A GEOMETRIC APPROACH TO ORLOV'S THEOREM

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Abstract. A famous theorem of D. Orlov describes the derived bounded category of coherent sheaves on projective hypersurfaces in terms of an algebraic construction called graded matrix factorizations. In this article, we work with E. Segal's graded D-branes and describe some equivalences and comparisons between categories of graded D-branes. We combine these with Segal's theorems to give a geometric proof of Orlov's theorem and describe a possible generalization to complete intersections.

Contents

1. Introduction

The main results of this paper stem from an attempt to give an alternative description of the bounded derived category of coherent sheaves on a projective complete intersection. Many theorems describing this type of derived category are now known. The first result in this direction was Serre's description of the abelian category of quasi-coherent sheaves on a projective algebraic variety. If Y is a projective algebraic variety with a very ample line bundle $\mathcal{O}(1)$ then we can consider the graded ring $A := \oplus_{n\geq 0} H^0(Y, \mathcal{O}(n))$. The abelian category of quasi-coherent sheaves on Y is equivalent to the quotient of the category of graded modules over A by the thick subcategory generated by the Artinian, or finite dimensional modules. Note that in this description there is a very nice category which has an "irrelevant" category that must be collapsed to get the description.

Another relevant prominent theorem is the theorem of Bernstein-Gel'fand-Gel'fand [BGG76] which describes an equivalence between the derived category of graded modules over the symmetric algebra $\mathrm{Sym}^\bullet V$ and the derived category of graded modules over the exterior algebra $\wedge^\bullet V^\vee$. The equivalence has the remarkable property that it interchanges the categories of Artinian and projective modules. Combining this with Serre's result we obtain a description of the derived category of quasi-coherent sheaves on $\mathbb{P}(V^\vee)$ as the triangulated quotient of the derived category of graded $\wedge^\bullet V^\vee$ modules by the full triangulated subcategory of projective graded $\wedge^\bullet V^\vee$ modules.

This correspondence has been studied extensively and very general formulations are known for quadratic algebras. Baranovsky [Bar05] showed that there is an analogous description of the derived category of

Date: December 24, 2010.

quasicoherent sheaves on a projective complete intersection, at the cost of replacing the dual algebra with a DG algebra, or A_∞ algebra. There is still an "irrelevant" category that must be factored out.

The quotient of the category of graded modules over a graded ring A by the full triangulated subcategory generated by graded projective modules is known as the category of graded singularities $D^{gr}_{Sg}(A)$. Working with smooth hypersurfaces in certain smooth projective varieties, Orlov [Orl09b] was able to show that the the category of singularities of its homogeneous coordinate ring is comparable to the derived category of quasi-coherent sheaves on the hypersurface. For smooth hypersurfaces in projective space, Orlov obtained a completely algebraic description of the category of singularities, which is remarkable in that the construction does not require taking the quotient by an "irrelevant" subcategory. When the hypersurface is Calabi-Yau, this algebraic construction gives a complete description of the derived bounded category of coherent sheaves.

The algebraic construction is a graded version of Eisenbud's matrix factorizations [Eis80]. View $\mathbb{C}[x_1, \dots, x_n]$ as a graded ring with $\deg(x_i) = 1$ and suppose that $W \in \mathbb{C}[x_1, \dots, x_n]$ is a homogeneous polynomial of degree n whose zero set is nonsingular away from the origin. Inspired by Eisenbud's definitions, Orlov constructed a triangulated category of *graded* matrix factorizations whose objects are 2-periodic diagrams of degree-zero maps

$$\cdots \xrightarrow{d_+} Q(-n) \xrightarrow{d_-} P \xrightarrow{d_+} Q \xrightarrow{d_-} P(n) \xrightarrow{d_+} Q(n) \longrightarrow \cdots$$

where P, Q are finitely generated, graded, free $\mathbb{C}[x_1, \dots, x_n]$ modules, (i) indicates a shift of i in the grading, $d_- \circ d_+ = W \cdot \mathrm{id}_P$, and $d_+ \circ d_- = W \cdot \mathrm{id}_Q$.

Theorem (3.1, [Orl09b])**.** *Let X be the smooth projective Calabi-Yau variety with homogeneous polynomial W. There is a triangulated equivalence between the derived bounded category of coherent sheaves on X and the triangulated category of graded matrix factorizations of W.*

Interested in applications to quantum field theory, E. Segal [Seg09] proposed an alternative method for proving Orlov's theorem that does not go through a BGG correspondence. He wanted to find a proof of Orlov's theorem at the DG level. It is now clear using results of Orlov-Luntz [OL10] or Căldăraru-Tu [CT10] that the original proof is sufficient for this purpose. Nonetheless, I was curious whether it was possible to implement Segal's method. This paper describes both how to carry out Segal's method in the case of a hypersurface and how to apply it toward a possible generalization of Orlov's theorem for smooth complete intersections in projective space.

Segal defined a DG category of graded matrix factorizations, which we refer to as the category of graded D-branes in keeping with the language used by the physics community. The input for his construction is a scheme or stack X with a suitable action of $\mathbb{C}^\times$, the multiplicative group, and a semi-invariant regular function W of degree 2. We use the notation $\mathrm{DBr}^{\mathrm{gr}}(X, W)$ to denote this category. Segal studies stacks that are obtained as quotients of the semistable sets of a Calabi-Yau action of $G = \mathbb{C}^\times$ on a vector space V. (In the background, we have another action of $\mathbb{C}^\times$ used to define the category of graded D-branes, but this will be suppressed for now.) There are two stability conditions for an action of G, corresponding to the identity and inverse characters. Let V_+ and V_- be the semistable loci for these stability conditions.

Theorem (3.3, [Seg09])**.** *For any G-invariant function F of degree 2, there is a family of quasi-equivalences between* $\mathrm{DBr}^{\mathrm{gr}}([V_+/G], F)$ *and* $\mathrm{DBr}^{\mathrm{gr}}([V_-/G], F)$*.*

A special case of this theorem is relevant in the context of Orlov's theorem. Let $X \subset \mathbb{P}$ be a smooth projective Calabi-Yau hypersurface. There is a vector space V, an action of $G = \mathbb{C}^\times$, and a G-invariant function W as in the theorem such that $[V_+/G] \cong K$, the canonical bundle of $\mathbb{P}$. Then the homotopy category of $\mathrm{DBr}^{\mathrm{gr}}([V_-/G], W)$ identifies naturally with the triangulated category of graded matrix factorizations of the equation of X. This is the starting point for the alternative approach to Orlov's theorem. Since X is smooth, the derived bounded category of coherent sheaves on X is equivalent to the homotopy category of $\mathfrak{Perf}(X)$, the DG category of perfect complexes on X. We will obtain a quasi-equivalence between $\mathfrak{Perf}(X)$ and $\mathrm{DBr}^{\mathrm{gr}}(K, W)$ in two steps.

First we endow the normal bundle $N_{X/K}$ of X embedded in K with a $\mathbb{C}^\times$-action and semi-invariant function, also called W by abuse of notation. By comparing $\mathrm{DBr}^{\mathrm{gr}}(K, W)$ and $\mathrm{DBr}^{\mathrm{gr}}(N_{X/K}, W)$ to a category of D-branes on a deformation of (K, W) to $(N_{X/K}, W)$ we obtain the following theorem.

Theorem. *There is a quasi-equivalence between* $\mathrm{DBr}^{\mathrm{gr}}(K, W)$ *and* $\mathrm{DBr}^{\mathrm{gr}}(N_{X/K}, W)$

Now, there is a projection $N_{X/K} \to X$. The fibers of the projection are two dimensional vector spaces. We can identify each fiber with $\operatorname{Spec} \mathbb{C}[x, y]$ in such a way that the restriction of W to each fiber can be identified with xy. A classical theorem of Knörrer says that there is one indecomposable object in the category of matrix factorizations of xy over $\mathbb{C}[x, y]$. We construct an object of $\mathrm{DBr}^{\mathrm{gr}}(N_{X/K}, W)$ whose restriction to every fiber is Knörrer's indecomposable object. Finally, we prove the follwing theorem.

Theorem. *Pulling a perfect complex back along the projection and tensoring with the Knörrer object defines a quasi-equivalence between* $\mathfrak{Perf}(X)$ *and* $\mathrm{DBr}^{\mathrm{gr}}(N_{X/K}, W)$.

