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POISSON ALGEBRAS AND SYMMETRIES OF BLOCK-UPPER-TRIANGULAR MATRICES

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Abstract. Using the structure of algebroid of block-upper-triangular matrices composed from blocks of size $m\times m$ we obtain the Poisson brackets on the entries of these matrices and construct the braid-group action that preserves the Poisson algebra in the case of arbitrary m. We extend these algebras to semiclassical twisted Yangian algebras and find their central elements.

1. Algebroid of block-upper-triangular matrices

Definition 1.1. Denote by $\mathbb{A}$ an $(nm)\times(nm)$-matrix composed from blocks $\mathbb{A}_{I,J}$, $I,J=1,\dots,n$, of size $m\times m$ with the block-upper-triangular structure: we impose the restrictions that $\mathbb{A}_{I,J}=0$ for $I>J$ and $\det\mathbb{A}_{I,I}=1$ for all $I=1,\dots,n$. Denote by $\mathcal{A}$ the set of all block-upper-triangular matrices.

The set $\mathcal{A}$ is a subset of the space $Mat_{nm}(\mathbb{C})$ of all bi-linear forms acting of $\mathbb{C}^{nm}$. The Lie group $GL(\mathbb{C}^{nm})$ acts on $Mat_{nm}(\mathbb{C})$ in the usual way:

$$\forall B\in GL(\mathbb{C}^{nm}),\ \forall A\in Mat_{nm}(\mathbb{C}),\qquad A\mapsto BAB^T.$$

This action of $GL(\mathbb{C}^{nm})$ does not preserve $\mathcal{A}$, however, for any element $\mathbb{A}\in\mathcal{A}$, we can take the subset $\mathcal{M}_{\mathbb{A}}\subset GL(\mathbb{C}^{nm})$ of elements that preserve the structure of $\mathbb{A}$, or in other words

$$\mathcal{M}_{\mathbb{A}}=\left\{B\in GL(\mathbb{C}^{nm})\,|\,\mathbb{A}\mapsto B\mathbb{A}B^T\in\mathcal{A}\right\}. \tag{1.1}$$

Let $(\mathcal{A},\mathcal{M})$ where $\mathcal{M}=\cup_{\mathbb{A}\in\mathcal{A}}\mathcal{M}_{\mathbb{A}}$ be the set of pairs $(\mathbb{A},B)$ such that $\mathbb{A}\in\mathcal{A}$ and $B\in\mathcal{A}_{\mathbb{A}}$. Following [2], we can define the identity morphism as

$$e=(\mathbb{A},\mathbb{1}),$$

the inverse as

$$i:(\mathbb{A},B)\to(B\mathbb{A}B^T,B^{-1}),$$

an obvious partial multiplication as:

$$m\left((B_1\mathbb{A}B_1^T,B_2),(\mathbb{A},B_1)\right)=(\mathbb{A},B_2B_1).$$

These rules define the structure of smooth algebraic groupoid on $(\mathcal{A},\mathcal{M})$ [2]. A smooth groupoid naturally defines a Lie algebroid $(\mathcal{A},\mathfrak{g})$, i.e. its infinitesimal version.

Definition 1.2. Denote

$$d\mathbb{A}:=\mathbb{A}g+g^T\mathbb{A}, \tag{1.2}$$

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where $d\mathbb{A}$ is an element of the tangent space $T\mathcal{A}$, i.e. $d\mathbb{A}$ is block-upper-triangular and

$$\operatorname{tr}\mathbb{A}_{I,I}^{-1}d\mathbb{A}_{I,I}=0, \tag{1.3}$$

so that we can put

$$\mathfrak{g}_{\mathbb{A}}:=\{g\in\mathfrak{gl}_{nm}(\mathbb{C}),|\,d\mathbb{A}\text{ is block-upper-triangular and satisfies (1.3)}\}.$$

Definition 1.2 allows defining in a natural way the *anchor map* a:

$$\begin{aligned} a:\mathfrak{g} &\to T\mathcal{A},\\ g &\mapsto dA. \end{aligned} \tag{1.4}$$

Lemma 1.3. *The general form of g such that $d\mathbb{A}\in T\mathcal{A}$ is*

$$g=P_{-,1/2}(w\mathbb{A})-P_{+,1/2}(w^T\mathbb{A}^T), \tag{1.5}$$

where $P_{\pm,1/2}$ are the projection operators:

$$P_{\pm,1/2}a_{i,j}=\frac{1\pm\epsilon(j-i)}{2}a_{i,j},\quad \epsilon(n)=\begin{cases}1, & n>0,\\ 0, & n=0,\\ -1, & n<0.\end{cases}, \tag{1.6}$$

$i,j=1,\dots,mn$, and the matrix w has block-lower-triangular form, that is, it is composed from $(m\times m)$-blocks $w_{I,J}$ with $w_{I,J}=0$ for $I<J$. In each equivalence class of the matrices w that define the same element g we can choose the representative such that

$$\operatorname{tr}\mathbb{A}_{I,I}^{-T}w_{I,I}=0. \tag{1.7}$$

Here and hereafter we let X^{-T} denote the matrix inverse transposed to X.

