}{y}+y\right)^{\alpha+2k+1} (1+|x|)^{\alpha+k+1},\end{aligned}$$

with $C_4 = 2^{\alpha+k+1} C_1 C_2$. We now analyze the case $h_0 < hr$ (resp. $hr < h_0$). Proposition 3.4 implies the existence of $k_1, l_1 \in \mathbb{N}$, $k_1 \geq k$, and C_5 such that

$$\begin{aligned}\left\|M_{\varphi}^{\mathbf{f}}\left(hx, hy\right)\right\| &< C_5 \left(\frac{1}{hy} + hy\right)^{k_1} (1+h|x|)^{l_1} \\ &< C_5 h^{\alpha} L(h) \left(\frac{1}{y}+y\right)^{k_1} (1+|x|)^{l_1} \frac{1}{h^{\alpha+k_1} L(h)} \\ &\left(\text{resp. } < C_5 h^{\alpha} L(h) \left(\frac{1}{y}+y\right)^{k_1} (1+|x|)^{l_1} \frac{h^{k_1+l_1}}{h^{\alpha} L(h)}\right) \\ &< C_3 C_5 h^{\alpha} L(h) \left(\frac{1}{y}+y\right)^{k_1} (1+|x|)^{l_1} \left(\frac{r}{h_0}\right)^{k_1+\alpha+1} \\ &\left(\text{resp. } < C_3 C_5 h^{\alpha} L(h) \left(\frac{1}{y}+y\right)^{k_1} (1+|x|)^{l_1} \left(\frac{h_0}{r}\right)^{k_1+l_1-\alpha+1}\right) \\ &< C_6 h^{\alpha} L(h) \left(\frac{1}{y}+y\right)^{\alpha+2k_1+1} (1+|x|)^{\alpha+l_1+k_1+1} \\ &\left(\text{resp. } < C_6 h^{\alpha} L(h) \left(\frac{1}{y}+y\right)^{2k_1+l_1-\alpha+1} (1+|x|)^{l_1}\right),\end{aligned}$$

with $C_6 = C_3 C_5 (2/h_0)^{\alpha+k_1+1}$ (resp. $C_6 = C_3 C_5 h_0^{k_1+l_1-\alpha+1}$). Therefore, if $C = \max\{C_4, C_6\}$, $k_2 > |\alpha| + 2k_1 + l_1 + 1$ and $l_2 > \alpha + l_1 + k_1 + 1$,

$$\left\|M_{\varphi}^{\mathbf{f}}\left(hx, hy\right)\right\| < C h^{\alpha} L(h) \left(\frac{1}{y}+y\right)^{k_2} (1+|x|)^{l_2},$$

for all $(x,y) \in \mathbb{H}^{n+1}$ and $0 < h \leq h_0$ (resp. $h_0 < h$).

Part (ii). Fix $(x,y) \in \mathbb{H}^{n+1}$ and write it as $(x,y) = (r\xi, r\cos\vartheta)$, where $(\xi, \cos\vartheta) \in \mathbb{H}^{n+1} \cap \mathbb{S}^n$. Then, as $h \to 0^+$ (resp. $h \to \infty$), we have

$$\begin{aligned}\frac{1}{h^{\alpha} L(h)} M_{\varphi}^{\mathbf{f}}(hr\xi, hr\cos\vartheta) &= \frac{L(hr)}{L(h)} r^{\alpha} \left(\frac{1}{(hr)^{\alpha} L(hr)} M_{\varphi}^{\mathbf{f}}(hr\xi, hr\cos\vartheta)\right) \\ &\longrightarrow 1 \cdot r^{\alpha} M_{\xi,\cos\vartheta}, \quad \text{in } E.\end{aligned}$$

□

We now state the Tauberian characterization of quasiasymptotics in the space $\mathcal{S}_0'(\mathbb{R}^n, E)$. The proof of the following theorem is a simple consequence of our previous work.

Theorem 5.2. *Let $\psi \in \mathcal{S}_0(\mathbb{R}^n)$ be a non-degenerate wavelet and let L be slowly varying at the origin (resp. at infinity).*

(i) *A necessary and sufficient condition for $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, E)$ to be quasiasymptotically bounded of degree α at the point x_0 (resp. at infinity) with respect to L in $\mathcal{S}'_0(\mathbb{R}^n, E)$ is the existence of $k \in \mathbb{N}$ such that*

$$\limsup_{\varepsilon \to 0^+} \sup_{|x|^2+y^2=1,\, y>0} \frac{y^k}{\varepsilon^\alpha L(\varepsilon)} \left\| \mathcal{W}_\psi \mathbf{f}\,(x_0+\varepsilon x, \varepsilon y)\right\| < \infty \tag{5.5}$$

$$\left(\textit{resp.}\ \limsup_{\lambda \to \infty} \sup_{|x|^2+y^2=1,\, y>0} \frac{y^k}{\lambda^\alpha L(\lambda)} \left\| \mathcal{W}_\psi \mathbf{f}\,(\lambda x, \lambda y)\right\| < \infty \right).$$

(ii) *The existence of the limits*

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^\alpha L(\varepsilon)} \mathcal{W}_\psi \mathbf{f}(x_0+\varepsilon x, \varepsilon y) = W_{x,y}\,, \quad \textit{for each } (x,y) \in \mathbb{H}^{n+1} \cap \mathbb{S}^n, \tag{5.6}$$

$$\left(\textit{resp.}\ \lim_{\lambda \to \infty} \frac{1}{\lambda^\alpha L(\lambda)} \mathcal{W}_\psi \mathbf{f}(\lambda x, \lambda y) = W_{x,y} \in E \right)$$

and the estimate (5.5), for some $k \in \mathbb{N}$, are necessary and sufficient for $\mathbf{f}$ to have quasiasymptotic behavior of degree α at the point x_0 (resp. at infinity) with respect to L in the space $\mathcal{S}'_0(\mathbb{R}^n, E)$.

Proof. The equivalence between the quasiasymptotic boundedness and the estimate (5.5) follows at once on combining Proposition 5.1 with Proposition 3.14 when considering the set (in $\mathcal{S}'_0(\mathbb{R}^n, E)$)

$$\left\{ \frac{1}{\varepsilon^\alpha L(\varepsilon)} \mathbf{f}(x_0+\varepsilon\,\cdot\,) : 0<\varepsilon\leq 1 \right\} \left(\text{resp.}\ \left\{ \frac{1}{\lambda^\alpha L(\lambda)}\, \mathbf{f}(\lambda\,\cdot\,) : 1 \leq \lambda \right\} \right),$$

while Part (ii) follows from Proposition 5.1 and Proposition 3.15. □

We will need the following proposition for future applications when studying Tauberian theorems for the non-wavelet case and wavelet transforms with respect to non-degenerate wavelets from $\mathcal{S}(\mathbb{R}^n) \setminus \mathcal{S}_0(\mathbb{R}^n)$. It tells us the quasiasymptotic properties of the projection of a tempered distribution onto $\mathcal{S}'_0(\mathbb{R}^n, E)$ when its transform $M^{\mathbf{f}}_\varphi$ has asymptotics as in Proposition 5.1.

Proposition 5.3. *Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be non-degenerate and let L be slowly varying at the origin (resp. at infinity). Suppose that $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, E)$.*

(i) *If there exists $k \in \mathbb{N}$ such that the estimate (5.1) holds, then $\mathbf{f}$ is quasiasymptotically bounded of degree α at the point x_0 (resp. at infinity) with respect to L in the space $\mathcal{S}'_0(\mathbb{R}^n, E)$.*

(ii) *If the limit (5.2) exists for each $(x,y) \in \mathbb{H}^{n+1} \cap \mathbb{S}^n$, and there is a $k \in \mathbb{N}$ such that the estimate (5.1) is satisfied, then $\mathbf{f}$ has quasiasymptotic behavior of degree α at the point x_0 (resp. at infinity) with respect to L in the space $\mathcal{S}'_0(\mathbb{R}^n, E)$.*

Proof. Translating $\mathbf{f}$, we can assume that $x_0 = 0$. Consider the non-degenerate wavelet $\psi \in \mathcal{S}_0(\mathbb{R}^n)$ given on Fourier side by $\hat{\psi}(u) = e^{-|u|-(1/|u|)}$. Set $\psi_1 = \check{\bar{\varphi}} * \psi$, then, $\psi_1 \in \mathcal{S}_0(\mathbb{R}^n)$ is also a non-degenerate wavelet. Indeed, $\hat{\psi}_1 = \widehat{\check{\bar{\varphi}}}\hat{\psi}$ and its partial derivatives of any order vanish at the origin. First notice that $\mathcal{W}_{\psi_1}\mathbf{f}$ is given by

$$\begin{aligned}\mathcal{W}_{\psi_1}\mathbf{f}(x,y) &= \left\langle \mathbf{f}(x+yt), \check{\varphi} * \bar{\psi}(t) \right\rangle = \left\langle \mathbf{f}(x+yt), \int_{\mathbb{R}^n} \bar{\psi}(u)\varphi(u-t)\mathrm{d}u \right\rangle \\ &= \int_{\mathbb{R}^n} \bar{\psi}(u) \left\langle \mathbf{f}(x+yt), \varphi(u-t) \right\rangle \mathrm{d}u \\ &= \int_{\mathbb{R}^n} \bar{\psi}(u)(\mathbf{f} * \varphi_y)(x+yu)\mathrm{d}u \\ &= \int_{\mathbb{R}^n} M^{\mathbf{f}}_{\varphi}(x+yu,y)\bar{\psi}(u)\mathrm{d}u.\end{aligned}$$

Part (i). By Proposition 5.1, (5.1) is equivalent to an estimate (4.1) (k may be a different number). Our strategy will be to show that $\mathcal{W}_{\psi_1}\mathbf{f}$ satisfies (5.5), and then the result would follow immediately from Theorem 5.2. Indeed, for all $(x,y) \in \mathbb{H}^{n+1} \cap \mathbb{S}^n$, and $0 < h \leq \varepsilon_0$ (resp. $\lambda_0 \leq h$) we have the estimate

$$\left\| M^{\mathbf{f}}_{\varphi}(hx+hyu,hy) \right\| \leq \frac{2^k C}{y^k} h^{\alpha} L(h) \left(1+|x|+y\,|u|\right)^l < \frac{C_1}{y^k} h^{\alpha} L(h)(1+|u|)^l, \tag{5.7}$$

with $C_1 = 2^{k+l}C$. Therefore, $\mathcal{W}_{\psi_1}\mathbf{f}$ satisfies (5.5), namely,

$$\sup_{|x|^2+y^2=1,\, y>0} y^k \left\| \mathcal{W}_{\psi_1}\mathbf{f}(hx,hy) \right\| < C_2 h^{\alpha} L(h),$$

where $C_2 = C_1 \int_{\mathbb{R}^n}(1+|u|)^l \left|\bar{\psi}(u)\right| \mathrm{d}u$.

Part (ii). If the limit (5.2) exists for each $(x,y) \in \mathbb{H}^{n+1} \cap \mathbb{S}^{n-1}$, then so does it for all $(x,y) \in \mathbb{H}^{n+1}$. The estimate (5.7) allows us to use the dominated convergence theorem for Bochner integrals and conclude that, for each fixed $(x,y) \in \mathbb{H}^{n+1} \cap \mathbb{S}^n$,

$$\begin{aligned}\frac{1}{h^{\alpha}L(h)}\mathcal{W}_{\psi}\mathbf{f}(hx,hy) &= \int_{\mathbb{R}^n} \frac{1}{h^{\alpha}L(h)} M^{\mathbf{f}}_{\varphi}(hx+hyu,hy)\bar{\psi}(u)\mathrm{d}u \\ &\longrightarrow \int_{\mathbb{R}^n} M_{x+yu,y}\,\bar{\psi}(u)\mathrm{d}u,\end{aligned}$$

as $h \to 0^+$ (resp. $h \to \infty$). Thus, Theorem 5.2 yields the result. □

6. Tauberian Theorems in $\mathcal{S}'(\mathbb{R}^n, E)$

We will state and prove in this section Tauberian theorems for quasi-asymptotics of tempered E-valued distributions.

6.1. **Associate Asymptotically Homogeneous and Homogeneously Bounded Functions.** We need to introduce a class of functions which is of great importance in the study of asymptotic properties of distributions. They appear naturally in the statements and proofs of our Tauberian theorems. The terminology is from [51, 52, 53, 54, 60] (see also de Haan theory in [1]).

Definition 6.1. *Let* $\mathbf{c} : (0, A) \to E$ *(resp.* $(A, \infty) \to E$*),* $A > 0$*, be a continuous* E*-valued function and let* L *be slowly varying function at the origin (resp. at infinity). We say that:*

(i) $\mathbf{c}$ *is associate asymptotically homogeneous of degree 0 with respect to* L *if for some* $\mathbf{v} \in E$

$$\mathbf{c}(a\varepsilon) = \mathbf{c}(\varepsilon) + L(\varepsilon) \log a \, \mathbf{v} + o(L(\varepsilon)) \quad \text{as } \varepsilon \to 0^+, \quad \text{for each } a > 0$$

$$(\text{resp. } \mathbf{c}(a\lambda) = \mathbf{c}(\lambda) + L(\lambda) \log a \, \mathbf{v} + o(L(\lambda)) \quad \text{as } \lambda \to \infty \,) \, .$$

(ii) $\mathbf{c}$ *is asymptotically homogeneously bounded of degree 0 with respect to* L *if*

$$\mathbf{c}(a\varepsilon) = \mathbf{c}(\varepsilon) + O(L(\varepsilon)) \quad \text{as } \varepsilon \to 0^+, \quad \text{for each } a > 0$$

$$(\text{resp. } \mathbf{c}(a\lambda) = \mathbf{c}(\lambda) + O(L(\lambda)) \quad \text{as } \lambda \to \infty \,) \, .$$

If $\mathbf{c}$ satisfies either condition (i) or (ii) of Definition 6.1, one can show as in [51, Prop. 2.3] that given any $\sigma > 0$

$$\|\mathbf{c}(\varepsilon)\| = o(\varepsilon^{-\sigma}) \quad \text{as } \varepsilon \to 0^+ \quad (\text{resp. } \|\mathbf{c}(\lambda)\| = o(\lambda^{\sigma}) \quad \text{as } \lambda \to \infty \,) \, .$$

6.2. **Tauberian Theorem for $\phi-$transforms.** The ensuing theorem characterizes quasiasymptotic boundedness in terms of the $\phi-$transform.

Theorem 6.2. *Let* $\phi \in \mathcal{S}(\mathbb{R}^n)$ *be such that* $\mu_0(\phi) = 1$ *and let* L *be slowly varying at the origin (resp. at infinity). A necessary and sufficient condition for* $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, E)$ *to be quasiasymptotically bounded of degree* $\alpha \in \mathbb{R}$ *at the point* $x_0 \in \mathbb{R}^n$ *(resp. at infinity) with respect to* L *is the existence of* $k \in \mathbb{N}$ *such that*

$$\limsup_{\varepsilon \to 0^+} \sup_{|x|^2 + y^2 = 1, \, y > 0} \frac{y^k}{\varepsilon^{\alpha} L(\varepsilon)} \left\| F_{\phi} \mathbf{f} \, (x_0 + \varepsilon x, \varepsilon y) \right\| < \infty \tag{6.1}$$

$$\left(\text{resp. } \limsup_{\lambda \to \infty} \sup_{|x|^2 + y^2 = 1, \, y > 0} \frac{y^k}{\lambda^{\alpha} L(\lambda)} \left\| F_{\phi} \mathbf{f} \, (\lambda x, \lambda y) \right\| < \infty \right) .$$

We shall present two different proofs of this theorem. The two methods of proof are applicable to both the case of behavior at infity and the one at finite points. We concentrate in showing the sufficiency because the necessity follows at once from the Abelian result (Proposition 4.1).

First proof of Theorem 6.2. We show the case of behavior at the point x_0 in this first proof. We first need to prove the following claim:

Claim 6.3. Given a set of distinct multi-indices $\{m_l\}_{l=1}^{q}$, a point $u = (u_1, \cdots, u_q) \in \mathbb{R}^q$, and an arbitrary positive number σ, there exists a test function ρ in the linear span of $\left\{y^{-n}\phi((\,\cdot\, - x)/y) : (x,y) \in \mathbb{H}^{n+1}\right\}$ such that

$$|u_l - \mu_{m_l}(\rho)| < \sigma, \quad l = 1, \cdots, q.$$

Proof of Claim 6.3. The linear continuous map

$$T : \eta \in \mathcal{S}(\mathbb{R}^n) \mapsto (\mu_{m_1}(\eta), \ldots, \mu_{m_q}(\eta)) \in \mathbb{R}^q,$$

is clearly surjective, as can be verified directly or by using general results (e.g., Borel theorem or results from [9, 11]) . Corollary 3.3 implies that the image under T of the linear span of $\left\{\phi((\,\cdot\, - x)/y) : (x,y) \in \mathbb{H}^{n+1}\right\}$ is dense in $\mathbb{R}^q$, from where we obtain the claimed approximation property. □

We now divide the proof of Theorem 6.2 into two cases.

Case $\alpha \notin \mathbb{N}$.

