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EXPLICIT SINGULAR VISCOSITY SOLUTIONS OF THE ARONSSON EQUATION

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Abstract. We establish that when $n \geq 2$ and $H \in C^1(\mathbb{R}^n)$ is a Hamiltonian such that some level set contains a line segment, the Aronsson equation

$$D^2u : H_p(Du) \otimes H_p(Du) \ = \ 0$$

admits explicit entire viscosity solutions. They are superpositions of a linear part plus a Lipschitz continuous everywhere differentiable singular part which in general is non-C^1 and nowhere twice differentiable. In particular, we supplement the C^1 regularity result of Wang and Yu [W-Y] by deducing that strict level convexity is necessary for C^1 regularity of solutions.

1. Introduction

Let $H \in C^1(\mathbb{R}^n)$ be a Hamiltonian function and $n \geq 2$. We discuss aspects of the C^1 regularity problem of viscosity solutions to the Aronsson equation, which is defined on smooth $u \in C^2(\mathbb{R}^n)$ by

$$\mathcal{A}[u] \ := D^2u : H_p(Du) \otimes H_p(Du) \ = \ 0. \tag{1}$$

Here, $\mathcal{A}[u]$ is understood as $\sum_{i,j=1}^n D^2_{ij}u\, H_{p_i}(Du)\, H_{p_j}(Du)$ and $H_{p_i} = D_{p_i}H$. Formula (1) defines a quasilinear highly degenerate elliptic PDE. It arises in L^∞ variational problems of the supremal functional $E_\infty(u,\Omega) := \|H(Du)\|_{L^\infty(\Omega)}$, as well as in other contexts (Barron-Evans-Jensen [BEJ]). When $H(p) = \frac{1}{2}|p|^2$, (1) reduces to the ∞-Laplacian:

$$\Delta_\infty u \ := \ D^2u : Du \otimes Du \ = \ 0. \tag{2}$$

Under reasonable convexity, coercivity and regularity assumptions on H, there exists a unique continuous solution of the Dirichlet problem with Lipschitz boundary data, interpreted in the viscosity sense of Crandall-Ishii-Lions [CIL]. Moreover, any continuous viscosity solution to (1) is actually Lipschitz continuous. The C^1 regularity problem for (1) however remains open. Wang and Yu [W-Y] established that when $n = 2$, H is in $C^2(\mathbb{R}^2)$ with $H \geq H(0) = 0$ and it is uniformly convex on the plane (i.e. there exists $a > 0$ such that $H_{pp} \geq aI$), then continuous viscosity solutions of (1) over $\Omega \subseteq \mathbb{R}^2$ are in $C^1(\Omega)$. When $n > 2$, viscosity solutions are linearly approximatable at all scales in the sense of De Pauw-Koeller [DePK], having approximate gradients. In the special case of Δ_∞ and when $n = 2$, solutions are $C^{1+\alpha}$ (Savin [S], Evans-Savin [E-S]). Recently, Evans and Smart established everywhere differentiability of ∞-Harmonic functions [E-Sm].

Herein we prove that when a level set $\{H = c\}$ of H contains a straight line segment, there exists an entire viscosity solution of (1) given as superposition of a

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linear term plus a rather arbitrary Lipschitz continuous everywhere once differentiable term. The latter may *not* be C^1; moreover, it may well be nowhere twice differentiable with Hessian realized only as a singular distribution and not even as a Radon measure, as we demonstrate by a concrete example.

We note that our *only* assumption is H being constant along a line segment but arbitrary otherwise. This suffices for these solutions to appear. Actually, they arise as everywhere differentiable solutions of the Hamilton-Jacobi equation

$$H(Du) \;=\; c. \tag{3}$$

In order to keep the proof self-contained and direct, we work with the second order PDE (1) ignoring the relation between viscosity solutions of (1) and differentiable solutions of (3). We just notice that in the classical C^2 context, the identity

$$D^2u : H_p(Du) \otimes H_p(Du) \;=\; H_p(Du)^\top D\big(H(Du)\big) \tag{4}$$

suffices to imply $\mathcal{A}[u] = 0$, whenever $H(Du) = c$. Let us now state our result.