The outline of this article is as follows. In section 2 we give a precise construction of categories of graded D-branes. We also define some DG functors between DG categories of graded D-branes. Sections 3-5 contain the main results. In section 3, we consider a smooth variety S with a vector bundle of the form $\mathcal{V} \oplus \mathcal{V}^\vee$. There is a canonical DG category of graded D-branes in this setting and we prove that it is quasi-equivalent to $\mathfrak{Perf}(S)$. In Section 4 we consider the problem of relating the category of graded D-branes on an LG pair (S, F) to the category of graded D-branes on the normal bundle to the critical locus of F, which also carries the structure of an LG pair. Section 5 is devoted to consequences of these quasi-equivalences for the derived bounded categories of coherent sheaves on smooth projective complete intersections. Finally, section 6 deals with formal completions of categories of graded D-branes. Given a smooth variety S and a function F we can consider the singular variety Z of the zero set of F. We construct a category of graded D-branes on the formal completion of S along Z and prove that it is quasi-equivalent to the category of graded D-branes associated to (S, F) when S is quasi-projective.

Remark 1.1. There is a parallel story unfolding concerning categories of graded singularities. The first work in this direction was completed in Mehrotra's thesis [Meh05]. Mehrotra considered the action of μ_n on $\mathbb{C}[x_1, \ldots, x_n]$ and constructed a fully faithful, triangulated functor from the μ_n-equivariant singularity category of $\mathbb{C}[x_1, \ldots, x_n]/(W)$ to the $\mathbb{C}^\times$-equivariant singularity category of the zero locus of W viewed as a function on K, the total space of the canonical bundle on $\mathbb{P}^{n-1}$. Then Quintero Velez [QV09] showed that this is an equivalence, and generalized it to the setting of the Bridgeland-King-Reid formulation of the McKay correspondence. Recently, Isik [Isi10] showed that if X is a smooth subvariety of a variety S given by a regular section of a vector bundle, then the perfect category of X is quasi-equivalent to the $\mathbb{C}^\times$ equivariant singularity category of the zero locus of the section, viewed as a function on the total space of the dual bundle. Then Baranovsky and Pecharich [BP10] generalized Quintero Velez's method and combined it with Isik's theorem to obtain a more general version of Theorem 5.1 in section 5 below for categories of graded singularities. Finally, Polishchuk and Vaintrob [PV] proved that the category of graded D-branes on (K, W) (notation as above) is equivalent to the category of graded singularities of $\{W = 0\} \subset K$. So it is now possible to prove Orlov's theorem using the work of Isik, Polishchuk-Vaintrob, and Segal.

Acknowledgements. My gratitude to V. Ginzburg for introducing me to graded matrix factorizations and for wonderful discussions and suggestions. It was his insight to use the deformation to the normal bundle. This idea is what brings coherence to the results in the paper. In addition, I greatly appreciate the time I spent talking with M. Umut Isik, when he told me about his result and much of the history of the results in this paper. Also, special thanks to E. Segal for pointing out a serious mistake in the first draft of this article. Finally, thanks to I. Dolgachev and M. Boyarchenko for bringing my attention to Sumihiro's theorem.

2. Construction of the category of graded D-branes

Let S be either a variety or a stack of the form $[Y/G]$ where Y is a variety and G is a finite group or a torus. In this setting, a variety is a separated integral scheme of finite type over $\mathbb{C}$. We intentionally work in less than full generality. Suppose that S has an action of the algebraic group $\mathbb{C}^\times$. If $S = [Y/G]$, this means that Y has an action of $\mathbb{C}^\times$ which commutes with the action of G. We assume that S admits a $\mathbb{C}^\times$-invariant open affine cover, and in this situation we say S is *graded*. If S is a normal variety, Sumihiro's theorem [Sum74] implies that such a cover exists for any action. For the most important situations below, $\mathbb{C}^\times$-invariant open affine covers will be abundant. When $S = [Y/G]$, a $\mathbb{C}^\times$-invariant open affine cover is a

$G \times \mathbb{C}^\times$-invariant open affine cover of Y. We say that S is *even graded* if $\{\pm 1\} \subset \mathbb{C}^\times$, acts trivially on S. A regular function $F \in \mathcal{O}(S)$ has degree n if $\lambda^* F = (\lambda)^n \cdot F$ for each $\lambda \in \mathbb{C}^\times$.

Definition 2.1 (LG pair)**.** A *Landau-Ginzburg (LG) pair* is a pair (S, F) where S is an even graded scheme or stack and F is a semi-invariant regular function of degree 2.

Suppose that (S, F) is an LG pair and $U \subset S$ is an invariant affine open set. Then $U = \operatorname{Spec}(R_\bullet)$ where $R_\bullet$ is a graded ring and the action of $\mathbb{C}^\times$ on U is generated by the Euler field of $R_\bullet$. The action is even if and only if $R_\bullet$ is concentrated in even degrees. Moreover, $F|_U$ corresponds to an element of R_2.

Example 2.2. The main example of an LG pair comes from a variety Y with a vector bundle $\mathcal{V}$ over it and a section s of $\mathcal{V}$. In this situation, S is the total space of $\mathcal{V}^\vee$, the dual vector bundle and F is the function on $\mathcal{V}^\vee$ that corresponds to s. This function is linear when restricted to each fiber of S over Y. Since S is the total space of a vector bundle it carries a $\mathbb{C}^\times$ action. However this action is not even, so we "double" it by letting $\lambda \in \mathbb{C}$ act by λ^2.

Definition 2.3 (Graded D-brane)**.** Suppose (S, F) is an LG pair. A *graded D-brane* on (S, F) is a $\mathbb{C}^\times$-equivariant vector bundle $\mathcal{E}$ of finite rank, together with an endomorphism $d_\mathcal{E}$ of degree 1 such that $d_\mathcal{E}^2 = F \cdot \mathrm{id}_\mathcal{E}$.

Let $U = \operatorname{Spec}(R_\bullet) \subset S$ be an invariant affine open set and $\mathcal{E}$ a graded D-brane on (S, F). Then $\mathcal{E}(U)$ is simply a graded, projective $R_\bullet$-module with an endomorphism that raises degrees by 1 and squares to multiplication by F. For any $\mathbb{C}^\times$-equivariant sheaf $\mathcal{F}$ on S, let σ be the endomorphism induced by the action of $-1 \in \mathbb{C}^\times$. The action of σ on a homogeneous $m \in \mathcal{F}(U)$ is by $\sigma(m) = (-1)^{\deg(m)} m$. Let $\mathcal{E}_1, \mathcal{E}_2$ be two graded D-branes on (S, F). We define an endomorphism on $\mathcal{H}om(\mathcal{E}_1, \mathcal{E}_2)$ by $d(\phi) = d_2 \circ \phi - \sigma(\phi)\phi \circ d_1$. Note that $d^2 = 0$ so the graded $R_\bullet$ module $\mathcal{H}om(\mathcal{E}_1, \mathcal{E}_2)(U)$ can be viewed as a complex of $\mathbb{C}$ vector spaces. We may define the $\mathbb{C}^\times$-equivariant coherent sheaf

$$\mathcal{H}(\mathcal{H}om(\mathcal{E}_1, \mathcal{E}_2)) := \ker(d)/\operatorname{im}(d).$$

We have $\mathcal{H}(\mathcal{H}om(\mathcal{E}_1, \mathcal{E}_2))(U) = H^\bullet(\mathcal{H}om(\mathcal{E}_1, \mathcal{E}_2)(U))$.

Let (S, F) be an LG pair. The *Jacobi ideal (sheaf)* $J(F)$ of F is defined to be the image of the map $\mathcal{T}_S \to \mathcal{O}_S$ given by contraction with dF, where $\mathcal{T}_S$ is the tangent sheaf. The *Tyurina ideal (sheaf)* is defined to be $\tau(F) := J(F) + F \cdot \mathcal{O}_S$. The Tyurina ideal sheaf defines the scheme theoretical singular locus of the zero locus of F. Let Z be the reduced subscheme associated to $\tau(F)$. Observe that $\tau(F)$ is $\mathbb{C}^\times$-equivariant and hence Z is invariant.

