ON ABSOLUTE CONVERGENCE OF MULTIPLE FOURIER INTEGRALS

E. LIFLYAND

Abstract. New sufficient conditions for representation of a function of several variables as an absolutely convergent Fourier integral are obtained in the paper.

1. Introduction

If

$$f(y)=\int\limits_{\mathbb{R}^d} g(x)e^{i(x,y)}dx, \qquad g\in L_1(\mathbb{R}^d),$$

we write $f\in A(\mathbb{R}^d)$, with $||f||_A=||g||_{L_1(\mathbb{R}^d)}$.

The possibility to represent a function via the absolutely convergent Fourier integral was studied by many mathematicians and is of importance in various problems of analysis. For example, belonging of a function $m(x)$ to $A(\mathbb{R}^d)$ makes it to be an $L_1\to L_1$ Fourier multiplier (or, equivalently, $L_\infty\to L_\infty$ Fourier multiplier); written $m\in M_1$ ($m\in M_\infty$, respectively). One of such m-s attracted much attention in 50-70s (see, e.g., [3] and [10, Ch.4, 7.4], and references therein):

$$m(x)=\theta(x)\frac{e^{i|x|^\alpha}}{|x|^\beta}, \tag{1.1}$$

where θ is a C^∞ function on $\mathbb{R}^d$, which vanishes near zero, and equals 1 outside a bounded set, and $\alpha,\beta>0$. In is known that for $d\ge 2$:

I) If $\frac{\beta}{\alpha}>\frac{d}{2}$, then $m\in M_1(M_\infty)$.

II) If $\frac{\beta}{\alpha}<\frac{d}{2}$, then $m\notin M_1(M_\infty)$.

The first assertion holds true for $d=1$ as well, while the second one only when $\alpha\ne 1$; however, the case $\alpha=d=1$ is obvious.

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Various sufficient conditions for absolute convergence of Fourier integrals were obtained by Titchmarsh, Beurling, Karleman, Sz.-Nagy, Stein, and many others. One can find more or less comprehensive and very useful survey on this problem in [9]. Let us mention also [7] and a couple of recent papers [1, 4].

New sufficient conditions of belonging to $A(\mathbb{R}^d)$ are obtained in this paper.

Let χ and η be d-dimensional vectors with the entries either 0 or 1 only. The inequality of vectors is meant coordinate wise. Here and in what follows $D^\chi f$ for $\chi = \mathbf{0} = (0, 0, ..., 0)$ or $\chi = \mathbf{1} = (1, 1, ..., 1)$ mean the function itself and the mixed derivative in each variable, respectively, where

$$D^\chi f(x) = \left(\prod_{j:\chi_j=1} \frac{\partial}{\partial x_j}\right) f(x).$$

Let us give a multidimensional result we are going, in a sense, to generalize (see [8]).

Theorem B. *Let $f \in L^1(\mathbb{R}^d)$. If all the mixed derivatives (in the distributional sense) $D^\chi f(x) \in L^p(\mathbb{R}^d)$, $\mathbf{0} < \chi < \mathbf{1}$, where $1 < p \le 2$, then $f \in A(\mathbb{R}^d)$.*

After fixing certain notation and conventions we formulate the results. In the next section we give the proofs.

We shall denote absolute constants by c or maybe by c with various subscripts, like c_1, c_2, etc., while $\gamma(...)$ will denote positive quantities depending only on the arguments indicated in the parentheses.

Our main result reads as follows.

Theorem 1.1. *Let $f \in C_0(\mathbb{R}^d)$ and let f and its partial derivatives $D^\eta f$, $\mathbf{0} \le \eta < \mathbf{1}$, be locally absolutely continuous on $(\mathbb{R} \setminus \{0\})^d$ in each variable.*

a) *Let $D^\chi f(x) \in L_{p_\chi}(\mathbb{R}^d)$, $1 < p_\chi < \infty$ (it is allowed $p_\mathbf{0} = 1$). If*

$$\sum_{\mathbf{0}\le\chi\le\mathbf{1}} \frac{1}{p_\chi} > 2^{d-1}, \tag{1.2}$$

then $f \in A(\mathbb{R}^d)$.

b) *If for the numbers $p_1, ..., p^{2^d}$, $1 \le p_1 < \infty$, $1 < p_2, ..., p_{2^d} < \infty$,*

$$\sum_{k=1}^{2^d} \frac{1}{p_k} < 2^{d-1}, \tag{1.3}$$

then there exists a function f *with* $D^\chi f(x) \in L_{p_\chi}(\mathbb{R}^d)$*, where each* $p\chi$ *is one and only one of the above* p_k*, but* $f \notin A(\mathbb{R}^d)$.

