NORM AND ANTI-NORM INEQUALITIES FOR POSITIVE SEMI-DEFINITE MATRICES

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Abstract. Some subadditivity results involving symmetric (unitarily invariant) norms are obtained. For instance, if $g(t)=\sum_{k=0}^{m} a_k t^k$ is a polynomial of degree m with non-negative coefficients, then, for all positive operators A, B and all symmetric norms,

$$\|g(A+B)\|^{1/m} \le \|g(A)\|^{1/m} + \|g(B)\|^{1/m}.$$

To give parallel superadditivity results, we investigate *anti-norms*, a class of functionals containing the Schatten q-norms for $q\in(0,1]$ and $q<0$. The results are extensions of the Minkowski determinantal inequality. A few estimates for block-matrices are derived. For instance, let $f:[0,\infty)\to[0,\infty)$ be concave and $p\in(1,\infty)$. If $f^p(t)$ is superadditive, then $\operatorname{Tr} f(A) \ge \left(\sum_{i=1}^{m} f^p(a_{ii})\right)^{1/p}$ for all positive $m\times m$ matrix $A=[a_{ij}]$. Furthermore, for the normalized trace τ, we consider functions $\varphi(t)$ and $f(t)$ for which the functional $A\mapsto \varphi\circ\tau\circ f(A)$ is convex or concave, and obtain a simple analytic criterion.

1. Introduction

Let $g(t)$ be a convex function on the positive half-line vanishing at 0. There exist several matrix versions of the obvious superadditivity property $g(a+b)\ge g(a)+g(b)$. It is less usual to mix convexity assumptions and subadditivity results. This paper points out some matrix subadditivity inequalities involving convex functions such as the first inequality stated in the abstract. In a parallel way, we also consider superadditivity inequalities with concave functions, implying some estimate such as the second inequality of the abstract. These lead us to study a wide class of functionals on the cone of positive definite matrices, that we call anti-norms.

The next section surveys a short list of known subadditivity and superadditivity inequalities. Section 3 introduces the class of anti-norms and Section 4 contains our new results. Finally in Section 5, we discuss some simple convexity/concavity criteria for a functional $A\mapsto\varphi\circ\tau\circ f(A)$, where the range of $f(t)$ is included in the domain of $\varphi(t)$ and τ is the normalized trace ($\tau(I)=1$).

By operator, we mean a linear operator on a finite-dimensional Hilbert space. We use interchangeably the terms operator and matrix. Especially, a positive operator means a positive (semi-definite) matrix. Consistently $\mathbb{M}_n$ denotes the set of operators on a space of dimension n and $\mathbb{M}_n^+$ stands for its positive part. Though we confine to $\mathbb{M}_n$, some extensions to the infinte dimensional setting and operator algebras are possible.

2000 *Mathematics Subject Classification.* Primary 15A60, 47A30, 47A60.

Key words and phrases. Matrix, operator, trace, symmetric norm, symmetric anti-norm, convex function, concave function, subadditivity, superadditivity, majorization.

2. Concave functions and unitary orbits

Here we recall some recent subadditivity properties for concave functions, and similarly superadditivity properties of convex functions. Section 4 will be devoted to our new inequalities contrasting by involving convex functions and subadditivity properties, or concave functions and superadditivity results. Given two Hermitian operators, the relation $X \le Y$ refers to the usual positive semi-definite ordering. From [1] we know:

Theorem 2.1. *Let $f : [0,\infty) \to [0,\infty)$ be concave and A, B be positive operators. Then, for some unitaries U, V,*

$$f(A+B) \le Uf(A)U^* + Vf(B)V^*.$$

Thus, the obvious scalar inequality $f(a+b) \le f(a)+f(b)$ can be extended to positive matrices A and B by considering elements in the unitary orbits of $f(A)$ and of $f(B)$. This inequality via unitary orbits considerably improves the famous Rotfel'd trace inequality for non-negative concave functions and positive operators,

$$\operatorname{Tr} f(A+B) \le \operatorname{Tr} f(A) + \operatorname{Tr} f(B), \tag{2.1}$$

and its symmetric norm version

$$\|f(A+B)\| \le \|f(A)\| + \|f(B)\|. \tag{2.2}$$

By definition, a symmetric norm $\|\cdot\|$ on $\mathbb{M}_n$ satisfies the unitary invariance condition $\|A\| = \|UA\| = \|AU\|$ for all A and all unitaries U.

We can employ Theorem 2.1 to get an inequality for positive block-matrices

$$\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \in \mathbb{M}_{n+m}^+, \qquad A \in \mathbb{M}_n^+,\ B \in \mathbb{M}_m^+,$$

which nicely extend (2.2). Combined with a useful decomposition for elements in $\mathbb{M}_{n+m}^+$ noticed in [4],

$$\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} = U \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} U^* + V \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix} V^* \tag{2.3}$$

for some unitaries U, $V \in \mathbb{M}_{n+m}$, Theorem 2.1 entails a recent theorem of Lee [9]:

Corollary 2.2. *Let $f(t)$ be a non-negative concave function on $[0,\infty)$. Then, given an arbitrary partitioned positive semi-definite matrix,*

$$\left\| f\left(\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right) \right\| \le \|f(A)\| + \|f(B)\|$$

for all symmetric norms.

Applied to $X = A^{1/2}B^{1/2}$, Lee's result yields the Rotfel'd inequalities (2.1) and (2.2). In case of the trace norm, the above result may be restated as a trace inequality without any non-negative assumption: For all concave function $f(t)$ on the positive half-line and for all positive block-matrices,

$$\operatorname{Tr} f\left(\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right) \le \operatorname{Tr} f(A) + \operatorname{Tr} f(B).$$

The case of $f(t) = \log t$ then gives *Fisher's inequality*

$$\det \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \le \det A \det B.$$

Theorem 2.1 may be used to extend another classical (superadditivity and concavity) property of the determinant, *Minkowski's inequality*, stating that for A, $B \in \mathbb{M}_n^+$,

$$\det{}^{1/n}(A+B) \ge \det{}^{1/n} A + \det{}^{1/n} B. \tag{2.4}$$

In fact, Theorem 2.1 is true for any *monotone* (i.e., non-decreasing or non-increasing) and concave function on $[0, \infty)$ such that $f(0) \ge 0$, see [1, Theorem 2.1]. Hence:

Corollary 2.3. *Let $g : [0,\infty) \to [0,\infty)$ be convex with $g(0) = 0$ and let A, B be positive operators. Then, for some unitaries U, V,*

$$g(A+B) \ge Ug(A)U^* + Vg(B)V^*.$$

This convexity version of Theorem 2.1 may then be used to refine (2.4) as

$$\det{}^{1/n} g(A+B) \ge \det{}^{1/n} g(A) + \det{}^{1/n} g(B) \tag{2.5}$$

for all A, $B \in \mathbb{M}_n^+$ and all non-negative convex functions $g(t)$ vanishing at 0.