Suppose that $\mathcal{E}$ is a graded D-brane on (S, F). We can trivialize $\mathcal{E}$ over a small open set so that $d = d_\mathcal{E}$ becomes a matrix. If v is a local vector field we can differentiate the entries of d to obtain an endomorphism $v(d)$. Now we have

$$v(F) \cdot \mathrm{id} = v(d^2) = dv(d) + v(d)d.$$

Hence multiplication by $v(F)$ is nullhomotopic. It follows that for any graded D-branes $\mathcal{E}, \mathcal{F}$, $\mathcal{H}(\mathcal{H}om(\mathcal{E}, \mathcal{F}))$ is annihilated by $\tau(F)$ and thus $\mathcal{H}(\mathcal{H}om(\mathcal{E}, \mathcal{F}))$ is supported on Z.

If $\{U_\alpha\}$ is a $\mathbb{C}^\times$-invariant open affine cover, $\check{C}^\bullet(S, \{U_\alpha\}, \mathcal{H}om(\mathcal{E}_1, \mathcal{E}_2))$ is a bicomplex. If $S = [Y/G]$ is a quotient stack and $\{U_\alpha\}$ is a $\mathbb{C}^\times$-invariant affine open cover then

$$\check{C}^\bullet(S, \{U_\alpha\}, \mathcal{H}om(\mathcal{E}_1, \mathcal{E}_2)) := \check{C}^\bullet(Y, \{U_\alpha\}, \mathcal{H}om(\mathcal{E}_1, \mathcal{E}_2))^G$$

From now on we restrict attention to finite $\mathbb{C}^\times$-invariant affine open covers.

If $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$ are graded D-branes, then the natural composition map $\mathcal{H}om(\mathcal{E}_2, \mathcal{E}_3) \otimes \mathcal{H}om(\mathcal{E}_1, \mathcal{E}_2) \to \mathcal{H}om(\mathcal{E}_1, \mathcal{E}_3)$ is compatible with the differentials. Hence this map induces a chain map

$$\check{C}^\bullet(S, \{U_\alpha\}, \mathcal{H}om(\mathcal{E}_2, \mathcal{E}_3)) \otimes_\mathbb{C} \check{C}^\bullet(S, \{U_\alpha\}, \mathcal{H}om(\mathcal{E}_1, \mathcal{E}_2)) \to \check{C}^\bullet(S, \{U_\alpha\}, \mathcal{H}om(\mathcal{E}_1, \mathcal{E}_3)). \tag{1}$$

Definition 2.4 (Category of graded D-branes)**.** Let (S, F) be an LG pair and $\{U_\alpha\}$ a $\mathbb{C}^\times$-invariant affine open cover of S. The DG category $\mathrm{DBr}^{\mathrm{gr}}(S, F, \{U_\alpha\})$ of graded D-branes is the DG category whose objects are graded D-branes and

$$\operatorname{Hom}^\bullet(\mathcal{E}_1, \mathcal{E}_2) := \check{C}^\bullet(S, \{U_\alpha\}, \mathcal{H}om(\mathcal{E}_1, \mathcal{E}_2))_\bullet$$

is the total complex of the bicomplex. Composition in this DG category is given by (1).

The quasi-equivalence class of $\mathrm{DBr}^{\mathrm{gr}}(S, F, \{U_\alpha\})$ does not depend on the specific choice of $\mathbb{C}^\times$-invariant affine open cover. First, the total complex of the bicomplex $\check{C}^\bullet(S, \{U_\alpha\}, \mathcal{H}om(\mathcal{E}_1, \mathcal{E}_2))$ has a finite filtration

$$F^i \check{C}^\bullet(S, \{U_\alpha\}, \mathcal{H}om(\mathcal{E}_1, \mathcal{E}_2)) := \bigoplus_{j \geq k} \check{C}^j(S, \{U_\alpha\}, \mathcal{H}om(\mathcal{E}_1, \mathcal{E}_2))$$

which gives rise to a convergent spectral sequence

$$H^i(S, \mathcal{H}(\mathcal{H}om(\mathcal{E}_1, \mathcal{E}_2)))_j \Rightarrow H^{i+j}(\check{C}^\bullet(S, \{U_\alpha\}, \mathcal{H}om(\mathcal{E}_1, \mathcal{E}_2))). \tag{2}$$

The other filtration, while not finite is locally finite since the Čech degree is bounded and therefore there is another convergent spectral sequence with first page

$$H^i(S, \mathcal{H}om(\mathcal{E}_1, \mathcal{E}_2))_j \Rightarrow H^{i+j}(\check{C}^\bullet(S, \{U_\alpha\}, \mathcal{H}om(\mathcal{E}_1, \mathcal{E}_2))). \tag{3}$$

Now if $\{V_\beta\}$ is another $\mathbb{C}^\times$-invariant open affine cover, then $\{U_\alpha \cap V_\beta\}$ is a common $\mathbb{C}^\times$-invariant open affine refinement. Moreover, there are comparison maps

$$\check{C}^\bullet(S, \{U_\alpha\}, \mathcal{H}om(\mathcal{E}_1, \mathcal{E}_2)) \longleftarrow \check{C}^\bullet(S, \{U_\alpha \cap V_\beta\}, \mathcal{H}om(\mathcal{E}_1, \mathcal{E}_2)) \longrightarrow \check{C}^\bullet(S, \{V_\beta\}, \mathcal{H}om(\mathcal{E}_1, \mathcal{E}_2))$$

which are compatible with the filtrations by Čech degree and thus (2) can be used to show that they are quasi-isomorphisms. In fact these comparison maps are compatible with (1) as well. So the comparison maps define quasi-equivalences

$$\mathrm{DBr}^{\mathrm{gr}}(S, F, \{U_\alpha\}) \longleftarrow \mathrm{DBr}^{\mathrm{gr}}(S, F, \{U_\alpha \cap V_\beta\}) \longrightarrow \mathrm{DBr}^{\mathrm{gr}}(S, F, \{V_\beta\}).$$

From now on, we write $\mathrm{DBr}^{\mathrm{gr}}(S, F)$, suppressing the choice of cover, since the ambiguity in defining the category is rectified by canonical quasi-equivalences.

For the rest of the section we will discuss additional structures on DG categories of D-branes and functors between them. The homotopy category $H^0 \mathrm{DBr}^{\mathrm{gr}}(S, F)$ carries a natural pretriangulated structure. The shift is defined at the level of $\mathrm{DBr}^{\mathrm{gr}}(S, F)$ as a shift in the equivariant structure. For a $\mathbb{C}^\times$-equivariant coherent sheaf $\mathcal{F}$ on S, let $\mathcal{F}[1]$ be the $\mathbb{C}^\times$-equivariant structure on $\mathcal{F}$ defined locally as $\mathcal{F}[1](U)_i = \mathcal{F}(U)_{i+1}$, whenever U is a $\mathbb{C}^\times$-invariant affine open set. Now, if $(\mathcal{E}, d_\mathcal{E})$ is a graded D-brane, the shift is given by $(\mathcal{E}, d_\mathcal{E})[1] = (\mathcal{E}[1], -d_\mathcal{E})$.