As a corollary, we present a different proof of **I)**.

Corollary 1.2. *If* $\frac{\beta}{\alpha} > \frac{d}{2}$*, then* $m \in A(\mathbb{R}^d)$.

The other corollary gives conditions on which exponent decay of a function f and its derivatives ensures $f \in A(\mathbb{R}^d)$.

Corollary 1.3. *If*

$$|D^\chi f(x)| \le c\frac{1}{(1+|x|)^{\gamma_\chi}}, \tag{1.4}$$

where $\gamma_\chi > 0$ *for all* χ, $\mathbf{0} \le \chi \le \mathbf{1}$, *and*

$$\sum_{\mathbf{0}\le\chi\le\mathbf{1}} \gamma_\chi > d2^{d-1}, \tag{1.5}$$

then $f \in A(\mathbb{R}^d)$.

2. Proofs

We give, step by step, proofs of the results formulated in Introduction. We need certain auxiliary results.

2.1. **Auxiliary results.** One of the basic tools is the following result (see Lemma 4 in [12] or Theorem 3 in [2]).

In order to formulate the next result, on which much in the proofs of our new results is based on, we denote $\Delta_u f(x) = \Delta_{u_1,\cdots,u_d} f(x) = \prod_{j=1}^{d} D_{u_j} f(x)$, where $D_{u_j} f$ is defined as

$$\Delta_{u_j} f(x) = f(x + u_j e_j^0) - f(x - u_j e_j^0), \quad 1 \le j \le d. \tag{2.1}$$

Theorem C. *Let* $f \in C_0(\mathbb{R}^d)$*. If*

$$\sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_d=-\infty}^{\infty} 2^{\frac{1}{2}\sum_{j=1}^d s_j} \|\Delta_{\frac{\pi}{2^{s_1}},\cdots,\frac{\pi}{2^{s_d}}}(f)\|_2 < \infty,$$

where the norm is that in $L_2(\mathbb{R}^d)$*, then* $f \in A(\mathbb{R}^d)$.

In dimension one the second basic tools was the following Steklov-Hardy type inequality (a partial case of the general result [5, Cor.3.14]):

For $F(s)\ge 0$ and $1<q\le Q<\infty$

$$\left(\int\limits_{\mathbb{R}}\left[\int\limits_{t-h}^{t+h}F(s)\,ds\right]^{Q}dt\right)^{1/Q}\le ch^{1/Q+1/q'}\left(\int\limits_{\mathbb{R}}F^{q}(t)\,dt\right)^{1/q}. \tag{2.2}$$

Here $\frac{1}{q}+\frac{1}{q'}=1$. Similarly $\frac{1}{p}+\frac{1}{p'}=1$.

Though there are multivariate versions of general Steklov-Hardy inequality (see, e.g., [11]), we need a simpler direct generalization of (2.2):

Lemma 2.1. *For $F(u)\ge 0$, $1\le k\le d$, and $1<q\le Q<\infty$*

$$\left(\int\limits_{\mathbb{R}^d}\left[\int\limits_{x_1-h_1}^{x_1+h_1}\dots\int\limits_{x_k-h_k}^{x_k+h_k}F(u_1,\dots,u_k,x_{k+1},\dots,x_d)\,du_1\dots du_k\right]^{Q}dx\right)^{1/Q}$$
$$\le c(h_1\dots h_d)^{1/Q+1/q'}(h_{k+1}\dots h_d)^{-1}\left(\int\limits_{\mathbb{R}^d}F^{q}(x)\,dx\right)^{1/q}. \tag{2.3}$$

