Blowup for the Euler and Euler-Poisson Equations with Repulsive Forces II

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Abstract

In this paper, we continue to study the blowup problem of the N-dimensional compressible Euler or Euler-Poisson equations with repulsive forces, in radial symmetry. In details, we extend the recent result of "M.W. Yuen, *Blowup for the Euler and Euler-Poisson Equations with Repulsive Forces*, Nonlinear Analysis Series A: Theory, Methods & Applications **74** (2011), 1465–1470.". We could further apply the integration method to obtain the more general results which the non-trivial classical solutions (ρ, V), with compact support in $[0, R]$, where $R > 0$ is a positive constant with $\rho(t, r) = 0$ and $V(t, r) = 0$ for $r \geq R$, under the initial condition

$$H_0 = \int_0^R r^n V_0 dr > 0 \tag{1}$$

where an arbitrary constant $n > 0$, blow up on or before the finite time $T = 2R^{n+2}/(n(n+1)H_0)$ for pressureless fluids or $\gamma > 1$. The results obtained here fully cover the previous known case for $n = 1$.

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1 Introduction

The compressible isentropic Euler ($\delta = 0$) or Euler-Poisson ($\delta = \pm 1$) equations can be written in the following form:

$$\begin{cases} \rho_t + \nabla \cdot (\rho u) = 0 \\ \rho[u_t + (u \cdot \nabla)u] + \nabla P = \rho \nabla \Phi \\ \Delta \Phi(t,x) = \delta \alpha(N) \rho \end{cases} \tag{2}$$

where $\alpha(N)$ is a constant related to the unit ball in R^N. As usual, $\rho = \rho(t,x) \geq 0$ and $u = u(t,x) \in \mathbf{R}^N$ are the density and the velocity respectively. $P = P(\rho)$ is the pressure function. The γ-law for the pressure term $P(\rho)$ could be applied:

$$P(\rho) = K\rho^{\gamma} \tag{3}$$

which the constant $\gamma \geq 1$. If $K > 0$, we call the system with pressure; if $K = 0$, we call it pressureless.

When $\delta = -1$, the system is self-attractive. The equations (2) are the Newtonian descriptions of a galaxy in astrophysics [3] and [6]. When $\delta = 1$, the system is the compressible Euler-Poisson equations with repulsive forces. It can be used as a semiconductor model [8]. For the compressible Euler equation with $\delta = 0$, it is a standard model in fluid mechanics [17]. And the Poisson equation $(2)_3$ could be solved by

$$\Phi(t,x) = \delta \int_{R^N} G(x-y)\rho(t,y)dy \tag{4}$$

where G is Green's function:

$$G(x) \doteq \begin{cases} |x|, & N = 1 \\ \log|x|, & N = 2 \\ \frac{-1}{|x|^{N-2}}, & N \geq 3. \end{cases} \tag{5}$$

For the construction of analytical solutions for the systems, interested readers could refer to [15], [18], [11], [16] and [24]. The local existence for the systems can be found in [17], [19], [2] and

[14]. The analysis of stabilities for the systems may be referred to [23], [1], [12], [20], [21], [22], [10], [11], [25], [5] and [4].

The solutions in radial symmetry could be in this form:

$$\rho = \rho(t,r) \text{ and } u = \frac{x}{r}V(t,r) =: \frac{x}{r}V \tag{6}$$

with the radius $r = \left(\sum_{i=1}^{N} x_i^2\right)^{1/2}$.

The Poisson equation $(2)_3$ becomes

$$r^{N-1}\Phi_{rr}(t,x) + (N-1)\, r^{N-2}\Phi_r {=} \alpha(N)\,\delta\rho r^{N-1} \tag{7}$$

$$\Phi_r = \frac{\alpha(N)\,\delta}{r^{N-1}}\int_0^r \rho(t,s)s^{N-1}ds. \tag{8}$$

By standard computation, the systems in radial symmetry can be rewritten in the following form:

$$\begin{cases} \rho_t + V\rho_r + \rho V_r + \dfrac{N-1}{r}\rho V = 0 \\ \rho\left(V_t + VV_r\right) + P_r(\rho) = \rho\Phi_r(\rho). \end{cases} \tag{9}$$

In literature, Makino, Ukai and Kawashima first studied the blowup of tame solutions [20] for the compressible Euler equations ($\delta = 0$). Then Makino and Perthame investigated the corresponding tame solutions for the system with gravitational forces [21]. After that Perthame [22] obtained the blowup results for the 3-dimensional pressureless system with repulsive forces ($\delta = 1$). There are other blowup results for the systems in [12], [13], [5] and [4].

