ON THE SPECTRAL THEORY FOR RICKART ORDERED *-ALGEBRAS

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Abstract. RO^*-algebras are defined and studied. For RO^*-algebra T, using properties of partial order, it is established that the set of bounded elements can be endowed with C^*-norm. The structure of commutative subalgebras of T is considered and the Spectral Theorem for any self-adjoint element of T is proven.

1. Introduction

The theory of AW^*-algebras and Baer *-ring was introduced and intensively studied by Kaplansky (see [7] and [8]). Those rings are defined axiomatically by an annihilator condition and have a rich algebraic structure.

Let us recall some definitions:

Definition 1. *A Baer *-ring is a *-ring A such that for any subset S of A, the right annihilator of S is the principle right ideal generated by a projection (a self-adjoint idempotent).*

The projections in Baer *-rings form a lattice and admit a classical equivalence relation. It has been proven that the set of all projections in a AW^*-algebra forms a complete lattice. A lot of important properties which are always fulfilled in W^*-algebras hold as well in the AW^*-algebras (polar decomposition, properties of partial isometries, etc.). A regular construction of measurable operators affiliated to a finite AW^*-algebra has been done by Berberian. This construction is executed in an elegant algebraic manner and the output can be interpreted as an algebra of unbounded linear operators satisfying the annihilator condition. All results mentioned above can be found in [4].

Rickart *-rings are σ-analogues of the Baer *-rings. Similarly to the Baer *-rings, Rickart *-rings are defined by an annihilator condition, but in another, more "modest" way:

Definition 2. *A Rickart *-ring is a *-ring A such that for any element x of A the right annihilator of x is the principle right ideal generated by a projection.*

The structure of Rickart C^*-algebras was studied in details in the papers of Kaplansky, Berberian, Maeda, Ara and the first author of the present paper (see [1]-[6]). It turns out so that the basic properties such as the equivalence of the

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right and left projections, the polar decomposition and the σ-normality also hold in Rickart C^*-algebras (see [1], [2], [4]).

Similarly to the AW^*-case, a finite Rickart C^*-algebra A enjoys a regular embedding into a ring of measurable operators (see [2] and for more details [6]). This ring of measurable operators plays a crucial role in the proof of the polar decomposition property in Rickart C^*-algebras.

In this paper we define more general objects which can be also interpreted as the algebra of unbounded operators endowed with Rickart's annihilator condition. In the next section of the present paper we define RO^*-algebras and establish their basic properties. Notice that in our axiomatic and technic we were strongly influenced by Chilin's work devoted to BO^*algebras (see [5]). In section 3 we prove that the algebra of all bounded elements M of a RO^*-algebra can be endowed with C^*-norm and that M is complete in that norm. Section 4 is devoted to commutative RO^*-algebras and their Spectral theory.

2. Rickart ordered *-algebras

Let T be a Rickart *-algebra,

$$T_h = \{x \in T : x^* = x\},$$

$$K = \{x = \sum_{k=1}^{n} x_k^* x_k\} : \; x_k \in T\},$$

and

$$K = \left\{ x \in T_h : x = \sum_{i=1}^{n} x_i^* x_i, \text{ for } n \in \mathbb{N}, \; x_i \in T \right\}.$$

We say that T satisfies the *positive square root axiom* (PSR) if for any $x \in K$ there exists

$$y \in K \cap \{x\}'',$$

such that

$$y^2 = x,$$

where x'' is a bicommutant of the element x.

Proposition 1. *Let T be a Rickart *-algebra satisfying the (PSR)-axiom. Then K is a proper cone.*

Proof. The properties

$$K + K \subset K,$$

and

$$\lambda K \subset K,$$

where $\lambda \geq 0$, follow immediately from the definition of K.