Theorem 1. *We assume that $H \in C^1(\mathbb{R}^n)$, $n \geq 2$ and there exists a straight line segment $[p', p''] \subseteq \mathbb{R}^n$ along which H is constant. Then, for any everywhere once differentiable locally Lipschitz function $f \in C^{0+1}(\mathbb{R})$ satisfying $\|f'\|_{C^0(\mathbb{R})} < 1$, the formula*

$$u(x) \;:=\; \left[\frac{p''+p'}{2}\right]^\top x \;+\; f\left(\left[\frac{p''-p'}{2}\right]^\top x\right), \quad x \in \mathbb{R}^n, \tag{5}$$

defines an entire viscosity solution $u \in C^{0+1}(\mathbb{R}^n)$ of the Aronsson equation

$$D^2u : H_p(Du) \otimes H_p(Du) \;=\; 0.$$

We deduce that the existence of the non-C^1 solutions (5) implies the following

Corollary 2. *Strict level convexity of the Hamiltonian H is necessary to obtain C^1 regularity of viscosity solutions to the Aronsson PDE in all dimensions $n \geq 2$.*

In particular, the uniform convexity assumption of Wang and Yu [W-Y] can not be relaxed to mere convexity, unless if strict level-convexity is additionally assumed.

We observe that C^1 regularity of solutions is not an issue of regularity of H; the singular solutions (5) persist even when $H \in C^\infty(\mathbb{R}^n)$. The sensitive dependence of regularity on the convexity of H is a result of the geometric degeneracy structure of the PDE which can be expressed by the perpendicularity condition

$$\mathcal{A}[u] \;=\; 0 \;\;\Leftrightarrow\;\; H_p(Du) \perp D\big(H(Du)\big). \tag{6}$$

Interestingly, the singular solutions persist for arbitrarily small straight line segments, as long as the segments do not trivialize to a single point.

2. Proofs

Let us recall from Crandall-Ishii-Lions [CIL] the definition of viscosity solutions to the Aronsson equation (1). Let $\mathbb{S}(n)$ denote the set of (real) symmetric matrices in $\mathbb{R}^{n\times n}$. Let $u \in C^0(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. The second order *super-jet set* $J^{2,+}u(x)$ of u at x is the set of generalized derivatives

$$J^{2,+}u(x) \;:=\; \Big\{(p,X) \in \mathbb{R}^n \times \mathbb{S}(n) \;\Big|\; \text{as } \; z \to x,$$

$$u(z) \;\leq\; u(x) + p^\top(z-x) + \frac{1}{2}X : (z-x)\otimes(z-x) + o\left(|z-x|^2\right)\Big\}. \tag{7}$$

The *sub-jet set* $J^{2,-}u(x)$ is defined by reversing the inequality in (7):

$$J^{2,-}u(x) \;:=\; \Big\{(p,X)\in\mathbb{R}^n\times\mathbb{S}(n)\;\Big|\;\text{ as }\; z\to x,$$

$$u(z) \;\geq\; u(x)+p^\top(z-x)+\frac{1}{2}X:(z-x)\otimes(z-x)+o\big(|z-x|^2\big)\Big\}. \tag{8}$$

A function $u\in C^0(\mathbb{R}^n)$ is a *viscosity subsolution* of the Aronsson equation (1) if for all $x\in\mathbb{R}^n$, it follows that

$$(p,X)\in J^{2,+}u(x) \quad\Longrightarrow\quad X:H_p(p)\otimes H_p(p)\;\geq\;0. \tag{9}$$

Similarly, $u\in C^0(\mathbb{R}^n)$ is a *viscosity supersolution* of (1) if for all $x\in\mathbb{R}^n$, it follows

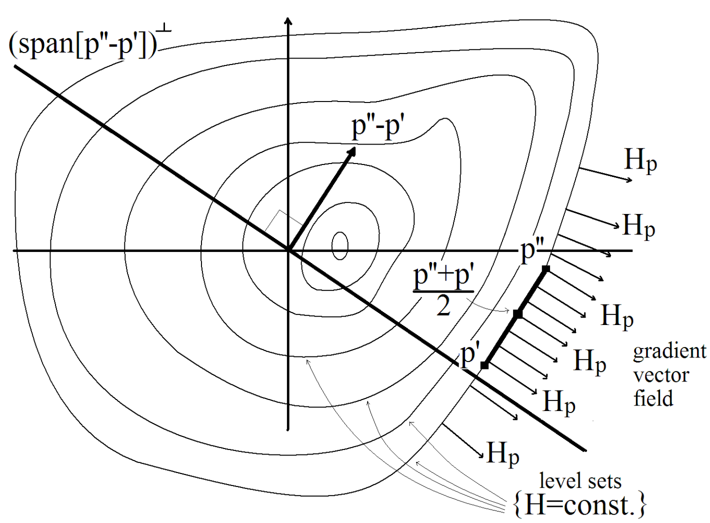
$$(p,X)\in J^{2,-}u(x) \quad\Longrightarrow\quad X:H_p(p)\otimes H_p(p)\;\leq\;0. \tag{10}$$