Very recently, Yuen [26] used the integration method to show that with the initial velocity

$$H_0 = \int_0^R rV_0 dr > 0, \tag{10}$$

the solutions with compact support to the Euler ($\delta = 0$) or Euler-Poisson equations with repulsive forces ($\delta = 1$) blow up in the finite time.

In this article, we can observe that the condition (10) could be more general to have the corresponding blowup results. In fact, we could further apply the integration method to extend Yuen's result as the following theorem:

Theorem 1 *Consider the Euler ($\delta = 0$) or Euler-Poisson equations with repulsive forces ($\delta = 1$) (2) in R^N. The non-trivial classical solutions (ρ, V), in radial symmetry, with compact support in*

$[0, R]$, where $R > 0$ is a positive constant ($\rho(t, r) = 0$ and $V(t, r) = 0$ for $r \geq R$) and the initial velocity:

$$H_0 = \int_0^R r^n V_0 dr > 0 \tag{11}$$

with an arbitrary constant $n > 0$,

blow up on or before the finite time $T = 2R^{n+2}/(n(n+1)H_0)$, for pressureless fluids ($K = 0$) or $\gamma > 1$.

We remark that the condition

$$\rho(t, r) = 0 \text{ and } V(t, r) = 0 \text{ for } r \geq R \tag{12}$$

in the theorem is called non-slip boundary condition [9] and [7].

2 Integration Method

We just follow the integration method which was designed in [26] to obtain the further results.

Proof. The density function $\rho(t, x(t; x))$ preserves its non-negative nature as we can integrate the mass equation $(2)_1$:

$$\frac{D\rho}{Dt} + \rho\nabla \cdot u = 0 \tag{13}$$

with the material derivative,

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (u \cdot \nabla) \tag{14}$$

to have:

$$\rho(t, x) = \rho_0(x_0(0, x_0)) \exp\left(-\int_0^t \nabla \cdot u(t, x(t; 0, x_0))dt\right) \geq 0 \tag{15}$$

for $\rho_0(x_0(0, x_0)) \geq 0$, along the characteristic curve.

Then we can manipulate the momentum equation $(9)_2$ for the non-trivial solutions in radial symmetry, $\rho_0 \neq 0$, to obtain:

$$V_t + VV_r + K\gamma\rho^{\gamma-2}\rho_r = \Phi_r \tag{16}$$

$$V_t + \frac{\partial}{\partial r}(\frac{1}{2}V^2) + K\gamma\rho^{\gamma-2}\rho_r = \Phi_r \tag{17}$$

$$r^n V_t + r^n \frac{\partial}{\partial r}(\frac{1}{2}V^2) + K\gamma r^n \rho^{\gamma-2}\rho_r = r^n \Phi_r \tag{18}$$

with multiplying the more general function r^n with $n > 0$, on the both sides.

We notice that this is the critical step in this paper to extend the blowup result in [26].

We could take the integration with respect to r, to equation (18), for $\gamma > 1$ or $K \geq 0$:

$$\int_0^R r^n V_t dr + \int_0^R r^n \frac{d}{dr}(\frac{1}{2}V^2) + \int_0^R K\gamma r^n \rho^{\gamma-2}\rho_r dr = \int_0^R r^n \Phi_r dr \tag{19}$$

$$\int_0^R r^n V_t dr + \int_0^R r^n \frac{d}{dr}(\frac{1}{2}V^2) + \int_0^R \frac{K\gamma r^n}{\gamma-1} d\rho^{\gamma-1} = \int_0^R \left[\frac{\alpha(N)\delta r^n}{r^{N-1}} \int_0^r \rho(t,s)s^{N-1}ds\right] dr \tag{20}$$

$$\int_0^R r^n V_t dr + \int_0^R r^n \frac{d}{dr}(\frac{1}{2}V^2) + \int_0^R \frac{K\gamma r^n}{\gamma-1} d\rho^{\gamma-1} \geq 0 \tag{21}$$

for $\delta \geq 0$.