Now, let

$$x \in K \cap (-K).$$

Then

$$x = u^* u = -v^* v.$$

Thus it follows that

$$u^* u + v^* v = \mathbf{0}.$$

There exist $y, z \in T_h$ such that

$$y^2 = u^* u,$$

$$z^2 = v^*v,$$

and

$$y^2 + z^2 = \mathbf{0}.$$

Therefore,

$$y^3 + yz^2 = \mathbf{0},$$
$$yz^2 = -y^3,$$

and

$$yz^2 = z^2y.$$

In just the same way we conclude that

$$y^2z = zy^2.$$

Hence,

$$y^2z^2 = yz^2y = (zy^*)(zy) \in K.$$

There exists

$$a \in \left\{y^2z^2\right\}'',$$

such that

$$a = a^* \text{ and } a^2 = y^2z^2.$$

Note that

$$y \in \left\{y^2z^2\right\}'',$$

and

$$ya = ay.$$

Further,

$$0 = y^2(y^2 + z^2) = y^2 + a^2 = (y^2 + ia)(y^2 - ia).$$

Consequently,

$$y^2 + ia = \mathbf{0}, \text{ and } x = y^2 = \mathbf{0}.$$

□

Remark 1. *The cone K defines a partial order on the algebra T :*

$$x \leq y \overset{def}{\Longleftrightarrow} y - x \in K.$$

Definition 3. *We say that T satisfies the Fisher-Riesz axiom (FR) if the following holds:*

If a sequence $\{x_n\} \subset T_h$ satisfies conditions

$$\mathbf{0} \leq x_n \leq \varepsilon_n \mathbf{1},$$

such that

$$\sum_{k=1}^{\infty} \varepsilon_k < \infty,$$

then there exists

$$\sup \sum_{k=1}^{n} x_k \in T_h.$$

Definition 4. *Rickart *-algebra T is called RO^*-algebra if T satisfies the axioms (PSR) and (FR).*

Definition 5. *Let T be a RO *-algebra. A *-subalgebra B of T is called a RO*-subalgebra, if B is RO*-algebra and*

$$RP_B(x) = RP_T(x),$$

for all $x \in B$.

Remark 2. *Recall that BO*-algebra is Baer $*$-algebra satisfying the axioms (PSR) and (FR) (see* [5] *for details).*

Example 1. *Let*

$$X = [0, 1],$$

and let F be a σ-algebra of X that contains each subset

$$A \subseteq X,$$

such that either A or $X \backslash A$ is countable. Define T to be a *-algebra of all measurable complex functions on (X, F). It is easily seen that T is a RO*-algebra which is not BO*-algebra.

3. The algebra of bounded elements in a RO^*-algebra

The order properties which are axiomatically defined on RO^*-algebra T allow us to specify a set of bounded elements of T.

In arbitrary RO^*-algebra T we will extract some C^*-subalgebra which contains all of bounded elements of T.

Definition 6. *An element*

$$a + ib,$$

$(a, b \in T_h)$ *is bounded, if there exists $\lambda \geq 0$ such that*

$$-\lambda \mathbf{1} \leq a, b \leq \lambda \mathbf{1}.$$

Proposition 2. *The set of all bounded elements of a RO*-algebra is a RO*-subalgebra. If*

$$x^* x \leq \mathbf{1},$$

then

$$x x^* \leq \mathbf{1}.$$

Proof. Let us denote by M the set of all bounded elements of T. Let $x, y \in M$,

$$x = a + ib, \ y = c + id,$$

$$-\lambda \mathbf{1} \leq a, b \leq \lambda \mathbf{1},$$

$$-\mu \mathbf{1} \leq c, d \leq \mu \mathbf{1},$$

for some $\lambda, \mu \geq 0$. Obviously,

$$x + y, x^*, \alpha x \in M,$$

where $\alpha \in \mathbb{C}$. We will prove that xy is also bounded. It will be sufficient to prove it for the case

$$x, y \in T_h,$$

i.e.