Minkowski's inequality means that $X \mapsto \det^{1/n} X$ is concave on $\mathbb{M}_n^+$. By using another estimate with unitary orbits, this concavity aspect of (2.4) can be generalized too. Recall that a linear map $\Phi : \mathbb{M}_n \to \mathbb{M}_m$ is called a unital positive linear map if Φ preserves positivity and identity. Denote by $\mathbb{M}_n\{\Omega\}$ the set of Hermitian operators in $\mathbb{M}_n$ with spectra in an interval $\Omega \subset \mathbb{R}$. Then, we have from [4], [3]:

Theorem 2.4. *Let $\Phi : \mathbb{M}_n \to \mathbb{M}_m$ be a unital positive linear map, let $f(t)$ be a concave function on an interval Ω, and let $A, B \in \mathbb{M}_n\{\Omega\}$. Then, for some unitaries U, $V \in \mathbb{M}_m$,*

$$f(\Phi(A)) \ge \frac{U\Phi(f(A))U^* + V\Phi(f(A))V^*}{2}.$$

If furthermore $f(t)$ is monotone, then we can take $U = V$.

This statement for positive maps contains several Jensen type inequalities. The simplest one is obtained by taking $\Phi : \mathbb{M}_{2n} \to \mathbb{M}_n$,

$$\Phi\left(\begin{bmatrix} A & X \\ Y & B \end{bmatrix} \right) := \frac{A+B}{2}.$$

With $X = Y = 0$, Theorem 2.4 then says: *If $A, B \in \mathbb{M}_n\{\Omega\}$ and $f(t)$ is a concave function on Ω, then, for some unitaries U, $V \in \mathbb{M}_n$,*

$$f\left(\frac{A+B}{2}\right) \ge \frac{1}{2}\left\{ U\frac{f(A)+f(B)}{2}U^* + V\frac{f(A)+f(B)}{2}V^* \right\}. \tag{2.6}$$

If furthermore $f(t)$ is monotone, then we can take $U = V$.

By combining (2.6) and (2.4) we obtain an extension of the concavity aspect of (2.4): *If $f(t)$ is a non-negative concave function on Ω and if $A, B \in \mathbb{M}_n\{\Omega\}$, then*

$$\det{}^{1/n} f\left(\frac{A+B}{2}\right) \ge \frac{\det^{1/n} f(A) + \det^{1/n} f(B)}{2}. \tag{2.7}$$

As another example of combination of Theorem 2.1 and (2.3), we have [4]:

Corollary 2.5. *Let $f : [0,\infty) \to [0,\infty)$ be concave and let $A = [a_{ij}]$ be a positive operator on a space of dimension m. Then, for some rank one ortho-projections $\{E_i\}_{i=1}^m$,*

$$f(A) \le \sum_{i=1}^{m} f(a_{ii})E_i.$$

This refines the standard majorization inequality $\operatorname{Tr} f(A) \le \sum_{i=1}^m f(a_{ii})$. The next sections propose some variations on this relation and other Minkowski type inequalities. For a detailed background on majorization and unitarily invariant norms, see for instance [2], [7], [10]. A proof of decomposition (2.3) will be given within the proof of Corollary 4.4 below for the convenience of the reader.

3. Anti-norms on positive operators

Symmetric norms on $\mathbb{M}_n$ can be defined by their restriction to the positive part $\mathbb{M}_n^+$. The following axioms are required: (a) $\|A\| = \|UAU^*\|$ for all unitary $U \in \mathbb{M}_n$ and all $A \in \mathbb{M}_n^+$; (b) $\|\lambda A\| = \lambda\|A\|$ for all real $\lambda \ge 0$ and $A \in \mathbb{M}_n^+$; (c) for all A, $B \in \mathbb{M}_n^+$,

$$\|A\| + \|B\| \ge \|A+B\| \ge \|A\|.$$

Indeed, a symmetric norm on $\mathbb{M}_n$ satisfies (a)–(c), and conversely, if a functional $\|\cdot\| : \mathbb{M}_n^+ \to [0,\infty)$ satisfies (a)–(c), then $\|X\| := \|\,|X|\,\|$ for $X \in \mathbb{M}_n$ is a symmetric norm on $\mathbb{M}_n$ as far as $\|\cdot\|$ is not identically zero.

The following definition then seems natural:

Definition 3.1. A functional on the cone $\mathbb{M}_n^+$ taking values in $[0,\infty)$, $A \mapsto \|A\|_!$, is called a *symmetric anti-norm* if the following two conditions are fulfilled:

1. It is homogeneous and concave (equivalently, superadditive), that is,
$$\|\lambda A\|_! = \lambda\|A\|_! \qquad \text{and} \qquad \|A+B\|_! \ge \|A\|_! + \|B\|_!$$
for all real $\lambda \ge 0$ and all A, $B \in \mathbb{M}_n^+$.
2. It is unitarily invariant (or symmetric), that is,
$$\|A\|_! = \|UAU^*\|_!$$
for all unitaries $U \in \mathbb{M}_n$ and all $A \in \mathbb{M}_n^+$.

For a general $X \in \mathbb{M}_n$, we may define $\|X\|_! = \|\,|X|\,\|_!$ and then obtain a symmetric anti-norm on the whole space $\mathbb{M}_n$. If furthermore $\|X\|_! = 0$ implies that $X = 0$, then $\|\cdot\|_!$ is called *regular*.

Similarly to the usual symmetric norms, symmetric anti-norms are defined by symmetric *anti-gauge* functions on $\mathbb{R}_+^n = [0,\infty)^n$. For $A \in \mathbb{M}_n^+$ we write $\lambda(A) = (\lambda_1(A), \dots, \lambda_n(A))$ for the eigenvalue vector of A arranged in decreasing order with multiplicities.

Proposition 3.2. *There is a bijective correspondence between the symmetric anti-norms $\|\cdot\|_!$ on $\mathbb{M}_n$ and the homogeneous and concave functions $\Phi_! : \mathbb{R}_+^n \to [0,\infty)$ that are invariant under coordinate permutations, determined by $\|A\|_! = \Phi_!(\lambda(A))$ for $A \in \mathbb{M}_n^+$.*

Proof. The proof is similar to the usual symmetric norm case (see [7, 4.4.3] for example), by using the Ky Fan majorization $\lambda(A+B) \prec \lambda(A)+\lambda(B)$ for $A, B \in \mathbb{M}_n^+$. The details are left to the reader. $\square$

Example 3.3. The trace norm is an anti-norm! More generally for $k = 1, \dots, n$, we define the *Ky Fan k-anti-norm* on $\mathbb{M}_n$ as the sum of the k smallest singular values, i.e.,

$$\|A\|_{\{k\}} := \sum_{j=1}^{k} \mu_{n+1-j}(A),$$

where $\mu_1(A) \geq \cdots \geq \mu_n(A)$ are the singular values of A in decreasing order with multiplicities. From the min-max principle, it is well-known that this functional is concave. The anti-norm $\|\cdot\|_{(k)}$ is not regular except for $k = n$ (the trace norm).