Then, the below equation can be showed by integration by part:

$$\begin{aligned} &\textstyle\int_0^R r^n V_t dr - \frac{1}{2}\int_0^R n r^{n-1}V^2 dr + \frac{1}{2}\left[R^n V^2(t,R) - 0^n \cdot V^2(t,0)\right] \\ &-\textstyle\int_0^R \frac{K\gamma n r^{n-1}}{\gamma-1}\rho^{\gamma-1}dr + \frac{K\gamma}{\gamma-1}\left[R^n \rho^{\gamma-1}(t,R) - 0^n \cdot \rho^{\gamma-1}(t,0)\right] \geq 0. \end{aligned} \tag{22}$$

The above inequality with the boundary condition ($V(t,R) = 0$ and $\rho(t,R) = 0$), becomes

$$\int_0^R r^n V_t dr - \frac{1}{2}\int_0^R n r^{n-1}V^2 dr - \int_0^R \frac{K\gamma n r^{n-1}}{\gamma-1}\rho^{\gamma-1}dr \geq 0 \tag{23}$$

$$\frac{d}{dt}\int_0^R r^n V dr - \frac{1}{2}\int_0^R n r^{n-1}V^2 dr - \int_0^R \frac{K\gamma n r^{n-1}}{\gamma-1}\rho^{\gamma-1}dr \geq 0 \tag{24}$$

$$\frac{d}{dt}\frac{1}{n+1}\int_0^R V dr^{n+1} - \frac{1}{2}\int_0^R \frac{n}{(n+1)r}V^2 dr^{n+1} \geq \int_0^R \frac{K\gamma n r^{n-1}}{\gamma-1}\rho^{\gamma-1}dr \geq 0 \tag{25}$$

for $n > 0$ and $\gamma > 1$ or $K = 0$.

For non-trivial initial density functions $\rho_0 \geq 0$, we obtain:

$$\frac{d}{dt}\frac{1}{n+1}\int_0^R V dr^{n+1} - \frac{1}{2}\int_0^R \frac{n}{(n+1)r}V^2 dr^{n+1} \geq 0 \tag{26}$$

$$\frac{d}{dt}\frac{1}{n+1}\int_0^R V dr^{n+1} \geq \int_0^R \frac{n}{2(n+1)r}V^2 dr^{n+1} \geq \frac{n}{2(n+1)R}\int_0^R V^2 dr^{n+1} \tag{27}$$

$$\frac{d}{dt}\int_0^R V dr^{n+1} \geq \frac{n}{2R}\int_0^R V^2 dr^{n+1}. \tag{28}$$

We denote

$$H := H(t) = \int_0^R r^n V dr = \frac{1}{n+1}\int_0^R V dr^{n+1} \tag{29}$$

and apply the Cauchy-Schwarz inequality:

$$\left|\int_0^R V\cdot 1dr^{n+1}\right| \leq \left(\int_0^R V^2dr^{n+1}\right)^{1/2}\left(\int_0^R 1dr^{n+1}\right)^{1/2} \tag{30}$$

$$\left|\int_0^R V\cdot 1dr^{n+1}\right| \leq \left(\int_0^R V^2dr^{n+1}\right)^{1/2}\left(R^{n+1}\right)^{1/2} \tag{31}$$

$$\frac{\left|\int_0^R Vdr^{n+1}\right|}{R^{\frac{n+1}{2}}} \leq \left(\int_0^R V^2dr^{n+1}\right)^{1/2} \tag{32}$$

$$\frac{(n+1)^2H^2}{R^{n+1}} \leq \int_0^R V^2dr^{n+1} \tag{33}$$

$$\frac{n(n+1)^2H^2}{2R^{n+2}} \leq \frac{n}{2R}\int_0^R V^2dr^{n+1} \tag{34}$$

for driving equation (28) to be

$$\frac{d}{dt}(n+1)H \geq \frac{n}{2R}\int_0^R V^2dr^{n+1} \geq \frac{n(n+1)^2H^2}{2R^{n+2}} \tag{35}$$

with inequality (34),

$$\frac{d}{dt}H \geq \frac{n(n+1)H^2}{2R^{n+2}} \tag{36}$$

$$H \geq \frac{-2R^{n+2}H_0}{n(n+1)H_0t-2R^{n+2}}. \tag{37}$$

Finally, we could require the initial condition

$$H_0 = \int_0^R r^nV_0dr > 0 \tag{38}$$

for showing that the solutions blow up on or before the finite time $T = 2R^{n+2}/(n(n+1)H_0)$.

This completes the proof. ■

We notice that the results in this paper fully cover the previous case for $n = 1$ [26]. Further researches are needed to have the corresponding results for the non-radial symmetric cases.

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