Standard triangles in $\mathrm{DBr}^{\mathrm{gr}}(S, F)$ are defined in a familiar way. Suppose that $\phi : \mathcal{E}_1 \to \mathcal{E}_2$ is a chain map in the sense that ϕ is $\mathbb{C}^\times$ equivariant and intertwines d_1 and d_2. Then we define

$$\mathrm{cone}(\phi) = \left(\mathcal{E}_1[1] \oplus \mathcal{E}_2, \begin{pmatrix} d_1[1] & 0 \\ \phi & d_2 \end{pmatrix} \right).$$

A standard triangle is then given by

$$\mathcal{E}_1 \xrightarrow{\phi} \mathcal{E}_2 \longrightarrow \mathrm{cone}(\phi) \longrightarrow \mathcal{E}_1[1] \longrightarrow \cdots$$

The pretriangulated structure is enough to prove the results in this article since for any chain map $\phi : \mathcal{E}_1 \to \mathcal{E}_2$ and graded D-brane $\mathcal{F}$ we obtain exact triangles

$$\mathrm{Hom}_{\mathrm{DBr}^{\mathrm{gr}}}(\mathcal{F}, \mathcal{E}_1) \to \mathrm{Hom}_{\mathrm{DBr}^{\mathrm{gr}}}(\mathcal{F}, \mathcal{E}_2) \to \mathrm{Hom}_{\mathrm{DBr}^{\mathrm{gr}}}(\mathcal{F}, \mathrm{cone}(\phi)) \to \mathrm{Hom}_{\mathrm{DBr}^{\mathrm{gr}}}(\mathcal{F}, \mathcal{E})[1] \to \cdots$$
$$\mathrm{Hom}_{\mathrm{DBr}^{\mathrm{gr}}}(\mathcal{E}_2, \mathcal{F}) \to \mathrm{Hom}_{\mathrm{DBr}^{\mathrm{gr}}}(\mathcal{E}_1, \mathcal{F}) \to \mathrm{Hom}_{\mathrm{DBr}^{\mathrm{gr}}}(\mathrm{cone}(\phi), \mathcal{F})[1] \to \mathrm{Hom}_{\mathrm{DBr}^{\mathrm{gr}}}(\mathcal{E}_2, \mathcal{F})[1] \to \cdots$$

in the homotopy category of complexes of $\mathbb{C}$-vector spaces. Hence, for example, if $\mathrm{cone}(\phi)$ is locally contractible, ϕ is an isomorphism in the homotopy category.

At the homotopy level, a distinguished triangle is one that is isomorphic to a standard triangle. In the case where the $\mathbb{C}^\times$ action on S is trivial, $\mathrm{DBr}^{\mathrm{gr}}(S, 0) \simeq \mathfrak{Perf}(S)$. When S is smooth, $H^0 \mathrm{DBr}^{\mathrm{gr}}(S, 0)$ is triangle equivalent to $D^b \mathrm{coh}(S)$, the derived bounded category of coherent sheaves on S, the sense that the triangulated structure on $D^b \mathrm{coh}(S)$ is defined by the pretriangulated structure on $H^0 \mathrm{DBr}^{\mathrm{gr}}(S, 0)$.

There is often a natural pullback functor associated to a morphism of LG pairs. Suppose that (S_1, F_1) and (S_2, F_2) are two LG pairs and that $\phi : S_1 \to S_2$ is a $\mathbb{C}^\times$-equivariant morphism such that $F_1 = \phi^* F_2$. Let $\{U_\alpha\}$ be a $\mathbb{C}^\times$-invariant affine open cover of S_2 and $\{V_\beta\}$ a $\mathbb{C}^\times$-invariant affine open cover refining $\{\phi^{-1}(U_\alpha)\}$.

Note that if $\mathcal{E}$ is a graded D-brane on (S_2, F_2) then $\phi^*\mathcal{E}$ is a graded D-brane on (S_1, F_1). In addition there are chain maps

$$\check{C}^\bullet(S_2, \{U_\alpha\}, \mathcal{H}om(\mathcal{E}_1, \mathcal{E}_2)) \to \check{C}^\bullet(S_1, \{V_\beta\}, \mathcal{H}om(\phi^*\mathcal{E}_1, \phi^*\mathcal{E}_2))$$

which are compatible with compositions. So there is a DG functor

$$\phi^* : \mathrm{DBr}^{\mathrm{gr}}(S_2, F_2) \to \mathrm{DBr}^{\mathrm{gr}}(S_1, F_1)$$

that is conpatible with the pretriangulated structures.

There is one last type of functorial construction that we can consider, the tensor product. Let S be an even graded scheme or stack and F_1, F_2 two semi-invariant regular functions of degree 2. Then there is a natural DG functor $\mathrm{DBr}^{\mathrm{gr}}(S, F_1) \otimes_{\mathbb{C}} \mathrm{DBr}^{\mathrm{gr}}(S, F_2) \to \mathrm{DBr}^{\mathrm{gr}}(S, F_1 + F_2)$. If $\mathcal{E}_1$ and $\mathcal{E}_2$ are graded D-branes on (S, F_1) and (S, F_2) respectively and we define an endomorphism of $\mathcal{E}_1 \otimes \mathcal{E}_2$ by

$$d = d_1 \otimes \mathrm{id} + \sigma \otimes d_2$$

then $(\mathcal{E}_1 \otimes \mathcal{E}_2, d)$ is a graded D-brane on $(S, F_1 + F_2)$. Now, if $\mathcal{E}_1, \mathcal{F}_1$ and $\mathcal{E}_2, \mathcal{F}_2$ are graded D-branes on (S, F_1) and (S, F_2) respectively, then there is a canonical isomorphism

$$\mathcal{H}om(\mathcal{E}_1 \otimes \mathcal{E}_2, \mathcal{F}_1 \otimes \mathcal{F}_2) \cong \mathcal{H}om(\mathcal{E}_1, \mathcal{F}_1) \otimes \mathcal{H}om(\mathcal{E}_2, \mathcal{F}_2)$$

and therefore a chain map

$$\check{C}^\bullet(S, \{U_\alpha\}, \mathcal{H}om(\mathcal{E}_1, \mathcal{F}_1)) \otimes_{\mathbb{C}} \check{C}^\bullet(S, \{U_\alpha\}, \mathcal{H}om(\mathcal{E}_2, \mathcal{F}_2)) \to \check{C}^\bullet(S, \{U_\alpha\}, \mathcal{H}om(\mathcal{E}_1 \otimes \mathcal{E}_2, \mathcal{F}_1 \otimes \mathcal{F}_2)).$$

This is compatible with the composition chain maps and therefore we obtain a DG functor

$$\mathrm{DBr}^{\mathrm{gr}}(S, F_1) \otimes_{\mathbb{C}} \mathrm{DBr}^{\mathrm{gr}}(S, F_2) \to \mathrm{DBr}^{\mathrm{gr}}(S, F_1 + F_2).$$

3. Global Knörrer periodicity

In this section we study certain special LG pairs. Let Z be a smooth variety which has the property that every coherent sheaf is a quotient of a coherent locally free sheaf. For example Z could be a smooth quasi-projective variety. This condition is called (ELF) in Orlov's work [Orl09b]. Let $\mathcal{V}$ be a vector bundle on Z of rank r.

The main variety that we will consider in this section is $S = \mathbb{V}(\mathcal{V} \oplus \mathcal{V}^\vee)$, the total space of $\mathcal{V} \oplus \mathcal{V}^\vee$ over Z. Let $p : S \to Z$ be the projection. Note that p is affine and flat, which implies that p_* and p^* are exact. If $\mathrm{Spec}(R) \cong U \subset Z$ is an affine open set such that $\mathcal{V}|_U$ is trivial we call U *small*. Whenever U is a small affine open in Z, we see that $p^{-1}(U) \cong \mathrm{Spec}(R[x_1, \ldots, x_r, y_1, \ldots, y_r])$, where $\{x_i\}, \{y_i\}$ correspond to dual trivializations of $\mathcal{V}$ and $\mathcal{V}^\vee$. There is a $\mathbb{C}^\times$ equivariant structure on S such that in these local coordinates the grading on $R[x_1, \ldots, x_r, y_1, \ldots, y_r]$ is $\deg(R[x_1, \ldots, x_r]) = 0$ and $\deg(y_i) = 2$ for each $i = 1, \ldots, r$. We can also define this equivariant structure in a coordinate free way. Endow $\mathbb{V}(\mathcal{V})$ with the trivial $\mathbb{C}^\times$ action. Take the natural action of $\mathbb{C}^\times$ coming from the linear structure on $\mathbb{V}(\mathcal{V}^\vee)$ and compose it with the squaring map $\lambda \mapsto \lambda^2$. Note that $S = \mathbb{V}(\mathcal{V}) \times_Z \mathbb{V}(\mathcal{V}^\vee)$ has a product $\mathbb{C}^\times$ action since both of the projections $\mathbb{V}(\mathcal{V}), \mathbb{V}(\mathcal{V}^\vee) \to Z$ are equivariant when Z has the trivial $\mathbb{C}^\times$ action.