Example 3.4. Let $q \in (0, 1)$. The *Schatten q-norms* (or q-quasi-norms) on $\mathbb{M}_n$

$$\|A\|_q := \left(\sum_{j=1}^{n} \mu_j^q(A)\right)^{1/q}$$

are symmetric regular anti-norms. The terminology Schatten q-anti-norms allows to distinguish with the norm case ($q > 1$). See Proposition 3.7 below for the proof of a more general result.

The next two examples are related to the classical geometric and harmonic means. These anti-norms are not regular.

Example 3.5. The Minkowski functional $A \mapsto \det^{1/n} A$ is a symmetric anti-norm on $\mathbb{M}_n^+$. This follows from the previous example by noticing that $\det^{1/n} A = \lim_{q\to 0} n^{-1/q}\|A\|_q$.

Example 3.6. The harmonic anti-norm on $\mathbb{M}_n$ is defined by

$$\|A\|_{-1} := \left(\sum_{j=1}^{n} \mu_j^{-1}(A)\right)^{-1}$$

if A is invertible, and vanishes on non-invertible operators. More generally for $r < 0$,

$$\|A\|_r := \left(\sum_{j=1}^{n} \mu_j^r(A)\right)^{1/r}$$

is a symmetric anti-norm of Schatten type with negative exponent. Concavity of these functions may be checked by arguing as in the next proof or from Proposition 3.2 and the fact that the function $a \in (0, \infty)^n \mapsto \left(\sum_{i=1}^n a_i^r\right)^{1/r}$ is concave for $r < 0$.

Proposition 3.7. *Let $A \mapsto \|A\|_!$ be a symmetric anti-norm on $\mathbb{M}_n^+$. Then, so is also $A \mapsto \|A^q\|_!^{1/q}$ for any $q \in (0, 1)$.*

Proof. The functional $A \mapsto \|A^q\|_!^{1/q}$ is homogeneous; let us check that it is also super-additive. Let A, $B \in \mathbb{M}_n^+$ and suppose $\|A^q\|_!^{1/q} = \|B^q\|_!^{1/q} = 1$. Since $t \mapsto t^q$ is operator concave, we have, for all $\lambda \in (0, 1)$,

$$(\lambda A + (1-\lambda)B)^q \geq \lambda A^q + (1-\lambda)B^q$$

so that

$$\|(\lambda A+(1-\lambda)B)^q\|_! \ge \|\lambda A^q+(1-\lambda)B^q\|_! \ge \lambda\|A^q\|_! + (1-\lambda)\|B^q\|_! = 1.$$

Hence,

$$\|(\lambda A+(1-\lambda)B)^q\|_!^{1/q} \ge 1. \tag{3.1}$$

Now, for general $X, Y \in \mathbb{M}_n^+$ with $\|X\|_!$, $\|Y\|_! > 0$, set

$$A = \frac{X}{\|X^q\|_!^{1/q}}, \qquad B = \frac{Y}{\|Y^q\|_!^{1/q}}$$

and pick

$$\lambda = \frac{\|X^q\|_!^{1/q}}{\|X^q\|_!^{1/q} + \|Y^q\|_!^{1/q}}.$$

Then (3.1) yields that

$$\|(X+Y)^q\|_!^{1/q} \ge \|X^q\|_!^{1/q} + \|Y^q\|_!^{1/q},$$

which also holds if $\|X^q\|_!$ or $\|Y^q\|_!$ vanishes. $\square$

Example 3.8. For $k = 1, \dots, n$, the functional

$$\Delta_k(A) := \left(\prod_{j=1}^{k} \mu_{n+1-j}(A)\right)^{1/k}$$

is a symmetric anti-norm on $\mathbb{M}_n$. Indeed, if $A \ge 0$, $\Delta_k(A) = \min \det^{1/k} A_{\mathcal{S}}$, where the minimum runs over the k-dimensional subspaces $\mathcal{S}$ and $A_{\mathcal{S}}$ stands for the compression onto $\mathcal{S}$. Hence, given $A\,, B \ge 0$,

$$\Delta_k(A+B) = \min \det{}^{1/k}(A+B)_{\mathcal{S}} = \min \det{}^{1/k}(A_{\mathcal{S}} + B_{\mathcal{S}})$$

and Minkowski's inequality implies $\Delta_k(A+B) \ge \Delta_k(A) + \Delta_k(B)$.

The geometric and harmonic means, $(a,b) \mapsto \sqrt{ab}$ and $(a,b) \mapsto 2(a^{-1}+b^{-1})^{-1}$, are homogeneous concave functions on the pairs of positive numbers. Hence:

Proposition 3.9. *If $\|\cdot\|_*$ and $\|\cdot\|_\circ$ are two symmetric anti-norms, then so are their geometric mean $\sqrt{\|\cdot\|_*\|\cdot\|_\circ}$ and harmonic mean $2(\|\cdot\|_*^{-1} + \|\cdot\|_\circ^{-1})^{-1}$.*

Indeed, more generally, if m is any homogeneous and jointly concave mean on non-negative numbers (this is the case for operator means [8]), then $\|\cdot\|_*\, m\, \|\cdot\|_\circ$ is also a symmetric anti-norm in the above situation.

Example 3.10. Given an non-decreasing sequence $0 \le w_1 \le w_2 \le \cdots \le w_n$, the maps

$$A \mapsto \sum_{k=1}^{n} w_k \mu_k(A) \qquad \text{and} \qquad A \mapsto \left(\prod_{k=1}^{n} \mu_k^{w_k}(A)\right)^{\frac{1}{w_1+\cdots+w_n}}$$

are symmetric anti-norms on $\mathbb{M}_n$. Indeed, positive sums and weighted geometric means of anti-norms are still symmetric anti-norms, and Examples 3.3 and 3.8 are used.

We now turn to a few consequences of the previous section. Corollary 2.3 implies:

Corollary 3.11. *Let $g : [0,\infty) \to [0,\infty)$ be convex, $g(0) = 0$, and let $A, B \in \mathbb{M}_n^+$. Then, for all symmetric anti-norms,*

$$\|g(A+B)\|_! \ge \|g(A)\|_! + \|g(B)\|_!.$$

In case of the trace norm, this is the convex function version of Rotfel'd inequality (2.1). The convexity requirement on $g(t)$ cannot be relaxed to a mere superadditivity assumption; indeed take for $s, t > 0$,

$$A = \frac{1}{2}\begin{bmatrix} s & \sqrt{st} \\ \sqrt{st} & t \end{bmatrix}, \qquad B = \frac{1}{2}\begin{bmatrix} s & -\sqrt{st} \\ -\sqrt{st} & t \end{bmatrix},$$

and observe that the trace inequality $\|g(A+B)\|_1 \ge \|g(A)\|_1 + \|g(B)\|_1$ combined with $g(0) = 0$ means that $g(t)$ is convex. For the functional of Example 3.5, we recapture (2.5) and, with $g(t) = t$, Minkowski's inequality.