Proof. The proof is inductive. For $d=1$, the result holds true: (2.2). Supposing that it is true for $d-1$, $d=2,3,\dots$, let us prove (2.3) with $k=d$. Applying inductive assumption for the first $d-1$ variables, we obtain

$$\left(\int\limits_{\mathbb{R}^d}\left[\int\limits_{x_1-h_1}^{x_1+h_1}\dots\int\limits_{x_d-h_d}^{x_d+h_d}F(u_1,\dots,u_d)\,du_1\dots du_d\right]^{Q}dx_1\dots dx_d\right)^{1/Q}$$
$$=\left(\int\limits_{\mathbb{R}}\left\{\int\limits_{\mathbb{R}^{d-1}}\left[\int\limits_{x_1-h_1}^{x_1+h_1}\dots\int\limits_{x_{d-1}-h_{d-1}}^{x_{d-1}+h_{d-1}}\int\limits_{x_d-h_d}^{x_d+h_d}F(u_1,\dots,u_d)\,du_1\dots du_d\right]^{Q}\right.\right.$$
$$\left.\left.dx_1\dots dx_{d-1}\right\}^{Q/Q}dx_d\right)^{1/Q}$$
$$\le c(h_1\dots h_{d-1})^{1/Q+1/q'}\left(\int\limits_{\mathbb{R}}\left\{\int\limits_{\mathbb{R}^{d-1}}\left[\int\limits_{x_d-h_d}^{x_d+h_d}F(x_1,\dots,x_{d-1},u_d)\,du_d\right]^{q}\right.\right.$$
$$\left.\left.dx_1\dots dx_{d-1}\right\}^{Q/q}dx_d\right)^{\frac{q}{Q}\frac{1}{q}}.$$

Applying now the generalized Minkowski inequality with exponent $Q/q \geq 1$, we bound the right-hand side by, times a constant,

$$(h_1...h_{d-1})^{1/Q+1/q'} \left(\int\limits_{\mathbb{R}^{d-1}} \left\{ \int\limits_{\mathbb{R}} \left[\int\limits_{x_d-h_d}^{x_d+h_d} F(x_1,...,x_{d-1},u_d)\,du_d \right]^Q dx_d \right\}^{q/Q} dx_1...dx_{d-1} \right)^{1/q}.$$

To obtain (2.3), it remains again to make use of (2.2) for the d-th variable.

If $k < d$, we just add the integration over $\int\limits_{x_{k+1}-h_{k+1}}^{x_{k+1}+h_{k+1}} ... \int\limits_{x_d-h_d}^{x_d+h_d}$, compensate it with $(2h_{k+1}...2h_d)^{-1}$, and apply the proved version for $k = d$. The proof is complete. □

2.2. **Proof of Theorem 1.1.** Let us first prove **b)**. Let us take m (see Introduction) such that $D^\chi m(x) \in L_{p_\chi}(\mathbb{R}^d)$. Since for each $k = \chi_1+...+\chi_d$ there holds $|D^\chi m(x)| \leq c|x|^{-\beta+k(\alpha-1)}$ for $|x|$ large, we must have, assuming $\beta - k(\alpha-1) > 0$ for all k, the inequality $(\beta - k(\alpha-1))p_j > d$ for $\binom{d}{k}$ numbers p_j. Then

$$\sum_{k=1}^{2^d} \frac{1}{p_k} < \frac{2^d\beta - (\alpha-1)\sum\limits_{k=1}^{d} k\binom{d}{k}}{d} = \frac{2^d\beta - (\alpha-1)d2^{d-1}}{d}.$$

Choosing also β and α such that the right-hand side is smaller than 2^{d-1}, we obtain

$$2^d\beta - (\alpha-1)d2^{d-1} < d2^{d-1},$$

which is equivalent to **II)**. Therefore such m cannot be in $A(\mathbb{R}^d)$.

For simplicity, let us give a two-dimensional proof of **a)**. First of all, this allows us to use much simpler notation. On the one hand, we will denote the derivatives by merely f_1, f_2, and f_{12}. On the other hand, we will correspondingly denote p_χ by p_0, p_1, p_2, and p_{12}. Analogously, we will denote Δ_{12}, Δ_1, and Δ_2. The two-dimensional version of the sum in Theorem C can be expressed as the 4 sums

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}2^{-k/2}2^{-l/2} \quad + \quad \sum_{k=1}^{\infty}\sum_{l=0}^{\infty}2^{-k/2}2^{l/2}$$
$$+\sum_{k=0}^{\infty}\sum_{l=1}^{\infty}2^{k/2}2^{-l/2} \quad + \quad \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}2^{k/2}2^{l/2}. \tag{2.4}$$

The second and third sums are estimated in absolutely the same way, therefore we will estimate only on of them.