Proposition 5.3 and Proposition A.2 imply the existence of an E-valued polynomial

$$\mathbf{P}(t) = \sum_{|m| \leq d} t^m \mathbf{w}_m$$

such that

$$\mathbf{f}(x_0 + \varepsilon t) = \mathbf{P}(\varepsilon t) + O\left(\varepsilon^{\alpha} L(\varepsilon)\right) \quad \text{in } \mathcal{S}'(\mathbb{R}^n, E).$$

We must show that $\mathbf{P}(\varepsilon t) = O\left(\varepsilon^{\alpha} L(\varepsilon)\right)$. We may assume that $d < \alpha$ because: $\varepsilon^{\nu-\alpha} = O(L(\varepsilon))$ whenever $\nu > \alpha$ [1, 44]. On the other hand, since $L(\varepsilon) = O(\varepsilon^{-\sigma})$, for any $\sigma > 0$, we obtain that

$$\mathbf{f}(x_0 + \varepsilon t) = \mathbf{P}(\varepsilon t) + O\left(\varepsilon^{d+\kappa}\right) \quad \text{in } \mathcal{S}'(\mathbb{R}^n, E), \tag{6.2}$$

where κ is chosen so that $0 < \kappa < \alpha - d$. Take ρ in the linear span of $\left\{y^{-n}\phi((\,\cdot\, - x)/y) : (x,y) \in \mathbb{H}^{n+1}\right\}$, this test function is fixed by the moment but its properties will be appropriately chosen later. The hypothesis (6.1) implies that $\|\langle f(x_0 + \varepsilon t), \rho(t)\rangle\| = O(\varepsilon^{d+\kappa})$. Evaluation of (6.2) at ρ and the last fact yield

$$\sum_{\nu=0}^{d} \varepsilon^{\nu} \sum_{|m|=\nu} \mu_m(\rho)\mathbf{w}_m = O(\varepsilon^{d+\kappa}),$$

which readily implies that,

$$\sum_{|m|=\nu} \mu_m(\rho)\mathbf{w}_m = 0, \quad \text{for } \nu = 0, 1, \cdots, d. \tag{6.3}$$

For a fixed index $0 \leq \nu \leq d$, let $q = q_\nu$ be the number of multi-indices such that $|m| = \nu$; moreover, index such multi-indices as $\{m_l\}_{l=1}^{q}$. Given an arbitrary $0 < \sigma < 1$, we select ρ as in Claim 6.3 with $u = e_l \in \mathbb{R}^q$, the vector with 1 in the lth component and zeros in the other entries. Then, (6.3) with this ρ gives

$$\|\mathbf{w}_{m_l}\| < \frac{\sigma}{1-\sigma} \sum_{i=1, i\neq l}^{q} \|\mathbf{w}_{m_i}\| ,$$

and taking $\sigma \to 0^+$, we conclude $\mathbf{w}_{m_l} = 0$. Since the argument works for all l and ν, it follows that $\mathbf{w}_m = 0$, for all $|m| \leq d$. This completes the proof in the first case.

Case $\alpha = p \in \mathbb{N}$.

In this case, Proposition 5.3 and Proposition A.2 imply the existence of $\mathbf{w}_j$, $|j| < p$, and asymptotically homogeneously bounded functions $\mathbf{c}_m$, $|m| = p$, of degree 0 with respect to L such that

$$\mathbf{f}(x_0 + \varepsilon t) = \sum_{|j|<p} \varepsilon^{|j|} t^j \mathbf{w}_j + \varepsilon^p \sum_{|m|=p} t^m \mathbf{c}_m(\varepsilon) + O\left(\varepsilon^p L(\varepsilon)\right) \quad \text{in } \mathcal{S}'(\mathbb{R}^n, E).$$

We have that each $\mathbf{c}_m$ satisfies $\mathbf{c}_m(\varepsilon) = O(\varepsilon^{-1/2})$ (cf. Subsection 6.1), and thus

$$\mathbf{f}(x_0 + \varepsilon t) = \sum_{|j|<p} \varepsilon^{|j|} t^j \mathbf{w}_j + O\left(\varepsilon^{p-1/2}\right) \quad \text{in } \mathcal{S}'(\mathbb{R}^n, E).$$

Proceeding as in the preceding case, we conclude that each $\mathbf{w}_j = 0$. Summarizing, we have shown so far

$$\mathbf{f}(x_0 + \varepsilon t) = \varepsilon^p \sum_{|m|=p} t^m \mathbf{c}_m(\varepsilon) + O\left(\varepsilon^p L(\varepsilon)\right) \quad \text{in } \mathcal{S}'(\mathbb{R}^n, E). \tag{6.4}$$

Let now q be the number of multi-indices such that $|m| = p$, once again, we index those multi-indices as $\{m_l\}_{l=1}^q$, and consider the vectors $e_l \in \mathbb{R}^q$ with 1 in the lth component and zeros in the other entries. Let $\sigma > 0$ be small enough such that if the $q \times q$ matrix $A = (a_{l,\nu})_{l,\nu}$ satisfies $|a_{l,\nu} - \delta_{l,\nu}| < \sigma$, then A is invertible ($\delta_{l,\nu}$ is the Kronecker delta). For each $1 \leq l \leq q$, find ρ_l satisfying the conclusions of Claim 6.3 for σ and e_l, that is, $|\mu_{m_\nu}(\rho_l) - \delta_{l,\nu}| < \sigma$. Then, the matrix $A := (\mu_{m_\nu}(\rho_l))_{l,\nu}$ is invertible. Evaluation of (6.4) at the ρ_l and the hypothesis (6.1) yield the $q \times q$ system of inequalities

$$\sum_{\nu=1}^{q} \mu_{m_\nu}(\rho_l) \mathbf{c}_{m_\nu}(\varepsilon) = O(L(\varepsilon)), \quad l = 1, \cdots, q.$$

Multiplication by A^{-1} implies that $\mathbf{c}_m(\varepsilon) = O(L(\varepsilon))$, for each $|m| = p$, which turns out to prove

$$\mathbf{f}(x_0 + \varepsilon t) = O\left(\varepsilon^p L(\varepsilon)\right) \quad \text{in } \mathcal{S}'(\mathbb{R}^n, E),$$

as required. □

Second proof of Theorem 6.2. In this second proof we only consider the behavior at infinity. As in the first proof, we can conclude the existence of an E-valued polynomial, which can be assumed to have the form $\mathbf{P}(t) = \sum_{\alpha<|m|\leq d} t^m \mathbf{w}_m$ such that $\mathbf{f}(\lambda t) = \mathbf{P}(\lambda t) + O(\lambda^\alpha L(\lambda))$ if $\alpha \in \mathbb{N}$, or $\mathbf{f}(\lambda t) =$

$\mathbf{P}(\lambda t)+\lambda^{p}\sum_{|m|=p}\mathbf{c}_{m}(\lambda)t^{m}+O(\lambda^{p}L(\lambda))$ if $\alpha=p\in\mathbb{N}$, where the $\mathbf{c}_m$ are asymptotically homogeneously bounded of degree 0 with respect to L; either asymptotic formula holding as $\lambda\to\infty$ in the space $\mathcal{S}'(\mathbb{R}^n,E)$. We first show that $\mathbf{P}=\mathbf{0}$. Select $\alpha<\kappa<[\alpha]+1$, since both $L(\lambda)$ and the $\mathbf{c}_m$ are $O(\lambda^{\kappa-\alpha})$, we obtain in either case that

$$\mathbf{f}(\lambda t)=\mathbf{P}(\lambda t)+O(\lambda^{\kappa})\quad\text{as } \lambda\to\infty \text{ in } \mathcal{S}'(\mathbb{R}^n).$$

If we use the estimate for $F_{\phi}\mathbf{f}(\lambda x,\lambda y)$, we have that for each (fixed) $(x,y)\in\mathbb{H}^{n+1}$,

$$F_{\phi}\mathbf{P}(\lambda x,\lambda y)=\sum_{\alpha<|m|\leq d}(-\lambda i)^{|m|}\frac{\partial^{|m|}}{\partial u^{m}}\left(e^{ix\cdot u}\hat{\phi}(-yu)\right)\Big|_{u=0}\mathbf{w}_{m}=O(\lambda^{\kappa}),$$

$\lambda\to\infty$. This allows us to conclude that, for each $(x,y)\in\mathbb{H}^{n+1}$ and $\alpha<\nu\leq d$,

$$\mathbf{0}=\sum_{|m|=\nu}\frac{\partial^{|m|}}{\partial u^{m}}\left(e^{ix\cdot u}\hat{\phi}(-yu)\right)\Big|_{u=0}\mathbf{w}_{m}=\sum_{|m|=\nu}(ix)^{m}\mathbf{w}_{m}+\sum_{q=1}^{\nu}(iy)^{q}\mathbf{R}_{q}(x),$$

for certain E-valued polynomials $\mathbf{R}_q$. But we can take $y\to0^{+}$ in the above equation, which implies that $\mathbf{w}_m=\mathbf{0}$ for $|m|=q$, and since the same holds for every $\alpha<\nu\leq d$, we have just shown that $\mathbf{P}=\mathbf{0}$. Therefore, Part (i) has been established. Part (ii) would now follow if we were able to prove that $\mathbf{C}(\lambda,t):=\sum_{|m|=p}t^{m}\mathbf{c}_{m}(\lambda)=O(L(\lambda))$ as $\lambda\to\infty$ in $\mathcal{S}'(\mathbb{R}^n,E)$. We keep $(x,y)\in B(0,1)\times(0,1)$, where $B(0,1)$ is the unit ball in $\mathbb{R}^n$. By Proposition 4.1, applied to $(\lambda^{-p}/L(\lambda))(\mathbf{f}(\lambda t)-\lambda^{p}\sum_{|m|=p}t^{m}\mathbf{c}_{m}(\lambda))$, and Proposition 5.1, applied to $F_{\phi}\mathbf{f}$, there are constants $\lambda_0, C>0$ and $l\in\mathbb{N}$ such that

$$\left\|\lambda^{p}\sum_{|m|=p}(-\lambda i)^{p}\frac{\partial^{|m|}}{\partial u^{m}}\left(e^{ix\cdot u}\hat{\phi}(-yu)\right)\Big|_{u=0}\mathbf{c}_{m}(\lambda)\right\|\leq\frac{C}{y^{l}}\lambda^{p}L(\lambda),$$

for all $(x,y)\in B(0,1)\times(0,1)$ and $\lambda_0\leq\lambda$, that is,

$$\left\|\mathbf{C}(\lambda,x)+\sum_{\nu=1}^{p}y^{\nu}\mathbf{C}_{\nu}(\lambda,x)\right\|\leq\frac{CL(\lambda)}{\lambda^{p}y^{l}},$$

for suitable E-valued functions $\mathbf{C}_q(\lambda,x)$. If we now select p points $0<y_1<y_2\dots<y_p<1$, we obtain a system of $q+1$ inequalities with Vandermonde matrix $A=(y_j^{\nu})_{j,\nu}$. Multiplying by A^{-1} and setting $x=t/(1+|t|)$, we can find a constant C_1 such that

$$\|\mathbf{C}(\lambda,t)\|\leq C_{1}(1+|t|)^{p}\frac{L(\lambda)}{\lambda^{p}},\quad\text{for all } t\in\mathbb{R}^n \text{ and } \lambda_0<\lambda.$$

This completes the proof. □

We now investigate the quasiasymptotic behavior.

Theorem 6.4. *Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\mu_0(\phi) = 1$ and let L be slowly varying at the origin (resp. at infinity). Then, the existence of the limits*

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^\alpha L(\varepsilon)} F_\phi \mathbf{f}(x_0 + \varepsilon x, \varepsilon y) = F_{x,y}\,, \quad \text{for each } (x,y) \in \mathbb{H}^{n+1} \cap \mathbb{S}^n, \tag{6.5}$$

$$\left(\text{resp. } \lim_{\lambda \to \infty} \frac{1}{\lambda^\alpha L(\lambda)} F_\phi \mathbf{f}(\lambda x, \lambda y) = F_{x,y} \in E \right),$$

and the estimate (6.1), for some $k \in \mathbb{N}$, are necessary and sufficient for $\mathbf{f}$ to have quasiasymptotic behavior in the space $\mathcal{S}'(\mathbb{R}^n, E)$, namely, the existence of an E-valued homogeneous distribution $\mathbf{g} \in \mathcal{S}'(\mathbb{R}^n, E)$ such that

$$\mathbf{f}(x_0 + \varepsilon t) \sim \varepsilon^\alpha L(\varepsilon) \mathbf{g}(t) \quad \text{as } \varepsilon \to 0^+ \quad \text{in } \mathcal{S}'(\mathbb{R}^n, E) \tag{6.6}$$

$$\big(\text{resp. } \mathbf{f}\,(\lambda t) \sim \lambda^\alpha L(\lambda) \mathbf{g}(t) \quad \text{as } \lambda \to \infty \quad \text{in } \mathcal{S}'(\mathbb{R}^n, E) \,\big)\,.$$

In such a case, $\mathbf{g}$ is completely determined by $F_\phi \mathbf{g}(x,y) = F_{x,y}$.

Proof. Theorem 6.2 gives the equivalence of (6.1) with the quasiasymptotic boundedness of $\mathbf{f}$. So, by the Banach-Steinhaus theorem, (6.6) is now equivalent to the convergence of $(\varepsilon^{-\alpha}/L(\varepsilon))\mathbf{f}(x_0 + \varepsilon\, \cdot\,)$ (resp. $(\lambda^{-\alpha}/L(\lambda))\mathbf{f}(\lambda\, \cdot\,)$) over a dense subset of $\mathcal{S}(\mathbb{R}^n)$. By Corollary 3.3, the linear span of the set $\left\{ y^{-n} \phi((\,\cdot\, - x)/y) : (x,y) \in \mathbb{H}^{n+1} \right\}$ is dense in $\mathcal{S}(\mathbb{R}^n)$, it remains only to observe that (6.5) gives precisely convergence over such a dense subset. □

Remark 6.5. We have stated the theorems of this subsection only for $\phi-$transforms, but they are obviously true for any non-wavelet transform $M^{\mathbf{f}}_\varphi$ if we just assume that $\mu_0 = \mu_0(\varphi) = \int_{\mathbb{R}^n} \varphi(t) \mathrm{d}t \neq 0$. Indeed, it follows simply by considering the $\phi-$transform with kernel $\phi = \mu_0^{-1} \check{\varphi}$.

6.3. **Tauberian Theorems for Wavelet Transforms.** We now present the Tauberian theorems for wavelet transforms. We begin with quasiasymptotic boundedness.

Theorem 6.6. *Let $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, E)$, let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be a non-degenerate wavelet, and let L be slowly varying at the origin (resp. at infinity). The estimate (5.5), for some $k \in \mathbb{N}$, is sufficient for the existence of an E-valued polynomial $\mathbf{P}$, of degree less than α (resp. of the form $\mathbf{P}(t) = \sum_{\alpha < |m| \leq d} t^m \mathbf{w}_m$, for some $d \in \mathbb{N}$), such that:*

(i) *If $\alpha \notin \mathbb{N}$, $\mathbf{f} - \mathbf{P}$ is quasiasymptotically bounded of degree α at the point x_0 (at infinity) with respect to L in the space $\mathcal{S}'(\mathbb{R}^n, E)$.*

(ii) *If $\alpha = p \in \mathbb{N}$, there exist asymptotically homogeneously bounded E-valued functions $\mathbf{c}_m$, $|m| = p$, of degree 0 with respect to L such that $\mathbf{f}$ has the following asymptotic expansion*

$$\mathbf{f}(x_0 + \varepsilon t) = \mathbf{P}(\varepsilon t) + \varepsilon^p \sum_{|m|=p} t^m \mathbf{c}_m(\varepsilon) + O\left(\varepsilon^p L(\varepsilon)\right)$$

$$\left(\text{resp. } \mathbf{f}\,(\lambda t) = \mathbf{P}(\lambda t) + \lambda^p \sum_{|m|=p} t^m \mathbf{c}_m(\lambda) + O\left(\lambda^p L(\lambda)\right) \right),$$

as $\varepsilon \to 0^+$ (resp. $\lambda \to \infty$) in the space $\mathcal{S}'(\mathbb{R}^n, E)$.

Moreover, denote by P_q the homogeneous terms of the Taylor polynomials of $\hat{\psi}$ at the origin, that is,

$$P_q(u) = \sum_{|m|=q} \frac{\hat{\psi}^{(m)}(0)u^m}{m!}, \quad q \in \mathbb{N}. \tag{6.7}$$

Then, the E-valued polynomial $\mathbf{P}$ must satisfy

$$\overline{P_q}\left(\frac{\partial}{\partial t}\right)\mathbf{P} = \mathbf{0}, \quad \text{for all } q \in \mathbb{N}. \tag{6.8}$$

Proof. By Proposition 5.3, (5.5) implies that $\mathbf{f}$ is quasiasymptotically bounded in the space $\mathcal{S}_0'(\mathbb{R}^n, E)$. The existence of the E-valued polynomial $\mathbf{P}$ is then a direct consequence of Proposition A.2. The assertion about the degree of $\mathbf{P}$ follows from the growth properties of L (in the case (ii) the terms of order $|m| = p$ can be assumed to be aborbed by the $\mathbf{c}_m$). It remains to establish that $\mathbf{P}$ satisfies the equations (6.8). We show this fact only in the case of infinity, the proof of the case of behavior at finite points is completely analogous. Suppose the E-valued polynomial has the form $\mathbf{P}(t) = \sum_{\alpha<|m|\leq d} t^m \mathbf{w}_m = \sum_{\nu=[\alpha]+1}^{d} \mathbf{Q}_\nu(t)$, where each $\mathbf{Q}_\nu$ is homogeneous of degree ν. Choose $\alpha < \kappa < [\alpha] + 1$. Then, since $L(\lambda) = O(\lambda^{\kappa-\alpha})$ and $\mathbf{c}_m(\lambda) = O(\lambda^{\kappa-\alpha})$, we obtain that

$$\mathbf{f}(\lambda t) = \mathbf{P}(\lambda t) + O(\lambda^\kappa) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{S}'(\mathbb{R}^n, E).$$

But, then, for each fixed $(x, y) \in \mathbb{H}^{n+1}$, the assumption on the size of $\mathcal{W}_\psi \mathbf{f}(\lambda x, \lambda y)$ implies that

$$\mathcal{W}_\psi \mathbf{P}(\lambda x, \lambda y) = \sum_{\alpha<|m|\leq d} (-\lambda i)^{|m|} \frac{\partial^{|m|}}{\partial u^m} \left(e^{ix\cdot u}\overline{\hat{\psi}}(yu)\right)\Big|_{u=0} \mathbf{w}_m = O(\lambda^\kappa),$$

$\lambda \to \infty$. Then, we infer that, for each $\alpha < \nu \leq d$ and each $(x, y) \in \mathbb{H}^{n+1}$,

$$\mathbf{0} = \sum_{|m|=\nu} \frac{\partial^{|m|}}{\partial u^m} \left(e^{ix\cdot u}\overline{\hat{\psi}}(yu)\right)\Big|_{u=0} \mathbf{w}_m = \sum_{q=0}^{\nu} (iy)^q (\overline{P_q\left(\partial/\partial x\right)}\mathbf{Q}_\nu)(x),$$

and thus

$$\overline{P_q}\left(\frac{\partial}{\partial x}\right)\mathbf{Q}_\nu = \mathbf{0}, \quad \text{for all } q, \nu \in \mathbb{N},$$

as required. □

We now consider the quasiasymptotic behavior.