Symmetric anti-norms behave well for positive linear maps between matrix spaces. The next two corollaries follow from Theorem 2.4. Recall that $\mathbb{M}_n\{\Omega\}$ stands for the set of Hermitian operators with spectra in an interval Ω. From (2.6) we have:

Corollary 3.12. *Let $f : \Omega \to [0,\infty)$ be concave and let $A, B \in \mathbb{M}_n\{\Omega\}$. Then, for all symmetric anti-norms,*

$$\left\| f\left(\frac{A+B}{2}\right)\right\|_! \ge \left\|\frac{f(A) + (B)}{2}\right\|_! \ge \frac{\|f(A)\|_! + \|f(B)\|_!}{2}.$$

Given a contraction $Z \in \mathbb{M}_n$, there exists some $K \in \mathbb{M}_n$ such that $Z^*Z + K^*K = I$. By using the unital positive map from $\mathbb{M}_{2n}$ to $\mathbb{M}_n$

$$\begin{bmatrix} A & X \\ Y & B \end{bmatrix} \mapsto Z^*AZ + K^*BK,$$

and letting $X = Y = B = 0$, we infer from Theorem 2.4:

Corollary 3.13. *Let $f : \Omega \to [0,\infty)$ be concave, $0 \in \Omega$, let $A \in \mathbb{M}_n\{\Omega\}$ and let $Z \in \mathbb{M}_n$ be a contraction. Then, for all symmetric anti-norms,*

$$\|f(Z^*AZ)\|_! \ge \|Z^*f(A)Z\|_!. \tag{3.2}$$

It is a matrix version of the obvious scalar inequality $f(za) \le zf(a)$ for $z \in [0,1]$, $a \in \Omega$. In case of the trace norm, it was noticed by Brown and Kosaki [6]. If $f(t)$ is non-negative, concave and *monotone* on Ω, Theorem 2.4 shows that (3.2) holds for symmetric norms too. If $f(t)$ is non-negative and *operator* concave on Ω, then $f(Z^*AZ) \ge Z^*f(A)Z$ so that once again (3.2) holds for symmetric norms too. However, it would be surprising if (3.2) were true in general for symmetric norms; thus an explicit counterexample would be desirable.

If $\Omega = [0,\infty)$ and if Z is no longer a contraction, but, in an opposite way, an expansive operator (i.e., its inverse is a contraction), then one might expect that a reverse inequality

$$\|f(Z^*AZ)\|_! \le \|Z^*f(A)Z\|_! \tag{3.3}$$

holds. This is not true, as shown by simple counterexamples, for the anti-norms of Example 3.3, except for the trace norm. In fact the symmetric norm version of (3.3) holds, see [5] and references therein. Nevertheless (3.3) might be true for Schatten q-anti-norms, $q \in (0,1)$.

Like symmetric norms, symmetric anti-norms are functions of the singular values as stated in Proposition 3.2. However, the notion of symmetric anti-norms is more flexible than that of symmetric norms as it is illustrated by Proposition 3.9 and the sample of previous examples. Thus, one cannot expect for symmetric anti-norms a set of inequalities as rich as in the symmetric norm case. For instance, (3.3) collapses for general anti-norms. The next section is successful in giving a few anti-norm estimates.

4. Subadditivity and superadditivity

The class of convex and subadditive functions $s : [0,\infty) \to [0,\infty)$ is small; such a function is a sum $s(t) = at + b(t)$ for some $a \geq 0$ and some non-increasing convex function $b(t) \geq 0$. Hence, the inequality for positive A, B,

$$\operatorname{Tr} s(A+B) \leq \operatorname{Tr} s(A) + \operatorname{Tr} s(B)$$

is trivial. Most of convex functions $g : [0,\infty) \to [0,\infty)$ are far from being subadditive; if $g(0) = 0$, they are automatically superadditive. To obtain subadditivity results, we will assume that composing $g(t)$ with $t \mapsto t^q$ for some $q \in (0,1)$ yields a subadditive function. A parallel approach yields superadditivity results for anti-norms.

Theorem 4.1. *Let $g, f : [0,\infty) \to [0,\infty)$, $0 < q < 1 < p$, and let A, $B \in \mathbb{M}_n^+$.*

(1) *If $g(t)$ is convex and $g^q(t)$ is subadditive, then for all symmetric norms,*

$$\|g(A+B)\|^q \leq \|g(A)\|^q + \|g(B)\|^q. \tag{4.1}$$

(2) *If $f(t)$ is concave and $f^p(t)$ is superadditive, then for all symmetric anti-norms,*

$$\|f(A+B)\|_!^p \geq \|f(A)\|_!^p + \|f(B)\|_!^p. \tag{4.2}$$

Note that (4.1) generalizes the fact that $X \mapsto \|\,|X|^{1/q}\|^q$ is a norm on $\mathbb{M}_n$. Indeed, standard majorizations extend (4.1) to $\mathbb{M}_n$ as follows: *Let $g : [0,\infty) \to [0,\infty)$ be convex and increasing, and let $q \in (0,1)$. If $g^q(t)$ is subadditive, then the map $X \mapsto \|g(|X|)\|^q$ is subadditive on $\mathbb{M}_n$.* Similarly, (4.2) is an extension of Proposition 3.7.

To prove (4.2) we will need a Ky Fan principle for anti-norms. Recall that the Ky Fan principle for symmetric norms on $\mathbb{M}_n^+$ states that $\|A\| \leq \|B\|$ for all symmetric norms if and only if the eigenvalues of A are weakly majorized by those of B, that is, $\|A\|_{(k)} \leq \|B\|_{(k)}$ for the Ky Fan k-norms, $1 \leq k \leq n$. We express it by writing $A \prec_w B$. By using the notation $A \prec^w B$ we mean that $\|A\|_{[k]} \geq \|B\|_{[k]}$ for every anti-norm of Example 3.3.