A *viscosity solution* is defined as a function which is both a viscosity sub- and supersolution of (1).

Lemma 3. *Let u be given by (5). Then, the range of its gradient Du is valued in the segment $[p',p'']$ and its image under the p-gradient of H is perpendicular to $[p',p'']$:*

$$Du(\mathbb{R}^n)\;\subseteq\;[p',p''], \tag{11}$$

$$H_p\big(Du(\mathbb{R}^n)\big)\;\subseteq\;\big(span[p''-p']\big)^\perp. \tag{12}$$



Proof of Lemma 3. Since f is assumed to be Lipschitz continuous and everywhere once differentiable on $\mathbb{R}$, the same holds for u on $\mathbb{R}^n$. By differentiating (5), for all $x\in\mathbb{R}^n$ we have

$$Du(x)\;=\;\frac{p''+p'}{2}\;+\;\frac{1}{2}\,f'\left(\left[\frac{p''-p'}{2}\right]^\top x\right)(p''-p'). \tag{13}$$

We rewrite (13) as

$$Du(x) \;=\; \frac{1}{2}\left\{1-\; f'\left(\left[\frac{p''-p'}{2}\right]^\top x\right)\right\}p' + \left(1-\frac{1}{2}\left\{1-\; f'\left(\left[\frac{p''-p'}{2}\right]^\top x\right)\right\}\right)p''. \tag{14}$$

Since by assumption $\|f'\|_{C^0(\mathbb{R})} < 1$, we have

$$\left|f'\left(\left[\frac{p''-p'}{2}\right]^\top x\right)\right| \;\leq\; \|f'\|_{C^0(\mathbb{R})} \;<\; 1 \tag{15}$$

and hence

$$0 \;<\; \frac{1}{2}\left\{1-\; f'\left(\left[\frac{p''-p'}{2}\right]^\top x\right)\right\} \;<\; 1. \tag{16}$$

By (14) and (16) we obtain that $Du(x)$ is for all $x \in \mathbb{R}^n$ a strict convex combination of p' and p''. Thus, since

$$[p',p''] \;=\; \big\{tp''+(1-t)p' \;:\; t\in[0,1]\big\}, \tag{17}$$

formula (11) follows. Now, since H is constant on $[p',p'']$, there is a $c \in \mathbb{R}$ such that, for all $t \in [0,1]$

$$H\big(tp''+(1-t)p'\big) \;=\; c. \tag{18}$$

Since $H \in C^1(\mathbb{R}^n)$, differentiation of (18) for $0<t<1$ implies

$$\frac{d}{dt}\Big(H\big(tp''+(1-t)p'\big)\Big) \;=\; (p''-p')^\top H_p\big(tp''+(1-t)p'\big). \tag{19}$$

By (19) and (18), we obtain

$$(p''-p')^\top H_p(\bar{p}) \;=\; 0, \tag{20}$$

for all $\bar{p} \in (p',p'') = \big\{tp''+(1-t)p' : t\in(0,1)\big\}$. Since by strictness $Du(x)$ does not touch the endpoints p', p'' of the segment and lies strictly inside it, we have

$$Du(\mathbb{R}^n) \;\subseteq\; (p',p''). \tag{21}$$

Hence, by combining (20) and (21), we obtain formula (12) and the lemma follows. □

The following lemma is irrelevant for the rest of the proof, but largely motivates the arguments that follow. It establishes the result in the case of twice differentiable solution (5) of the PDE.