Theorem 6.7. *Let $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, E)$, let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be a non-degenerate wavelet, and let L be slowly varying at the origin (resp. at infinity). Suppose that the estimate (5.5) holds for some $k \in \mathbb{N}$, and the limits (5.6) exist. Then, there exist an E-valued tempered distribution $\mathbf{g}$, which satisfies $\mathcal{W}_\psi \mathbf{g}(x, y) = W_{x,y}$, and an E-valued polynomial $\mathbf{P}$, of degree less than α (resp. of the form $\mathbf{P}(t) = \sum_{\alpha<|m|\leq d} t^m \mathbf{w}_m$, for some $d \in \mathbb{N}$), such that:*

(i) *If* $\alpha \notin \mathbb{N}$, $\mathbf{g}$ *is homogeneous of degree* α *and*

$$\mathbf{f}(x_0+\varepsilon t)-\mathbf{P}(\varepsilon t)\sim\varepsilon^{\alpha}L(\varepsilon)\mathbf{g}(t) \quad \text{as } \varepsilon\to 0^{+} \quad \text{in } \mathcal{S}'(\mathbb{R}^n,E)$$

$$\left(\text{resp. } \mathbf{f}\,(\lambda t)-\mathbf{P}(\lambda t)\sim\lambda^{\alpha}L(\lambda)\mathbf{g}(t) \quad \text{as } \lambda\to\infty \quad \text{in } \mathcal{S}'(\mathbb{R}^n,E)\right).$$

(ii) *If* $\alpha = p \in \mathbb{N}$, *there exist associate asymptotically homogeneous E-valued functions* $\mathbf{c}_m$, $|m| = p$, *of degree 0 with respect to L such that* $\mathbf{f}$ *has the following asymptotic expansion*

$$\mathbf{f}(x_0+\varepsilon t)=\mathbf{P}(\varepsilon t)+\varepsilon^{p}L(\varepsilon)\mathbf{g}(t)+\varepsilon^{p}\sum_{|m|=p}t^{m}\mathbf{c}_m(\varepsilon)+o\left(\varepsilon^{p}L(\varepsilon)\right)$$

$$\left(\text{resp. } \mathbf{f}\,(\lambda t)=\mathbf{P}(\lambda t)+\lambda^{p}L(\lambda)\mathbf{g}(t)+\lambda^{p}\sum_{|m|=p}t^{m}\mathbf{c}_m(\lambda)+o\left(\lambda^{p}L(\lambda)\right)\right),$$

as $\varepsilon \to 0^+$ *(resp.* $\lambda \to \infty$*) in the space* $\mathcal{S}'(\mathbb{R}^n, E)$.

Furthermore, $\mathbf{P}$ *satisfies the equations (6.8).*

Proof. Proposition 5.3, under the assumptions (5.5) and (5.6), implies that $\mathbf{f}$ has quasiasymptotic behavior. An application of Proposition A.1 yields now the existence of $\mathbf{g}$ and $\mathbf{P}$. That $\mathbf{P}$ sastisfies the equations (6.8) actually follows from Theorem 6.6. □

When $\alpha \notin \mathbb{N}$ in Theorem 6.7, the condition $\mathcal{W}_\psi \mathbf{g}(x, y) = W_{x,y}$ uniquely determines $\mathbf{g}$, in view of its homogeneity. On the other hand, if $\alpha \in \mathbb{N}$, the prescribed values of $\mathcal{W}_\psi \mathbf{g}$ can only determine $\mathbf{g}$ modulo polynomials which are homegeneous of degree α.

At this point it is worth to point out that the use of non-degenerate wavelets in Theorem 6.6 and Theorem 6.7 is absolutely imperative. Clearly, if $\hat{\psi}$ identically vanishes on a ray through the origin, then there are distributions for which $\mathcal{W}_\psi \mathbf{f}$ is identically zero and hence for those distributions the hypothesis (5.5) is satisfied for all α. However, it is easy to find explicit examples of such $\mathbf{f}$ for which the conclusion of Theorem 6.6 does not hold for a given α.

Observe that if $\psi \in \mathcal{S}_0(\mathbb{R}^n)$ in Theorem 6.6 and Theorem 6.7, then the converses are also true, as follows from the moment vanishing properties of ψ. This is actually the content of Theorem 5.2.

7. Tauberian Class Estimates

In this section we show that the estimate of type

$$\left\| M^{\mathbf{f}}_{\varphi}(x,y)\right\| \leq C\frac{(1+y)^{k}\,(1+|x|)^{l}}{y^{k}}, \quad (x,y)\in\mathbb{H}^{n+1}, \tag{7.1}$$

characterizes the space $\mathcal{S}'(\mathbb{R}^n, E)$. We call (7.1) a *global class estimate*, and we may say that it has a Tauberian nature. Specifically, we prove that if $\mathbf{f}$ takes values in a "broad" locally convex space which contains the narrower Banach space E, and if $\mathbf{f}$ satisfies (7.1) for a non-degenerate test function

φ, then, there is a distribution $\mathbf{G}$ with values in the broad space such that $\operatorname{supp}\hat{\mathbf{G}} \subseteq \{0\}$ and $\mathbf{f}-\mathbf{G} \in \mathcal{S}'(\mathbb{R}^n, E)$. In case when the broad space is a normed one, $\mathbf{G}$ reduces simply to a polynomial. This will be done in Subsection 7.1.

We shall also investigate in Subsection 7.2 the consequences of (7.1) when it is only assumed to hold for $(x,y) \in \mathbb{R}^n \times (0,1]$, we call it then a *local class estimate*. In this case the situation is slightly different and we obtain that $\mathbf{f}-\mathbf{G} \in \mathcal{S}'(\mathbb{R}^n, E)$, where $\hat{\mathbf{G}}$ has compact support but its support may not be any longer the origin. We may take $\mathbf{G}$ with $\operatorname{supp}\hat{\mathbf{G}} \subseteq \{0\}$ if we employ the wavelets introduced in the Example 3.7 (Subsection 7.4). For the $\phi-$transform $\mathbf{G}$ does not occur (Subsection 7.3).

We point out that the results of this section extend in several directions those of Drozhzhinov and Zavialov from [5, 6].

Throughout this section, unless specified, X is assumed to be a (arbitrary) Hausdorff locally convex topological vector space such that $E \subset X$, where the embedding is linear and continuous. Observe that the transform (3.1) makes sense for X-valued distributions as well. Measurability for E-valued functions is meant in the sense of Bochner (i.e., a.e. pointwise limits of E-valued continuous functions); likewise, integrals for E-valued functions are taken in the Bochner sense.

7.1. **Global Class Estimates.** We begin with wavelets in $\mathcal{S}_0(\mathbb{R}^n)$.

Proposition 7.1. *Let* $\mathbf{f} \in \mathcal{S}_0'(\mathbb{R}^n, X)$ *and let* $\psi \in \mathcal{S}_0(\mathbb{R}^n)$ *be a non-degenerate wavelet. The following three conditions,*

$$\mathcal{W}_\psi \mathbf{f}(x,y) \in E, \ \text{for almost all } (x,y) \in \mathbb{H}^{n+1}, \tag{7.2}$$

$$\mathcal{W}_\psi \mathbf{f} \text{ is measurable as an } E\text{-valued function,} \tag{7.3}$$

there are constants $k,l \in \mathbb{N}$ *and* $C>0$ *such that*

$$\|\mathcal{W}_\psi \mathbf{f}(x,y)\| \le C\left(\frac{1}{y}+y\right)^k (1+|x|)^l, \ \text{for almost all } (x,y) \in \mathbb{H}^{n+1}, \tag{7.4}$$

are necessary and sufficient for $\mathbf{f} \in \mathcal{S}_0'(\mathbb{R}^n, E)$.

Proof. The necessity is clear (Proposition 3.14). We show the sufficience. Let η be a reconstruction wavelet for ψ. We apply the wavelet synthesis operator to $\mathbf{K}(x,y) = \mathcal{W}_\psi \mathbf{f}(x,y)$, this is valid because our assumptions (7.2)–(7.4) ensure that $\mathbf{K} \in \mathcal{S}'(\mathbb{H}^{n+1}, E)$. So, set $\tilde{\mathbf{f}} := \mathcal{M}_\eta \mathbf{K} \in \mathcal{S}_0'(\mathbb{R}^n, E) \subset \mathcal{S}_0'(\mathbb{R}^n, X)$. We must therefore show $\tilde{\mathbf{f}} = \mathbf{f}$. Let $\rho \in \mathcal{S}_0(\mathbb{R}^n)$. We have, by definition, (3.12), and (3.14),

$$\langle \tilde{\mathbf{f}}, \rho\rangle = \frac{1}{c_{\psi,\eta}} \int_0^\infty \int_{\mathbb{R}^n} \left\langle f(t), \frac{1}{y^n}\bar{\psi}\left(\frac{t-x}{y}\right) \mathcal{W}_{\bar{\eta}}\rho(x,y)\right\rangle \frac{\mathrm{d}x\mathrm{d}y}{y}$$

and

$$\begin{aligned}\langle \mathbf{f}, \rho\rangle &= \frac{1}{c_{\psi,\eta}} \left\langle \mathbf{f}, \mathcal{M}_{\bar{\psi}} \mathcal{W}_{\bar{\eta}} \rho \right\rangle \\ &= \frac{1}{c_{\psi,\eta}} \left\langle f(t), \int_0^\infty \int_{\mathbb{R}^n} \frac{1}{y^n} \bar{\psi}\left(\frac{t-x}{y}\right) \mathcal{W}_{\bar{\eta}}\rho(x,y) \right\rangle \frac{\mathrm{d}x \mathrm{d}y}{y} .\end{aligned}$$

Thus, with $\Phi(x,y;t) = y^{-n-1}\bar{\psi}\left((t-x)/y\right) \mathcal{W}_{\bar{\eta}}\rho(x,y)$, our problem reduces to justify the interchange of the integrals with the dual pairing in

$$\int_0^\infty \int_{\mathbb{R}^n} \langle f(t), \Phi(x,y;t)\rangle \,\mathrm{d}x\mathrm{d}y = \left\langle f(t), \int_0^\infty \int_{\mathbb{R}^n} \Phi(x,y;t)\mathrm{d}x\mathrm{d}y \right\rangle . \tag{7.5}$$

To show (7.5), we verify that

$$\left\langle \mathbf{w}^*, \int_0^\infty \int_{\mathbb{R}^n} \langle f(t), \Phi(x,y;t)\rangle \,\mathrm{d}x\mathrm{d}y \right\rangle = \left\langle \mathbf{w}^*, \left\langle f(t), \int_0^\infty \int_{\mathbb{R}^n} \Phi(x,y;t)\mathrm{d}x\mathrm{d}y \right\rangle\right\rangle , \tag{7.6}$$

for arbitrary $\mathbf{w}^* \in X'$ (here is where the local convexity of X plays a role). Since the integral involved in the left hand side of the above expression is a Bochner integral in E and the restriction of $\mathbf{w}^*$ to E belongs to E', we obtain at once the exchange formula

$$\left\langle \mathbf{w}^*, \int_0^\infty \int_{\mathbb{R}^n} \langle f(t), \Phi(x,y;t)\rangle \,\mathrm{d}x\mathrm{d}y \right\rangle = \int_0^\infty \int_{\mathbb{R}^n} \langle \mathbf{w}^*, \langle f(t), \Phi(x,y;t)\rangle\rangle \,\mathrm{d}x\mathrm{d}y. \tag{7.7}$$

On the other hand, we may write $\int_0^\infty \int_{\mathbb{R}^n} \Phi(x,y;t)\mathrm{d}x\mathrm{d}y$ as the limit of Riemann sums, convergent in $\mathcal{S}_0(\mathbb{R}^n_t)$, we then easily justify the exchanges that yield

$$\left\langle \mathbf{w}^*, \left\langle f(t), \int_0^\infty \int_{\mathbb{R}^n} \Phi(x,y;t)\mathrm{d}x\mathrm{d}y \right\rangle\right\rangle = \int_0^\infty \int_{\mathbb{R}^n} \langle \mathbf{w}^*, \langle f(t), \Phi(x,y;t)\rangle\rangle \,\mathrm{d}x\mathrm{d}y. \tag{7.8}$$

The equality (7.6) follows now by comparing (7.7) and (7.8). □

Proposition 7.1 provides a full characterization of $\mathcal{S}_0'(\mathbb{R}^n, E)$.

We now abord the general wavelet case. The $\phi-$transform will be studied separately in Subsection 7.3 because a stronger result holds for it.

Theorem 7.2. *Let $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, X)$ and let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be a non-degenerate wavelet. Sufficient conditions for the existence of an X-valued distribution $\mathbf{G} \in \mathcal{S}'(\mathbb{R}^n, X)$ such that $\mathbf{f} - \mathbf{G} \in \mathcal{S}'(\mathbb{R}^n, E)$ and* $\operatorname{supp} \hat{\mathbf{G}} \subseteq \{0\}$ *are:*

- (i) *$\mathcal{W}_\psi \mathbf{f}(x,y)$ takes values in E for almost all $(x,y) \in \mathbb{H}^{n+1}$ and is measurable as an E-valued function.*
- (ii) *There exist constants $k, l \in \mathbb{N}$ and $C > 0$ such that (7.4) holds.*

Proof. Let $\psi_1 \in \mathcal{S}_0(\mathbb{R}^n)$ be the non-degenerate wavelet given by $\hat{\psi}(u) = e^{-|u|-(1/|u|)}$. Set $\psi_2 = \psi_1 * \psi$, then, $\psi_2 \in \mathcal{S}_0(\mathbb{R}^n)$ is also a non-degenerate wavelet. Using the same argument as in the proof of Proposition 7.1, the exchange of integral and dual paring performed in the proof of Proposition 5.3 is valid and so we have the formula

$$\mathcal{W}_{\psi_2}\mathbf{f}(x,y)=\int_{\mathbb{R}^n}\mathcal{W}_{\psi}\mathbf{f}(x+yu,y)\overline{\psi}_1(u)\mathrm{d}u,$$

where the integral is taken in the sense of Bochner. Thus, the restriction of $\mathbf{f}$ to $\mathcal{S}_0(\mathbb{R}^n)$ satiesfies the hypotheses of Proposition 7.1, and hence there exists $\mathbf{g}\in\mathcal{S}'(\mathbb{R}^n,E)$ such that $\langle\mathbf{f}-\mathbf{g},\rho\rangle=0$ for all $\rho\in\mathcal{S}_0(\mathbb{R}^n)$. This gives at once that $\mathbf{G}=\mathbf{f}-\mathbf{g}$ satisfies $\operatorname{supp}\hat{\mathbf{G}}\subseteq\{0\}$ and $\mathbf{f}-\mathbf{G}\in\mathcal{S}'(\mathbb{R}^n,E)$. □

When X is a normed space, we obviously have that the only X-valued distributions with support at the origin are precisely those having the form $\sum_{|m|\leq N}\delta^{(m)}\mathbf{w}_m$, $\mathbf{w}_m\in X$. Thus, we have:

Corollary 7.3. *Let X be a normed space. Then, the conditions (i) and (ii) of Theorem 7.2 imply the existence of an X-valued polynomial $\mathbf{P}$ such that $\mathbf{f}-\mathbf{P}\in\mathcal{S}'(\mathbb{R}^n,E)$.*

Moreover, if P_q denote the homogeneous terms of the Taylor polynomials of $\hat{\psi}$ at the origin (cf. (6.7)), then

$$\overline{P_q}\left(\frac{\partial}{\partial t}\right)\mathbf{f}\in\mathcal{S}'(\mathbb{R}^n,E),\quad \textit{for all } q\in\mathbb{N}. \tag{7.9}$$

Proof. By Theorem 7.2, one can find $\mathbf{P}\in\mathcal{S}'(\mathbb{R}^n,X)$ such that $\mathbf{f}-\mathbf{P}\in\mathcal{S}'(\mathbb{R}^n,E)$ and $\operatorname{supp}\hat{\mathbf{P}}\subseteq\{0\}$. Since X is normed, the point support property of $\hat{\mathbf{P}}$ implies that $\mathbf{P}$ is a polynomial. Next, write $\mathbf{P}(t)=\sum_{|m|\leq N}i^{|m|}t^m\mathbf{w}_m$, with $\mathbf{w}_m\in X$. The relation (7.9) would follow immediately if we show that $\overline{P_q\,(\partial/\partial t)}\mathbf{P}$ is an E-valued polynomial. Observe that the hypotheses imply that $\mathcal{W}_\psi\mathbf{P}(x,y)\in E$, for almost all (x,y). Hence, for almost all (x,y),

$$\begin{aligned}\mathcal{W}_\psi\mathbf{P}(x,y)&=\frac{1}{(2\pi)^n}\left\langle\hat{\mathbf{P}}(u),e^{ix\cdot u}\overline{\hat{\psi}}(yu)\right\rangle=\sum_{|m|\leq N}\frac{\partial^{|m|}}{\partial u^m}\left(e^{ix\cdot u}\overline{\hat{\psi}}(yu)\right)\Big|_{u=0}\mathbf{w}_m\\&=\sum_{q=0}^{N}y^q\sum_{|m|=q}\overline{\hat{\psi}^{(m)}(0)}\sum_{|j|\leq N-q}\binom{m+j}{m}(ix)^j\mathbf{w}_{m+j}\\&=\sum_{q=1}^{N}(iy)^q(\overline{P_q\,(\partial/\partial x)}\mathbf{P})(x)\in E.\end{aligned}$$