Proof. Let us introduce the set of block–projection operators: the operators $\Pi_\pm$ project on the subspace of strictly block upper- or lower-triangular matrices and the operators $D_{I,I}$ $(I=1,\dots,n)$ project on the diagonal block with the number I; we also introduce the diagonal projection operator $D\equiv\sum_{I=1}^n D_{I,I}$. The conditions to be satisfied by $\mathbb{A}g+g^T\mathbb{A}$ can be then written as

$$\Pi_-(\mathbb{A}g+g^T\mathbb{A})=0,\qquad \operatorname*{tr}_{I,I}\mathbb{A}_{I,I}^{-1}D_{I,I}(\mathbb{A}g+g^T\mathbb{A})=0\text{ for }I=1,\dots,n. \tag{1.8}$$

We consider the decomposition $g=\Pi_-(g)+D(g)+\Pi_+(g)$. Then, because of invertibility of $\mathbb{A}=\Pi_+(\mathbb{A})+D(\mathbb{A})$, we can always assume

$$\Pi_-(g)=\Pi_-\big(\Pi_-(w)\mathbb{A}\big), \tag{1.9}$$

where w is any matrix (its block-upper-triangular and block-diagonal parts do not contribute to $\Pi_-(g)$). For brevity, let us write $w_-\equiv\Pi_-(w)$ and plug it in the first condition in (1.8):

$$\Pi_-(\mathbb{A}g+g^T\mathbb{A})=\Pi_-\left(\mathbb{A}\Pi_-(g)+\Pi_+(g)^T\mathbb{A}\right)=\Pi_-\left(\mathbb{A}(w_-\mathbb{A})+\Pi_+(g)^T\mathbb{A}\right)=0,$$

where we have used that since $\mathbb{A}$ is block-upper-triangular, for any matrix C, $\Pi_-(\mathbb{A}\Pi_-(C))=\Pi_-(\mathbb{A}C)$. Now, by transposing everything, we obtain

$$\Pi_+\left(\mathbb{A}^Tw_-^T\mathbb{A}^T+\mathbb{A}^T\Pi_+(g)\right)=D(\mathbb{A}^T)w_-^T\mathbb{A}^T+D(\mathbb{A}^T)\Pi_+(g)=0,$$

where we have used that for any matrix C, $\Pi_+\left(\mathbb{A}^T\Pi_+(C)\right) = D(\mathbb{A})\Pi_+(C)$. Because $D(\mathbb{A})$ is invertible, the above equation implies that in order to ensure the first condition in (1.8) we must have

$$\Pi_+(g) = -\Pi_+\left(w_-^T\mathbb{A}^T\right). \tag{1.10}$$

Note that the first condition in (1.8) then takes the form

$$\Pi_-\left(\mathbb{A}\Pi_-(w_-\mathbb{A}) - \Pi_-(\mathbb{A}w_-)\mathbb{A}\right) = 0,$$

but under the sign of projection we can replace the inner projection operators by the identity operators thus obtaining the trivial identity

$$\Pi_-\left(\mathbb{A}w_-\mathbb{A} - \mathbb{A}w_-\mathbb{A}\right) = 0,$$

Let us now impose the second condition in (1.8), in particular:

$$\begin{aligned} D_{I,I}\left(\mathbb{A}g + g^T\mathbb{A}\right) &= D_{I,I}\left(\mathbb{A}(\Pi_-(w_-\mathbb{A}) - \Pi_+(w_-^T\mathbb{A}^T) + D(g)) + \right.\\ &+ \left.(\Pi_-(w_-\mathbb{A}) - \Pi_+(w_-^T\mathbb{A}^T) + D(g))^T\mathbb{A}\right), \end{aligned}$$

and by using the fact that for any matrix C, $D(\mathbb{A}\Pi_+(C)) = \mathbb{O}$, we obtain:

$$\begin{aligned} D_{I,I}\left(\mathbb{A}g + g^T\mathbb{A}\right) &= D_{I,I}\left(\mathbb{A}w_-\mathbb{A} - \mathbb{A}D(w_-\mathbb{A}) - \mathbb{A}w_-^T\mathbb{A}^T + D(w_-^T\mathbb{A}^T) + \mathbb{A}D(g) + \right.\\ &+ \left.\mathbb{A}^Tw_-^T\mathbb{A} - D(\mathbb{A}^Tw_-^T)\mathbb{A} - \mathbb{A}w_-\mathbb{A} + D(\mathbb{A}w_-)\mathbb{A}\right). \end{aligned}$$

After cancelations, we come to the condition:

$$\operatorname*{tr}_{I,I}\mathbb{A}_{I,I}^{-1}\left(-\mathbb{A}D(w_-\mathbb{A}) + D(\mathbb{A}w_-)\mathbb{A}) + \mathbb{A}D(g) + D(g^T)\mathbb{A}\right) = 0. \tag{1.11}$$

Now, since all the projections are diagonal one, we can replace $\mathbb{A}$ in the brackets outside the projection sign by $A_{I,I}$ and the general diagonal projection D by $D_{I,I}$ thus obtaining

$$\operatorname*{tr}_{I,I}D_{I,I}\left(-w_-\mathbb{A} + \mathbb{A}w_- + D(g) + D(g)^T\right) = 0, \tag{1.12}$$

which imposes the only restriction

$$\operatorname*{tr}_{I,I}D_{I,I}(g) = \frac{1}{2}\operatorname*{tr}_{I,I}D_{I,I}(w_-\mathbb{A} - \mathbb{A}w_-), \tag{1.13}$$

whereas the traceless part of $D_{I,I}(g)$ is arbitrary.