Lemma 4.2. *Let A, $B \in \mathbb{M}_n^+$. If $A \prec^w B$, then $\|A\|_! \geq \|B\|_!$ for all symmetric anti-norms.*

Proof. Denote by $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$ the eigenvalues of A arranged in decreasing order. By assumption,

$$\begin{gathered}\lambda_n(A) \geq \lambda_n(B), \\ \lambda_n(A) + \lambda_{n-1}(A) \geq \lambda_n(B) + \lambda_{n-1}(B), \\ \vdots \\ \lambda_n(A) + \lambda_{n-1}(A) + \cdots + \lambda_1(A) \geq \lambda_n(B) + \lambda_{n-1}(B) + \cdots + \lambda_1(B).\end{gathered}$$

Hence, replacing $\lambda_1(B)$ with $\lambda_1(B) + r$ for some $r \geq 0$, we may obtain a majorization

$$A \prec C$$

where C is the diagonal matrix $C = \operatorname{diag}(\lambda_1(B) + r, \lambda_2(B), \cdots, \lambda_n(B))$. We then have some unitaries $\{U_i\}_{i=1}^m$ and some non-negative scalars $\{\alpha_i\}_{i=1}^m$ of sum 1 (we may take $m = n$, see [11]) such that

$$A = \sum_{i=1}^{m} \alpha_i U_i C U_i^*$$

so that by concavity and unitary invariance of anti-norms, $\|A\|_! \geq \|C\|_! \geq \|B\|_!$. $\square$

We turn to the proof of the theorem.

Proof of Theorem 4.1. Let $A^{\downarrow}$ denote the diagonal matrix listing the eigenvalues of A arranged in decreasing order.

(1) From the Ky-Fan majorization

$$A + B \prec A^{\downarrow} + B^{\downarrow} \tag{4.3}$$

and the convexity of $g(t)$ we infer the weak majorization

$$g(A+B) \prec_w g(A^{\downarrow} + B^{\downarrow}),$$

that is,

$$g(A+B) \prec_w (g^q(A^{\downarrow} + B^{\downarrow}))^{1/q}.$$

By the subadditivity assumption $g^q(A^{\downarrow} + B^{\downarrow}) \leq g^q(A^{\downarrow}) + g^q(B^{\downarrow})$ combined with the previous weak-majorization, we obtain

$$g(A+B) \prec_w \{g^q(A^{\downarrow}) + g^q(B^{\downarrow})\}^{1/q}.$$

Thus

$$\|g(A+B)\| \leq \|\{g^q(A^{\downarrow}) + g^q(B^{\downarrow})\}^{1/q}\|$$

so that

$$\begin{aligned}\|g(A+B)\|^q &\leq \|\{g^q(A^{\downarrow}) + g^q(B^{\downarrow})\}^{1/q}\|^q \\ &\leq \|g(A)\|^q + \|g(B)\|^q,\end{aligned}$$

where the last step follows from the well-known fact that $X \mapsto \|X^{1/q}\|^q$ is concave on $\mathbb{M}_n^+$ (this may also be proved in a similar way to Proposition 3.7).

(2) From the majorization (4.3) and the concavity of $f(t)$ we infer the super-majorization

$$f(A+B) \prec^w f(A^{\downarrow} + B^{\downarrow}),$$

that is,

$$f(A+B) \prec^w \{f^p(A^{\downarrow}+B^{\downarrow})\}^{1/p}.$$

Combine this with the superadditivity assumption $f^p(A^{\downarrow}+B^{\downarrow}) \geq f^p(A^{\downarrow})+f^p(B^{\downarrow})$ to obtain

$$f(A+B) \prec^w \{f^p(A^{\downarrow})+f^p(B^{\downarrow})\}^{1/p}.$$

Thus, by Lemma 4.2,

$$\|f(A+B)\|_! \geq \|\{f^p(A^{\downarrow})+f^p(B^{\downarrow})\}^{1/p}\|_!,$$

and applying Proposition 3.7 yields the result. ∎

The most important special case of (4.1) is:

Corollary 4.3. *Let $g(t)=\sum_{k=0}^m a_k t^k$ be a polynomial of degree m with all non-negative coefficients. Then, for all positive operators A, B and all symmetric norms,*

$$\|g(A+B)\|^{1/m} \leq \|g(A)\|^{1/m} + \|g(B)\|^{1/m}.$$

Proof. It suffices to show that if $g(t)=\sum_{k=0}^m a_k t^k$ is a polynomial of degree m with non-negative coefficients, then $g^{1/m}(t)$ is subadditive (on the positive half-line). To this end, note that for $u\in(0,1)$, we have $g^{1/m}(ut)\geq ug^{1/m}(t)$ so that $t\mapsto g^{1/m}(t)/t$ is non-increasing (this decreasing property of $w(t)/t$ is called *quasi-concavity* for $w(t)$). Thus for all positive reals a, b,

$$g^{1/m}(a) \geq \frac{a}{a+b}g^{1/m}(a+b) \qquad \text{and} \qquad g^{1/m}(b) \geq \frac{b}{a+b}g^{1/m}(a+b)$$

so that

$$g^{1/m}(a)+g^{1/m}(b) \geq g^{1/m}(a+b).$$

∎

The assumptions of Corollary 4.3 do not ensure that $A\mapsto\|g^{1/m}(A)\|$ is subadditive on $\mathbb{M}_n^+$. Indeed, for a non-negative function $h(t)$ vanishing at 0, the subadditivity of $A\mapsto\|h(A)\|_1$, the trace norm, implies that $h(t)$ is convex. A lot of polynomial $g(t)$ in Corollary 4.3 may satisfy $g(0)=0$ and $g^{1/m}(t)$ is not convex; for instance, if $m=3$ and $g(t)=t+t^3$.

Another example satisfying the assumptions of Theorem 4.1 (1), with $q=1/2$, is

$$g(t)=t+(t-1)_+,$$

where $(t-1)_+ := \max\{0,t-1\}$. Again, this follows from the fact that $\sqrt{g(t)}$ is quasi-concave.

Let $q\in(0,1)$. If $g_1,g_2:[0,\infty)\to[0,\infty)$ are convex functions such that $g_1^q(t)$ and $g_2^q(t)$ are quasi-concave, then g_1+g_2 shares the same properties. This provides a subclass, closed under sum, of the class of functions satisfying the assumptions of Theorem 4.1 (1).

Thus, there are several examples of functions $g(t)$ for which Theorem 4.1 (1) and its corollaries below are available.

Now we turn to some applications of (4.1) to block-matrices.

Corollary 4.4. *Let $g : [0, \infty) \to [0, \infty)$ be convex, $g(0) = 0$, and let $q \in (0, 1)$. If $g^q(t)$ is subadditive, then*

$$\left\| g\left(\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right) \right\| \le \left(\|g(A)\|^q + \|g(B)\|^q \right)^{1/q}$$

for all partitioned positive operators and all symmetric norms.