Let us start with the first sum. We have

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}2^{-k/2}2^{-l/2}\biggl(\int\limits_{\mathbb{R}^2}|\Delta_{12}f(x,y)|^2\,dx\,dy\biggr)^{1/2}.$$

Here $\Delta_{12}f(x,y)=f(x+h(-k),y+h(-l))-f(x+h(-k),y-h(-l))-f(x-h(-k),y+h(-l))+f(x-h(-k),y-h(-l))$, with $h(q)=\pi 2^{-q}$. For this sum, Δ_1 and Δ_2 will be used with the same steps. In what follows, we will use the steps $h(k)$ or $h(-k$ when $2^{k/2}$ or $2^{-k/2}$, respectively, are summed up; the same for $h(\pm l)$.

Now, we represent the inner integral as

$$\int\limits_{\mathbb{R}^2}|\Delta_{12}f(x,y)|^{1/2}\Biggl|\int\limits_{x-h(-k)}^{x+h(-k)}\Delta_2 f_1(u,y)\,du\Biggr|^{1/2}$$
$$\times\quad\Biggl|\int\limits_{y-h(-l)}^{y+h(-l)}\Delta_1 f_2(x,v)\,dv\Biggr|^{1/2}\Biggl|\int\limits_{x-h(-k)}^{x+h(-k)}\int\limits_{y-h(-l)}^{y+h(-l)}f_{12}(u,v)\,du\,dv\Biggr|^{1/2}dx\,dy.$$

Further, we apply Hölder's inequality with 4 parameters $2p_0$, $2q_1\ge 2p_1$, $2q_2\ge 2p_2$, and $2q_{12}\ge 2p_{12}$ satisfying

$$\frac{1}{2p_0}+\frac{1}{2q_1}+\frac{1}{2q_2}+\frac{1}{2q_{12}}=1. \tag{2.5}$$

This is possible because of (1.2) for $d=2$. We obtain 4 norms, the first one is dominated by $\|f\|_{p_0}^{1/4}$. We then apply Lemma 2.1 to the next 3 norms with the exponents q_1 and p_1, q_2 and p_2, q_{12} and p_{12}, respectively. The bound for the sum in question will then be, times a constant,

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} 2^{-k/2}2^{-l/2}(2^{k/2}2^{l/2})^{(\frac{1}{2q_1}+\frac{1}{2p'_1})+(\frac{1}{2q_2}+\frac{1}{2p'_2})-\frac{1}{2}+(\frac{1}{2q_{12}}+\frac{1}{2p'_{12}})}$$
$$\times\|f\|_{p_0}^{1/4}\|f_1\|_{p_1}^{1/4}\|f_2\|_{p_2}^{1/4}\|f_{12}\|_{p_{12}}^{1/4}.$$

We thus have $2^{-k/2}$ (and $2^{-l/2}$) in the power

$$\begin{aligned} 1 \quad &- \quad [(\frac{1}{2q_1}+\frac{1}{2p'_1})+(\frac{1}{2q_2}+\frac{1}{2p'_2})-\frac{1}{2}+(\frac{1}{2q_{12}}+\frac{1}{2p'_{12}})] \\ &= \quad [1-\frac{1}{2q_1}-\frac{1}{2q_2}-\frac{1}{2q_{12}}]-1+\frac{1}{2p_1}+\frac{1}{2p_2}+\frac{1}{2p_{12}}. \end{aligned}$$

By (2.5), the expression in the brackets on the right is $\frac{1}{2p_0}$, and by (1.2) the whole right-hand side is positive. Therefore, the sums converge, as required.

Let us now prove the finiteness of the sum

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} 2^{k/2}2^{l/2}\Big(\int\limits_{\mathbb{R}^2}|\Delta_{12}f(x,y)|^2\,dx\,dy\Big)^{1/2}.$$

We remind that the steps here are $h(k)$ and $h(l)$. In this case we represent the inner integral as

$$\int\limits_{\mathbb{R}^2}|\Delta_{12}f(x,y)|^{\frac{(1-\delta)^2}{2}}\Big|\int\limits_{x-h(-k)}^{x+h(-k)}\Delta_2 f_1(u,y)\,du\Big|^{\frac{1-\delta^2}{2}}$$
$$\Big|\int\limits_{y-h(-l)}^{y+h(-l)}\Delta_1 f_2(x,v)\,dv\Big|^{\frac{1-\delta^2}{2}}\Big|\int\limits_{x-h(-k)}^{x+h(-k)}\int\limits_{y-h(-l)}^{y+h(-l)}f_{12}(u,v)\,du\,dv\Big|^{\frac{(1+\delta)^2}{2}}\,dx\,dy.$$