We now demonstrate that the parametrization (1.5) satisfies (1.9), (1.10), and (1.13) for any block-lower-triangular matrix w. That the strictly lower- and upper-block-triangular parts of the matrix (1.5) are of the respective forms (1.9) and (1.10) is obvious, whereas for the condition (1.13) we have

$$\begin{aligned} \operatorname*{tr}_{I,I}D_{I,I}(g) &= \operatorname*{tr}_{I,I}D_{I,I}\left(P_{-,1/2}(w\mathbb{A}) - P_{+,1/2}(w^T\mathbb{A}^T)\right)\\ &= \frac{1}{2}\operatorname*{tr}_{I,I}(w\mathbb{A} - w^T\mathbb{A}^T) = \frac{1}{2}\operatorname*{tr}_{I,I}(w\mathbb{A} - \mathbb{A}w)\\ &= \frac{1}{2}\operatorname*{tr}_{I,I}(w_-\mathbb{A} - \mathbb{A}w_-) + \frac{1}{2}\operatorname*{tr}_{I,I}(D(w)\mathbb{A} - \mathbb{A}D(w)), \end{aligned} \tag{1.14}$$

because we are able to transpose under the sign of a partial trace $\mathrm{tr}_{I,I}$. The first term in the right-hand side of (1.14) is just the right-hand side of (1.13) whereas the second term in the right-hand side of (1.14) vanishes being equal to the one-half of a standard trace of the commutator $[w_{I,I}, \mathbb{A}_{I,I}]$ of two $(m \times m)$-matrices.

We now want to determine the freedom in w. Assume we have that two matrices w_1 and w_2 give rise to the same g. In particular this implies that for the difference

$\Delta w = w_1 - w_2$ we have $\Pi_-(\Delta w \mathbb{A}) = \Pi_-((\Delta w)_- \mathbb{A}) = 0$ and since $\mathbb{A}$ is block-upper-triangular with all blocks $\mathbb{A}_{I,I}$ invertible, it is easy to see by induction that $(\Delta w)_- = \mathbb{O}$.[1] This automatically ensures that

$$\Big[\Pi_+\big((\Delta w)^T \mathbb{A}^T\big)\Big]^T = \Pi_-\big(\mathbb{A}(\Delta w)_-\big) = \mathbb{O}.$$

It now only remains to solve the equation

$$P_{-,1/2}\big((\Delta w)_{I,I}\mathbb{A}_{I,I}\big) - P_{+,1/2}\big((\Delta w)^T_{I,I}\mathbb{A}^T_{I,I}\big) = 0$$

w.r.t. $\Delta w_{I,I}$ in each diagonal block. Transposing the second term we rewrite this equation as the system

$$P_-(\Delta w_{I,I}\mathbb{A}_{I,I}) = P_-(\mathbb{A}_{I,I}\Delta w_{I,I}) = 0,$$
$$\mathrm{diag}\,(\Delta w_{I,I}\mathbb{A}_{I,I}) = \mathrm{diag}\,(\mathbb{A}_{I,I}\Delta w_{I,I}).$$

Whence $\Delta w_{I,I} = (S_d + S_+)A^{-1}_{I,I} = A^{-1}_{I,I}(S_d + \widetilde{S}_+)$ or

$$S_d + \widetilde{S}_+ = A_{I,I}(S_d + S_+)A^{-1}_{I,I},$$

where S_d is a diagonal and S_+ and $\widetilde{S}_+$ strictly upper-triangular matrices. Introducing the lower- and upper-triangular decomposition $\mathbb{A}_{I,I} = (\mathbb{E} + L_-)Z_d(\mathbb{E} + R_+)$ with L_- strictly lower-triangular matrix, R_+ strictly upper-triangular matrix, Z_d diagonal matrix with all nonzero entries, and $\mathbb{E}$ the $m \times m$ unit matrix, and

$$S_d + \widehat{S}_+ := Z_d(\mathbb{E} + R_+)(S_d + S_+)(\mathbb{E} + R_+)^{-1}Z_d^{-1},$$

we obtain the condition

$$S_d + \widetilde{S}_+ = (\mathbb{E} + L_-)(S_d + \widehat{S}_+)(\mathbb{E} + L_-)^{-1}, \tag{1.15}$$

where S_d, $\widetilde{S}_+$, and $\widehat{S}_+$ can be arbitrary. If the matrix L_- is of the general form (which we assume), then the only solution of (1.15) is $S_d = c_I\mathbb{E}$, $\widetilde{S}_+ = \widehat{S}_+ = \mathbb{O}$ with arbitrary $c_I \in \mathbb{C}$.[2]

So, once we fix g, the only freedom remaining is the freedom of adding to $w_{I,I}$ the matrix $\mathbb{A}^{-1}_{I,I}$ with any constant coefficient c_I. The condition (1.7) plays the role of the gauge, or normalization condition, i.e., it uniquely determines c_I. The equation $\mathrm{tr}\,\mathbb{A}^{-T}_{I,I}(w_{I,I} + c_I\mathbb{A}^{-1}_{I,I}) = 0$ w.r.t. c_I always has the unique solution because the coefficient of c_I, $\mathrm{tr}\,\mathbb{A}^{-T}_{I,I}\mathbb{A}^{-1}_{I,I}$, is the sum of squares of all the entries of the matrix $\mathbb{A}^{-1}_{I,I}$ and is therefore always strictly positive. □

Let $\Omega^1_{\mathcal{A}} = T^\star\mathcal{A}$ be the bundle of diffenential 1–forms on $\mathcal{A}$. By defining the natural pairing:

$$\begin{aligned} \langle\cdot,\cdot\rangle \quad : \quad & T\mathcal{A} \times \Omega^1_{\mathcal{A}} \to \mathbb{C} \\ & (\xi,\omega) \to \mathrm{tr}(\xi\omega) \end{aligned} \tag{1.16}$$

we see that

$$\Omega^1_{\mathcal{A}} = \Big\{\omega \in \mathfrak{gl}_{nm}(\mathbb{C})|\, \omega_{I,J} = 0 \text{ for } I < J, \text{ and } \mathrm{tr}\,\mathbb{A}^{-T}_{I,I}\omega_{I,I} = 0,\Big\}$$

[1] We begin with that $(\Delta w\mathbb{A})_{I,1} = (\Delta w)_{I,1}\mathbb{A}_{1,1} = \mathbb{O}_{I,1}$ for $I = 2, \dots, n$ and, therefore, all blocks $(\Delta w)_{I,1}$ with $I > 1$ vanish; we then consider the second block-column and find that all $(\Delta w)_{I,2}$ with $I > 2$ vanish, etc.