Proof. The corollary is a straightforward consequence of the combination of (4.1) with the decomposition (2.3) in $\mathbb{M}_{n+m}^+$,

$$\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} = U \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} U^* + V \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix} V^*$$

for some unitaries $U, V \in \mathbb{M}_{n+m}$. To prove this, factorize positive matrices as a square of positive matrices,

$$\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} = \begin{bmatrix} C & Y \\ Y^* & D \end{bmatrix} \begin{bmatrix} C & Y \\ Y^* & D \end{bmatrix}$$

and observe that it can be written as

$$\begin{bmatrix} C & 0 \\ Y^* & 0 \end{bmatrix} \begin{bmatrix} C & Y \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & Y \\ 0 & D \end{bmatrix} \begin{bmatrix} 0 & 0 \\ Y^* & D \end{bmatrix} = T^*T + S^*S.$$

Then, use the fact that T^*T and S^*S are unitarily congruent to

$$TT^* = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad SS^* = \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix}.$$

□

Of course, a version of this corollary holds for partitioned operators in m^2 blocks. In particular, in case of the trace norm and an $m \times m$ matrix:

Corollary 4.5. *Let $g : [0, \infty) \to [0, \infty)$ be convex, $g(0) = 0$, and let $q \in (0, 1)$. If $g^q(t)$ is subadditive, then*

$$\operatorname{Tr} g(A) \le \left(\sum_{i=1}^{m} g^q(a_{ii}) \right)^{1/q}$$

for all positive $m \times m$ matrix $A = [a_{ij}]$.

Corollary 4.5 is unusual as it reverses the standard majorization inequality

$$\operatorname{Tr} g(A) \ge \sum_{i=1}^{m} g(a_{ii}).$$

Corollary 4.4 is understood with the natural convention that a symmetric norm $\| \cdot \|$ on $\mathbb{M}_{n+m}$ induces a symmetric norm on $\mathbb{M}_n$, denoted with the same symbol, by setting for all $A \in \mathbb{M}_n$,

$$\|A\| := \left\| \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \right\|.$$

In the opposite way, starting from a symmetric norm $\| \cdot \|$ on $\mathbb{M}_n$, we can extend it to a symmetric norm $\| \cdot \|_\wedge$ on $\mathbb{M}_{n+m}$ by setting for all $A \in \mathbb{M}_{n+m}$,

$$\|A\|_\wedge := \|A^\wedge\|$$

where $A^\wedge$ denotes the $n\times n$ diagonal matrix whose entries down to the diagonal are the n largest singular values of A. With this convention, we may remove the assumption $g(0)=0$ in Corollary 4.4:

Corollary 4.6. *Let $g:[0,\infty)\to[0,\infty)$ be convex and $q\in(0,1)$. If $g^q(t)$ is subadditive, then*
$$\left\| g\left(\begin{bmatrix} A & X \\ X^* & B\end{bmatrix}\right)\right\|_\wedge \le \left(\|g(A)\|^q+\|g(B)\|^q\right)^{1/q}$$
for all positive operators partitioned in blocks of same size and all symmetric norms.

The remaining part of this section deals with consequences of (4.2).

Corollary 4.7. *Let $f(t)=a_1t+a_2t^{1/2}+\cdots+a_mt^{1/m}$ with all non-negative a_k's. Then, for all positive operators A, B and all symmetric anti-norms,*
$$\|f(A+B)\|_!^m \ge \|f(A)\|_!^m+\|f(B)\|_!^m.$$

Proof. The proof is similar to that of Corollary 4.3: One first checks that $t\mapsto f^m(t)/t$ is increasing on $(0,\infty)$ (this property is called *quasi-convexity*). □

Another example satisfying to the assumptions of (4.2), with $p=2$, is
$$f(t)=t-\frac{(t-1)_+}{2}.$$
This follows from the fact that $f^2(t)$ is quasi-convex.

Let $p\in(1,\infty)$. If $f_1,f_2:[0,\infty)\to[0,\infty)$ are concave functions such that $f_1^p(t)$ and $f_2^p(t)$ are quasi-convex, then f_1+f_2 shares the same properties. This provides a sub-class, closed under sum, of the class of functions satisfying the assumptions of Theorem 4.1 (2). Furthermore, under the above assumptions, $\sqrt{f_1(t)f_2(t)}$ is also concave and $\{\sqrt{f_1(t)f_2(t)}\}^p$ is quasi-convex. Thus, this sub-class of concave functions with quasi-convex p-powers is invariant for both arithmetic and geometric means.

The next corollary is an extension on Minkowski's inequality (when $h(t)=t$).

Corollary 4.8. *Let $h:[0,\infty)\to[0,\infty)$ be superadditive. Assume that $h(t)$ is strictly positive and C^2 on $(0,\infty)$ and that $(\log h(t))''<0$ for all $t>0$ (hence $h(t)$ is strictly log-concave). Then, for all positive operators A, B on an n-dimensional space,*
$$\det{}^{1/n}h(A+B)\ge \det{}^{1/n}h(A)+\det{}^{1/n}h(B).$$

Proof. We may assume that A, B and $A+B$ are invertible, thus with spectra lying on a compact interval $[a,b]\subset(0,\infty)$. There exists a $q\in(0,1)$ small enough to ensure that $\exp\{q\log h(t)\}=h^q(t)$ is concave on $[a,b]$. Indeed, this is obvious since
$$(h^q(t))''=qh^q(t)\left(q\{(\log h(t))'\}^2+(\log h(t))''\right).$$
Note that to apply Theorem 4.1 (2) to the operators A, B and some function $f(t)$, it suffices to have concavity of $f(t)$ on $[a,b]$ and superadditivity of $f^p(t)$ on $[a,\infty)$. Thus, we may apply Theorem 4.1 (2) to $f(t)=h^q(t)$ and $p=1/q\in(1,\infty)$. This proves the corollary. □

Corollary 4.9. *Let $f : [0, \infty) \to [0, \infty)$ be concave and $p \in (1, \infty)$. If $f^p(t)$ is superadditive, then, for all partitioned positive operators and all Schatten r-anti-norms,*

$$\left\| f\left(\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}\right)\right\|_r \geq \left(\|f(A)\|_r^p + \|f(B)\|_r^p\right)^{1/p}.$$

This can be extended to all anti-norms $\|\cdot\|_!$ on $\mathbb{M}_{n+m}$ with the convention that it induces an anti-norm on $\mathbb{M}_n$ by replacing $A \in \mathbb{M}_n$ with $A \oplus 0 \in \mathbb{M}_{n+m}$.

Corollary 4.10. *Let $f : [0, \infty) \to [0, \infty)$ be concave and $p \in (1, \infty)$. If $f^p(t)$ is superadditive, then*

$$\operatorname{Tr} f(A) \geq \left(\sum_{i=1}^{m} f^p(a_{ii})\right)^{1/p}$$

for all positive $m \times m$ matrix $A = [a_{ij}]$.

Corollary 4.10 is unusual as it reverses the standard majorization inequality

$$\operatorname{Tr} f(A) \leq \sum_i f(a_{ii})$$

that has been strengthened in Corollary 2.5.