But the latter readily implies that $(\overline{P_q\,(\partial/\partial x)}\mathbf{P})(x)\in E$, for all $0\leq q\leq N$ and $x\in\mathbb{R}^n$. □

In general, it is not possible to replace the $\mathbf{G}$ by an X-valued polynomial in Theorem 7.2. However, we know some valuable information about $\hat{\mathbf{G}}$. Since it is supported by the origin, it is easy to show that

$$\hat{\mathbf{G}}=\sum_{m\in\mathbb{N}^n}\frac{(-1)^{|m|}\delta^{(m)}}{m!}\mu_m(\hat{\mathbf{G}}),$$

where $\mu_m(\hat{\mathbf{G}}) = \left\langle \hat{\mathbf{G}}(u), u^m \right\rangle \in X$ are its moments and the series is convergent in $\mathcal{S}'(\mathbb{R}^n, X)$. This series is "weakly finite", in the sense that for each $\mathbf{w}^* \in X'$ there exists $N_{\mathbf{w}^*} \in \mathbb{N}$ such that

$$\langle \mathbf{w}^*, \langle \hat{\mathbf{G}}, \rho \rangle \rangle = \sum_{|m| \leq N_{\mathbf{w}^*}} \frac{\rho^{(m)}(0)}{m!} \left\langle \mathbf{w}^*, \mu_m(\hat{\mathbf{G}}) \right\rangle, \quad \text{for all } \rho \in \mathcal{S}(\mathbb{R}^n).$$

Furthermore, given any continuous seminorm $\mathfrak{p}$ on X, one can find an $N_{\mathfrak{p}}$ such that $\mathfrak{p}(\langle \hat{G}, \rho \rangle - \sum_{|m| \leq N_{\mathfrak{p}}} (\rho^{(m)}(0)/m!) \mu_m(\hat{\mathbf{G}})) = 0$, for all $\rho \in \mathcal{S}(\mathbb{R}^n)$. Finally, we remark that $\mathbf{G}$, its inverse Fourier transform, can be naturally identified with an entire X-valued function (cf. Subsection 7.2).

Example 7.4. We consider $X = C(\mathbb{R})$ and $E = C_b(\mathbb{R})$, the space of continuous bounded functions. Let $\chi_\nu \in C(\mathbb{R})$ be non-trival such that $\operatorname{supp} \chi_\nu \subset (\nu, \nu + 1)$, $\nu \in \mathbb{N}$. Furthermore, for each $\nu \in \mathbb{N}$ find a harmonic homogeneous polynomial Q_ν of degree ν, i.e., $\Delta Q_\nu = 0$. Consider the E-valued distribution

$$\mathbf{G}(t, \xi) = \sum_{\nu=0}^{\infty} Q_\nu(t) \chi_\nu(\xi) \in \mathcal{S}'(\mathbb{R}^n_t, C(\mathbb{R}_\xi)) \setminus \mathcal{S}'(\mathbb{R}^n_t, C_b(\mathbb{R}_\xi)).$$

Its Fourier transform is given by an infinite multipole series supported at the origin, i.e.,

$$\hat{\mathbf{G}}(u, \xi) = (2\pi)^n \sum_{\nu=0}^{\infty} \left(Q_\nu \left(i\partial/\partial u \right) \delta \right)(u)\, \chi_\nu(\xi).$$

Let $\mathbf{h} \in \mathcal{S}'(\mathbb{R}^n, C_b(\mathbb{R}))$ and let $\psi \in \mathcal{S}(\mathbb{R}^n) \setminus \mathcal{S}_0(\mathbb{R}^n)$ be a non-degenerate wavelet such that its Fourier transform satisfies $\hat{\psi}(u) = |u|^2 + O(|u|^N)$ as $u \to 0$, for all $N > 2$. If $\mathbf{f} = \mathbf{h} + \mathbf{G} \in \mathcal{S}'(\mathbb{R}^n, C(\mathbb{R}))$, then $\mathcal{W}_\psi \mathbf{f}(x, y) = \mathcal{W}_\psi \mathbf{h}(x, y)$ for all $(x, y) \in \mathbb{H}^{n+1}$. Thus, $\mathbf{f}$ satisfies all the hypotheses of Theorem 7.2; however, there is no $C(\mathbb{R})$-valued polynomial $\mathbf{P}$ such that $\mathbf{f} - \mathbf{P} \in \mathcal{S}'(\mathbb{R}, C_b(\mathbb{R}))$.

7.2. **Local Class Estimates.** We now proceed to study local class estimates, namely, (7.1) only assumed for $(x, y) \in \mathbb{R}^n \times (0, 1]$. Let us start by pointing out that $M^{\mathbf{f}}_\varphi(x, y)$ may sometimes be trivial for $y \in (0, 1)$, this may happen even if φ is non-degenerate:

Example 7.5. Let $\omega \in \mathbb{S}^{n-1}$ and $r \in \mathbb{R}_+$. Denote $[0, r\omega] = \{\sigma\omega : \sigma \in [0, r]\}$. Suppose that $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, X)$ is such that $\operatorname{supp} \hat{\mathbf{f}} \subset [0, r\omega]$ and $\psi \in \mathcal{S}_0(\mathbb{R}^n)$ is any wavelet satisfying $\operatorname{supp} \hat{\psi} \subset \mathbb{R}^n \setminus [0, r\omega]$, then

$$\mathcal{W}_\psi \mathbf{f}(x, y) = \frac{1}{(2\pi)^n} \left\langle \hat{\mathbf{f}}(u), e^{ixu} \overline{\hat{\psi}}(yu) \right\rangle = 0, \quad \text{for all } y \in (0, 1).$$

Fortunately, we will show that the only distributions $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, X) \setminus \mathcal{S}'(\mathbb{R}^n, E)$ that may satisfy a local class estimate, with respect to a non-degenerate wavelet, are those whose Fourier transforms are compactly supported.

We need to introduce some terminology in order to move further on. We will make use of weak integrals for X-valued functions as defined, for example, in [39, p. 77]. We say that a tempered X-valued distribution $\mathbf{g} \in \mathcal{S}'(\mathbb{R}^n, X)$ is *weakly regular* if there exists an X-valued function $\tilde{\mathbf{g}}$ such that $\rho\tilde{\mathbf{g}}$ is weakly integrable over $\mathbb{R}^n$ for all $\rho \in \mathcal{S}(\mathbb{R}^n)$ and

$$\langle \mathbf{g}, \rho \rangle = \int_{\mathbb{R}^n} \rho(t)\tilde{\mathbf{g}}(t) \in X,$$

where the last integral is taken in the weak sense. We identify $\mathbf{g}$ with $\tilde{\mathbf{g}}$, so, as usual, we write $\mathbf{g} = \tilde{\mathbf{g}}$. The same notion makes sense, modulo X-valued polynomials, on $\mathcal{S}_0'(\mathbb{R}^n, X)$.

Let us recall some facts about (vector valued) compactly supported distributions. Let $\mathbf{g} \in \mathcal{S}'(\mathbb{R}^n, X)$ have support in $\overline{B(0,r)}$, the closed ball of radius r. Then, the following version of the Schwartz-Paley-Wiener theorem holds: $\mathbf{G}(z) = \left\langle \mathbf{g}(u), e^{-iz\cdot u} \right\rangle, z \in \mathbb{C}^n$, is an X-valued entire function which defines a weakly regular tempered distribution, and $\mathbf{G}(\xi) = \hat{\mathbf{g}}(\xi), \xi \in \mathbb{R}^n$; moreover, $\mathbf{G}$ is of weakly exponential type, i.e., for all $\mathbf{w}^* \in X'$ one can find constants $C_{\mathbf{w}^*} > 0$ and $N_{\mathbf{w}^*} \in \mathbb{N}$ with

$$|\langle \mathbf{w}^*, \mathbf{G}(z) \rangle| \leq C_{\mathbf{w}^*}(1+|z|)^{N_{\mathbf{w}^*}} e^{r|\Im m\, z|}, \quad z \in \mathbb{C}^n. \tag{7.10}$$

Conversely, if $\mathbf{G}$ is an X-valued entire function which defines a weakly regular tempered distribution and for all $w^* \in X^*$ there exist $C_{w^*} > 0$ and $N_{w^*} \in \mathbb{N}$ such that (7.10) holds, then $\hat{\mathbf{G}} = \mathbf{g}$, where $\mathbf{g} \in \mathcal{S}'(\mathbb{R}^n, X)$ and $\operatorname{supp} \mathbf{g} \subseteq \overline{B(0,r)}$.

The following concept for non-degenerate test functions is of much relevance for the problem under consideration.

Definition 7.6. *Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be non-degenerate. Given $\omega \in \mathbb{S}^{n-1}$, consider the function of one variable $R_\omega(r) = \hat{\varphi}(r\omega) \in C^\infty[0,\infty)$. We define the index of non-degenerateness of φ as the (finite) number*

$$\tau = \inf\left\{ r \in \mathbb{R}_+ : \operatorname{supp} R_\omega \cap [0,r] \neq \emptyset, \forall \omega \in \mathbb{S}^{n-1} \right\}.$$

We first study local class estimates for wavelets in $\mathcal{S}_0(\mathbb{R}^n)$.

Proposition 7.7. *Let $\mathbf{f} \in \mathcal{S}_0'(\mathbb{R}^n, X)$ and let $\psi \in \mathcal{S}_0(\mathbb{R}^n)$ be a non-degenerate wavelet with index of non-degenerateness τ. Suppose that $\mathcal{W}_\psi \mathbf{f}(x,y)$ takes values in E for almost every $(x,y) \in \mathbb{R}^n \times (0,1]$ and is measurable as an E-valued function on $\mathbb{R}^n \times (0,1]$. Furthermore, assume that it satisfies the local class estimate*

$$\|\mathcal{W}_\psi \mathbf{f}(x,y)\| \leq C \frac{(1+|x|)^l}{y^k}, \quad \textit{for almost all } (x,y) \in \mathbb{R}^n \times (0,1].$$

Let $r > \tau$. Then there exists an X-valued entire function $\mathbf{G}$, which defines a weakly regular tempered distribution and satisfies (7.10), such that

$$\mathbf{f} - \mathbf{G} \in \mathcal{S}'_0(\mathbb{R}^n, E).$$

Proof. Let r_1 be such that $\tau < r_1 < r$. A similar argument to that given in the proof of Proposition 3.10 shows the existence of a reconstruction wavelet η for ψ with the property $\operatorname{supp}\hat{\eta} \subset B(0, r_1)$. Observe now that if $\operatorname{supp}\hat{\rho} \subseteq \mathbb{R}^n \setminus B(0, r_1)$, then

$$\mathcal{W}_{\bar{\eta}}\rho(x,y) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot u}\hat{\rho}(u)\hat{\eta}(-yu)\mathrm{d}u = 0, \quad \text{for all } y \in [1,\infty).$$

Hence, the same argument applied in Proposition 7.1 applies to show

$$\langle \mathbf{f}, \rho\rangle = \frac{1}{c_{\psi,\eta}}\int_0^1\int_{-\infty}^{\infty} \mathcal{W}_{\psi}\mathbf{f}(x,y)\mathcal{W}_{\bar{\eta}}\rho(x,y)\,\frac{\mathrm{d}x\mathrm{d}y}{y}\,, \tag{7.11}$$

for all $\rho \in \mathcal{S}_0(\mathbb{R}^n)$ with $\operatorname{supp}\hat{\rho} \subseteq \mathbb{R}^n \backslash B(0, r_1)$. Choose $\chi \in C^\infty(\mathbb{R}^n)$ such that $\chi(u) = 1$ for all $u \in \mathbb{R}^n \backslash B(0, r)$ and $\operatorname{supp}\chi \in \mathbb{R}^n \backslash B(0, r_1)$. Now, $\hat{\chi} * \mathbf{f}$ is well defined since $\hat{\chi} \in \mathcal{O}'_C(\mathbb{R}^n)$ (the space of convolutors, cf. [43]), and actually (7.11) implies that $(2\pi)^{-n}\hat{\chi} * \mathbf{f} \in \mathcal{S}'_0(\mathbb{R}^n, E)$. Therefore, $\mathbf{G} = \mathbf{f} - (2\pi)^{-n}\hat{\chi} * \mathbf{f}$ satisfies the requirements because $\operatorname{supp}\hat{\mathbf{G}} \subseteq \overline{B(0, r)}$. □

The following theorem deals with the general case, we state it in terms of the transform $M^{\mathbf{f}}_{\varphi}$.

Theorem 7.8. *Let $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, X)$ and let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be a non-degenerate test function with index of non-degenerateness τ. Assume:*

- (i) *$M^{\mathbf{f}}_{\varphi}(x, y)$ takes values in E for almost all $(x, y) \in \mathbb{R}^n \times (0, 1]$ and is measurable as an E-valued function on $\mathbb{R}^n \times (0, 1]$.*
- (ii) *There exist constants $k, l \in \mathbb{N}$ and $C > 0$ such that (7.1) holds for almost all $(x, y) \in \mathbb{R}^n \times (0, 1]$.*

Then, for any $r > \tau$, there exists an X-valued entire function $\mathbf{G}$, which defines a weakly regular tempered distribution and satisfies (7.10), such that

$$\mathbf{f} - \mathbf{G} \in \mathcal{S}'(\mathbb{R}^n, E).$$

Proof. Let $\psi \in \mathcal{S}_0(\mathbb{R}^n)$ be given by $\hat{\psi}(u) = e^{-|u|-(1/|u|)}$. Set $\psi_1 = \check{\bar{\varphi}} * \psi$, then, $\psi_1 \in \mathcal{S}_0(\mathbb{R}^n)$ is also a non-degenerate wavelet and we have the formula

$$\mathcal{W}_{\psi_1}\mathbf{f}(x,y) = \int_{\mathbb{R}^n} M^{\mathbf{f}}_{\varphi}(x + yu, y)\bar{\psi}(u)\mathrm{d}u,$$

and so the restriction of $\mathbf{f}$ to $\mathcal{S}_0(\mathbb{R}^n)$ satisfies the hypotheses of Proposition 7.7; consequently there exists $\mathbf{G}_1 \in \mathcal{S}'(\mathbb{R}^n, X)$ with $\operatorname{supp}\hat{\mathbf{G}}_1 \subset B(0, r)$ such that the restriction of $\mathbf{f} - \mathbf{G}_1$ to $\mathcal{S}_0(\mathbb{R}^n)$ belongs to $\mathcal{S}'_0(\mathbb{R}^n, E)$. Finally, one can find $\mathbf{G}_2 \in \mathcal{S}'(\mathbb{R}^n, X)$ whose Fourier transform is supported at the origin and $\mathbf{f} - \mathbf{G_1} - \mathbf{G_2} \in \mathcal{S}'_0(\mathbb{R}^n, E)$, and therefore, $\mathbf{G} = \mathbf{G}_2 + \mathbf{G}_1$ satisfies all the requirements. □

One may be tempted to think that in Proposition 7.7 and Theorem 7.8 it is possible to take $\mathbf{G}$ with support in $\overline{B(0,\tau)}$; however, this is not true, in general, as the following counterexample shows.

Example 7.9. Let X, E, and the sequence $\{\chi_\nu\}_{\nu=1}^{\infty}$ be as in Example 7.4. We work in dimension $n=1$. We assume additionally that $\sup_\xi |\chi_\nu(\xi)| = 1$, for all $\nu \in \mathbb{N}$. Let $\tau \geq 0$, the wavelet ψ, given by $\hat{\psi}(u) = e^{-|u|-(1/(|u|-\tau))}$ for $|u| > \tau$ and $\hat{\psi}(u) = 0$ for $|u| \leq \tau$, has index of non-degenerateness τ. Consider the $C(\mathbb{R})$-valued distribution

$$\mathbf{f}(t,\xi) = \sum_{\nu=1}^{\infty} e^{\nu + i\left(\tau+\frac{1}{n}\right)t}\chi_\nu(\xi) \in \mathcal{S}'(\mathbb{R}_t, C(\mathbb{R}_\xi)) \setminus \mathcal{S}'(\mathbb{R}_t, C_b(\mathbb{R}_\xi)).$$

Then,

$$\mathcal{W}_\psi \mathbf{f}(x,y)(\xi) = \sum_{1\leq \nu < \frac{y}{\tau(1-y)}} e^{\nu + (ix-y)(\tau+1/\nu) - \frac{\nu}{y-\nu\tau(1-y)}}\chi_\nu(\xi), \quad 0<y<1.$$

and hence, $\|\mathcal{W}_\psi \mathbf{f}(x,y)\|_{C_b(\mathbb{R})} \leq 1$, for all $0<y<1$. Therefore, the hypotheses of both Proposition 7.7 and Theorem 7.8 are fully satisfied, however, $\mathbf{f}-\mathbf{G} \notin \mathcal{S}'(\mathbb{R}, C_b(\mathbb{R}))$, for any $\mathbf{G} \in \mathcal{S}'(\mathbb{R}, C(\mathbb{R}))$ with $\operatorname{supp} \hat{\mathbf{G}} \subseteq [-\tau,\tau]$.