There is a canonical semi-invariant regular function on S of degree 2. Indeed note that $p^*\mathcal{V}$ and $p^*\mathcal{V}^\vee$ have canonical sections s_y and s_x. There is a tautological bilinear form $\langle -, - \rangle : p^*\mathcal{V} \otimes p^*\mathcal{V}^\vee \to \mathcal{O}_S$ and we set $F = \langle s_y, s_x \rangle$. Over a small affine open $\mathrm{Spec}(R) \cong U \subset S$ we can write $p^{-1}(U) \cong \mathrm{Spec}(R[x_1, \ldots, x_r, y_1, \ldots, y_r])$ and $F = \sum_{i=1}^r x_i y_i$.

Definition 3.1 (Knörrer D-brane)**.** Let $\mathcal{S} = \bigwedge^\bullet(p^*\mathcal{V}[1])$. Let $s_x \wedge$ and $s_y \vee$ denote the operators of left multiplication by s_x and contraction with s_y respectively. Define a endomorphism on $\mathcal{S}$ by $d = s_x \wedge + s_y \vee$ and observe that d has degree 1 and $d^2 = F \cdot \mathrm{id}$. We call the graded D-brane $(\mathcal{S}, d)$ the *Knörrer D-brane*.

The reason for the name is that the restriction of $(\mathcal{S}, d)$ to each fiber of the projection is the unique indecomposable matrix factorization of $\sum_{i=1}^r x_i y_i$. The following is the main result of this section and the rest of the section will be dedicated to its proof. It is a generalization of Theorem 3.1 in [Knö87], hence the name of the section.

Theorem 3.2. *The functor* $\Phi = \mathcal{S} \otimes p^* : \mathfrak{Perf}(Z) \to \mathrm{DBr}^{\mathrm{gr}}(S, F)$ *is a quasi-equivalence.*

Before we prove the theorem we need some preparation. Let $\mathcal{E}$ be a graded D-brane on (S, F). There is a canonical evaluation $\mathrm{eval} : \mathcal{S} \otimes p^* p_* \mathcal{H}om(\mathcal{S}, \mathcal{E}) \to \mathcal{E}$ which intertwines the differentials. Morover, the Knörrer D-brane $\mathcal{S}$ contains a summand $\mathcal{O}_S$ by construction. So we can restrict the evaluation to this summand, restrict $\mathcal{E}$ to $\mathcal{V}^\vee$, and push everything down to Z to derive a chain map $\mathrm{eval}_1 : p_* \mathcal{H}om(\mathcal{S}, \mathcal{E}) \to p_* \mathcal{E}|_{\mathcal{V}^\vee}$. By abuse of notation we also denote by p the projection $\mathcal{V}^\vee \to Z$.

Lemma 3.3. *The natural map* $\mathrm{eval}_1 : p_* \mathcal{H}om(\mathcal{S}, \mathcal{E}) \to p_* \mathcal{E}|_{\mathcal{V}^\vee}$ *is a quasi isomorphism of complexes of quasi-coherent sheaves on* Z*. Moreover the natural evaluation map* $\mathrm{eval} : \mathcal{S} \otimes p^* p_* \mathcal{H}om(\mathcal{S}, \mathcal{E}) \to \mathcal{E}$ *induces, by restriction, a quasi-isomorphism*

$$(\mathcal{S} \otimes p^* p_* \mathcal{H}om(\mathcal{S}, \mathcal{E}))|_Z \cong \wedge^\bullet(\mathcal{V}[1]) \otimes p_* \mathcal{H}om(\mathcal{S}, \mathcal{E}) \to \mathcal{E}|_Z,$$

of complexes of quasi-coherent sheaves on Z*.*

Proof. We will prove the first claim by induction on r. Locally on Z, we will construct an idempotent endomorphism of $p_* \mathcal{H}om(\mathcal{S}, \mathcal{E})$ that is homotopic to the identity. Let $U \cong \mathrm{Spec}(R)$ be a small affine open set in Z. Then $p^{-1}(U) \cong \mathrm{Spec}(R[x_1, \ldots, x_r, y_1, \ldots, y_r])$. The vector bundle $\mathcal{E}$ corresponds to a graded, projective module E and we identify $\mathcal{S}$ with $R[x_i, y_i] \otimes \wedge^\bullet \mathbb{C}\{e_1, \ldots, e_r\}$. We will prove the local version of the statement even when E is not finitely generated, since we are going to need this case to perform the induction. We choose a graded splitting of the exact sequence

$$0 \to E \to E \to E/(x_r)E \to 0$$

to get an $R[x_1, \ldots, x_{r-1}, y_1, \ldots, y_i]$-linear left inverse T of muliplication by x_r on E. Define a $R[x_i, y_i]$ linear endomorphism h of $\mathcal{H}om(\mathcal{S}, \mathcal{E})$ by the recipe

$$h(\phi)(e_I) = 0, \quad r \notin I$$
$$h(\phi)(e_r \wedge e_J) = (-1)^\phi T\phi(e_J), \quad r \notin J$$

We compute that

$$(dh + hd)(\phi)(e_I) = x_r T\phi(e_I), \quad r \notin I$$
$$(dh + hd)(\phi)(e_r \wedge e_J) = \phi(e_r \wedge e_J) + (-1)^\phi (Td_E - d_E T)\phi(e_J), \quad r \notin J$$

Now, $\mathrm{id} - x_r T$ is idempotent by construction and $(Td_E - d_E T)x_r m = 0$ for any $m \in E$. Therefore $\Psi := \mathrm{id} - (dh + hd)$ is idempotent. Observe that the kernel of Ψ consists of exactly those ϕ such that $\phi(e_I) \in x_r E$ whenever $r \notin I$. Let $\mathcal{S}'$ be the submodule of $\mathcal{S}$ generated by $1, e_1, \ldots, e_{r-1}$. Consider the composite

$$\mathrm{Hom}(\mathcal{S}, E) \to \mathrm{Hom}(\mathcal{S}, E/x_r E) \to \mathrm{Hom}(\mathcal{S}', E/x_r E)$$

and note that the second map is actually a morphism of complexes since x_r acts by zero. The kernel of this map is equal to the kernel of Ψ and hence Ψ factors as

$$\mathrm{Hom}(\mathcal{S}, E) \to \mathrm{Hom}(\mathcal{S}', E/x_r E) \to \mathrm{Hom}(\mathcal{S}, E)$$

and both arrows are quasi-isomorphisms. Now, y_r does not appear in the differential of $\mathcal{S}'$. Write $\mathcal{S}_{r-1}$ for the Knörrer D-brane associated to $R[x_1, \ldots, x_{r-1}, y_1, \ldots, y_{r-1}]$ and notice that $\mathrm{Hom}_{R[x_i, y_i]_{i=1}^r}(\mathcal{S}', E/x_r E) \cong \mathrm{Hom}_{R[x_i, y_i]_{i=1}^{r-1}}(\mathcal{S}_{r-1}, E/x_r E)$. If $r > 1$ then by induction the evaluation map

$$\mathrm{Hom}(\mathcal{S}_{r-1}, E/x_r E) \to E/(x_1, \ldots, x_r)E$$

is a quasi-isomorphism. Thus our original evaluation map factors as the composite of two evaluation maps. In the case $r = 1$, $\mathcal{S}' = R[y_1]$ and $\mathrm{Hom}(\mathcal{S}', E/x_1 E) = E/x_1 E$, and the map above was already the desired quasi-isomorphism.