5. Convexity and concavity criteria for trace functionals

Let $\tau := (1/n)\mathrm{Tr}$ denote the normalized trace on $\mathbb{M}_n$. In the preceding section we have studied functionals on $\mathbb{M}_n^+$ of the type $X \mapsto \|h(X)\|^r$ or $X \mapsto \|h(X)\|_!^r$ and obtained superadditivity or subadditivity results according to that $h^r(x)$ is superadditive or subadditive. We may also address the question of convexity/concavity properties of these functionals. In case of the trace norm, this leads us to consider conditions on functions $\varphi(t)$ ensuring that $A \mapsto \varphi \circ \tau \circ f(A)$ is convex (resp., concave) on $\mathbb{M}_n^+$ whenever $\varphi \circ f(t)$ is convex (resp., concave) on $[0, \infty)$. In this section we will treat the question in the setting of a general interval $\Lambda \subset \mathbb{R}$. Let $\mathbb{D}_n\{\Xi\}$ denote the diagonal part of $\mathbb{M}_n\{\Xi\}$ for an interval Ξ.

Proposition 5.1. *Let $\varphi(t)$ be a strictly increasing continuous function on an interval Λ and $\Xi := \varphi(\Lambda)$. If the functional $A \mapsto \varphi \circ \tau \circ \varphi^{-1}(A)$ is convex (resp., concave) on $\mathbb{M}_n\{\Xi\}$, or on $\mathbb{D}_n\{\Xi\}$, then φ is necessarily concave (resp., convex) on Λ.*

Proof. The assumption of convexity (resp., concavity) on $\mathbb{D}_n\{\Xi\}$ means the following condition:

(c′) The n-variable function

$$(x_1, \dots, x_n) \in \Xi^n \mapsto \varphi\left(\frac{1}{n}\sum_{i=1}^{n} \varphi^{-1}(x_i)\right)$$

is convex (resp., concave).

Thus, we may show that convexity condition (c′) implies the concavity of φ (the other case is similar). For $\vec{x} = (x_1, \dots, x_n) \in \Xi^n$ we use the notations $\tau(\vec{x}) := n^{-1}\sum_{i=1}^{n} x_i$ and $\varphi^{-1}(\vec{x}) := (\varphi^{-1}(x_1), \dots, \varphi^{-1}(x_n))$. Consider the cyclic permutations $\vec{x}^{(0)} := \vec{x}$

and $\vec{x}^{(k)} := (x_{k+1}, \dots, x_n, x_1, \dots, x_k)$ for $k = 1, 2, \dots, n-1$. Since $n^{-1}\sum_{k=0}^{n} \vec{x}^{(k)} = (\tau(\vec{x}), \dots, \tau(\vec{x}))$, we have

$$\tau(\vec{x}) = \varphi \circ \tau \circ \varphi^{-1}\left(\frac{1}{n}\sum_{k=0}^{n-1} \vec{x}^{(k)}\right) \le \frac{1}{n}\sum_{k=0}^{n-1} \varphi \circ \tau \circ \varphi^{-1}(\vec{x}^{(k)}) = \varphi \circ \tau \circ \varphi^{-1}(\vec{x}).$$

Since φ^{-1} is increasing on Ξ, we have $\varphi^{-1} \circ \tau(\vec{x}) \le \tau \circ \varphi^{-1}(\vec{x})$, which means that φ^{-1} is convex on Ξ, hence φ is concave on Λ. □

The above proposition shows that the concavity or convexity of $\varphi(t)$ is necessary to state our criteria in the next theorem. Thus, there is no serious loss of generality by assuming that $\varphi(t)$ is smooth with the strictly negative or positive second derivative. This assumption yields a simple analytic criterion.

Theorem 5.2. *Let $\varphi(t)$ be a continuous function on an interval Λ and $\Xi := \varphi(\Lambda)$. Assume that $\varphi(t)$ is C^2 on Λ°, the interior of Λ, and that $\varphi'(t) > 0$ and $\varphi''(t) < 0$ (resp., $\varphi''(t) > 0$) on Λ°. Then the following conditions are equivalent:*

(a) *The function $\varphi'(t)/\varphi''(t)$ is convex (resp., concave) on Λ°.*
(b) *The functional $A \mapsto \varphi \circ \tau \circ \varphi^{-1}(A)$ is convex (resp., concave) on $\mathbb{M}_n\{\Xi\}$.*
(c) *The functional $A \mapsto \varphi \circ \tau \circ \varphi^{-1}(A)$ is convex (resp., concave) on $\mathbb{D}_n\{\Xi\}$.*
(d) *For any interval Ω and any function $f : \Omega \to \Lambda$ such that $\varphi \circ f(t)$ is convex (resp, concave), the functional $A \mapsto \varphi \circ \tau \circ f(A)$ is convex (resp., concave) on $\mathbb{M}_n\{\Omega\}$.*

Proof. We prove the convexity case in (a)–(d). First, assume only that $\varphi(t)$ is a strictly increasing continuous function on Λ, and we show that each of (b), (c), and (d) are equivalent to condition (c′) in the proof of Proposition 5.1. It is obvious that (d) ⇒ (b) ⇒ (c) ⇔ (c′). To prove (c′) ⇒ (d), let f be as stated in (d). Then $f = \varphi^{-1} \circ \varphi \circ f$ is automatically convex on Ω since φ^{-1} is increasing and convex on Ξ as shown in the proof of Proposition 5.1. Let $A, B \in \mathbb{M}_n\{\Omega\}$. By the Ky Fan majorization $(A+B)/2 \prec (A^\downarrow + B^\downarrow)/2$ in the same notation as in the proof of Theorem 4.1, we have

$$\begin{aligned} f\left(\frac{A+B}{2}\right) &\prec_w f\left(\frac{A^\downarrow + B^\downarrow}{2}\right) = \varphi^{-1} \circ \varphi \circ f\left(\frac{A^\downarrow + B^\downarrow}{2}\right) \\ &\le \varphi^{-1}\left(\frac{\varphi \circ f(A^\downarrow) + \varphi \circ f(B^\downarrow)}{2}\right) \end{aligned}$$

so that, by using (c′),

$$\begin{aligned} \varphi \circ \tau \circ f\left(\frac{A+B}{2}\right) &\le \varphi \circ \tau \circ \varphi^{-1}\left(\frac{\varphi \circ f(A^\downarrow) + \varphi \circ f(B^\downarrow)}{2}\right) \\ &\le \frac{\varphi \circ \tau \circ f(A^\downarrow) + \varphi \circ \tau \circ f(B^\downarrow)}{2} \\ &= \frac{\varphi \circ \tau \circ f(A) + \varphi \circ \tau \circ f(B)}{2}. \end{aligned}$$

Hence it has been proved that (b)–(d) and (c′) are equivalent.