[2] This follows from that if $S_d + \widehat{S}_+$ is a solution of (1.15) then any matrix polynomial $P_k(S_d + \widehat{S}_+)$ is a solution of (1.15) as well. The completer proof will be presented in a more detailed publication.

so that by Lemma 1.3 we have defined an isomorphism

$$\begin{array}{rcl}\phi:\ \Omega^1_{\mathcal{A}} & \to & \mathfrak{g} \\ \omega & \mapsto & P_{-,1/2}(\omega\mathbb{A}) - P_{+,1/2}(\omega^T\mathbb{A}^T).\end{array} \tag{1.17}$$

2. The Poisson structure

We define the following Poisson bi–vector on $\Omega^1_{\mathcal{A}}$:

$$P(\omega_1,\omega_2) := \langle a(\phi(\omega_1)), \omega_2\rangle.$$

This is well defined because ϕ maps one forms to elements g in the algebroid and a maps these to tangent vectors, so that $a(\phi(\omega_1)) \in T\mathcal{A}$.

Since we can regard $\mathcal{A}$ as a subvariety of $\mathcal{M}$ by the embedding e, we can obtain the Poisson bracket on $\mathcal{A}$ by projecting P from $\mathcal{M}$ to $\mathcal{A}$. This corresponds to evaluating

$$S := P(d\mathbb{A}^T, d\mathbb{A}^T),$$

which gives

$$\begin{aligned} S = \operatorname{tr}\big((d\mathbb{A})^T\mathbb{A}P_{-,1/2}[(d\mathbb{A})^T\mathbb{A}] - (d\mathbb{A})^T\mathbb{A}P_{+,1/2}[d\mathbb{A}\mathbb{A}^T] \\ +\mathbb{A}(d\mathbb{A})^T P_{+,1/2}[\mathbb{A}^T d\mathbb{A}] - \mathbb{A}(d\mathbb{A})^T P_{-,1/2}[\mathbb{A}(d\mathbb{A})^T]\big).\end{aligned}$$

Evaluating $\frac{\partial}{\partial da_{i,j}} \wedge \frac{\partial}{\partial da_{k,l}} S$, we obtain the Poisson brackets on the set of the algebra generators $a_{i,j}$:

$$\{a_{i,j}, a_{k,l}\} = \big(\epsilon(j-l)+\epsilon(i-k)\big)a_{i,l}a_{k,j} + \big(\epsilon(j-k)+1\big)a_{j,l}a_{i,k} + \big(\epsilon(i-l)-1\big)a_{l,j}a_{k,i}. \tag{2.18}$$

These are the brackets obtained by Molev, Ragoucy, and Sorba [4] in the case of the extended twisted (quantum) enveloping algebra $U'_q(\mathfrak{sp}_{2n})$ (which corresponds to the case $m = 2$). Note that the brackets (2.18) do not depend on the size of the $m \times m$ blocks. And, for any size of the block, the obtained algebra is Poissonian by construction of the block-upper-triangular algebroid. We come therefore to the following theorem.

Theorem 2.1. *The restriction of the brackets (2.18) to the block-upper-triangular matrices $\mathbb{A}$ from Definition 1.1 is Poissonian for any m.*

Remark 2.2. In fact, even more general statement is true: let us consider block-upper-triangular matrices from Definition 1.1 allowing the sizes of blocks depending on the index, that is, let us consider an arbitrary partition of N (previously equal to mn) into n positive integers, $N = m_1 + \cdots + m_n$, and let $A_{I,J}$ be a matrix of size $m_I \times m_J$. The statement of Theorem 2.1 remains valid for these block-upper-triangular structures.

In this, more general as compared to Definition 1.1 case, the action of the braid group from the next section ceases to exist, so we consider the case of equal-size blocks in the next section.

Remark 2.3. A general form of a Poissonian restriction for the brackets (2.18) is as follows. Let us introduce *arbitrary* broken line that splits the square into two Young type tableauxes as shown in Fig. 1 (it begins anywhere and may go only right and down). Setting all the elements $a_{i,j}$ below the broken line (domain colored white in the figure) to be zeros, we obtain a Poissonian reduction. Moreover, for the pivotal

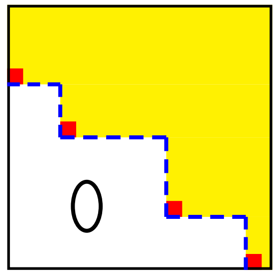

FIGURE 1. A general Poisson reduction of the algebra (2.18). All the items below the dashed broken line are zeros. The pivotal elements at the corners are marked by dark squares.