This is true for any $\delta\in(0,1)$, and the sum of the exponents is 2. Applying similarly Hölder's inequality, this time with the parameters $\frac{(1-\delta)^2}{2p_0}$, $2q_1\ge 2p_1$, $2q_2\ge 2p_2$, and $2q_{12}\ge 2p_{12}$ satisfying

$$\frac{(1-\delta)^2}{2p_0}+\frac{1}{2q_1}+\frac{1}{2q_2}+\frac{1}{2q_{12}}=1. \tag{2.6}$$

We again obtain 4 norms, the first one is dominated by $\|f\|_{p_0}^{(1-\delta)^2/4}$. We then apply Lemma 2.1 to the next 3 norms with the exponents

$(1-\delta^2)q_1$ and p_1, $(1-\delta^2)q_2$ and p_2, $(1+\delta)^2 q_{12}$ and p_{12}, respectively. The bound for the sum in question will then be, times a constant,

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} 2^{k/2}2^{l/2}(2^{-k/2}2^{-l/2})^{(\frac{1}{2q_1}+\frac{1-\delta^2}{2p'_1})+(\frac{1}{2q_2}+\frac{1-\delta^2}{2p'_2})-\frac{1-\delta^2}{2}+(\frac{1}{2q_{12}}+\frac{(1+\delta)^2}{2p'_{12}})}$$
$$\times \quad \|f\|_{p_0}^{(1-\delta)^2/4}\|f_1\|_{p_1}^{(1-\delta^2)/4}\|f_2\|_{p_2}^{(1-\delta^2)/4}\|f_{12}\|_{p_{12}}^{(1+\delta)^2/4}.$$

We thus have $2^{k/2}$ (and $2^{l/2}$) in the power

$$\begin{aligned} 1 \quad &- \quad [(\frac{1}{2q_1}+\frac{1-\delta^2}{2p'_1})+(\frac{1}{2q_2}+\frac{1-\delta^2}{2p'_2})-\frac{1-\delta^2}{2}+(\frac{1}{2q_{12}}+\frac{(1+\delta)^2}{2p'_{12}})] \\ &= \quad [1-\frac{1}{2q_1}-\frac{1}{2q_2}-\frac{1}{2q_{12}}]-1-\delta+\frac{1-\delta^2}{2p_1}+\frac{1-\delta^2}{2p_2}+\frac{(1+\delta)^2}{2p_{12}}. \end{aligned}$$

By (2.6), the expression in the brackets on the right is $\frac{(1-\delta)^2}{2p_0}$. We must have the final expression

$$\frac{(1-\delta)^2}{2p_0}+\frac{1-\delta^2}{2p_1}+\frac{1-\delta^2}{2p_2}+\frac{(1+\delta)^2}{2p_{12}}-1-\delta$$

to be negative. We cannot use (1.2) immediately as above: the choice of δ is in order. Since for $\delta=1$ the inequality $\frac{4}{2p_{12}}<2$ is true (we remind that $1<p_{12}<\infty$), one can choose by continuity certain $\delta<1$ to continue keeping the inequality true. This completes the proof of the case in question.

It remains to prove the finiteness of the sum

$$\sum_{k=1}^{\infty}\sum_{l=0}^{\infty} 2^{-k/2}2^{l/2}\Bigg(\int_{\mathbb{R}^2}|\Delta_{12}f(x,y)|^2\,dx\,dy\Bigg)^{1/2},$$

with the steps $h(-k)$ and $h(l)$ now. In this case we represent the inner integral as

$$\int_{\mathbb{R}^2}|\Delta_{12}f(x,y)|^{\frac{1-\delta}{2}}\Bigg|\int_{x-h(-k)}^{x+h(-k)}\Delta_2 f_1(u,y)\,du\Bigg|^{\frac{1-\delta}{2}}$$
$$\Bigg|\int_{y-h(-l)}^{y+h(-l)}\Delta_1 f_2(x,v)\,dv\Bigg|^{\frac{1+\delta}{2}}\Bigg|\int_{x-h(-k)}^{x+h(-k)}\int_{y-h(-l)}^{y+h(-l)} f_{12}(u,v)\,du\,dv\Bigg|^{\frac{1+\delta}{2}}\,dx\,dy.$$

This is true for any $\delta \in (0,1)$, and the sum of the exponents is 2. Applying similarly Hölder's inequality, this time with the parameters $\frac{1-\delta}{2p_0}$, $2q_1 \ge 2p_1$, $2q_2 \ge 2p_2$, and $2q_{12} \ge 2p_{12}$ satisfying

$$\frac{1-\delta}{2p_0} + \frac{1}{2q_1} + \frac{1}{2q_2} + \frac{1}{2q_{12}} = 1. \tag{2.7}$$