7.3. **The $\phi-$transform.** Theorem 7.8 can be improved for the $\phi-$transform. Observe that the index of non-degenerateness of ϕ is now $\tau = 0$. Remarkably, one gets a full characterization of the space $\mathcal{S}'(\mathbb{R}^n, E)$.

Theorem 7.10. *Let $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, X)$ and let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\mu_0(\phi) = 1$. Necessary and sufficient conditions for $\mathbf{f}$ to belong to the space $\mathcal{S}'(\mathbb{R}^n, E)$ are:*

(i) *$F_\phi \mathbf{f}(x,y)$ takes values in E for almost all $(x,y) \in \mathbb{R}^n \times (0,1]$ and is measurable as an E-valued function on $\mathbb{R}^n \times (0,1]$, and,*
(ii) *There exist constants $k, l \in \mathbb{N}$ and $C>0$ such that*

$$\|F_\phi \mathbf{f}(x,y)\| \leq C\frac{(1+|x|)^l}{y^k}, \quad \textit{for almost all } (x,y) \in \mathbb{R}^n \times (0,1].$$

Proof. By Theorem 7.8, one may assume that $\operatorname{supp} \hat{\mathbf{f}} \subseteq \overline{B(0,1)}$. So, $\mathbf{f}$ is given by an entire function of weakly exponential type which defines a weakly regular X-valued tempered distribution. Let $\rho \in \mathcal{S}(\mathbb{R}^n)$ such that $\rho(u) = 1$ for $u \in B(0,3/2)$ and $\operatorname{supp}\rho \subset B(0,2)$. Choose $2\sigma < 1$ such that $|\hat{\phi}(u)| > 0$ for all $u \in \overline{B(0,2\sigma)}$. For a fixed $x \in \mathbb{R}^n$, the function $\hat{\chi}_x(u) = e^{ixu}\rho(u)/\hat{\phi}(-\sigma u)$ defines an element of $\mathcal{S}(\mathbb{R}^n)$. Thus,

$$\begin{aligned}\mathbf{f}(x) &= \frac{1}{(2\pi)^n}\left\langle \hat{\mathbf{f}}(u), e^{ix\cdot u}\right\rangle = \frac{1}{(2\pi)^n}\left\langle \hat{\mathbf{f}}(u), \hat{\chi}_x(u)\hat{\phi}(-\sigma u)\right\rangle \\ &= \frac{1}{\sigma^n}\left\langle \mathbf{f}(t), \int_{\mathbb{R}^n}\chi_x(\xi)\phi\left(\frac{t+\xi}{\sigma}\right)\mathrm{d}\xi\right\rangle = \int_{\mathbb{R}^n}\chi_x(\xi)F_\phi\mathbf{f}(-\xi,\sigma)\mathrm{d}\xi \in E,\end{aligned}$$

where the exchange with the integral sign can be established as in the proof of Proposition 7.1. Hence the entire *function* $\mathbf{f}$ takes values in E. Moreover,

$$\|\mathbf{f}(x)\| < \frac{C}{\sigma^k}\int_{\mathbb{R}^n}(1+|\xi|)^l\,|\chi_x(\xi)|\,\mathrm{d}\xi \leq C_1(1+|x|)^N, \quad \text{for all } x\in\mathbb{R}^n,$$

for some constants $C_1 > 0$ and $N \in \mathbb{N}$. Clearly, the last E-norm estimate over the growth of $\mathbf{f}$ implies that $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, E)$, as required. □

7.4. **Local Class Estimates and Strongly Non-degenerate Wavelets.** A strengthened version of both Theorem 7.2 and Theorem 7.8 holds if we restrict the non-degenerate wavelets to those fulfilling the requirements of the following definition.

Definition 7.11. *Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be a wavelet. We call ψ strongly non-degenerate if there exist constants $N \in \mathbb{N}$, $r > 0$, and $C > 0$ such that*

$$C\,|u|^N \leq |\hat{\psi}(u)|\ , \quad \textit{for all } |u| \leq r. \tag{7.12}$$

Theorem 7.12. *Let $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, X)$ and let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be a strongly non-degenerate wavelet. Assume:*

(i) *$\mathcal{W}_\psi\mathbf{f}(x,y)$ takes values in E for almost all $(x,y) \in \mathbb{R}^n \times (0,1]$ and is measurable as an E-valued function on $\mathbb{R}^n \times (0,1]$.*
(ii) *There exist constants $k, l \in \mathbb{N}$ and $C > 0$ such that*

$$\|\mathcal{W}_\psi\mathbf{f}(x,y)\| \leq C\frac{(1+|x|)^l}{y^k}, \quad \textit{for almost all } (x,y)\in\mathbb{R}^n\times(0,1].$$

Then, there exists $\mathbf{G} \in \mathcal{S}'(\mathbb{R}^n, X)$ such that $\mathbf{f} - \mathbf{G} \in \mathcal{S}'(\mathbb{R}^n, E)$ and $\operatorname{supp}\hat{\mathbf{G}} \subseteq \{0\}$.

Proof. As in Theorem 7.10, we may assume that $\operatorname{supp}\hat{\mathbf{f}} \subseteq \overline{B(0,1)}$. Let $\rho \in \mathcal{S}(\mathbb{R}^n)$ be the same as in the proof of Theorem 7.10. We can find $\sigma, C_1 > 0$ and $N \in \mathbb{N}$ such that $2\sigma \leq 1$ and $C_1|u|^N \leq |\hat{\psi}(u)|$, for all $u \in \overline{B(0,2\sigma)}$. Given $\eta \in \mathcal{S}_0(\mathbb{R}^n)$, then $\hat{\chi}(u) = \hat{\chi}_\eta(u) = \rho(u)\hat{\eta}(-u)/\overline{\hat{\psi}}(\sigma u)$ defines an element of $\mathcal{S}(\mathbb{R}^n)$ in a continuous fashion, consequently, the mapping $\gamma : \mathcal{S}_0(\mathbb{R}^n) \mapsto [0,\infty)$ given by $\gamma(\eta) = \int_{\mathbb{R}^n}(1+|\xi|)^l\,|\chi(\xi)|\,\mathrm{d}\xi$ is a continuous seminorm over $\mathcal{S}_0(\mathbb{R}^n)$. Now, for any $\eta \in \mathcal{S}_0(\mathbb{R}^n)$,

$$\langle \mathbf{f}, \eta\rangle = \frac{1}{(2\pi)^n}\left\langle \hat{\mathbf{f}}(u), \hat{\chi}(u)\overline{\hat{\psi}}(\sigma u)\right\rangle = \int_{\mathbb{R}^n}\chi(\xi)\mathcal{W}_\psi\mathbf{f}(-\xi,\sigma)\mathrm{d}\xi.$$

Therefore, $\|\langle \mathbf{f}, \eta\rangle\| \leq (C/\sigma^k)\gamma(\eta)$, for all $\eta \in \mathcal{S}_0(\mathbb{R}^n)$, and the latter implies that the restriction of $\mathbf{f}$ to $\mathcal{S}_0(\mathbb{R}^n)$ belongs to $\mathcal{S}_0'(\mathbb{R}^n, E)$. The standard argument (see the proof of Theorem 7.2) yields the existence of $\mathbf{G}$ satisfying all the requirements.

□

It should be noticed that the class of strongly non-degenerate wavelets coincides with that of Drozhzhinov-Zavialov wavelets, introduced in Example

3.7. Indeed, the condition (7.12) holds if and only if the Taylor polynomial of order N at the origin of $\hat{\psi}$ is non-degenerate in the sense of Example 3.7.

In dimension $n = 1$, there is no distinction between non-degenerateness and strong non-degenerateness, whenever we consider wavelets in $\mathcal{S}(\mathbb{R}) \setminus \mathcal{S}_0(\mathbb{R})$. Actually, a stronger result than Theorem 7.12 holds in the one-dimensional case.

Proposition 7.13. *Let* $\mathbf{f} \in \mathcal{S}'(\mathbb{R}, X)$ *and let* $\psi \in \mathcal{S}(\mathbb{R})$ *be a wavelet with* $\mu_d(\psi) \neq 0$*, for some* $d \in \mathbb{N}$*. If the conditions (i) and (ii) of Theorem 7.12 are satisfied, then there exists an* X*-valued polynomial* $\mathbf{P}$ *of degree at most* $d-1$ *such that* $\mathbf{f} - \mathbf{P} \in \mathcal{S}'(\mathbb{R}, E)$*.*

Proof. There exists $\phi \in \mathcal{S}(\mathbb{R})$ such that $\overline{\phi^{(d)}} = (-1)^d \psi$, and we may assume that $\mu_0(\phi) = 1$. Then,

$$F_{\phi}(\mathbf{f}^{(d)})(x, y) = \frac{1}{y^d} \mathcal{W}_{\psi} \mathbf{f}(x, y),$$

hence, an application of Theorem 7.10 gives that $\mathbf{f}^{(d)} \in \mathcal{S}'(\mathbb{R}, E)$, and this clearly implies the existence of $\mathbf{P}$ with the desired properties. □

Observe that the conclusion of Proposition 7.13 does not hold for multi-dimensional wavelets, in general, even if they are strongly non-degenerate. This fact is shown by Example 7.4.

Naturally, if X is a normed space in Theorem 7.12, then $\mathbf{G}$ must be an X-valued polynomial, this fact is stated in the next corollary. Corollary 7.14 extends an important result of Drozhzhinov and Zavialov [6, Thm. 2.1].

Corollary 7.14. *Let the hypotheses of Theorem 7.10 be satisfied. If* X *is a normed space, then there is an* X*-valued polynomial* $\mathbf{P}$ *such that* $\mathbf{f} - \mathbf{P} \in \mathcal{S}'(\mathbb{R}^n, E)$*. Moreover, if* P_q*,* $q \in \mathbb{N}$*, are the homogeneous terms of the Taylor polynomials of* $\hat{\psi}$ *at the origin (cf. (6.7)), then* $\overline{P_q}\,(\partial/\partial t)\,\mathbf{P}$ *is an* E*-valued polynomial, for each* $q \in \mathbb{N}$*.*

Proof. The existence of the polynomial is clear. The proof of the remaining assertion is identically the same as that of Corollary 7.3. □

In general, the degree of the the polynomial $\mathbf{P}$ occurring in Corollary 7.14 depends merely on $\mathbf{f}$, and not on the wavelet. However, when the Taylor polynomials of the wavelet $\hat{\psi}$ posses a rich algebraic structure, it is possible to say more about the degree of $\mathbf{P}$. This fact was already observed in [6, Thm. 2.2] for Banach spaces X. We denote by $\mathbb{P}_d(\mathbb{R}^n)$ the ideal of (scalar-valued) polynomials of the form $Q(t) = \sum_{d \leq |m| \leq N} a_m t^m$, for some $N \in \mathbb{N}$.

Corollary 7.15. *Let the hypotheses of Corollary 7.14 be satisfied. If there exists* $d \in \mathbb{N}$ *such that* $\mathbb{P}_d(\mathbb{R}^n)$ *is contained in the ideal generated by the polynomials* $P_1, P_2, \ldots, P_d$*, where the* P_q*,* $q \in \mathbb{N}$*, are the homogeneous terms of the Taylor polynomials of* $\hat{\psi}$ *at the origin (cf. (6.7)), then there exists an* X*-valued polynomial* $\mathbf{P}$ *of degree at most* $d-1$ *such that* $\mathbf{f} - \mathbf{P} \in \mathcal{S}'(\mathbb{R}^n, E)$*.*

Proof. Corollary 7.14 yields the existence of an X-valued polynomial $\tilde{\mathbf{P}}(t) = \mathbf{P}(t) + \sum_{d\leq|m|\leq N} \mathbf{w}_m t^m$ such that $\mathbf{f} - \tilde{\mathbf{P}} \in \mathcal{S}'(\mathbb{R}^n, E)$ and $\mathbf{P}$ has degree at most $d-1$. Then, we must show that $\mathbf{w}_m \in E$, for $d \leq |m| \leq N$. But Corollary 7.14 also implies that $\overline{P_q(\partial/\partial t)}\tilde{\mathbf{P}}$ is an E-valued polynomial for $q = 1, \dots, d$, and since $\mathbb{P}_d(\mathbb{R}^n)$ is also contained in the ideal generated by $\overline{P}_1, \dots, \overline{P}_d$, we obtain at once that $\mathbf{w}_m = m!((\partial^{|m|}/\partial t^m)\tilde{\mathbf{P}})(0) \in E$, for $d \leq |m| \leq N$. □

8. Several Applications

8.1. **Distributionally Small Distributions at Infinity.** Estrada [10] has characterized the class of distributions which are distributionally small at infinity (cf. Example 2.3), that is, the ones which have an asymptotic expansion

$$\mathbf{f}(\lambda t) \sim \sum_{|m|=0}^{\infty} \frac{(-1)^{|m|}}{m!\lambda^{|m|+n}} \delta^{(m)}(t)\mathbf{w}_m \quad \text{as } \lambda \to \infty \ \text{ in } \mathcal{S}'(\mathbb{R}^n, E), \tag{8.1}$$

for some multi-sequence $\{\mathbf{w}_m\}_{m\in\mathbb{N}^n}$ in E.

The distributionally small distributions are precisely the elements of the space $\mathcal{K}'(\mathbb{R}^n, E)$, where $\mathcal{K}(\mathbb{R}^n)$ is test function space of the so-called GLS symbols [17], defined as follows. Given $\beta \in \mathbb{R}$, the space $\mathcal{K}_\beta(\mathbb{R}^n)$ is formed by those smooth functions ρ such that $\rho^{(m)}(t) = O(|t|^{\beta-|m|})$ as $|t| \to \infty$ for each $m \in \mathbb{N}^n$, provided with the topology generated by the seminorms

$$\max\{\sup_{|t|\leq 1} |\rho^{(m)}(t)|, \sup_{|t|\geq 1} |t|^{|m|-\beta} |\rho^{(m)}(t)|\}.$$

Then, $\mathcal{K}(\mathbb{R}^n) = \operatorname{ind} \lim_{\beta\to\infty} \mathcal{K}_\beta(\mathbb{R}^n)$. Observe that the elements of $\mathcal{K}(\mathbb{R}^n)$ are indeed symbols of pseudodifferential operators.

Using the Theorem 6.6 and Fourier transforming (8.1), we obtain the following wavelet characterization of $\mathcal{K}'(\mathbb{R}^n, E)$. Let $\psi \in \mathcal{S}_0(\mathbb{R}^n)$ be a non-degerate wavelet. *Then, a tempered E-valued distribution $\mathbf{f}$ belongs to $\mathcal{K}'(\mathbb{R}^n, E)$ if and only if there exists a sequence $\{\nu_p\}_{p=0}^{\infty}$ of non-negative integers such that for each $p \in \mathbb{N}$*

$$\limsup_{\varepsilon\to 0^+} \sup_{|x|^2+y^2=1,\, y>0} \frac{y^{\nu_p}}{\varepsilon^p} \left\| \mathcal{W}_\psi \hat{\mathbf{f}}(\varepsilon x, \varepsilon y) \right\| < \infty.$$

Example 8.1. We will find the complete distributional asymptotic expansion of Riemann's "nondifferentiable function" at $t = 0$ and $t = 1$, i.e.,

$$\sum_{n=1}^{\infty} \frac{\sin(\pi n^2 t)}{n^2}.$$

Let us consider the distribution

$$f(t) = \sum_{n=1}^{\infty} e^{i\pi n^2 t} - \frac{e^{i\frac{\pi}{4}}}{2(t+i0)^{\frac{1}{2}}},$$

where $(t+i0)^\alpha$ is the distributional boundary value of the analytic function z^α, $\Im m\, z>0$. Then, if $\psi\in\mathcal{S}_0(\mathbb{R})$ and $k\in\mathbb{N}$, there exists C_k such that

$$|\mathcal{W}_\psi f(\varepsilon x,\varepsilon y)| = \left|\sum_{n=1}^{\infty} e^{i\varepsilon x\pi n^2}\overline{\hat{\psi}}\left(\varepsilon y\pi n^2\right) - \int_0^\infty e^{i\varepsilon x\pi u^2}\overline{\hat{\psi}}\left(\varepsilon y\pi u^2\right)\mathrm{d}u\right| \leq \frac{C_k}{y}\varepsilon^k,$$

for all $y,\varepsilon\in(0,1)$ and $|x|\leq 1$, as shown by the Euler-Maclaurin summation formula [15]. This implies that $\hat{f}\in\mathcal{K}'(\mathbb{R})$ and thus satisfies the moment asymptotic expansion,

$$\hat{f}(\lambda u)\sim\sum_{m=0}^{\infty}\frac{(-1)^m\mu_m}{m!\lambda^{m+1}}\delta^{(m)}(u)\quad\text{as }\lambda\to\infty\text{ in }\mathcal{K}'(\mathbb{R}),$$

where the moments of $\hat{f}$ can be evaluated in the Cesàro sense [10, 15]. If H denotes the Heaviside function, ζ the Riemann zeta function, and (C) stands for limits in the Cesàro sense, then

$$\begin{aligned}\frac{1}{2\pi}\mu_m &= \frac{1}{2\pi}\left\langle\hat{f}(u),u^m\right\rangle = \pi^m\left\langle\sum_{n=1}^{\infty}\delta(\xi-n)-H(\xi),\xi^{2m}\right\rangle\\ &= \pi^m\lim_{x\to\infty}\left(\sum_{1\leq n\leq x}n^{2m}-\int_0^x\xi^{2m}\mathrm{d}\xi\right) = \pi^m\zeta(-2m)\quad\text{(C)},\end{aligned}$$

and hence $\mu_0=2\pi\zeta(0)=-\pi$, and $\mu_m=\zeta(-2m)=0$ for every $m\geq 1$. Consequently,

$$\hat{f}(\lambda u) = -\pi\frac{\delta(u)}{\lambda}+o\left(\frac{1}{\lambda^\infty}\right)\quad\text{as }\lambda\to\infty\text{ in }\mathcal{S}'(\mathbb{R}),$$

where $o\left(\frac{1}{\lambda^\infty}\right)$ means $o\left(\frac{1}{\lambda^k}\right)$ for every $k\in\mathbb{N}$. Taking Fourier inverse transform and integrating [60] the resulting asymptotic expression, we have

$$W(\varepsilon t) := \sum_{n=1}^{\infty}\frac{e^{i\pi n^2\varepsilon t}}{n^2} = \frac{\pi^2}{6}+\varepsilon^{\frac{1}{2}}i\pi e^{i\frac{\pi}{4}}(t+i0)^{\frac{1}{2}}-\frac{i\pi}{2}\varepsilon t+o(\varepsilon^\infty)\quad\text{in }\mathcal{S}'(\mathbb{R})$$

as $\varepsilon\to 0^+$. Finally, if we split into real and imaginary parts, we obtain the ensuing distributional asymptotic formulas at the origin:

$$\sum_{n=1}^{\infty}\frac{\sin(\pi n^2\varepsilon t)}{n^2} = \frac{\sqrt{2}\pi}{2}\operatorname{sgn}(t)\,|\varepsilon t|^{\frac{1}{2}}-\frac{\pi}{2}\varepsilon t+o(\varepsilon^\infty)\quad\text{as }\varepsilon\to 0^+\text{ in }\mathcal{S}'(\mathbb{R})$$

and

$$\sum_{n=1}^{\infty}\frac{\cos(\pi n^2\varepsilon t)}{n^2} = \frac{\pi^2}{6}-\frac{\sqrt{2}\pi}{2}\,|\varepsilon t|^{\frac{1}{2}}+o(\varepsilon^\infty)\quad\text{as }\varepsilon\to 0^+\text{ in }\mathcal{S}'(\mathbb{R}).$$