Next we will argue the second point in the Lemma. Notice that the first part implies we have a diagram

$$\wedge^\bullet \mathcal{V}[1] \otimes \pi_* \mathcal{E}|_{\mathcal{V}^\vee} \xleftarrow{\simeq} \wedge^\bullet \mathcal{V}[1] \otimes \pi_* \mathcal{H}om(\mathcal{S}, \mathcal{E}) \longrightarrow \mathcal{E}|_Z$$

We will check that this is a quasi-isomorphism over every small affine open set U. As above, over such a set U with $\mathcal{O}_Z(U) = R$ we have $\pi^{-1}(U) \cong \mathrm{Spec}\, R[x_1, \ldots, x_r, y_1, \ldots, y_r]$. For each $1 \leq i \leq r$ set

$$A_i = \pi_*(\mathcal{E}|_{\mathcal{V}^\vee}/(y_1, \ldots, y_i)\mathcal{E}|_{\mathcal{V}^\vee}),$$

and

$$B_i = \pi_*(\mathcal{H}om(\mathcal{S},\mathcal{E})/(y_1,\dots,y_i)\mathcal{H}om(\mathcal{S},\mathcal{E})).$$

We have diagrams

$$\begin{array}{ccccc}
\wedge^\bullet\mathcal{V}[1]\otimes A_i & \xleftarrow{\simeq} & \wedge^\bullet\mathcal{V}[1]\otimes B_i & \xrightarrow{\epsilon_i} & \wedge^\bullet V_i[1]\otimes\mathcal{E}_Z \\
\downarrow{\scriptstyle y_{i+1}} & & \downarrow{\scriptstyle y_{i+1}} & & \downarrow{\scriptstyle 0} \\
\wedge^\bullet\mathcal{V}[1]\otimes A_i & \xleftarrow[\simeq]{} & \wedge^\bullet\mathcal{V}[1]\otimes B_i & \xrightarrow{\epsilon_i} & \wedge^\bullet V_i[1]\mathcal{E}_Z \\
\downarrow & & \downarrow & & \downarrow \\
\wedge^\bullet\mathcal{V}[1]\otimes A_{i+1} & \xleftarrow[\simeq]{} & \wedge^\bullet\mathcal{V}[1]\otimes B_{i+1} & \xrightarrow{\epsilon_{i+1}} & \wedge^\bullet V_{i+1}[1]\otimes\mathcal{E}_Z
\end{array}$$

where the columns are exact triangles. We can use nullhomotopies of $y_i\cdot\mathrm{id}\in\mathrm{End}(E)$ to construct nullhomotopies h_{i+1}, g_{i+1} of multiplication by y_{i+1} fitting into a commutative diagram

$$\begin{array}{ccc}
\wedge^\bullet\mathcal{V}[1]\otimes A_i & \xleftarrow{\simeq} & \wedge^\bullet\mathcal{V}[1]\otimes B_i \\
\downarrow{\scriptstyle h_{i+1}} & & \downarrow{\scriptstyle g_{i+1}} \\
\wedge^\bullet\mathcal{V}[1]\otimes A_i[1] & \xleftarrow[\simeq]{} & \wedge^\bullet\mathcal{V}[1]\otimes B_i[1]
\end{array}$$

Therefore we can construct a diagram

$$\begin{array}{ccccc}
\wedge^\bullet\mathcal{V}[1]\otimes A_i\oplus\wedge^\bullet\mathcal{V}[1]\otimes A_i[1] & \xleftarrow{\simeq} & \wedge^\bullet\mathcal{V}[1]\otimes B_i\oplus\wedge^\bullet\mathcal{V}[1]\otimes B_i[1] & \xrightarrow{\epsilon_i\oplus\epsilon_i[1]} & \wedge^\bullet V_i[1]\otimes\mathcal{E}_Z\oplus\wedge^\bullet V_i[1]\otimes\mathcal{E}_Z[1] \\
\downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \cong} \\
\wedge^\bullet\mathcal{V}[1]\otimes A_{i+1} & \xleftarrow[\simeq]{} & \wedge^\bullet\mathcal{V}[1]\otimes B_{i+1} & \xrightarrow{\epsilon_{i+1}} & \wedge^\bullet V_{i+1}[1]\otimes\mathcal{E}_Z
\end{array}$$

So ϵ_i is a quasi-isomorphism if and only if ϵ_{i+1} is a quasi-isomorphism. This means that For $i=r$ the corresponding diagram is just

$$\wedge^\bullet\mathcal{V}[1]\otimes\mathcal{E}_Z \xleftarrow{\simeq} \wedge^\bullet\mathcal{V}[1]\otimes\pi_*\mathcal{E}|_{\mathcal{V}} \longrightarrow \wedge^\bullet V\otimes\mathcal{E}_Z$$

Of course, $\wedge^\bullet\mathcal{V}[1]\otimes\mathcal{E}_Z\cong\wedge^\bullet V[1]\otimes\mathcal{E}_Z$ on our open set and the two arrows of the previous display are isomorphic to one another. It follows that our original arrow $\wedge^\bullet\mathcal{V}[1]\otimes\pi_*\mathcal{H}om(\mathcal{S},\mathcal{E})\to\mathcal{E}|_Z$ is a quasi-isomorphism. □

Proof of 3.2. First we will see that Φ is quasi-fully faithful. Then we'll check that it is quasi-essentially surjective. Let P, Q be perfect complexes on Z. Consider the commutative diagram

$$\begin{array}{ccc}
\check{C}^\bullet(Z,\{U_\alpha\},\mathcal{H}om(P,Q)) & \xrightarrow{\Phi_{P,Q}} & \check{C}^\bullet(S,\{p^{-1}(U_\alpha)\},\mathcal{H}om(\mathcal{S}\otimes p^*P,\mathcal{S}\otimes p^*Q)) \\
\downarrow & & \uparrow{\scriptstyle \cong} \\
\check{C}^\bullet(S,\{p^{-1}(U_\alpha)\},\mathcal{H}om(\mathcal{S},\mathcal{S})\otimes p^*\mathcal{H}om(P,Q)) & \xleftarrow[\cong]{} & \check{C}^\bullet(S,\{p^{-1}(U_\alpha)\},\mathcal{S}^\vee\otimes\mathcal{S}\otimes p^*P^\vee\otimes p^*Q)
\end{array}$$

By the projection formula, the left vertical arrow can be identified with

$$\check{C}^\bullet(Z,\{U_\alpha\},\mathcal{H}om(P,Q))\to\check{C}^\bullet(Z,\{U_\alpha\},p_*\mathcal{H}om(\mathcal{S},\mathcal{S})\otimes\mathcal{H}om(P,Q))$$

and this is induced by the quasi-isomorphism

$$\mathcal{O}_Z\otimes\mathcal{H}om(P,Q)\to p_*\mathcal{O}_S\otimes\mathcal{H}om(P,Q)\to p_*\mathcal{H}om(\mathcal{S},\mathcal{S})\otimes\mathcal{H}om(P,Q).$$

Hence $\Phi_{P,Q}$ must be a quasi-isomorphism.

Now we will prove that Φ is quasi-essentially surjective. Let $\mathcal{E}$ be a graded D-brane on (S,F) and consider $\mathcal{H}om(\mathcal{S},\mathcal{E})$. Note that since p is affine, p_* is exact, and $p_*\mathcal{H}(\mathcal{H}om(\mathcal{S},\mathcal{E}))\cong\mathcal{H}(p_*\mathcal{H}om(\mathcal{S},\mathcal{E}))$. Now, $\mathcal{H}(\mathcal{H}om(\mathcal{S},\mathcal{E}))$ is coherent over $\mathcal{O}_S$ and supported on Z. Hence $\mathcal{H}(p_*\mathcal{H}om(\mathcal{S},\mathcal{E}))$ is coherent over Z. Since Z is smooth and (ELF), this implies that there is a perfect complex P and a quasi-isomorphism

$P \to p_*\mathcal{H}om(\mathcal{S}, \mathcal{E})$. The morphism p is flat so $p^*P \to p^*p_*\mathcal{H}om(\mathcal{S}, \mathcal{E})$ is a quasi-isomorphism. We obtain a morphism $\Phi(P) \to \mathcal{S} \otimes p^*p_*\mathcal{H}om(\mathcal{S}, \mathcal{E})$ and there is a natural evaluation chain map

$$\mathcal{S} \otimes p^*p_*\mathcal{H}om(\mathcal{S}, \mathcal{E}) \to \mathcal{E}.$$

This gives a chain map $\Phi(P) \to \mathcal{E}$. We will show that

$$\mathrm{DBr}^{\mathrm{gr}}(\mathcal{E}, \Phi(P)) \to \mathrm{DBr}^{\mathrm{gr}}(\mathcal{E}, \mathcal{E})$$

is a quasi-isomorphism. It follows immediately that $\Phi(P) \to \mathcal{E}$ is an isomorphism in the homotopy category.