Lemma 4. *If f'' exists on $\mathbb{R}$, then (5) defines a twice differentiable solution of the Aronsson equation (1).*

Proof of Lemma 4. By (5) and our assumption, the Hessian $D^2u(x)$ exists for all $x \in \mathbb{R}^n$ and by differentiating (13)

$$D^2u(x) \;=\; \frac{1}{4}\, f''\left(\left[\frac{p''-p'}{2}\right]^\top x\right)(p''-p')\otimes(p''-p'). \tag{22}$$

We calculate, utilizing (11), (12), (6), (22) and (1):

$$
\begin{aligned}
\mathcal{A}[u](x) &= D^2u(x) : H_p\big(Du(x)\big) \otimes H_p\big(Du(x)\big) \\
&= \frac{1}{4}\, f''\left(\left[\frac{p''-p'}{2}\right]^\top x\right)(p''-p') \otimes (p''-p') : \\
&\quad : H_p\left(\frac{p''+p'}{2} \;+\; \frac{1}{2}\, f'\left(\left[\frac{p''-p'}{2}\right]^\top x\right)(p''-p')\right) \otimes \\
&\quad \otimes H_p\left(\frac{p''+p'}{2} \;+\; \frac{1}{2}\, f'\left(\left[\frac{p''-p'}{2}\right]^\top x\right)(p''-p')\right).
\end{aligned}
\tag{23}
$$

Hence, by employing Lemma 3, we have

$$
\begin{aligned}
\mathcal{A}[u](x) &= \left\{(p''-p')^\top H_p\left(\frac{p''+p'}{2} + \frac{1}{2}\, f'\left(\left[\frac{p''-p'}{2}\right]^\top x\right)(p''-p')\right)\right\}^2 \cdot \\
&\qquad \cdot \frac{1}{4}\, f''\left(\left[\frac{p''-p'}{2}\right]^\top x\right) \\
&= 0,
\end{aligned}
\tag{24}
$$

and as a result u is a twice differentiable solution or the PDE. □

Hence, u solves (1) because its Hessian D^2u is perpendicular to $H_p(Du) \otimes H_p(Du)$ in the space of symmetric matrices. Now we return to the general case of merely Lipschitz everywhere differentiable $f \in C^{0+1}(\mathbb{R})$. The next lemma states that the symmetry of u along $\big(\mathrm{span}[p''-p']\big)^\perp$ is reflected to its Jets $J^{2,\pm}u$.

Lemma 5. *If u is as in (5), then for all $x \in \mathbb{R}^n$,*

$$
h \in \big(span[p''-p']\big)^\perp \quad\Longrightarrow\quad J^{2,\pm}u(x) \;=\; J^{2,\pm}u(x+h). \tag{25}
$$

Proof of Lemma 5. It suffices to consider only the case of $J^{2,+}u(x)$ and to establish only that

$$
J^{2,\pm}u(x) \;\subseteq\; J^{2,\pm}u(x+h). \tag{26}
$$

Indeed, if (26) holds for all $x \in \mathbb{R}^n$ and all $h \perp p''-p'$, by replacing x with $x-h$ and then h with $-h$ we obtain

$$
J^{2,\pm}u(x) \;\supseteq\; J^{2,\pm}u(x+h). \tag{27}
$$

Moreover, we may assume that $J^{2,+}u(x) \neq \emptyset$, otherwise (26) follows trivially. Since u is everywhere once differentiable, we have $u(z) = u(x)+Du(x)^\top(z-x)+o(|z-x|)$ as $z \to x$. Hence, it follows from (7) that if $(p, X) \in J^{2,+}u(x)$, then

$$
p \;=\; Du(x). \tag{28}
$$

Suppose $(Du(x), X) \in J^{2,+}u(x)$. Then,

$$
u(z) \;\leq\; u(x) \;+\; Du(x)^\top(z-x) + \;\frac{1}{2}X : (z-x)\otimes(z-x) + o\left(|z-x|^2\right), \tag{29}
$$

as $z \to x$ in $\mathbb{R}^n$.