We can also determine the distributional asymptotic expansions of these functions at $t=1$. Observe that $W(1+t)=(1/2)W(4t)-W(t)$, thus the behavior at origin implies that

$$W(1+\varepsilon t) = -\frac{\pi^2}{6} - \frac{i\pi}{2}\varepsilon t + o(\varepsilon^\infty) \quad \text{as } \varepsilon \to 0^+ \text{ in } \mathcal{S}'(\mathbb{R}),$$

and hence,

$$\sum_{n=1}^{\infty} \frac{\sin(\pi n^2(1+\varepsilon t))}{n^2} = -\frac{\pi}{2}\varepsilon t + o(\varepsilon^\infty) \quad \text{as } \varepsilon \to 0^+ \text{ in } \mathcal{S}'(\mathbb{R})$$

and

$$\sum_{n=1}^{\infty} \frac{\cos(\pi n^2(1+\varepsilon t))}{n^2} = -\frac{\pi^2}{6} + o(\varepsilon^\infty) \quad \text{as } \varepsilon \to 0^+ \text{ in } \mathcal{S}'(\mathbb{R}).$$

It should be noticed that all these expansions actually hold in the ordinary sense, and not just distributionally, when considered up to order $o(\varepsilon)$ (cf. [19, 20]).

8.2. **Applications to Regularity Theory in Algebras of Generalized Functions.** In this section we show how the Tauberian theorems for the wavelet transform can be used as a standard device to derive results in the regularity theory for algebras of generalized functions.

First, we consider the algebra of tempered generalized functions which contains $\mathcal{S}'(\mathbb{R}^n)$ as a proper subspace. Let $\mathcal{O}_M(\mathbb{R}^n)$ be the space of multipliers of $\mathcal{S}(\mathbb{R}^n)$ [43], that is, the space of smooth functions whose derivatives are bounded by polynomials, of possible different degress. Colombeau [2] defined the algebra of tempered generalized functions as the quotient $\mathcal{G}_\tau(\mathbb{R}^n) = \mathcal{E}_{M,\tau}(\mathbb{R}^n)/\mathcal{N}_\tau(\mathbb{R}^n)$, where $\mathcal{E}_{M,\tau}(\mathbb{R}^n)$ is the algebra of nets $(f_\varepsilon)_\varepsilon \in \mathcal{O}_M(\mathbb{R}^n)^{(0,1)}$

$$(\forall m \in \mathbb{R}^n)(\exists N \in \mathbb{N})(\sup_{x\in\mathbb{R}^n}(1+|x|)^{-N}|f_\varepsilon^{(m)}(x)| = O(\varepsilon^{-N}))$$

while its ideal $\mathcal{N}_\tau(\mathbb{R}^n)$ consists of those such that

$$(\forall m \in \mathbb{R}^n)(\exists N \in \mathbb{N})(\forall b > 0)(\sup_{x\in\mathbb{R}^n}(1+|x|)^{-N}|f_\varepsilon^{(m)}(x)| = O(\varepsilon^{b})).$$

We can embed $\mathcal{S}'(\mathbb{R}^n)$ into $\mathcal{G}_\tau(\mathbb{R}^n)$ via $\iota(f) = [(f*\phi_\varepsilon)_\varepsilon]$, where ϕ satisfies (3.8).

The algebra of regular tempered generalized functions $\mathcal{G}_\tau^\infty(\mathbb{R}^n)$ consists of those nets in $\mathcal{O}_M(\mathbb{R}^n)^{(0,1)}$ such that

$$(\exists a \in \mathbb{R})(\forall m \in \mathbb{R}^n)(\exists N \in \mathbb{N})(\sup_{x\in\mathbb{R}^n}(1+|x|)^{-N}|f_\varepsilon^{(m)}(x)| = O(\varepsilon^{-a})). \tag{8.2}$$

We will show the regularity theorem for $\mathcal{G}_\tau^\infty(\mathbb{R}^n)$; it originally appeared in [23].

Theorem 8.2. $\mathcal{S}'(\mathbb{R}^n) \cap \mathcal{G}_\tau^\infty(\mathbb{R}^n) = \mathcal{O}_M(\mathbb{R}^n)$.

This equality means that if $f \in \mathcal{S}'(\mathbb{R}^n)$ and $f_\varepsilon = f * \phi_\varepsilon, \varepsilon \in (0,1)$, determines an element of $\mathcal{G}_\tau^\infty(\mathbb{R}^n)$, then $f \in \mathcal{O}_M(\mathbb{R}^n)$.

Proof. The inclusion $\mathcal{O}_M(\mathbb{R}^n) \subseteq \mathcal{S}'(\mathbb{R}^n) \cap \mathcal{G}^\infty_\tau(\mathbb{R}^n)$ is obvious. Let $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $\iota(f) \in \mathcal{G}^\infty_\tau(\mathbb{R}^n)$, that is, the net $f_\varepsilon = f * \phi_\varepsilon$ satisfies (8.2). We should show that $f^{(m)}$ is continuous of polynomial growth for each $m \in \mathbb{N}^n$. Let $\nu \in \mathbb{N}$ be such that $\beta = 2\nu - a > 0$. Then, there exists $N_0 \in \mathbb{N}$ such that

$$\sup_{x\in\mathbb{R}^n} (1+|x|)^{-N_0} \left|\mathcal{W}_\psi f^{(m)}(x,y)\right| = O(y^\beta), \quad 0 < y < 1, \tag{8.3}$$

where $\psi = \overline{\Delta^\nu \check{\phi}}$, a non-degenerate wavelet. Define $\mathbf{h}$ by $\langle \mathbf{h}, \rho \rangle = f^{(m)} * \check{\rho}$, for $\rho \in \mathcal{S}(\mathbb{R}^n)$, then there exists $N > N_0$ such that $\mathbf{h} \in \mathcal{S}'(\mathbb{R}^n, E)$, where E is the Banach space of continuous functions $v \in C(\mathbb{R}^n)$ such that $\|v\| := \sup_{\xi\in\mathbb{R}^n} (1+|\xi|)^{-N} |v(\xi)| < \infty$, provided with the norm $\|\cdot\|$. Since $\mathcal{W}_\psi \mathbf{h}(x,y)(\xi) = \mathcal{W}_\psi f^{(m)}(\xi + x, y)$, the estimate (8.3) gives now

$$\limsup_{\varepsilon\to 0^+} \sup_{|x|\le 1,\, 0<y<1} \varepsilon^{-\beta} \|\mathcal{W}_\psi \mathbf{h}(\varepsilon x, \varepsilon y)\| < \infty.$$

The Theorem 6.6 implies, in particular, that $\mathbf{h}$ has a distributional point value at the origin (cf. Example 2.2), say $\mathbf{h}(0) = v \in E$, i.e., for each test function ρ, $\lim_{\varepsilon\to 0^+} f^{(m)} * \check{\rho}_\varepsilon = \lim_{\varepsilon\to 0^+} \langle \mathbf{h}(\varepsilon t), \rho(t)\rangle = v \int_{\mathbb{R}^n} \rho(t)\mathrm{d}t$, where the limit holds in E. But if we take $\rho = \check{\phi}$, we obtain in particular that $\lim_{\varepsilon\to 0^+} (f * \phi_\varepsilon)(\xi) = v(\xi)$ uniformly for ξ in compacts of $\mathbb{R}^n$, and this means exactly that $f^{(m)} = v$ is a continuous function of at most polynomial growth. □

Remark 8.3. Recall [2], the Colombeau algebra of generalized functions is defined as $\mathcal{G}(\Omega) = \mathcal{E}_M(\Omega)/\mathcal{N}(\Omega)$, where $\mathcal{E}_M(\Omega)$, $\mathcal{N}(\Omega)$, consist of nets of smooth functions in Ω, $(f_\varepsilon)_{\varepsilon\in(0,1)}$, with the properties

$$(\forall \omega \subset\subset \Omega)(\forall \nu \in \mathbb{N})(\exists N \in \mathbb{N})(\sup_{|m|\le\nu, x\in\omega} |f_\varepsilon^{(m)}(x)| = O(\varepsilon^{-N})),$$

$$(\forall \omega \subset\subset \Omega)(\forall b \in \mathbb{R})(\forall \nu \in \mathbb{N})(\sup_{|m|\le\nu, x\in\omega} |f_\varepsilon^{(m)}(x)| = O(\varepsilon^{b})).$$

The embedding of the Schwartz distribution space $\mathcal{E}'(\Omega)$ is realized through the sheaf homomorphism $\mathcal{E}'(\Omega) \ni f \mapsto \iota(f) = [(f * \phi_\varepsilon|\Omega)_\varepsilon] \in \mathcal{G}(\Omega)$, where $\phi \in \mathcal{S}(\mathbb{R}^n)$ is as before. This sheaf homomorphism, extended onto $\mathcal{D}'$, gives the embedding of $\mathcal{D}'(\Omega)$ into $\mathcal{G}(\Omega)$. The embedding respects the multiplication of smooth functions.

The generalized algebra of "smooth generalized functions" $\mathcal{G}^\infty(\Omega)$ is defined in [35] as the quotient of algebras $\mathcal{E}_M^\infty(\Omega)$ and $\mathcal{N}(\Omega)$, where $\mathcal{E}_M^\infty(\Omega)$, consists of nets of smooth functions in Ω with the property

$$(\forall \omega \subset\subset \Omega)(\exists a \in \mathbb{R})(\forall \nu \in \mathbb{N})(\sup_{|m|\le\nu, x\in\omega} |f_\varepsilon^{(m)}(x)| = O(\varepsilon^{-a})),$$

Note that $\mathcal{G}^\infty$ is a subsheaf of $\mathcal{G}$. Roughly speaking, it has the same role as C^∞ in $\mathcal{D}'$.

Similarly as above, one can prove the following well known assertion [35]:

Theorem 8.4. $\mathcal{D}'(\Omega) \cap \mathcal{G}^\infty(\Omega) = C^\infty(\Omega)$.

8.3. **Asymptotic Stabilization in Time for Cauchy Problems.** We retain in this subsection the notation from Example 3.8, that is, U is the unique solution to the Cauchy problem (3.9) and $\phi = (2\pi)^{-n}\hat{\eta}$, where $\eta \in \mathcal{S}(\mathbb{R}^n)$ satisfies $\eta(u) = e^{P(iu)}$, $u \in \Gamma$; thus, U is given by (3.10). We apply Theorem 6.2 to find sufficient geometric conditions for the stabilization in time of the solution to the Cauchy problem (3.9), namely, we study conditions which ensure the existence of a function $T : (A, \infty) \to \mathbb{R}_+$ and a constant $\ell \in \mathbb{C}$ such that the following limits exist

$$\lim_{t\to\infty} \frac{U(x,t)}{T(t)} = \ell, \quad \text{for each } x \in \mathbb{R}^n. \tag{8.4}$$

Let L be slowly varying at infinity and $\alpha \in \mathbb{R}$. We shall say that U *stabilizes along d-curves* (at infinity), relative to $\lambda^{\alpha}L(\lambda)$, if the following two conditions hold:

(1) There exist the limits

$$\lim_{\lambda\to\infty} \frac{U(\lambda x, \lambda^d t)}{\lambda^{\alpha}L(\lambda)} = U_0(x,t), \quad (x,t) \in \mathbb{S}^n \cap \mathbb{H}^{n+1}; \tag{8.5}$$

(2) There are constants $M \in \mathbb{R}_+$ and $l \in \mathbb{N}$ such that

$$\left| \frac{U(\lambda x, \lambda^d t)}{\lambda^{\alpha}L(\lambda)} \right| \leq \frac{M}{t^l}, \quad (x,t) \in \mathbb{S}^n \cap \mathbb{H}^{n+1}. \tag{8.6}$$

Theorem 8.5. *The solution U to the Cauchy problem (3.9) stabilizes along d-curves, relative to $\lambda^{\alpha}L(\lambda)$, if and only if f has quasiasymptotic behavior of degree α at infinity with respect to L.*

Proof. We have that $U(x,t) = F_{\phi}f(x,y)$, with $y = t^{1/d}$, then, conditions (8.5) and (8.6) translate directly into conditions (6.1) and (6.5), with $F_{x,y} = U_0(x, t^{1/d})$ and $k = dl$. Therefore, Theorem 6.4 yields the desired equivalence. □

Corollary 8.6. *If U stabilizes along d-curves, relative to $\lambda^{\alpha}L(\lambda)$, then U stabilizes in time with respect to $T(t) = t^{\alpha/d}L(t^{1/d})$. Moreover, the limit (8.4) holds uniformly for x in compacts of $\mathbb{R}^n$.*

Proof. By Theorem 8.5, there exists $g \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$f(\lambda\xi) \sim \lambda^{\alpha}L(\lambda)g(\xi) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{S}'(\mathbb{R}^n).$$

If $K \subset \mathbb{R}^n$ is compact, then,

$$\lim_{t\to\infty} \frac{U(x,t)}{T(t)} = \lim_{t\to\infty} \frac{1}{t^{\alpha/d}L(t^{1/d})} \left\langle f(t^{1/d}\xi), \phi\left(\xi - \frac{x}{t^{1/d}}\right)\right\rangle = \langle g(\xi), \phi(\xi)\rangle,$$

uniformly for $x \in K$ because $\phi\left(\xi - x/t^{1/d}\right) \to \phi(\xi)$ in $\mathcal{S}(\mathbb{R}^n)$, as $t \to \infty$. □

Example 8.7. *The heat equation.* When $\Gamma = \mathbb{R}^n$ and $P(\partial/\partial x) = \Delta$, we obtain that stabilization along parabolas (i.e., $d = 2$) is sufficient for stabilization in time of the solution to the Cauchy problem for the heat equation. This particular case of Corollary 8.6 was studied in [5, 6, 8].

8.4. **Tauberian Theorems for Laplace Transforms.** We now apply the results from Subsection 6.2 to Laplace transforms. As in Example 3.9, Γ is assumed to be a closed convex acute cone with vertex at the origin; we set $C_\Gamma = \operatorname{int}\Gamma^*$ and $T^{C_\Gamma} = \mathbb{R}^n + iC_\Gamma$. The following Tauberian theorems for the Laplace transform were originally obtained in [3, 64] under the additional assumption that Γ is a regular cone (i.e., its Cauchy-Szegö kernel is a divisor of the unity in the Vladimirov algebra $H(T^{C_\Gamma})$ [63, 64]); we will not make use of such a hypothesis over the cone Γ.