We again obtain 4 norms, the first one is dominated by $\|f\|_{p_0}^{(1-\delta)/4}$. We then apply Lemma 2.1 to the next 3 norms with the exponents $(1-\delta)q_1$ and p_1, $(1+\delta)q_2$ and p_2, $(1+\delta)q_{12}$ and p_{12}, respectively. The bound for the sum in question will then be, times a constant,

$$\sum_{k=1}^{\infty} 2^{-k/2}(2^{k/2})^{(\frac{1}{2q_1}+\frac{1-\delta}{2p'_1})+(\frac{1}{2q_2}+\frac{1+\delta}{2p'_2})-\frac{1+\delta}{2}+(\frac{1}{2q_{12}}+\frac{1+\delta}{2p'_{12}})}$$
$$\sum_{l=0}^{\infty} 2^{l/2}(2^{-l/2})^{(\frac{1}{2q_1}+\frac{1-\delta}{2p'_1})+(\frac{1}{2q_2}+\frac{1+\delta}{2p'_2})-\frac{1-\delta}{2}+(\frac{1}{2q_{12}}+\frac{1+\delta}{2p'_{12}})}$$
$$\|f\|_{p_0}^{(1-\delta)/4}\|f_1\|_{p_1}^{(1-\delta)/4}\|f_2\|_{p_2}^{(1+\delta)/4}\|f_{12}\|_{p_{12}}^{(1+\delta)/4}.$$

To ensure the convergence of both sums, we arrive at the inequality

$$1 < \frac{1-\delta}{2p_0} + \frac{1-\delta}{2q_1} + \frac{1+\delta}{2q_2} + \frac{1+\delta}{2q_{12}} < 1+\delta \tag{2.8}$$

to be valid for certain δ. The left inequality follows from the estimates in k, while the right one from estimates in l. Recording the sum in the left-hand side of (1.2) as $1+\epsilon$, $0<\epsilon<1$, we can rewrite (2.8) as

$$0 < \epsilon - \delta A + \delta B < \delta, \tag{2.9}$$

with obvious meaning for A and B. We note meantime that $0 < A, B < 1$ and $A + B = 1$.

1) Let first $B > A$. In this case the left inequality in (2.9) holds automatically, we discuss the right one. It is equivalent to the inequality $\frac{\epsilon}{2A-\epsilon} < \delta$. Since $A > \epsilon$, there holds $\frac{\epsilon}{2A-\epsilon} < 1$, and such δ does exist.

2) When $B < A$, both inequalities should be justified. The right one is treated as above. The left one is equivalent to $\delta < \frac{\epsilon}{2A-1-\epsilon}$. If the value on the right is not smaller than 1, the needed inequality holds true for any $\delta < 1$. If it is smaller than one, it suffices to take δ just a little bit smaller than $\frac{\epsilon}{2A-1-\epsilon}$, but such that it is still greater than $\frac{\epsilon}{2A-\epsilon}$.

3) Of course, the situation when $A = B$ is trivial.

The proof is complete.

2.3. **Proof of Corollary 1.2.** The result, in fact, follows from Corollary 1.3 and estimates for the derivatives of m as in the proof of **b)** of the theorem.

2.4. **Proof of Corollary 1.3.** Let us rewrite (1.5) as

$$\sum_{\mathbf{0}\le\chi\le\mathbf{1}}\gamma_\chi=d2^{d-1}+\epsilon.$$

For each χ, let us choose p_χ so that $\gamma_\chi p_\chi=d+\frac{\epsilon}{2^d}$. Then

$$\sum_{\mathbf{0}\le\chi\le\mathbf{1}}\frac{d+\epsilon/2^d}{p_\chi}>d2^{d-1}+\epsilon.$$

Since

$$\sum_{\mathbf{0}\le\chi\le\mathbf{1}}\frac{\epsilon/2^d}{p_\chi}<\epsilon,$$

there holds

$$\sum_{\mathbf{0}\le\chi\le\mathbf{1}}\frac{d}{p_\chi}>d2^{d-1}.$$

This is equivalent to (1.2), and hence $f\in A(\mathbb{R}^d)$.

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Department of Mathematics, Bar-Ilan University, 52900 Ramat-Gan, Israel

E-mail address: liflyand@math.biu.ac.il