elements at the corners, i.e., for $a_{i,j}$ such that $a_{i,k}=0$ for $k<j$, $a_{l,j}=0$ for $l>i$, and $a_{l,k}=0$ for $l>i$ and $k<j$, we have that the only nonzero brackets are

$$\begin{array}{lll} \{a_{i,j},a_{i,l}\}=-a_{i,j}a_{i,l}, & \{a_{i,j},a_{l,i}\}=-a_{i,j}a_{l,i}, & l\neq i, \\ \{a_{i,j},a_{k,j}\}=a_{i,j}a_{k,j}, & \{a_{i,j},a_{j,k}\}=a_{i,j}a_{j,k}, & k\neq j, \\ \{a_{i,j},a_{i,i}\}=-2a_{i,j}a_{i,i}, & \{a_{i,j},a_{j,j}\}=2a_{i,j}a_{j,j}, & \end{array}$$

and $\{a_{i,j},a_{j,i}\}=0$. This enables us to consistently impose the restriction that $a_{i,j}=e^{\phi_{i,j}}\neq 0$, and $\phi_{i,j}$ has in turn linear brackets $\{\phi_{i,j},a_{l,k}\}=(\delta_{j,l}+\delta_{j,k}-\delta_{i,l}-\delta_{i,k})a_{l,k}$ provided $a_{l,k}\neq 0$. We can therefore introduce the element $a_{i,j}^{-1}=e^{-\phi_{i,j}}$.

3. Braid-group transformations

The braid-group transformations $\beta_{I,I+1}$, $I=1,\dots,n-1$, are transformations from the groupoid (1.1) preserving the form of the matrix $\mathbb{A}$, so by construction they must preserve the Poisson structure (2.18). They act of $\mathbb{A}$ as follows:

$$\beta_{I,I+1}[\mathbb{A}]=B_{I,I+1}\mathbb{A}B_{I,I+1}^T\equiv\widetilde{\mathbb{A}}, \tag{3.19}$$

where the matrix $B_{I,I+1}$ has the block form

$$B_{I,I+1}=\begin{array}{c} \\ \\ \vdots \\ I \\ I+1 \\ \vdots \\ \\ \end{array}\left[\begin{array}{cccccccc} \mathbb{E} & & & & & & & \\ & \ddots & & & & & & \\ & & \mathbb{E} & & & & & \\ & & & \mathbb{A}_{I,I+1}^T\mathbb{A}_{I,I}^{-T} & -\mathbb{E} & & & \\ & & & \mathbb{A}_{I,I}\mathbb{A}_{I,I}^{-T} & \mathbb{O} & & & \\ & & & & & \mathbb{E} & & \\ & & & & & & \ddots & \\ & & & & & & & \mathbb{E} \end{array}\right], \tag{3.20}$$

where, as above, $\mathbb{E}$ and $\mathbb{O}$ are the respective $m\times m$ unit and zero matrices.

It is straightforward to verify that the transformation (3.19) preserves the form of the matrix $\mathbb{A}$ with

$$\begin{array}{l} \widetilde{\mathbb{A}_{I,I}}=\mathbb{A}_{I+1,I+1},\quad \widetilde{\mathbb{A}_{I+1,I+1}}=\mathbb{A}_{I,I},\quad \widetilde{\mathbb{A}_{I,I+1}}=\mathbb{A}_{I,I+1}^T \\ J<I:\ \widetilde{\mathbb{A}_{J,I}}=\mathbb{A}_{J,I}\mathbb{A}_{I,I}^{-1}\mathbb{A}_{I,I+1}-\mathbb{A}_{J,I+1},\quad \widetilde{\mathbb{A}_{J,I+1}}=\mathbb{A}_{J,I}\mathbb{A}_{I,I}^{-1}\mathbb{A}_{I,I}^T, \\ J>I+1:\ \widetilde{\mathbb{A}_{I,J}}=\mathbb{A}_{I,I+1}^T\mathbb{A}_{I,I}^{-T}\mathbb{A}_{I,J}-\mathbb{A}_{I+1,J},\quad \widetilde{\mathbb{A}_{I+1,J}}=\mathbb{A}_{I,I}\mathbb{A}_{I,I}^{-T}\mathbb{A}_{I,J} \end{array} \tag{3.21}$$

and with all other blocks retaining their form.

We have two theorems concerning the transformations (3.19), (3.20).

Theorem 3.1. *The transformations (3.19), (3.20) are automorphisms of the Poisson structure (2.18) restricted to the block-upper-triangular matrices* $\mathbb{A}$ *from Definition 1.1.*

The statement follows from that the transformation (3.19), (3.20) is a transformation from the groupoid of block-upper-triangular matrices.

Theorem 3.2. *The transformations (3.19), (3.20) satisfy the braid-group relation,*

$$\beta_{I,I+1}\beta_{I+1,I+2}\beta_{I,I+1}[\mathbb{A}] = \beta_{I+1,I+2}\beta_{I,I+1}\beta_{I+1,I+2}[\mathbb{A}],\ \ I = 1,\dots,n-2. \tag{3.22}$$

The proof of this theorem and of the following proposition is the direct calculation.

Proposition 3.3. *We have that* $\big(\beta_{n-1,n}\cdots\beta_{2,3}\beta_{1,2}\big)^n[\mathbb{A}] = \widetilde{\mathbb{A}}$*, where* $\widetilde{\mathbb{A}_{I,J}} = \mathbb{A}_{I,I}\mathbb{A}^{-T}_{I,I}\mathbb{A}_{I,J}\mathbb{A}^{-1}_{I,I}\mathbb{A}^{T}_{I,I}$ *and, in particular,* $\widetilde{\mathbb{A}_{I,I}} = \mathbb{A}_{I,I}$.