Given $0 \leq \kappa$, we denote by $\Omega^\kappa \subset \mathbb{H}^{n+1}$ the set

$$\Omega^\kappa = \left\{(x,\sigma) \in \mathbb{H}^{n+1} : \, |x| \leq \sigma^\kappa \text{ and } 0 < \sigma \leq 1\right\}. \tag{8.7}$$

Theorem 8.8. *Let* $\mathbf{h} \in \mathcal{S}'_\Gamma(E)$ *and let* L *be slowly varying at infinity. Then,* $\mathbf{h}$ *is quasiasymptotically bounded of degree* α *at infinity with respect to* L *if and only if there exist numbers* $k \in \mathbb{N}$ *and* $0 \leq \kappa < 1$ *and a vector* $\omega \in C_\Gamma$ *such that*

$$\limsup_{\varepsilon \to 0^+} \sup_{(x,\sigma)\in\partial\Omega^\kappa,\, \sigma>0} \frac{\sigma^k \varepsilon^{n+\alpha}}{L(1/\varepsilon)} \left\|\mathcal{L}\left\{\mathbf{h}; \varepsilon\left(x + i\sigma\omega\right)\right\}\right\| < \infty. \tag{8.8}$$

Proof. Set $\hat{\mathbf{f}} = (2\pi)^n \mathbf{h}$ and keep the notation from Example 3.9. Clearly, $\mathbf{h}$ is quasiasymptotically bounded of degree α at infinity with respect to L if and only if $\mathbf{f}$ is quasiasymptotically bounded of degree $-\alpha - n$ at the origin with respect to $L(1/\varepsilon)$. The latter holds, by (3.11) and Theorem 6.2, if and only if there exists $k_1 \in \mathbb{N}$ such that

$$\limsup_{\varepsilon \to 0^+} \sup_{\substack{|x|^2 + (\cos\vartheta)^2 = 1 \\ \vartheta \in [0,\pi/2)}} \frac{(\cos\vartheta)^{k_1} \, \varepsilon^{n+\alpha}}{L(1/\varepsilon)} \left\|F_{\phi_\omega}\mathbf{f}(\varepsilon x, \varepsilon\cos\vartheta)\right\| < \infty. \tag{8.9}$$

Thus, we shall show the equivalence between (8.8) and (8.9). By part (i) of Proposition 5.1, (8.9) implies (8.8). Assume now (8.8), namely, there exist C_1 and $0 < \varepsilon_0 < 1$ such that

$$\left\|F_{\phi_\omega}\mathbf{f}(\varepsilon x', \varepsilon\sigma)\right\| < \frac{C_1}{\sigma^k}\varepsilon^{-\alpha-n} L\left(1/\varepsilon\right), \quad \varepsilon \leq \varepsilon_0, \; (x',\sigma) \in \Omega^\kappa. \tag{8.10}$$

We may assume that $k \geq \alpha + n + 1$ and L satisfies (5.3) and (5.4) (the case at infinity). We keep arbitrary $\varepsilon < \varepsilon_0$, $\vartheta \in (0, \pi/2)$ and $x \in \mathbb{R}^n$ with $|x|^2 + (\cos\vartheta)^2 = 1$. Set $r = |x|^{\frac{1}{1-\kappa}} / (\cos\vartheta)^{\frac{\kappa}{1-\kappa}}$, $x' = x/r$, and $\sigma = (\cos\vartheta)/r$; observe that $(x', \sigma) \in \partial\Omega^\kappa$. Assume first that $r\varepsilon \leq \varepsilon_0$, then, in view of (8.10) and (5.3),

$$\begin{aligned} \left\|F_{\phi_\omega}\mathbf{f}(\varepsilon x, \varepsilon\cos\vartheta)\right\| &< \frac{C_1}{(\cos\vartheta/r)^k}(r\varepsilon)^{-\alpha-n} L\left(1/(r\varepsilon)\right) \\ &\leq 4C_1C_2\varepsilon^{-\alpha-n} L\left(1/\varepsilon\right)\left(\cos\vartheta\right)^{-k-\frac{\kappa}{1-\kappa}(k-\alpha-n+1)}; \end{aligned}$$

on the other hand, if now $\varepsilon_0 < r\varepsilon$, Proposition 3.4 implies that for some $k_2 \in \mathbb{N}$, $k_2 \leq k$ and $C_4 > 0$,

$$\begin{aligned}\|F_{\phi_\omega}\mathbf{f}(\varepsilon x, \varepsilon\cos\vartheta)\| &< \frac{C_4}{(\varepsilon\cos\vartheta)^{k_2}} = \frac{C_4}{(\cos\vartheta)^{k_2}}\varepsilon^{-\alpha-n}L\left(1/\varepsilon\right)\frac{(1/\varepsilon)^{k_2-\alpha-n}}{L(1/\varepsilon)}\\ &< \frac{C_4C_3}{(\cos\vartheta)^{k_2}}\varepsilon^{-\alpha-n}L\left(1/\varepsilon\right)\left(\frac{r}{\varepsilon_0}\right)^{k_2+1-\alpha-n}\\ &< \frac{C_4C_3}{\varepsilon_0^{k_2+1-\alpha-n}}\varepsilon^{-\alpha-n}L\left(1/\varepsilon\right)(\cos\vartheta)^{-k_2-\frac{\kappa}{1-\kappa}(k_2-\alpha-n+1)},\end{aligned}$$

where we have used (5.4). Therefore, (8.9) is satisfied with $k_1 \geq k_2 + \kappa(k_2 - \alpha - n + 1)/(1-\kappa)$. □

We obtain as a corollary the so-called general Tauberian theorem for Laplace transforms [64, p. 84].

Corollary 8.9. *Let $\mathbf{h} \in \mathcal{S}'_\Gamma(E)$ and let L be slowly varying at infinity. Then, an estimate (8.8), for some $k \in \mathbb{N}$ and $\omega \in C_\Gamma$, and the existence of a solid cone $C' \subset C_\Gamma$ (i.e., $\operatorname{int} C' \neq \emptyset$) such that*

$$\lim_{\varepsilon\to 0^+}\frac{\varepsilon^{\alpha+n}}{L(1/\varepsilon)}\mathcal{L}\left\{\mathbf{h}; i\varepsilon\xi\right\} = \mathbf{G}(i\xi), \ \text{ in } E, \quad \text{for each } \xi \in C', \tag{8.11}$$

are necessary and sufficient for $\mathbf{h}$ to have quasiasymptotic behavior at infinity of degree α, i.e.,

$$\mathbf{h}(\lambda u) \sim \lambda^\alpha L(\lambda)\mathbf{g}(u) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{S}'(\mathbb{R}^n, E), \quad \text{for some } \mathbf{g} \in \mathcal{S}'_\Gamma(E).$$

In such a case, $\mathbf{G}(z) = \mathcal{L}\left\{\mathbf{g}; z\right\}$, $z \in T^{C_\Gamma}$.

Proof. Recall [63] that $\mathcal{S}'_\Gamma(E)$ is canonically isomorphic to $\mathcal{S}'(\Gamma, E) = L_b(\mathcal{S}(\Gamma), E)$. By the injectivity of the Laplace transform and the uniqueness property of holomorphic functions, the linear span of $\left\{e^{i\xi\cdot u} : \xi \in C'\right\}$ is dense in $\mathcal{S}(\Gamma)$; observe that (8.11) gives precisely convergence of $(1/(\lambda^\alpha L(\lambda)))\mathbf{h}(\lambda\,\cdot)$ over such a dense subset. To conclude the proof, it suffices to apply Theorem 8.8 and the Banach-Steinhaus theorem. □

Example 8.10. *Littlewood's Tauberian theorem.* The classical Tauberian theorem of Littlewood [18, 27, 29] states that if

$$\lim_{\varepsilon\to 0^+}\sum_{n=0}^{\infty} c_n e^{-\varepsilon n} = \beta \tag{8.12}$$

and if the Tauberian hypothesis $c_n = O(1/n)$ is satisfied, then the numerical series is convergent, i.e., $\sum_{n=0}^{\infty} c_n = \beta$.

We give a quick proof of this theorem based on Corollary 8.9. We first show that the distribution $h(u) = \sum_{n=0}^{\infty} c_n\delta(u-n)$ has the quasiasymptotic behavior

$$h(\lambda u) = \sum_{n=0}^{\infty} c_n\delta(\lambda u - n) \sim \beta\frac{\delta(u)}{\lambda} \quad \text{as } \lambda \to \infty \text{ in } \mathcal{S}'(\mathbb{R}). \tag{8.13}$$

Observe that (8.11) is an immediate consequence of (8.12) (here $n = 1$, $\alpha = -1$, $L \equiv 1$). We verify (8.8) with $\kappa = 0$, actually, on the rectangle $\Omega_0 = [-1, 1] + i(0, 1]$. Indeed, (8.12) and the Tauberian hypothesis imply that for suitable constants $C_1, C_2, C_3, C_4 > 0$, independent of $(x, \sigma) \in \Omega^0$,

$$|\mathcal{L}\{h; \varepsilon(x + i\sigma)\}| = \left|\sum_{n=0}^{\infty} c_n e^{-\varepsilon\sigma n} e^{i\varepsilon x n}\right| \le C_1 + C_2 \sum_{n=1}^{\infty} \frac{e^{-\varepsilon\sigma n}}{n} \left|e^{i\varepsilon x n} - 1\right|$$

$$< C_1 + C_3\varepsilon \sum_{n=1}^{\infty} e^{-\varepsilon\sigma n} < \frac{C_4}{\sigma}, \quad (x, \sigma) \in \Omega^0, \ 0 < \varepsilon \le 1.$$

Consequently, Corollary 8.9 yields (8.13). Finally, it is well known that (8.13) and $c_n = O(1/n)$ imply the convergence of the series; in fact, this is true under more general Tauberian hypotheses (cf. [56, Sec. 3]). We can proceed as follows. Let $\sigma > 1$ be arbitrary. Choose $\rho \in \mathcal{D}(\mathbb{R})$ such that $0 \le \rho \le 1$, $\rho(u) = 1$ for $u \in [0, 1]$, and $\operatorname{supp}\rho \subset [-1, \sigma]$, then, evaluation of (8.13) at ρ gives, for some constant C_5,

$$\limsup_{\lambda\to\infty} \left|\sum_{0\le n\le\lambda} c_n - \beta\right| \le \limsup_{\lambda\to\infty} \left|\sum_{\lambda\le n} c_n \rho\left(\frac{n}{\lambda}\right)\right| < C_5 \limsup_{\lambda\to\infty} \sum_{1<\frac{n}{\lambda}<\sigma} \frac{\rho\left(\frac{n}{\lambda}\right)}{n}$$

$$= C_5 \int_1^\sigma \frac{\rho(x)}{x}\, \mathrm{d}x < C_5(\sigma - 1),$$

and so, taking $\sigma \to 1^+$, we conclude $\sum_{n=0}^{\infty} c_n = \beta$.

Remark 8.11. We refer to the monograph [64] (and references therein) for the numerous applications of Corollary 8.9 in mathematical physics, especially in quantum field theory. Corollary 8.9 can also be used to easily recover Vladimirov multidimensional generalization [62] of the Hardy-Littlewood-Karamata Tauberian theorem (cf. [3, 64]).

8.5. **Relation between Quasiasymptotics in $\mathcal{D}'(\mathbb{R}^n, E)$ and $\mathcal{S}'(\mathbb{R}^n, E)$.** If a tempered E-valued distribution has quasiasymptotic behavior in the space $\mathcal{S}'(\mathbb{R}^n, E)$ then, clearly, it has the same quasiasymptotic behavior in $\mathcal{D}'(\mathbb{R}^n, E)$. The converse is also well known in the case of scalar-valued distributions, but the true of this result is less obvious. There have been several proofs of such a converse result and, remarkably, none of them is simple (cf. [36, 59, 60] and especially [72, Lem. 6] for the general case). We provide a new proof of this fact, which will actually be derived as an easy consequence of the results from Subsection 6.2.

We begin with quasiasymptotic boundedness. Let L be slowly varying at the origin (resp. at infinity)

Proposition 8.12. *Let $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, E)$. If $\mathbf{f}$ is quasiasymptotically bounded of degree α at the point x_0 (resp. at infinity) with respect to L in the space $\mathcal{D}'(\mathbb{R}^n, E)$, so is $\mathbf{f}$ in the space $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, E)$.*

Proof. We may assume that $x_0 = 0$. The Banach-Steinhaus theorem implies the existence of $\nu \in \mathbb{N}$, $C > 0$, and $h_0 > 0$ such that

$$|\langle \mathbf{f}(ht), \rho(t)\rangle| \leq Ch^{\alpha}L(h) \sup_{|x|\leq 1,\, |m|\leq \nu} \left|\rho^{(m)}(t)\right|, \quad \text{for all } \rho \in \mathcal{D}(B(0,3))$$

and all $0 < h < h_0$ (resp. $h_0 < h$), where $B(0,3)$ is the ball of radius 3. Let now $\phi \in \mathcal{D}(B(0,1))$. If we take $\rho(t) = y^{-n}\phi(y^{-1}(t-x))$ in the above estimate, where $0 < y < 1$ and $|x| \leq 1$, we then obtain at once that (6.1) is satisfied with $k = \nu + n$, and consequently the Theorem 6.2 implies the result. □

Proposition 8.12, the Banach-Steinhaus theorem, and the density of $\mathcal{D}(\mathbb{R}^n)$ in $\mathcal{S}(\mathbb{R}^n)$ immediately yield what we wanted:

Corollary 8.13. *If* $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, E)$ *has quasiasymptotic behavior in* $\mathcal{D}'(\mathbb{R}^n, E)$*, so does* $\mathbf{f}$ *have the same quasiasymptotic behavior in the space* $\mathcal{S}'(\mathbb{R}^n, E)$.

9. Further Extensions

We indicate in this section some useful extentions and variants of the Tauberian results from the previous sections.

9.1. **Other Tauberian Conditions.** The Tauberian conditions (5.5) and (6.1), occurring in Theorems 5.2 – 6.7, can be replaced by estimates of the form (8.8), that is, one may use the boundary of some set Ω^{κ}, $0 \leq \kappa < 1$ (cf. (8.7)), instead of the upper half sphere $\mathbb{H}^{n+1} \cap \mathbb{S}^n$. Specifically, the same argument given in proof of Theorem 8.8 applies to show that (5.1) (and hence (4.1)) is equivalent to the estimate

$$\limsup_{\varepsilon\to 0^+} \sup_{(x,y)\in\partial\Omega^{\kappa},\, y>0} \frac{y^k}{\varepsilon^{\alpha}L(\varepsilon)} \left\| M^{\mathbf{f}}_{\varphi}\left(x_0+\varepsilon x, \varepsilon y\right)\right\| < \infty$$

$$\left(\text{resp. } \limsup_{\lambda\to\infty} \sup_{(x,y)\in\partial\Omega^{\kappa},\, y>0} \frac{y^k}{\lambda^{\alpha}L(\lambda)} \left\| M^{\mathbf{f}}_{\varphi}\left(\lambda x, \lambda y\right)\right\| < \infty\right)$$

for some $0 \leq \kappa < 1$ and $k \in \mathbb{N}$ (the k may be different numbers).

9.2. **Distributions with Values in DFS Spaces.** All the results from Sections 3–7 hold if we replace the Banach space E by a Silva [47, 28] inductive limit of Banach spaces $E_n, n \in \mathbb{N}$, that is, $E = \bigcup_{n=1}^{\infty} E_n = \operatorname{ind}\lim_{n\to\infty}(E_n, ||\cdot||_n)$, where $E_1 \subset E_2 \subset ...$ and each injection $E_n \to E_{n+1}$ is compact. These spaces is an DFS spaces (strong duals of Fréchet-Schwartz spaces). Particular examples are $E = \mathcal{S}'(\mathbb{R}^n), \mathcal{S}'_0(\mathbb{R}^n), \mathcal{D}'(Y)$, where Y is a compact manifold, among many other important spaces arising in applications. In this situation E is regular, namely, for any bounded set $\mathfrak{B}$ there exists $n_0 \in \mathbb{N}$ such that $\mathfrak{B}$ is bounded in E_{n_0}.

Thus, our Tauberian theorems from Sections 3–6 for E-valued distributions are valid if we replace the norm estimates by memberships in bounded sets of E. For instance, a condition such as (5.1) should be replaced by one of the form: There exist $k \in \mathbb{N}$, $\varepsilon_0 > 0$, and a bounded set $\mathfrak{B} \subset E$ such that

$$\frac{y^k}{\varepsilon^\alpha L(\varepsilon)} M^{\mathbf{f}}_\varphi(x_0 + \varepsilon x, \varepsilon y) \in \mathfrak{B}, \quad \text{for all } 0 < \varepsilon \le \varepsilon_0 \text{ and } |x|^2 + y^2 = 1; \tag{9.1}$$

and similarly for all other conditions occurring within these sections. As already observed, (9.1) is equivalent to an estimate of the form (5.1) in some norm $\|\cdot\|_{n_0}$, but the existence of the n_0 would be extremely hard to verify in applications and thus such a Tauberian condition would have no value in concreate situations. It is therefore desirable to have more realistic Tauberian conditions. We can achieve this if we use the Mackey theorem [50, Thm. 36.2], because the condition (9.1) is then equivalent to the following one: There exists $k \in \mathbb{N}$ such that for each $e^* \in E'$

$$\limsup_{\varepsilon \to 0^+} \sup_{|x|^2 + y^2 = 1,\, y > 0} \frac{y^k}{\varepsilon^\alpha L(\varepsilon)} \left| \left\langle e^*, M^{\mathbf{f}}_\varphi(x_0 + \varepsilon x, \varepsilon y) \right\rangle \right| < \infty. \tag{9.2}$$

Since E is a Montel space [47, 50], the limit condition (5.2) can be replaced be the equivalent one: There exist the limits

$$\lim_{\varepsilon^\alpha \to 0^+} \frac{1}{\varepsilon L(\varepsilon)} \left\langle e^*, M^{\mathbf{f}}_\varphi(x_0 + \varepsilon x, \varepsilon y) \right\rangle \in \mathbb{C}, \tag{9.3}$$

for all $e^* \in E'$ and $(x, y) \in \mathbb{H}^{n+1}$, and likewise for all other limit conditions.

Furthermore, the results from Section 7 are also valid in this context, if we use suitable hypotheses. For example, Theorem 7.2 remains true if we replace the hypotheses (i) and (ii) by:

(i)′ $\mathcal{W}_\psi \mathbf{f}(x, y) \in E$ *for all* $(x, y) \in \mathbb{H}^{n+1}$.

(ii)′ There exist $k, l \in \mathbb{N}$ such that

$$\sup_{(x,y) \in \mathbb{H}^{n+1}} \left(\frac{1}{y} + y \right)^{-k} (1 + |x|)^{-l} \left| \langle e^*, \mathcal{W}_\psi \mathbf{f}(x, y) \rangle \right| < \infty, \quad \text{for each } e^* \in E'.$$

The other results are true under similar considerations.

Note that one can find in [28] an overview of results concerning regular inductive limits (extensions of Silva's results) which are also Montel spaces. Since the Montel property of Silva spaces is actually what we used above, the comments of this section are also valid in more general situations.

Let us discuss an example in order to illustrate the ideas of this subsection.