Next, assume that $\varphi(t)$ is smooth as stated in the theorem with $\varphi''(t) < 0$, and we prove the equivalence between (c′) and (a). Let $t_i \in \Lambda^\circ$ and $s_i := \varphi(t_i) \in \Xi^\circ$ for

$1 \le i \le n$. For every $x_i \in \mathbb{R}$, $1 \le i \le n$, one can directly compute the second derivative

$$\frac{d^2}{du^2}\,\varphi\Biggl(\frac{1}{n}\sum_{i=1}^n \phi^{-1}(s_i+ux_i)\Biggr)\Bigg|_{u=0}$$
$$=\varphi''\Biggl(\frac{1}{n}\sum_{i=1}^n t_i\Biggr)\Biggl(\frac{1}{n}\sum_{i=1}^n \frac{x_i}{\varphi'(t_i)}\Biggr)^2-\varphi'\Biggl(\frac{1}{n}\sum_{i=1}^n t_i\Biggr)\Biggl(\frac{1}{n}\sum_{i=1}^n \frac{\varphi''(t_i)x_i^2}{\varphi'(t_i)^3}\Biggr).$$

Hence condition (c$'$) is satisfied if and only if

$$\Biggl(\frac{1}{n}\sum_{i=1}^n \frac{x_i}{\varphi'(t_i)}\Biggr)^2 \le \frac{\varphi'\bigl(\frac{1}{n}\sum_{i=1}^n t_i\bigr)}{\varphi''\bigl(\frac{1}{n}\sum_{i=1}^n t_i\bigr)}\Biggl(\frac{1}{n}\sum_{i=1}^n \frac{\varphi''(t_i)x_i^2}{\varphi'(t_i)^3}\Biggr) \tag{5.1}$$

holds for all $t_i \in \Lambda^\circ$ and $x_i \in \mathbb{R}$. If this holds, then letting $x_i := \varphi'(t_i)^2/\varphi''(t_i)$ gives, thanks to $\varphi'(t)/\varphi''(t) < 0$,

$$\frac{1}{n}\sum_{i=1}^n \frac{\varphi'(t_i)}{\varphi''(t_i)} \ge \frac{\varphi'\bigl(\frac{1}{n}\sum_{i=1}^n t_i\bigr)}{\varphi''\bigl(\frac{1}{n}\sum_{i=1}^n t_i\bigr)},$$

which means that $\varphi'(t)/\varphi''(t)$ is convex on Λ°. Conversely, if $\varphi'(t)/\varphi''(t)$ is convex on Λ°, then we have (5.1) as follows:

$$\Biggl(\frac{1}{n}\sum_{i=1}^n \frac{x_i}{\varphi'(t_i)}\Biggr)^2 = \Biggl(\frac{1}{n}\sum_{i=1}^n \frac{\varphi'(t_i)^{1/2}}{\{-\varphi''(t_i)\}^{1/2}}\cdot\frac{\{-\varphi''(t_i)\}^{1/2}x_i}{\varphi'(t_i)^{3/2}}\Biggr)^2$$
$$\le \frac{\varphi'\bigl(\frac{1}{n}\sum_{i=1}^n t_i\bigr)}{\varphi''\bigl(\frac{1}{n}\sum_{i=1}^n t_i\bigr)}\Biggl(\frac{1}{n}\sum_{i=1}^n \frac{\varphi''(t_i)x_i^2}{\varphi'(t_i)^3}\Biggr)$$

by using the Schwarz inequality. Thus (c$'$) follows.

The concavity case is similarly proved, or else one can reduce the assertion to the convexity case by considering $-\varphi(-t)$ on $-\Lambda$ and $-f(t)$ on Ω. ☐

For given functions φ on Λ and $f:\Omega\to\Lambda$, if we define $\tilde\varphi(t) := \varphi(-t)$ on $-\Lambda$ and $\tilde f(t) := -f(t)$ on $-\Lambda$, then $\tilde\varphi\circ\tilde f = \varphi\circ f$ and $\tilde\varphi\circ\tau\circ\tilde f = \varphi\circ\tau\circ f$ on Ω. By taking account of these, we see that Theorem 5.2 holds true also when the assumption $\varphi'(t) > 0$ is replaced with $\varphi'(t) < 0$.

The following examples are applications of Theorem 5.2:

Example 5.3. When $\varphi(t) = \log t$ on $(0,\infty)$, $\varphi'(t) > 0$, $\varphi''(t) < 0$ and $\varphi'(t)/\varphi''(t) = -t$. Hence, if $f:\Omega\to\mathbb{R}$ is convex, then $\log\tau\bigl(e^{f(A)}\bigr)$ (also $\log\operatorname{Tr} e^{f(A)}$) is convex on $\mathbb{M}_n\{\Omega\}$. In particular, $\log\operatorname{Tr} e^A$ is called the pressure of $A = A^*$ and its convexity in A is well-known.

Example 5.4. In case of $\varphi(t) = e^t$ on $\mathbb{R}$, $\varphi'(t)/\varphi''(t) = 1$. Hence, if $f:\Omega\to(0,\infty)$ is concave, then $\det^{1/n} f(A) = \exp\tau(\log f(A))$ is concave on $\mathbb{M}_n\{\Omega\}$. By continuity this is true if f is non-negative and concave on Ω; hence (2.7) is recaptured.

Example 5.5. When $\varphi(t) = t^r$ on $[0,\infty)$ with $r\in(0,\infty)\setminus\{1\}$, $\varphi'(t)/\varphi''(t) = (r-1)^{-1}t$. Hence, $\|f(A)\|_{1/r} = \{\operatorname{Tr} f(A)^{1/r}\}^r$ is convex (resp., concave) on $\mathbb{M}_n\{\Omega\}$ if $r\in(0,1)$ (resp., $r\in(1,\infty)$) and $f:\Omega\to[0,\infty)$ is convex (resp., concave). We thus have the convexity of Schatten norms and the concavity of Schatten anti-norms involving a convex/concave function $f(t)$. A stronger statement has been obtained in Corollary 3.12.

Finally, we state an abstract version of Theorem 4.1. In fact, in case of $\varphi(t) = t^r$, combined with Proposition 3.7 and its norm version, it becomes the superadditivity part (when $r \in (1, \infty)$) of Theorem 4.1 and its subadditivity version (when $r \in (0, 1)$). The proof is essentially the same as that of Theorem 4.1.

Proposition 5.6. *Let φ be a strictly increasing continuous function from $[0, \infty)$ onto itself. Let $\|\cdot\|$ (resp., $\|\cdot\|_!$) be a symmetric norm (resp., anti-norm) and Φ (resp., $\Phi_!$) be the corresponding gauge (resp., anti-gauge, see Proposition 3.2) function on $\mathbb{R}_n^+$. Assume that the n-variable function $\varphi \circ \Phi \circ \varphi^{-1}(\vec{x})$ (resp., $\varphi \circ \Phi_! \circ \varphi^{-1}(\vec{x})$) is convex (resp., concave) on $\mathbb{R}_+^n$. Then for any convex (resp., concave) function $f : [0, \infty) \to [0, \infty)$ such that $\varphi \circ f$ is subadditive (resp., superadditive), the functional $A \mapsto \varphi(\|f(A)\|)$ (resp., $\varphi(\|f(A)\|_!)$) is subadditive (resp., superadditive) on $\mathbb{M}_n^+$.*

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