4. (Anti)automorphisms of the Poisson algebra

Another symmetry relation of the algebra (2.18) is provided by the following mapping. We let $N = nm$ denote the total size of the matrix $\mathbb{A}$. Then the transformation

$$P[\mathbb{A}] = \widetilde{\mathbb{A}},\quad \widetilde{a}_{i,j} = a_{N+1-j,N+1-i} \tag{4.23}$$

is an antiautomorphism of the Poisson algebra (2.18), that is,

$$\big\{\widetilde{a}_{i,j},\widetilde{a}_{k,l}\big\} = -\big\{a_{i,j},a_{k,l}\big\}\Big|_{a\mapsto\widetilde{a}}. \tag{4.24}$$

Besides it we have the scaling transformation, which obviously leaves invariant the algebra (2.18):

$$a_{i,j}\mapsto e^{\phi_i+\phi_j}a_{i,j},\qquad \phi_i = \phi_{N+1-i}, \tag{4.25}$$

where we impose the restriction on ϕ_i in order to ensure the transformation (4.25) to be consistent with the antiautomorphism (4.23). We also impose that $\sum_{i=Jm+1}^{Jm+m}\phi_i = 0$, $J = 1,\dots,n$, to ensure the preservation of the determinant condition $\det A_{J,J} = 1$ for any J.

5. Twisted Yangian extension of the algebras of block-upper-triangular matrices

Definition 5.1. We introduce the generating function

$$\mathcal{G}_{i,j}(\lambda) := G^{(0)}_{i,j} + \sum_{p=1}^{\infty} G^{(p)}_{i,j}\lambda^{-p}, \tag{5.26}$$

where $G^{(0)} := \mathbb{A}$ is the block-upper-triangular matrix from Definition 1.1 and $G^{(p)}$ are full-size matrices. We postulate the Poisson brackets to be [1]

$$\begin{aligned}\{\mathcal{G}_{i,j}(\lambda),\mathcal{G}_{k,l}(\mu)\} \;=\; &\left(\epsilon(i-k) - \frac{\lambda+\mu}{\lambda-\mu}\right)\mathcal{G}_{k,j}(\lambda)\mathcal{G}_{i,l}(\mu) + \\ &+\left(\epsilon(j-l) + \frac{\lambda+\mu}{\lambda-\mu}\right)\mathcal{G}_{k,j}(\mu)\mathcal{G}_{i,l}(\lambda) +\end{aligned}$$

$$+\left(\epsilon(j-k)-\frac{1+\lambda\mu}{1-\lambda\mu}\right)\mathcal{G}_{i,k}(\lambda)\mathcal{G}_{j,l}(\mu)+$$
$$+\left(\epsilon(i-l)+\frac{1+\lambda\mu}{1-\lambda\mu}\right)\mathcal{G}_{l,j}(\lambda)\mathcal{G}_{k,i}(\mu). \tag{5.27}$$

We call the index p the level of the corresponding element; elements of $\mathbb{A}$ are then zero-level elements. We let $G^{(p)}_{i,j}$ denote the entries of the matrix $G^{(p)}$ for $p>0$.

Theorem 5.2. *The algebra (5.27) is an abstract infinite-dimensional Poisson algebra for the matrix $\mathbb{A}$ having any block-upper-triangular form depicted in Fig. 1.*

Proof. The proof of the Jacobi relations in Appendix A of [1] used only combinatorial properties and was independent on possible reductions. So, it remains only to prove the closeness of the algebra. For this, let us calculate the bracket between elements of the zeroth and $k>0$ levels. From (5.27), we have (one can obtain the formula below by taking a formal limit $\lambda\to\infty$)

$$\begin{aligned}\{a_{i,j},G^{(p)}_{k,l}\} &= (\epsilon(i-k)-1)a_{k,j}G^{(p)}_{i,l}+(\epsilon(j-l)+1)G^{(p)}_{k,j}a_{i,l}\\ &\quad +(\epsilon(j-k)+1)a_{i,k}G^{(p)}_{j,l}+(\epsilon(i-l)-1)a_{l,j}G^{(p)}_{k,i}.\end{aligned} \tag{5.28}$$

It is now straightforward to see that every term in the right-hand side is nonzero (due to combinations of ϵ-factors) only provided it contains either $a_{i,s}$ with $s\le j$ or $a_{q,j}$ with $q\ge i$, which we set to be zero as soon as we assume $a_{i,j}$ to be zero. This proves the consistency between our reduction and the algebra (5.27).

5.1. **Extension of the braid group action in the twisted Yangian case.** As in the case of the standard twisted Yangian algebra (see [1]), we have the extension of the braid-group action in the case where the matrix $\mathbb{A}$ has the original block-upper-triangular form with all blocks having the same size $m\times m$.

Proposition 5.3. *The extended braid group transformations for the algebra (5.27) in the case where the matrix $\mathbb{A}$ has the block-upper-triangular form described in Definition 1.1 admits the following matrix representation in terms of the matrix $\mathcal{G}(\lambda)$ (5.26):*

$$\beta_{I,I+1}[\mathcal{G}(\lambda)]=B_{I,I+1}\mathcal{G}(\lambda)B^T_{I,I+1},\qquad I=1,\dots,n-1 \tag{5.29}$$

where the matrices $B_{I,I+1}$ have the form (3.20).

The action of $\beta_{n,1}$ is

$$\beta_{n,1}[\mathcal{G}(\lambda)]=B_{n,1}(\lambda)\mathcal{G}(\lambda)\big(B_{n,1}(\lambda^{-1})\big)^T, \tag{5.30}$$

where the matrix $B_{n,1}(\lambda)$ has the block form

$$B_{n,1}(\lambda)=\begin{bmatrix}\mathbb{O} & & & & \lambda\mathbb{A}_{n,n}\mathbb{A}^{-T}_{n,n}\\ & \mathbb{E} & & & \\ & & \ddots & & \\ & & & \mathbb{E} & \\ -\lambda^{-1}\mathbb{E} & & & & \big[\mathbb{G}^{(1)}_{n,1}\big]^T\mathbb{A}^{-T}_{n,n}\end{bmatrix} \tag{5.31}$$

in which $\mathbb{G}^{(1)}_{n,1}$ is the $m\times m$ block in the lower left corner of the $mn\times mn$ matrix $G^{(1)}$.