Fix now h in $\big(\mathrm{span}[p''-p']\big)^\perp$ and set $y:=z+h$ to (29). Then,

$$
\begin{split}
u(y-h)\;-\;u(x)\;\le\;& Du(x)^\top\big(y-(x+h)\big)\\
&+\frac{1}{2}X:\big(y-(x+h)\big)\otimes\big(y-(x+h)\big)\\
&+o\Big(\big|y-(x+h)\big|^2\Big).
\end{split}
\tag{30}
$$

Since $h\in\big(\mathrm{span}[p''-p']\big)^\perp$, we have $(p''-p')^\top h=0$ and hence (5) implies

$$
\begin{split}
u(y-h)\;-\;u(x)\;&=\;\frac{p''+p'}{2}(y-h)\;+\;f\left(\left[\frac{p''-p'}{2}\right]^\top(y-h)\right)\\
&\quad-\;\frac{p''+p'}{2}x\;-\;f\left(\left[\frac{p''-p'}{2}\right]^\top x\right)\\
&=\;\frac{p''+p'}{2}y\;+\;f\left(\left[\frac{p''-p'}{2}\right]^\top y\right)\\
&\quad-\;\frac{p''+p'}{2}(x+h)\;-\;f\left(\left[\frac{p''-p'}{2}\right]^\top (x+h)\right)\\
&=\;u(y)\;-\;u(x+h).
\end{split}
\tag{31}
$$

By plugging (31) into (30), we obtain

$$
\begin{split}
u(y)\;\le\;\;u(x+h)\;+\;& Du(x)^\top\big(z-(x+h)\big)\\
&+\;\frac{1}{2}X:\big(y-(x+h)\big)\otimes\big(y-(x+h)\big)\\
&+\;o\Big(\big|y-(x+h)\big|^2\Big),
\end{split}
\tag{32}
$$

as $y\to x+h$. But (32) means that $(Du(x),X)\in J^{2,+}u(x+h)$. Moreover, we have $(Du(x),X)=(Du(x+h),X)$ since for $h\in\big(\mathrm{span}[p''-p']\big)^\perp$, (13) implies

$$
\begin{split}
Du(x)\;&=\;\frac{p''+p'}{2}\;+\;\frac{1}{2}\,f'\left(\left[\frac{p''-p'}{2}\right]^\top x+\left[\frac{p''-p'}{2}\right]^\top h\right)(p''-p')\\
&=\;\frac{p''+p'}{2}\;+\;\frac{1}{2}\,f'\left(\left[\frac{p''-p'}{2}\right]^\top (x+h)\right)(p''-p')\\
&=\;Du(x+h).
\end{split}
\tag{33}
$$

Thus, we conclude that $J^{2,\pm}u(x)\subseteq J^{2,\pm}u(x+h)$ and the lemma follows. $\square$

Proof of Theorem 1. Let $x\in\mathbb{R}^n$ be such that $(Du(x),X)\in J^{2,+}u(x)$ and fix $\varepsilon>0$ and $\xi\in\big(\mathrm{span}[p''-p']\big)^\perp$. By setting $z:=x+\varepsilon\xi$ in (7), we have

$$
u(x+\varepsilon\xi)\;-\;u(x)\;\le\varepsilon Du(x)^\top\xi\;+\;\frac{\varepsilon^2}{2}X:\xi\otimes\xi\;+\;o(\varepsilon^2).
\tag{34}
$$

By Lemma (5) we have $J^{2,+}u(x+\varepsilon\xi) = J^{2,+}u(x) \neq \emptyset$. Hence, by (7),

$$\begin{aligned} u(z) \ - \ u(x+\varepsilon\xi) \ &\leq Du(x+\varepsilon\xi)^{\top}\big(z-(x+\varepsilon\xi)\big) \\ &+ \ \frac{1}{2}X : \big(z-(x+\varepsilon\xi)\big) \otimes \big(z-(x+\varepsilon\xi)\big) \\ &+ \ o\left(\big|z-(x+\varepsilon\xi)\big|^2\right), \end{aligned} \tag{35}$$

as $z \to x+\varepsilon\xi$. By taking $z := x$ in (37) and adding (34) and (37), we obtain

$$\varepsilon\Big(Du(x+\varepsilon\xi) \ - \ Du(x)\Big) \ \leq \varepsilon^2\left(\frac{X+X}{2}\right) : \xi \otimes \xi \ + \ o(\varepsilon^2). \tag{36}$$