$$x = a, \ y = c.$$

If

$$x, y \geq \mathbf{0},$$

then, since

$$\sqrt{x} \in \{x\}'',$$
$$\sqrt{y} \in \{y\}'',$$

we have

$$xy \geq \mathbf{0}.$$

Therefore, the inequalities

$$\lambda\mathbf{1}{+}x \geq \mathbf{0} \text{ and } \lambda\mathbf{1}{-}x \geq \mathbf{0},$$

imply

$$x^2 \leq \lambda^2\mathbf{1}.$$

In just the same way

$$y^2 \leq \mu^2\mathbf{1}.$$

Since

$$(x+y)^2 \geq \mathbf{0},$$

we have

$$-(x^2+y^2) \leq (xy+yx) \leq (x^2+y^2).$$

Consequently,

$$-(\lambda^2+y^2)\mathbf{1} \leq (xy+yx) \leq (\lambda^2+\mu^2)\mathbf{1}.$$

Similarly,

$$-(\lambda^2+y^2)\mathbf{1} \leq i(xy-yx) \leq (\lambda^2+\mu^2)\mathbf{1}.$$

On the other hand,

$$xy = \frac{(xy+yx)}{2} - i\left(\frac{\mathbf{1}}{2i}\right)(xy-yx),$$

hence, $xy \in M$.

Thus, M is a *-subalgebra of T.

Let now $x \in M$,

$$R_M(x) = \{y \in M : xy = \mathbf{0}\},$$

be a right annihilator of x. There exists a projection

$$e \in M,$$

such that

$$R(x) = \{x \in T : xy = \mathbf{0}\} = eT.$$

It's clear that

$$R_M(x) = R(x) \cap M.$$

On the other hand,

$$\mathbf{0} \leq e \leq \mathbf{1},$$

therefore $e \in M$, and

$$RP_M(x) = RP_T(x).$$

If

$$x \in M \cap K,$$

then there exists

$$y \in K \cap \{x\}'',$$

such that

$$y^2 = x.$$

Since

$$-\mathbf{1}-y^2 \leq 2y \leq \mathbf{1}+y^2,$$

we have

$$-\frac{(\mathbf{1}+x)}{2} \leq y \leq \frac{(\mathbf{1}+x)}{2},$$

i.e. y is a bounded element. Thus, M satisfies (PSR).

Let now a sequence $\{x_n\}_{n\in N}$ satisfies conditions

$$\mathbf{0} \leq x_n \leq \varepsilon_n \mathbf{1},$$

and

$$\sum_{n=1}^{\infty} \varepsilon_n < +\infty.$$

Obviously,

$$\sum_{n=1}^{k} x_n \leq \left(\sum_{n=1}^{k} \varepsilon_n\right)\mathbf{1} \leq \left(\sum_{n=1}^{\infty} \varepsilon_n\right)\mathbf{1},$$

for all $k \in \mathbb{N}$. Then

$$\mathbf{0} \leq \sup \sum_{n=1}^{k} x_n \leq \left(\sum_{n=1}^{\infty} \varepsilon_n\right)\mathbf{1},$$

i.e., the algebra M satisfies the axiom (RF). □

Theorem 1. *Let T be a RO^*-algebra, M denotes a subalgebra of all bounded elements of T. Then there exists a norm on M such that M is (relatively to this norm) a C^*-algebra.*

Proof. Define

$$\|x\| = \inf\{\lambda \geq 0 : -\lambda\mathbf{1} \leq x \leq \lambda\mathbf{1}\}.$$

One can easily check that $\|\cdot\|$ is a norm on M_h. We will prove that $(M_h, \|\cdot\|)$ is a Banach space. Let

$$\{x_n\}_{n\in N} \subset M_h,$$

$$\sum_{n=1}^{\infty} x_n < +\infty.$$

Let us denote

$$a_n = \|x_n\|\,\mathbf{1} + x_n,\; b_n = \|x_n\|\,\mathbf{1} - x_n.$$

Then

$$x_n = \frac{1}{2}(a_n - b_n),$$

$$\|a_n\| \leq 2\|x_n\|,$$

$$\|b_n\| \leq 2\|x_n\|.$$

Therefore

$$\mathbf{0} \leq a_n \leq 2\|x_n\|\,\mathbf{1},\; \mathbf{0} \leq b_n \leq 2\|x_n\|\,\mathbf{1}.$$