Theorem 5.4. *The transformations (5.29), (5.30) satisfy the braid-group relation,*

$$\beta_{I,I+1}\beta_{I+1,I+2}\beta_{I,I+1}[\mathcal{G}(\lambda)] = \beta_{I+1,I+2}\beta_{I,I+1}\beta_{I+1,I+2}[\mathcal{G}(\lambda)],\ I=1,\dots,n \quad \text{mod } n. \tag{5.32}$$

5.2. **Poisson reductions of the twisted Yangian extension.**

Definition 5.5. We call the k-level reduction of the algebra (5.27) the mapping

$$\mathcal{G}(\lambda) \mapsto \mathbb{A} + \lambda^{-1}\widehat{G}^{(1)} + \cdots + \lambda^{-k+1}\widehat{G}^{(k-1)} + \lambda^{-k}\mathbb{A}^T, \tag{5.33}$$

where

$$\widehat{G}^{(i)} = \left[\widehat{G}^{(k-i)}\right]^T \tag{5.34}$$

and $\widehat{G}^{(i)} = G^{(i)}$ for $i = 1,\dots,(k-1)/2$ for odd k and for $i = 1,\dots,k/2-1$ for even k and $\widehat{G}^{(k/2)}_{i,j} = G^{(k/2)}_{i,j}$ for $i \geq j$ for even k, whereas the other entries of $\widehat{G}^{(i)}$ for $i \geq k/2$ are defined by the symmetry condition (5.34).

Theorem 5.6. *The mapping (5.33) defines a surjective homomorphism of the algebra (5.27) to the corresponding algebra of the elements* $\mathbb{A}_{i,j}$ *and* $\widehat{G}^{(l)}_{i,j}$, $l = 1,\dots,k-1$, *for any* $k \in \mathbb{Z}_+$.

6. Central elements

In this section, we construct all the central elements of the algebra of block-upper-triangular matrices and of its twisted Yangian extensions.

6.1. **"Traditional" central elements.** In this subsection, we construct a part of central elements that can be obtained by standard methods based on algebra symmetries as, say, in [2].

Theorem 6.1. *The polynomial functions of the elements of the algebra (5.27) in the infinite-series expansion of* $\det\mathcal{G}(\mu)$ *in powers of* μ^{-1} *are central elements of the algebra provided* $\det\mathbb{A}$ *is nonzero.*

Proof. Although it follows from the more general statement of Molev and Ragoucy [3] on the central elements of the (quantum) Yangian algebra, we can easily verify it directly using that

$$\{\mathcal{G}_{i,j}(\lambda), \det\mathcal{G}(\mu)\} = \sum_{k,l=1}^{nm} \{\mathcal{G}_{i,j}(\lambda), \mathcal{G}_{k,l}(\mu)\}\mathcal{G}^{-1}_{l,k}(\mu)$$

(the invertibility of $\mathbb{A}$ ensures the existence of the inverse matrix $\mathcal{G}^{-1}(\mu)$). We now substitute the bracket (5.27) and use the obvious identities $\sum_{l=1}^{nm}\mathcal{G}_{x,l}(\mu)\mathcal{G}^{-1}_{l,k}(\mu) = \delta_{x,k}$ and $\sum_{k=1}^{nm}\mathcal{G}_{k,x}(\mu)\mathcal{G}^{-1}_{l,k}(\mu) = \delta_{l,x}$ for $x = i,j$ therefore obtaining zero.

We formulate the statement of Theorem 5.6 for $k = 1$ as the lemma.

Lemma 6.2. *The mapping*

$$\mathcal{G}(\mu) \mapsto \mathbb{A} + \mu^{-1}\mathbb{A}^T \tag{6.35}$$

defines a surjective homomorphism of the algebra (5.27) to the algebra (2.18).

The *proof* is by a direct substitution of expression (6.35) into (5.27) using the algebra (2.18). We now use Theorem 6.1 and the homomorphism (6.35) to formulate the corollary.

Corollary 6.3. Every term in the μ^{-1}-expansion of $\det(\mathbb{A}+\mu^{-1}\mathbb{A}^T)$ is a central element of the Poisson algebra (2.18) restricted to the block-upper-triangular matrices $\mathbb{A}$.

6.2. **New central elements.** The central elements in Corollary 6.3 do not however exhaust all the central elements of the algebra of entries of $\mathbb{A}$.

We begin constructing the new central elements by considering the case of the nonrestricted Poisson algebra (2.18) assuming that all the elements $a_{i,j}$ can be in principle nonzero. We then have the following generalization of the observation in Remark 2.3 concerning the pivotal elements.