Since $\xi \in \big(\mathrm{span}[p''-p']\big)^{\perp}$, we have by (33) that $Du(x+\varepsilon\xi) - Du(x) = 0$. Thus, we obtain

$$X : \xi \otimes \xi \ \geq \ o(1), \tag{37}$$

as $\varepsilon \to 0^+$. By Lemma 3, we may choose

$$\xi \ := \ H_p\big(Du(x)\big) \tag{38}$$

and by plugging (38) into (37) and passing to the limit as $\varepsilon \to 0^+$, we conclude

$$X : H_p\big(Du(x)\big) \otimes H_p\big(Du(x)\big) \ \geq \ 0. \tag{39}$$

Thus, u is a viscosity subsolution of the Aronsson equation (1). The supersolution property follows similarly. □

2.1. **A Concrete Example of Singular Solution.**

In the paper [K] we demonstrated a singular function $K : \mathbb{R} \longrightarrow [0,1]$ which is $C^{0+\alpha}$ Hölder continuous for any $0<\alpha<1$ but pointwisely nowhere differentiable on $\mathbb{R}$. Moreover, the derivative of K is a singular first order distribution, not given against integration over a signed Radon measure. K is given by

$$K(x) \ := \ \sum_{k=0}^{\infty} 2^{-\alpha\mu k}\phi(2^{\mu k}x), \tag{40}$$

where $\mu \in \mathbb{N}$ with $\mu > 1/(1-\alpha)$ and ϕ is a sawtooth function, given by $\phi(x) := |x|$ when $x \in [-1,1]$ and extended on $\mathbb{R}$ as a periodic function by setting $\phi(x+2) := \phi(x)$. Under the assumption on H of Theorem 1, a singular solution u of the Aronsson PDE can be given by formula (5), by taking f to be the Lipschitz function

$$f(t) \ := \ \int_0^t \big(\chi_{(-\infty,0]}(s) - \chi_{(0,+\infty)}(s)\big)K(s)ds. \tag{41}$$

Then, we obtain a C^{0+1} everywhere once differentiable but non-C^1 solution, which moreover is nowhere twice differentiable with genuinely distributional Hessian.

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References

Ar1. G. Aronsson, *On Certain Singular Solutions of the Partial Differential Equation* $u^2xu_{xx} + 2u_xu_yu_{xy} + u^2yu_{yy} = 0$, Manuscripta Math. 47 (1984), no 1-3, 133 - 151.

Ar2. G. Aronsson, *Construction of Singular Solutions to the p-Harmonic Equation and its Limit Equation for* $p = \infty$, Manuscripta Math. 56 (1986), 135 - 158.

BEJ. E. N. Barron, L. C. Evans, R. Jensen, *The Infinity Laplacian, Aronsson's Equation and their Generalizations*, Transactions of the AMS, Vol. 360, Nr 1, Jan 2008, electr/ly published on July 25, 2007.

CIL. M. G. Crandall, H. Ishii, P.-L. Lions, *User's Guide to Viscosity Solutions of 2nd Order Partial Differential Equations*, Bulletin of the AMS, Vol. 27, Nr 1, Pages 1 - 67, 1992.

DePK. T. De Pauw, A. Koeller, *Linearly Approximatable Functions*, Proc. of the AMS, April 2009, 1347 - 1356, electr. published on Oct. 6, 2008.

E-S. L. C. Evans, O. Savin, $C^{1,\alpha}$ *Regularity for Infinity Harmonic Functions in Two Dimensions*, Calc. Var. 32, 325 - 347, (2008).

E-Sm. L. C. Evans, C. K. Smart, *Everywhere differentiability of Infinity Harmonic Functions*, preprint.

GWY. R. Gariepy, Ch. Wang, Y. Yu, *Generalized Cone Comparison Principle for Viscosity Solutions of the Aronsson Equation and Absolute Minimizers*, Communications in PDE, 31, 1027 - 1046, 2006.

K. N. I. Katzourakis, *A Hölder Continuous Nowhere Differentiable Function with Derivative Singular Distribution*, preprint, 2010.

S. O. Savin, C^1 *Regularity for Infinity Harmonic Functions in Two Dimensions*, Arch. Rational Mech. Anal. 176, 351 - 361, (2005).

W-Y. C. Wang, Y. Yu C^1 *Regularity of the Aronsson Equation in* $\mathbb{R}^2$, Ann. Inst. H. Poincaré, AN 25, 659 - 678, (2008).

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