Because of axiom the (RF) axiom, there exist

$$\sup_n \sum_{n=1}^{k} a_n = a, \text{ and } \sup_n \sum_{n=1}^{k} b_n = b.$$

Since
$$\sum_{k=1}^{n} a_k \leq 2\sum_{k=1}^{\infty} \|x_k\| \mathbf{1},$$
we have inequalities
$$\mathbf{0} \leq a \leq 2\sum_{k=1}^{\infty} \|x_k\| \mathbf{1},$$
i.e. $a \in M$. In just the same way we obtain the fact that $b \in M$.

We will prove now that
$$||a - \sum_{n=1}^{k} a_n|| \to 0, \text{ when } k \to \infty.$$
In fact,
$$\left\| a - \sum_{n=1}^{k} a_n \right\| = \sup_{s \geq k} \left(\sum_{i=1}^{s} a_n - \sum_{n=1}^{k} a_n \right) = \sup_{s \geq k} \sum_{n=k+1}^{s} a_n$$
Note that
$$\mathbf{0} \leq \sum_{n=k+1}^{s} a_n \leq \left(2 \sum_{n=k+1}^{\infty} ||x_n|| \right) \mathbf{1}.$$
Hence
$$a - \sum_{n=1}^{k} a_n \leq \left(2 \sum_{n=k+1}^{\infty} ||x_n|| \right) \mathbf{1},$$
and
$$\left\| a - \sum_{n=1}^{k} a_n \right\| \leq 2 \sum_{n=k+1}^{\infty} ||x_n|| \to 0,$$
as $k \to \infty$. In the analogous way
$$\left\| b - \sum_{n=1}^{k} b_n \right\| \to 0,$$
as $k \to \infty$.

Now,
$$\sum_{n=1}^{k} x_n = \frac{1}{2} \sum_{n=1}^{k} (a_n - b_n) \to \frac{1}{2}(a - b),$$
as $k \to \infty$. Thus, $(M_h, \|\cdot\|)$ is a Banach space.

The next claim is to prove that the cone
$$K \cap M,$$
is closed in the norm $\|\cdot\|$ Let $x \in K$, and
$$\|x\| \leq 1.$$
Then
$$\mathbf{0} \leq x \leq \mathbf{1},$$
$$0 \leq \mathbf{1} - x \leq \mathbf{1},$$
$$\|\mathbf{1} - x\| \leq 1.$$
Conversely, if $x \in M_h$,
$$\|x\| \leq 1,$$

$$\|\mathbf{1}-x\| \leq 1,$$

then

$$\mathbf{1}-x \leq \mathbf{1},$$

$$x \geq \mathbf{0}.$$

Thus,

$$\{x \in K : \|x\| \leq 1\} = \{x \in M_h : \|x\| \leq 1,\ \|\mathbf{1}-x\| \leq 1\}.$$

Let

$$x_n \geq \mathbf{0},$$

$$\lim_{n\to\infty} x_n = x.$$

Then

$$\lim_{n\to\infty} \|x_n\| = \|x\|.$$

It is easy to see that there exists

$$\lambda > 0,$$

such that

$$\|\lambda x_n\| \leq 1.$$

Since

$$\lambda x_n \geq \mathbf{0},$$

we have that

$$\|\mathbf{1}-\lambda x_n\| \leq 1.$$

Obviously,

$$\|\lambda x\| \leq 1,$$

and

$$\|\mathbf{1}-\lambda x\| \leq 1.$$

Therefore

$$\lambda x \in K,$$

and

$$x \in K.$$

Thus,

$$K \cap M_h,$$

is closed in $(M_h, \|\cdot\|)$. Assume now that

$$x \in M,$$

$$x \neq 0,$$

and

$$y = -x^* x.$$

Then

$$y \neq \mathbf{0},$$

and

$$y \notin K.$$

By the Hahn-Banach Theorem there exists a continuous functional φ on M_h which is non-negative on K and

$$\varphi(y) < 0.$$

We now extend the functional φ up to a functional on M in the usual linear manner:

$$\varphi(a+ib)=\varphi(a)+i\varphi(b),$$

for any $a,b\in M_h$. Therefore φ is the state on M, i.e.