Lemma 6.4. *For the nonrestricted $N\times N$ matrix $\mathbb{A}$ let M_d be the minor of size $d\times d$ located at the lower-left corner; then* $\det M_d$ *and* $\det M_{N-d}$ *have* exactly the same *commutation relations with all of $a_{k,l}$ and with all of $g^{(p)}_{k,l}$ for $p\ge 1$ in the twisted Yangian case; denoting $a_{k,l}=G^{(0)}_{k,l}$ and introducing the function $\theta(x)=\{0,\ x\le 0,\ 1,\ x>0; x\in\mathbb{Z}\}$, we obtain*

$$\{\det M_d, G^{(p)}_{k,l}\} = c^d_{k,l} G^{(p)}_{k,l}\det M_d \text{ for } p=0,1,\dots, \tag{6.36}$$

where

$$c^d_{k,l} = -\theta(k+d-n)+\theta(d-l+1)+\theta(d-k+1)-\theta(l+d-n). \tag{6.37}$$

Proof. We use the relation (5.28) and expand the obtained determinants over rows for the first and fourth terms in the r.h.s. and over columns for the second and third terms in the r.h.s. Let us consider the first term: when commuting with the determinant of $a_{i,j}$ with $i=n-d+1,\dots,n$ and $j=1,\dots,d$ we obtain the sum of d determinants enumerated by the index i and such that in each of the corresponding matrices the ith row vector is replaced by the kth row vector multiplied by $(\epsilon(i-k)-1)G^{(p)}_{i,l}$ for $i\le k$ or by zero for $i>k$. If $i<k$, the corresponding determinant is zero (the matrix then contains two proportional row vectors), so the only nonzero contribution occurs when $i=k$, which is possible only if $k>n-d$, and this contribution is $-G^{(p)}_{k,l}\det M_d$, which gives the first term in the r.h.s. of (6.37). Using the same reasonings we obtain that the three remaining terms in (5.28) ensure the three other terms in the sum in the r.h.s. of formula (6.37). Because $\theta(x)=1-\theta(x+1)$ for $x\in\mathbb{Z}$, we easily obtain from (6.37) that $c^d_{k,l}=c^{n-d}_{k,l}$, which completes the proof.

From here, we immediately come to the following conclusion.

Lemma 6.5. *Providing* $\det M_d$ *is nonzero, we have that* $\det M_{N-d}/\det M_d$ *is a central element of the algebra $\mathbb{A}$.*

Including $\det\mathbb{A}$ itself, we have $\left[\frac{N+1}{2}\right]$ central elements described by Lemma 6.5 and $\left[\frac{N}{2}\right]$ central elements from Corollary 6.3, so, in the general case of a nonrestricted algebra $\mathbb{A}$, we have exactly N central elements.

Remark 6.6. Elementary, but lengthy calculations demonstrate that the highest Poisson leaf dimension is not less than N^2-N. Here we only briefly outline the way of proving it. For this it suffices to consider the case where all $a_{i,j}$ with $i\ne j$ are ϵ-small as compared to all $a_{i,i}$ and to retain only terms of order ϵ in the Poisson relations (2.18) neglecting all the terms of order ϵ^2. Introducing then the combination $b_{i,j}=a_{i,j}-a_{j,i}$ and retaining the elements $a_{i\ge j}$ with $i\ge j$ we observe that in the limit of small ϵ, all the $b_{i,j}$ commute with all the $a_{i\ge j}$, and each subalgebra constitute (literally or with small modifications) the Nelson–Regge algebra; the highest Poisson dimension of the b-algebra will be $\frac{N(N-1)}{2}-\left[\frac{N}{2}\right]$ and that of the $a_{i\ge j}$-algebra will be $\frac{N(N+1)}{2}-\left[\frac{N+1}{2}\right]$, so the highest rank of the Poisson relations (2.18) will be not less than N^2-N, as expected.

We eventually formulate the theorem describing new central elements in the general case of the block-upper-triangular matrix $\mathbb{A}$ and its possible Yangian extensions.

We begin with the definition.

Definition 6.7. For an block-upper-triangular matrix $\mathbb{A}$ with n blocks of sizes $m_i\times m_i$, $i=1,\dots,n$ on the diagonal, let $M_d^{(i)}$, $d=0,\dots,m_i$, $i=1,\dots,n$, be minors of size $d\times d$ located at lower-left corners of the corresponding diagonal blocks of the matrix $\mathbb{A}$.

Theorem 6.8. *Provided* $\det M_d^{(i)}$ *is nonzero, all the quotients* $\det M_{m_i-d}^{(i)}/\det M_d^{(i)}$ *are central elements of both the algebra (2.18) restricted to the block-upper-triangular matrix* $\mathbb{A}$ *and of its Yangian extension (5.26)–(5.27). These central elements remain central for all the* k*-level reductions (5.33).*

The **proof** is analogous to the proof of Lemma 6.4 with the only distinction that now some of the row- or column vectors will be zero because of the Poissonian restrictions.

Remark 6.9. Note that the constructed central elements are of two, very different, sorts. Those generated by $\det(\mathbb{A}+\mu^{-1}\mathbb{A}^T)$ are invariant under the transformation (4.25) whereas, providing all $\det M_d^{(i)}$ are nonzero, we can use transformations (4.25) to set all the central elements $\det M_{m_i-d}^{(i)}/\det M_d^{(i)}$ equal to ± 1 (in the case of real parameters ϕ_s). Then, the whole group of (anti)automorphisms of the Poisson algebra (2.18) in the case of the block-upper-triangular matrices from Definition 1.1 is presumably generated by the braid group transformations (3.19) with $I=1,\dots,n-1$ and by the antiautomorphism P from (4.23).

If we address the problem of the highest dimension of a Poisson leaf of the obtained algebra, then, first, note that this dimension is always even in the case of the standard upper-triangular matrix $\mathbb{A}$ (with units on the diagonal). In the case of the block-upper-triangular matrix $\mathbb{A}$, we have $k_i(k_i+1)/2$ new elements on and below the diagonal in the ith block. At the same time, we have $\left[\frac{k_i+1}{2}\right]$ new central elements constructed on the base of the very same block. It is easy to see that the added Poisson dimension $k_i(k_i+1)/2-\left[\frac{k_i+1}{2}\right]$ is even for any k_i and so the total Poisson dimension will be also even for any choice of the block sizes k_i.

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