It suffices to show that the induced map

$$\mathcal{H}om(\mathcal{E}, \Phi(P)) \to \mathcal{H}om(\mathcal{E}, \mathcal{E})$$

induces an isomorphism on cohomology sheaves and to establish this we can show that the map

$$p_*\mathcal{H}om(\mathcal{E}, \Phi(P)) \to p_*\mathcal{H}om(\mathcal{E}, \mathcal{E})$$

is a quasi-isomorphism. Using an argument similar to the one in the proof of Lemma 3.3, we find that the arrow

$$\wedge^\bullet \mathcal{V}[1] \otimes p_*\mathcal{H}om(\mathcal{E}, \Phi(P)) \to \wedge^\bullet \mathcal{V}[1] \otimes p_*\mathcal{H}om(\mathcal{E}, \mathcal{E})$$

is locally quasi-isomorphic to

$$\mathcal{H}om(\mathcal{E}, \Phi(P))|_Z \to \mathcal{H}om(\mathcal{E}, \mathcal{E})|_Z$$

This arrow factors as

$$\mathcal{H}om(\mathcal{E}|_Z, \wedge^\bullet \mathcal{V}[1] \otimes P) \to \mathcal{H}om(\mathcal{E}|_Z, \wedge^\bullet \mathcal{V}[1] \otimes p_*\mathcal{H}om(\mathcal{S}, \mathcal{E})) \to \mathcal{H}om(\mathcal{E}|_Z, \mathcal{E}|_Z)$$

where both of the arrows are induced by quasi-isomorphisms, the first by construction and the second by Lemma 3.3. Since $\mathcal{E}|_Z$ is perfect, the induced arrows are quasi-isomorphisms. □

4. Deformation to the normal bundle

Suppose that (S, F) is an LG pair. Let Z be the reduced subscheme defined by $\tau(F)$. We assume that S and Z are smooth. Suppose that Z is $\mathbb{C}^\times$ invariant and that $\mathcal{I}_Z$ is the ideal sheaf defining Z. Let $N_{Z/S}$ be the normal bundle, the spectrum of the sheaf of algebras $\mathrm{Sym}^\bullet \mathcal{I}_Z/\mathcal{I}_Z^2 \cong \bigoplus_n \mathcal{I}_Z^n/\mathcal{I}_Z^{n+1}$. Since $\mathcal{I}_Z$ is $\mathbb{C}^\times$ equivariant, there is a natural $\mathbb{C}^\times$ action on $N_{Z/S}$ inherited from S. Let d be the largest natural number such that $F \in \mathcal{I}_Z^d$, so F defines a nonzero section of $\mathcal{I}_Z^d/\mathcal{I}_Z^{d+1} \subset \mathcal{O}_{N_{Z/S}}$. By abuse of notation we denote this regular function on $N_{Z/S}$ by F. Under the inherited $\mathbb{C}^\times$ action, F has degree 2. Hence we obtain a new LG pair $(N_{Z/S}, F)$.

Consider the sheaf of algebras on S given by

$$\mathcal{O}_S[t, t^{-1}\mathcal{I}_Z] = \cdots \oplus t^{-2}\mathcal{I}_Z^2 \oplus t^{-1}\mathcal{I}_Z \oplus \mathcal{O}_S \oplus t\mathcal{O}_S \oplus \cdots$$

and let $\widetilde{S}$ be the spectrum of this sheaf of algebras. Note that $\widetilde{S}$ admits a map $\pi : \widetilde{S} \to \mathbb{A}^1$. Write $\widetilde{S}_\lambda = \pi^{-1}(\lambda)$ for any point $\lambda \in \mathbb{A}^1$. This map has the property that $\widetilde{S}_\lambda \cong S$ for any $\lambda \neq 0$ while $\widetilde{S}_0 = N_{Z/S}$. For this reason $\widetilde{S}$ is called the *deformation to the normal bundle*.

Since $\mathcal{I}_Z$ is $\mathbb{C}^\times$-equivariant, the sheaf of algebras $\mathcal{O}_S[t, t^{-1}\mathcal{I}_Z]$ is $\mathbb{C}^\times$-equivariant. Hence $\widetilde{S}$ carries a $\mathbb{C}^\times$ action. Each fiber $\widetilde{S}_\lambda$ is $\mathbb{C}^\times$ invariant and the induced $\mathbb{C}^\times$ actions on $\widetilde{S}_1$ and $\widetilde{S}_0$ agree with the actions we have already considered. Observe that $t^{-d}F$ is regular function on $\widetilde{S}$ having degree 2 for the $\mathbb{C}^\times$ action on $\widetilde{S}$. So we obtain an LG pair $(\widetilde{S}, t^{-d}F)$. The function $t^{-d}F$ has the property that its restrictions to $\widetilde{S}_1$ and to $\widetilde{S}_0$ are the functions we are calling F. Hence the inclusions $(S, F) \to (\widetilde{S}, t^{-d}F)$ and $(N_{Z/S}, F) \to (\widetilde{S}, t^{-d}F)$ are morphisms of LG pairs.

Assume now that $d = 2$. We write $G = \mathbb{C}^\times$ and we will consider an action of G on the LG pair $(\widetilde{S}, t^{-d}F)$. The notation is to avoid confusion between the two $\mathbb{C}^\times$ actions that exist in this setting. First note that there is a $\mathbb{C}^\times$ action on $\widetilde{S}$ lifting the $\mathbb{C}^\times$ action on $\mathbb{A}^1$. This corresponds to the graded structure on the sheaf of algebras $\mathcal{O}_S[t, t^{-1}\mathcal{I}_Z]$ where $\deg(t) = 1$. To obtain the G action we combine this action with the extant action. We have constructed a $\mathbb{C}^\times \times \mathbb{C}^\times$ action on $\widetilde{S}$ and we let G act via the diagonal homomorphism $G \to \mathbb{C}^\times \times \mathbb{C}^\times$, $\lambda \to (\lambda, \lambda)$. Under this action $t^{-2}F$ is G invariant.

We now consider the category

$$\mathrm{DBr}^{\mathrm{gr}}(\widetilde{S}, t^{-2}F)^G$$

whose objects are G-equivariant graded D-branes and where the complex of morphisms between two G-equivariant graded D-branes $\mathcal{E}, \mathcal{F}$ is

$$\mathrm{Hom}_{\mathrm{DBr}^{\mathrm{gr}}}(\mathcal{E}, \mathcal{F})^G,$$

the subcomplex of G-invariants. We can consider the natural pullback functors

$$\mathrm{DBr}^{\mathrm{gr}}(\widetilde{S}, t^{-2}F)^G \to \mathrm{DBr}^{\mathrm{gr}}(S, F), \mathrm{DBr}^{\mathrm{gr}}(N_{Z/S}, F)$$

Suppose that (S, F) is obtained as in Example 2.2. This means that there is a smooth variety Y, which we assume is projective for this application, and a vector bundle $\mathcal{V}$ over Y with a regular section s. Then S is the total space of $\mathcal{V}^\vee$ and F is the function corresponding to the section s. The $\mathbb{C}^\times$ action on S is derived from the natural action of $\mathbb{C}^\times$ on $\mathcal{V}^\vee$ by having λ in the new action act by λ^2 in the old action. Let $Z \subset Y$ be the zero locus of s and view Z as embedded in S along the zero section.

Consider the LG pair $(N_{Z/S}, F)$. In this situation

$$N_{Z/S} = N_{Z/Y} \oplus \mathcal{V}^\vee|_Z \cong \mathcal{V}|_Z \oplus \mathcal{V}^\vee|_Z$$

and the induced grading comes from doubling the natural action of $\mathbb{C}^\times$ by scaling the $\mathcal{V}^\vee$ summand and fixing the $\mathcal{V}$ summand. Moreover, the function F on S comes from contracting a point of $\mathcal{V}^\vee$ with the section s. Since s vanishes along Z, the induced function on the normal bundle comes from contracting the $\mathcal{V}|_Z$ summand with the $\mathcal{V}|_Z$ summand. Hence the LG pair $(N_{Z/S}, F)$ has the form considered in Knörrer periodicity.