$$\varphi(x^*x)>0,$$

and we can put

$$\varphi(\mathbf{1})=1.$$

Let π_φ be a representation of a *-algebra M in the algebra of all bounded linear operators on some Hilbert space H_φ (the Gelfand-Naimark-Segal construction). It is well known that for all $x\in M$ the following inequality is valid:

$$\varphi(y^*x^*xy)\leq\|x^*x\|\,\varphi(y^*y).$$

Hence $\pi_\varphi(x)$ is a bounded linear operator on H_φ, that is, π_φ is a *-representation of M into the *-algebra $B(H_\varphi)$.

Now, let

$$\pi=\bigoplus_{\varphi\in\Phi}\pi_\varphi,$$

be a direct sum of the all states on M, where Φ denotes the set of all states on M. Then Φ is a faithful representation of M into the algebra $B\left(\bigoplus_{\varphi\in\Phi}H_\varphi\right)$. It is clear now that the norm

$$q(x)=\|\pi(x)\|\,,$$

induces the norm $\|\cdot\|$ on M_h. □

4. Commutative RO^*-algebras and the Spectral Theorem

In this section we describe the structure of commutative RO^*-algebras and prove the Spectral Theorem for self-adjoint element of a RO^*-algebra.

Theorem 2. *Let T be a commutative RO^*-algebra. Then T_h is a conditionally σ-complete lattice.*

Proof. Assume that $x\in T_h$ and x is non-comparable with $\mathbf{0}$. There exists

$$y\geq\mathbf{0},$$

such that

$$y^2=x^2.$$

Put

$$a=\frac{1}{2}\left(y+x\right),$$

and

$$b=\frac{1}{2}\left(y-x\right).$$

Since x is non-comparable with $\mathbf{0}$, we can conclude that $a,b\neq\mathbf{0}$.

Note that

$$b\in r(b+x),$$

and

$$a\in r(x-a).$$

Because T is Rickart *-algebra, there exist projections e and f, such that
$$r(x+b) = eT,$$
and
$$r(a-x) = fT.$$
Obviously,
$$xe = ye,$$
$$xf = -yf,$$
and
$$(\mathbf{1}-e)(\mathbf{1}-f) = \mathbf{0}.$$
Therefore,
$$f - ef = \mathbf{1} - e,$$
and
$$ye - x = -x(1-e) = -x(f-ef) = y(f-ef) \geq \mathbf{0}.$$
Thus,
$$ye \geq x.$$
Now, let
$$d \in T_h,$$
$$d \geq \mathbf{0},$$
and
$$d \geq x.$$
Then
$$d = de + d(1-e) \geq ede \geq exe = ye.$$
This yields
$$ye = \sup\{x, \mathbf{0}\} = x \vee \mathbf{0}.$$
We have showed that T_h is a vector lattice.

To prove the conditional σ-completeness let us consider an increasing sequence $\{x_n\}$, where
$$x_n \geq \mathbf{0}$$
and
$$x_n \leq v.$$
The element
$$w = \sqrt{v^2 + \mathbf{1}} + v,$$
has an inverse
$$w^{-1} = \sqrt{v^2 + \mathbf{1}} - v.$$
The sequence $\left\{w^{-1}x_nw^{-1}\right\}$ is increasing and bounded by w^{-1}. It is easy to see that
$$w^{-1} \leq \mathbf{1}.$$
Thus
$$w^{-1}x_nw^{-1} \in M_h.$$
By Theorem 1 M is a Rickart C^*-algebra, consequently, M_h is a conditionally σ-complete lattice (see, for example, [1]). Therefore, there exists
$$x = \sup\left\{w^{-1}x_nw^{-1}\right\}.$$
It is clear that
$$wxw = \sup\{x_n\},$$

belongs to T_h. □

Corollary 1. *Let T be a commutative RO^*-algebra. Then T_h is K_σ-space.*

Theorem 3. *Let T be a RO^*-algebra and let B be some maximal commutative subalgebra of T. Then B is a RO^*-subalgebra of T.*