Example 9.1. *Fixation of variables in tempered distribumaner1tions.* Let $f \in \mathcal{S}'(\mathbb{R}^n_t \times \mathbb{R}^m_\xi)$ and $t_0 \in \mathbb{R}^n$. Following Łojasiewicz [32], we say that the variable $t = t_0 \in \mathbb{R}^n$ can be fixed in $f(t, \xi)$ if there exists $g \in \mathcal{S}'(\mathbb{R}^m_\xi)$ such that

$$\lim_{\varepsilon \to 0^+} \langle f(t_0 + \varepsilon t, \xi), \eta(t, \xi) \rangle = \int_{\mathbb{R}^n} \langle g(\xi), \eta(t, \xi) \rangle \quad \text{for each } \eta \in \mathcal{S}(\mathbb{R}^n_t \times \mathbb{R}^m_\xi).$$

We write $f(t_0, \xi) = g(\xi)$, *distributionally.* The nuclearity of the Schwartz spaces [50, 48] implies that $\mathcal{S}'(\mathbb{R}^n_t \times \mathbb{R}^m_\xi)$ is isomorphic to $\mathcal{S}'(\mathbb{R}^n_t, E)$, where

$E = \mathcal{S}'(\mathbb{R}^m_\xi)$, a DFS space. Actually, the latter tells us that fixation of variables is nothing but the notion of Łojasiewicz point values itself for E-valued distributions (cf. Example 2.2). Therefore, the DSF space valued version of Theorem 6.4 implies that if $\phi \in \mathcal{S}(\mathbb{R}^n_t)$ with $\mu_0(\phi) = 1$, then the variable $t = t_0$ can be fixed in $f(t, \xi)$ if and only if there exists k such that for each $\rho \in \mathcal{S}'(\mathbb{R}^m_\xi)$

$$\limsup_{\varepsilon \to 0^+} \sup_{\substack{(x,y)\in\mathbb{H}^{n+1} \\ |x|^2+y^2=1}} y^k \left|\left\langle f\left(t_0 + \varepsilon x + \varepsilon y t, \xi\right), \phi(t)\rho(\xi)\right\rangle\right| < \infty,$$

and

$$\lim_{\varepsilon \to 0^+} \left\langle f\left(t_0 + \varepsilon x + \varepsilon y t, \xi\right), \phi(t)\rho(\xi)\right\rangle \text{ exists for all } (x, y) \in \mathbb{H}^{n+1} \cap \mathbb{S}^n.$$

Remark 9.2. It is well known [21] that the projection $\pi : \mathbb{R}^n_t \times \mathbb{R}^m \to \{t_0\} \times \mathbb{R}^m$, $\pi(t, \xi) = (t_0, \xi)$, defines the pull-back $\mathcal{S}'(\mathbb{R}^n_t \times \mathbb{R}^m_\xi) \ni f(t, \xi) \to f(t_0, \xi) := \pi^* f(\xi) \in \mathcal{S}'(\mathbb{R}^m_\xi)$ if the wave front set of f satisfies

$$WFf \cap \{(t_0, \xi, \omega, 0); \xi \in \mathbb{R}^m, \omega \in \mathbb{R}^n\} = \emptyset.$$

Thus the result given in Example 9.1 is interesting since we give the necessary and sufficient condition for the existence of this pull-back.

A. Appendix
Relation Between Quasiasymptotics in $\mathcal{S}'_0(\mathbb{R}^n, E)$ and $\mathcal{S}'(\mathbb{R}^n, E)$

The purpose of this Appendix is to show two propositions which establish the precise connection between quasiasymptotics in the spaces $\mathcal{S}'_0(\mathbb{R}^n, E)$ and $\mathcal{S}'(\mathbb{R}^n, E)$. Observe that such a relation was crucial for the arguments given in Section 6.

Propositions A.1 and A.2 below are multidimensional generalizations of the results from [61, Sec. 4] and their proofs are based on recent structural theorems from [54].

Proposition A.1. *Let L be slowly varying at the origin (resp. at infinity) and let $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, E)$ have quasiasymptotic behavior of degree α at the point x_0 (resp. at infinity) with respect to L in $\mathcal{S}'_0(\mathbb{R}^n, E)$, i.e., for each $\varphi \in \mathcal{S}_0(\mathbb{R}^n)$ the following limit exists*

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^\alpha L(\varepsilon)} \left\langle \mathbf{f}(x_0 + \varepsilon t), \varphi(t)\right\rangle \quad \text{in } E \tag{A.1}$$

$$\left(\text{resp. } \lim_{\lambda \to \infty} \frac{1}{\lambda^\alpha L(\lambda)} \left\langle \mathbf{f}(\lambda t), \varphi(t)\right\rangle\right).$$

Then, there is $\mathbf{g} \in \mathcal{S}'(\mathbb{R}^n, E)$ such that:

(i) *If $\alpha \notin \mathbb{N}$, $\mathbf{g}$ is homogeneous of degree α and there exists an E-valued polynomial $\mathbf{P}$ such that*

$$\mathbf{f}(x_0 + \varepsilon t) - \mathbf{P}(\varepsilon t) \sim \varepsilon^\alpha L(\varepsilon)\mathbf{g}(t) \quad \text{as } \varepsilon \to 0^+ \quad \text{in } \mathcal{S}'(\mathbb{R}^n, E) \tag{A.2}$$

$$\left(\text{resp. } \mathbf{f}(\lambda t) - \mathbf{P}(\lambda t) \sim \lambda^\alpha L(\lambda)\mathbf{g}(t) \quad \text{as } \lambda \to \infty \quad \text{in } \mathcal{S}'(\mathbb{R}^n, E)\right).$$

(ii) *If $\alpha = p \in \mathbb{N}$, $\mathbf{g}$ is associate homogeneous of order 1 and degree $-n-p$ (cf.* [15, p. 74], [46]*) satisfying*

$$\mathbf{g}(ax) = a^p\mathbf{g}(x) + a^p \log a \sum_{|m|=p} t^m \mathbf{v}_m, \quad \text{for each } a > 0, \tag{A.3}$$

for some vectors $\mathbf{v}_m \in E$, $|m| = p$, and there exist an E-valued polynomial $\mathbf{P}$ and associate asymptotically homogeneous E-valued functions $\mathbf{c}_m$, $|m| = p$, of degree 0 with respect to L such that

$$\mathbf{c}_m(a\varepsilon) = \mathbf{c}(\varepsilon) + L(\varepsilon)\log a\, \mathbf{v}_m + o(L(\varepsilon)) \quad \text{as } \varepsilon \to 0^+, \quad \text{for each } a > 0, \tag{A.4}$$

$$(\text{resp. } \mathbf{c}_m(a\lambda) = \mathbf{c}(\lambda) + L(\lambda)\log a\, \mathbf{v}_m + o(L(\lambda)) \quad \text{as } \lambda \to \infty\ )$$

and $\mathbf{f}$ has the following asymptotic expansion

$$\mathbf{f}(x_0 + \varepsilon t) = \mathbf{P}(\varepsilon t) + \varepsilon^p L(\varepsilon)\mathbf{g}(t) + \varepsilon^p \sum_{|m|=p} t^m \mathbf{c}_m(\varepsilon) + o\left(\varepsilon^p L(\varepsilon)\right) \tag{A.5}$$

$$\left(\text{resp. } \mathbf{f}\left(\lambda t\right) = \mathbf{P}(\lambda t) + \lambda^p L(\lambda)\mathbf{g}(t) + \lambda^p \sum_{|m|=p} t^m \mathbf{c}_m(\lambda) + o\left(\lambda^p L(\lambda)\right)\right),$$

as $\varepsilon \to 0^+$ (resp. $\lambda \to \infty$) in the space $\mathcal{S}'(\mathbb{R}^n, E)$.

Proof. Let $\mathcal{S}^0(\mathbb{R}^n)$ be the image under Fourier transform of $\mathcal{S}_0(\mathbb{R}^n)$. Then, $\mathcal{S}^0(\mathbb{R}^n)$ is precisely the closed subspace of $\mathcal{S}(\mathbb{R}^n)$ consisting of test functions which vanish at the origin together with their partial derivatives of any order. Thus, if we Fourier transform (A.1) and employ the Banach-Steinhaus theorem, we obtain the existence of $\mathbf{h} \in \mathcal{S}^{0\prime}(\mathbb{R}^n, E)$ such that the restriction of $\mathbf{f}$ to $\mathcal{S}^0(\mathbb{R}^n)$ satisfies

$$\exp(i\varepsilon^{-1}u \cdot x_0)\hat{\mathbf{f}}(\varepsilon^{-1}u) \sim \varepsilon^{n+\alpha}L(\varepsilon)\hat{\mathbf{g}}(u) \quad \text{as } \varepsilon \to 0^+ \ \text{ in } \mathcal{S}^{0\prime}(\mathbb{R}^n, E)$$

$$\left(\text{resp. } \hat{\mathbf{f}}(\lambda^{-1}u) \sim \lambda^{n+\alpha}L(\lambda)\hat{\mathbf{g}}(u) \quad \text{as } \lambda \to \infty \ \text{ in } \mathcal{S}^{0\prime}(\mathbb{R}^n, E)\right).$$

Setting $\tilde{\mathbf{f}}(u) = e^{i\,u\cdot x_0}\hat{\mathbf{f}}(u)$ (resp. $\tilde{\mathbf{f}}(u) = \hat{\mathbf{f}}(u)$), $\tilde{L}(y) = L(1/y)$, $\beta = -n-\alpha$ and replacing ε by λ^{-1}, we have that the restriction of $\tilde{\mathbf{f}}$ to $\mathcal{S}^0(\mathbb{R}^n)$ has the quasiasymptotic behavior

$$\tilde{\mathbf{f}}(\lambda u) \sim \lambda^{\beta}\tilde{L}(\lambda)\mathbf{h}_0(u) \quad \text{as } \lambda \to \infty \ \text{ in } \mathcal{S}^{0\prime}(\mathbb{R}^n, E) \tag{A.6}$$

$$\left(\text{resp. } \tilde{\mathbf{f}}(\varepsilon u) \sim \varepsilon^{\beta}\tilde{L}(\varepsilon)\mathbf{h}_0(u) \quad \text{as } \varepsilon \to 0^+ \ \text{ in } \mathcal{S}^{0\prime}(\mathbb{R}^n, E)\right),$$

for some $\mathbf{h}_0 \in \mathcal{S}^{0\prime}(\mathbb{R}^n, E)$. We now apply the results from [54].

Case (i): $\alpha \notin \mathbb{N}$. By (A.6) and [54, Part (i) of Thm. 3.1], there are an E-valued distribution $\mathbf{h} \in \mathcal{S}'(\mathbb{R}^n, E)$, which is homogeneous of degree $\beta = -n-\alpha$, an natural number $d \in \mathbb{N}$, and $\mathbf{w}_m \in E$, $|m| \le d$, such that

$$\tilde{\mathbf{f}}(\lambda u) = \lambda^{\beta}\tilde{L}(\lambda)\mathbf{h}(u) + \sum_{|m|\le d} \frac{\delta^{(m)}(u)}{\lambda^{n+|m|}}\mathbf{w}_m + o\left(\lambda^{\beta}\tilde{L}(\lambda)\right) \quad \text{as } \lambda \to \infty$$

$$\left(\text{resp. } \tilde{\mathbf{f}}(\varepsilon u) = \varepsilon^{\beta}\tilde{L}(\varepsilon)\mathbf{h}(u) + \sum_{|m|\leq d} \frac{\delta^{(m)}(u)}{\varepsilon^{n+|m|}}\mathbf{w}_m + o\left(\varepsilon^{\beta}\tilde{L}(\varepsilon)\right) \quad \text{as } \varepsilon \to 0^+ \right)$$

in $\mathcal{S}'(\mathbb{R}^n, E)$. Finally, by setting $\hat{\mathbf{g}} = \mathbf{h}$, taking Fourier inverse transform and replacing λ by ε^{-1} (resp. ε by λ^{-1}), the last relation shows that $\mathbf{f}$ satisfies (A.2) with $\mathbf{P}(t) = (1/2\pi)^n \sum_{|m|\leq d}(-it)^m \mathbf{w}_m$.

Case (ii): $\beta = -n-p$, $p \in \mathbb{N}$. The quasiasymptotics (A.6) and [54, Part (ii) of Thm. 3.1] yield the existence of $d \in \mathbb{N}$, $\mathbf{w}_m \in E$ (for $|m| \leq d$), $\tilde{\mathbf{v}}_m \in E$ (for $|m| = p$), continuous functions $\tilde{\mathbf{c}}_m : \mathbb{R}_+ \to E$ (for $|m| = p$), and a tempered E-valued distribution $\mathbf{h} \in \mathcal{S}'(\mathbb{R}^n, E)$ such that $\tilde{\mathbf{f}}$ has the following asymptotic expansion in $\mathcal{S}'(\mathbb{R}^n, E)$ as $\lambda \to \infty$ (resp. $\varepsilon \to 0^+$)

$$\tilde{\mathbf{f}}(\lambda u) = \frac{\tilde{L}(\lambda)}{\lambda^{n+p}}\mathbf{h}(u) + \sum_{|m|\leq d} \frac{\delta^{(m)}(u)}{\lambda^{n+|m|}}\mathbf{w}_m + \sum_{|m|=p} \frac{\delta^{(m)}(u)}{\lambda^{n+p}}\tilde{\mathbf{c}}_m(\lambda) + o\left(\frac{\tilde{L}(\lambda)}{\lambda^{n+p}}\right),$$

respectively

$$\tilde{\mathbf{f}}(\varepsilon u) = \frac{\tilde{L}(\varepsilon)}{\varepsilon^{n+p}}\mathbf{h}(u) + \sum_{|m|\leq d} \frac{\delta^{(m)}(u)}{\varepsilon^{n+|m|}}\mathbf{w}_m + \sum_{|m|=p} \frac{\delta^{(m)}(u)}{\varepsilon^{n+p}}\tilde{\mathbf{c}}_m(\varepsilon) + o\left(\frac{\tilde{L}(\varepsilon)}{\varepsilon^{n+p}}\right),$$

where $\mathbf{h}$ satisfies

$$\mathbf{h}(au) = a^{-n-p}\mathbf{h}(u) + a^{-n-p}\log a \sum_{|m|=p} \delta^{(m)}(u)\tilde{\mathbf{v}}_m,$$

for each $a > 0$, while the $\tilde{\mathbf{c}}_m$ fulfill

$$\tilde{\mathbf{c}}_m(a\lambda) = \tilde{\mathbf{c}}_m(\lambda) + \tilde{L}(\lambda)\log a\, \tilde{\mathbf{v}}_m + o\left(\tilde{L}(\lambda)\right), \quad |m| = p,$$

$$\left(\text{resp. } \tilde{\mathbf{c}}_m(a\varepsilon) = \tilde{\mathbf{c}}_m(\varepsilon) + \tilde{L}(\varepsilon)\log a\, \tilde{\mathbf{v}}_m + o\left(\tilde{L}(\varepsilon)\right)\right).$$

Then, Fourier inverse transforming the quasiasymptotic expansion of $\tilde{\mathbf{f}}$, we convince ourselves that $\mathbf{f}$ satisfies (A.5) with the polynomial $\mathbf{P}(t) = (2\pi)^{-n}\sum_{|m|\leq d}(-it)^m \mathbf{w}_m$, the functions $\mathbf{c}_m(y) = (-i)^p(2\pi)^{-n}\tilde{\mathbf{c}}_m(y^{-1})$ and $\mathbf{g}$ given by $\hat{\mathbf{g}} = \mathbf{h}$. In addition, the relations (A.3) and (A.4) hold with $\mathbf{v}_m = -(-i)^p(2\pi)^{-n}\tilde{\mathbf{v}}_m$, $|m| = p$.

□

The proof of the following proposition is completely analogous to that of Proposition A.1, but now making use of [54, Thm. 3.2] instead of [54, Thm. 3.1]; we therefore omit it.

Proposition A.2. *Let L be slowly varying at the origin (resp. at infinity) and let $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, E)$ be quasiasymptotically bounded of degree α at the point x_0 (at infinity) with respect to L in $\mathcal{S}'_0(\mathbb{R}^n, E)$. Then:*

(i) *If $\alpha \notin \mathbb{N}$, there exists an E-valued polynomial $\mathbf{P}$ such that $\mathbf{f} - \mathbf{P}$ is quasiasymptotically bounded of degree α at the point x_0 (at infinity) with respect to L in the space $\mathcal{S}'(\mathbb{R}^n, E)$.*

(ii) *If $\alpha = p \in \mathbb{N}$, there exist an E-valued polynomial $\mathbf{P}$ and asymptotically homogeneously bounded E-valued functions $\mathbf{c}_m$, $|m| = p$, of degree 0 with respect to L such that $\mathbf{f}$ has the following asymptotic expansion*

$$\mathbf{f}(x_0 + \varepsilon t) = \mathbf{P}(\varepsilon t) + \varepsilon^p \sum_{|m|=p} t^m \mathbf{c}_m(\varepsilon) + O\left(\varepsilon^p L(\varepsilon)\right)$$

$$\left(\textit{resp. } \mathbf{f}\left(\lambda t\right) = \mathbf{P}(\lambda t) + \lambda^p \sum_{|m|=p} t^m \mathbf{c}_m(\lambda) + O\left(\lambda^p L(\lambda)\right) \right),$$

as $\varepsilon \to 0^+$ (resp. $\lambda \to \infty$) in the space $\mathcal{S}'(\mathbb{R}^n, E)$.

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Department of Mathematics and Informatics, University of Novi Sad, Trg Dositeja Obradovića 4, 21000 Novi Sad, Serbia

E-mail address: pilipovic@dmi.uns.ac.rs

Department of Mathematics, Ghent University, Krijgslaan 281 Gebouw S22, B 9000 Gent, Belgium

E-mail address: jvindas@cage.Ugent.be