Proof. The proof that B is a Rickart $*$-algebra is analogous to [5]. Further, let

$$x \in K_B = \left\{ \sum_{i=1}^{n} x_i^* x_i : x_i \in B, \text{ where } i = 1, 2, ..., n \right\}.$$

There exists

$$y \in K_B \cap \{x\}'',$$

such that

$$y^2 = x.$$

Thus the algebra B satisfies the(PSR) axiom. To prove the (RF) axiom, we take

$$x_n \in B,$$

$$\mathbf{0} \leq x_n \leq \varepsilon_n \mathbf{1},$$

and

$$\sum_{k=1}^{\infty} \varepsilon_k < \infty.$$

One can see there exists

$$x = \sup \sum_{k=1}^{n} x_k,$$

in T_h. Since

$$x \leq \left(\sum_{k=1}^{\infty} \varepsilon_k \right) \mathbf{1},$$

we can conclude that x belongs to the Rickart C^*-algebra of bounded elements of T. We have seen above (Theorem 1) that

$$\|x - s_n\| \to 0,$$

as $n \to \infty$, where

$$s_n = \sum_{k=1}^{n} x_k,$$

and therefore

$$x \in (B \cap M)'.$$

Now let

$$a \in B_h,$$

and

$$a \geq \mathbf{0}.$$

Then

$$y = \sqrt{x^2 + \mathbf{1}} - x \geq a + \mathbf{1},$$

and y is invertible. Notice that

$$\mathbf{0} \leq y^{-1} \leq \mathbf{1},$$

and

$$\mathbf{0} \leq y^{-1} a y^{-1} \leq y^{-1}.$$

Thus

$$y^{-1}ay^{-1} \in B \cap M.$$

Since

$$y^{-1} \in B \cap M,$$

we have

$$xy^{-1} = y^{-1}x.$$

Therefore

$$y^{-1}xay^{-1} = y^{-1}axy^{-1},$$

and

$$xa = ax.$$

As any element $c \in B$ is linear combination of positive ones, we obtain that

$$xc = cx.$$

Since B is a maximal one we can conclude that $x \in B$.

To finish the proof, notice that

$$x = \sup s_n,$$

in T_h. Hence

$$x = \sup s_n,$$

in B_h. Thus B is a RO^*-algebra. □

Now we can proceed to formulate the main result of this section.

Theorem 4 (Spectral Theorem)**.** *Let T be a RO^*-algebra, $x \in T_h$. There exists a unique family of projections $\{e_\lambda\}_{\lambda \in \mathbf{R}}$ satisfying the following properties:*

(a)

$$e_\lambda \leq e_\mu \textit{ if } \lambda \leq \mu;$$

(b)

$$\inf e_\lambda = \mathbf{0};$$

(c)

$$\sup e_\lambda = \mathbf{1};$$

(d)

$$\sup_{\mu < \lambda} e_\mu = e_\lambda;$$

(e)

$$e_\lambda \cdot x \leq \lambda e_\lambda,\ e_{\underset{\lambda}{\perp}} \cdot x \geq \lambda e_{\underset{\lambda}{\perp}}.$$

Moreover, for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left\| x - \sum_{i=1}^{n} \xi_i \left(\lambda_i - \lambda_{i-1} \right) \right\| < \varepsilon,$$

for any partition

$$\{\lambda_i\}_{i=0}^{\infty},$$

of the real line with

$$\sup \left(\lambda_i - \lambda_{i-1} \right) < \delta,$$

and

$$\xi_i \in \left[\lambda_{i-1}, \lambda_i \right].$$

Proof. One obtains the proof by using the classical proof of the Spectral Theorem for K_σ-spaces in [9], and previous results of this section. □

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