Recall that there is a canonical graded D-brane $\mathcal{S}$ on $(N_{Z/S}, F)$. There is an analogous graded D-brane in the model (S, F). Indeed, we consider the pullback of $\mathcal{V}$ to S under the natural projection $S \to Y$. By abuse of notation we denote the pullback by $\mathcal{V}$. Now, the section s pulls back and over S, $\mathcal{V}$ has a canonical cosection $\alpha : \mathcal{V} \to \mathcal{O}$. We form the Koszul graded D-brane

$$\mathcal{S}_S = (\wedge^\bullet \mathcal{V}, s\wedge + \alpha\vee).$$

In fact, we can extend this to $\widetilde{S}$. Indeed we can form

$$\mathcal{S}_{\widetilde{S}} = (\wedge^\bullet \mathcal{V}, t^{-1}s\wedge + t^{-1}\alpha\vee)$$

where $\mathcal{V}$ here denotes the pullback of $\mathcal{V}$ to $\widetilde{S}$ under the natural map $\widetilde{S} \to S \to Y$. This is equivariant since s and α both have degree 1 for the LG $\mathbb{C}^\times$ action. We have $\mathcal{S}_{\widetilde{S}}|_{\widetilde{S}_1} = \mathcal{S}_S$ and $\mathcal{S}_{\widetilde{S}}|_{\widetilde{S}_0} = \mathcal{S}$. Therefore $\mathcal{S}_{\widetilde{S}}$ deforms $\mathcal{S}_S$ into $\mathcal{S}$. Let $\mathcal{O}(1)$ denote a very ample line bundle on $\widetilde{S}$ and let $\langle \mathcal{S}_{\widetilde{S}}(i)\rangle_{i\in\mathbb{Z}}$ denote the full DG subcategory of $\mathrm{DBr}^{\mathrm{gr}}(\widetilde{S}, t^{-1}F)^G$ containing all of the $\mathcal{S}_{\widetilde{S}}(i)$ and closed under shifts and cones on G-equivariant morphisms of graded D-branes.

Theorem 4.1. *The restriction functor*

$$\langle \mathcal{S}_{\widetilde{S}}(i)\rangle \to \mathrm{DBr}^{\mathrm{gr}}(N_{Z/S}, F)$$

is a quasi-equivalence and

$$\langle \mathcal{S}_{\widetilde{S}}(i)\rangle \to \mathrm{DBr}^{\mathrm{gr}}(S, F)$$

is quasi-fully faithful.

Proof. To show that these functors are quasi-fully faithful it suffices to show that

$$\mathrm{Hom}(\mathcal{S}_{\widetilde{S}}(i), \mathcal{S}_{\widetilde{S}}(j)) \to \mathrm{Hom}(\mathcal{S}(i), \mathcal{S}(j)),$$
$$\mathrm{Hom}(\mathcal{S}_{\widetilde{S}}(i), \mathcal{S}_{\widetilde{S}}(j)) \to \mathrm{Hom}(\mathcal{S}_S(i), \mathcal{S}_S(j)).$$

are quasi-isomorphisms. Now observe that there is a G-equivariant quasi-isomorphism $\mathcal{E}nd(\mathcal{S}_{\widetilde{S}}) \to \mathcal{O}_{Z\times\mathbb{A}^1}$, since $\mathcal{E}nd(\mathcal{S}_{\widetilde{S}})$ is a Koszul complex. Moreover, the restriction of this quasi-isomorphism gives quasi-isomorphisms

$$\mathcal{E}nd(\mathcal{S}_S) \to \mathcal{O}_Z,$$
$$\mathcal{E}nd(\mathcal{S}) \to \mathcal{O}_Z.$$

Using the main spectral sequence (2) we see that the induced maps on the second page are

$$\begin{array}{ccc} & H^\bullet(\widetilde{S}, \mathcal{O}_{Z\times\mathbb{A}^1}(j-i))^G_\bullet & \\ \swarrow & & \searrow \\ H^\bullet(Z, \mathcal{O}_Z(j-i)) \cong H^\bullet(S, \mathcal{O}_Z(j-i))_\bullet & & H^\bullet(N_{Z/S}, \mathcal{O}_Z(j-i))_\bullet \cong H^\bullet(Z, \mathcal{O}_Z(j-i)) \end{array}$$

Of course $H^\bullet(\widetilde{S}, \mathcal{O}_{Z\times\mathbb{A}^1}(j-i))_\bullet = H^\bullet(Z, \mathcal{O}_Z(j-i))_\bullet[t]$. The G-invariants are just $H^\bullet(Z, \mathcal{O}_Z(j-i))$ and we see that these maps are isomophisms at the second page of the relevant spectral sequences. Hence these maps are quasi-isomorphisms.

Note that the essential image of the functor $\langle \mathcal{S}_{\widetilde{S}}(i)\rangle \to \mathrm{DBr}^{\mathrm{gr}}(N_{Z/S}, F)$ contains all of the $\mathcal{S}(i)$ by construction. Now under the equivalence $\mathfrak{Perf}(Z) \to \mathrm{DBr}^{\mathrm{gr}}(N_{Z/S}, F)$ the line bundles $\mathcal{O}_Z(i)$ go to the $\mathcal{S}(i)$. Since $\mathfrak{Perf}(Z)$ is generated by $\mathcal{O}_Z(i)$ for $i \in \mathbb{Z}$, we conclude that $\mathrm{DBr}^{\mathrm{gr}}(N_{Z/S}, F)$ is generated by $\mathcal{S}(i)$ and hence our restriction functor is quasi-essentially surjective. □

Isik's result [Isi10] suggests that the following conjecture should be true.

Conjecture 1. *The restriction functor*

$$\langle \mathcal{S}_{\widetilde{S}}(i)\rangle_{i\in\mathbb{Z}} \to \mathrm{DBr}^{\mathrm{gr}}(S, F)$$

is a quasi-equivalence.

We can reformulate this to say that every object of $H^0\,\mathrm{DBr}^{\mathrm{gr}}(S, F)$ is isomorphic to an object in the smallest full pretriangulated subcategory containing all of the $\mathcal{S}_S(i)$.

Theorem 4.2. *The objects $\mathcal{S}_S(i)$ generate $H^0\,\mathrm{DBr}^{\mathrm{gr}}(S, F)$ in the weak sense that if $H^*\,\mathrm{Hom}(\mathcal{S}_S(i), \mathcal{E}) = 0$ for all i then $\mathcal{E} \cong 0$ in $H^0\,\mathrm{DBr}^{\mathrm{gr}}(S, F)$.*

Proof. Note that since $\mathcal{H}om(\mathcal{S}_S, \mathcal{E})$ is coherent, for $i \ll 0$ the higher cohomology vanishes so $H^j(S, \mathcal{H}om(\mathcal{S}_S(i), \mathcal{E})) = H^j(S, \mathcal{H}om(\mathcal{S}_S, \mathcal{E})(-i)) = 0$ for $j > 0$. It follows that the spectral sequence (3) collapses at the first page. We have

$$H^*H^0(S, \mathcal{H}om(\mathcal{S}_S(i), \mathcal{E})) = H^*H^0(S, \mathcal{H}om(\mathcal{S}_S, \mathcal{E})(-i)) = H^0(S, \mathcal{H}(\mathcal{H}om(\mathcal{S}_S, \mathcal{E}))(i))$$

and therefore for $i \ll 0$

$$H^0(S, \mathcal{H}(\mathcal{H}om(\mathcal{S}_S, \mathcal{E}))(i)) = 0$$

which means that $\mathcal{H}(\mathcal{H}om(\mathcal{S}_S, \mathcal{E})) = 0$. Consider

$$\mathcal{H}om(\mathcal{S}_S, \mathcal{S}_S) \otimes \mathcal{H}om(\mathcal{E}, \mathcal{E}) \cong \mathcal{H}om(\mathcal{E}, \mathcal{S}_S) \otimes \mathcal{H}om(\mathcal{S}_S, \mathcal{E})$$

We wish to show that on the one hand we have a quasi isomorphism

$$\mathcal{H}om(\mathcal{S}_S, \mathcal{S}_S) \otimes \mathcal{H}om(\mathcal{E}, \mathcal{E}) \to \mathcal{H}om(\mathcal{E}, \mathcal{E})|_Z$$

and on the other hand we have a quasi isomorphism

$$0 \to \mathcal{H}om(\mathcal{E}, \mathcal{S}_S) \otimes \mathcal{H}om(\mathcal{S}_S, \mathcal{E}).$$

Now $\mathcal{H}om(\mathcal{E}, \mathcal{S}_S), \mathcal{H}om(\mathcal{S}_S, \mathcal{S}_S), \mathcal{H}om(\mathcal{S}_S, \mathcal{E})$, and $\mathcal{H}om(\mathcal{E}, \mathcal{E})$ are graded D-branes on $(S, 0)$. In general suppose that $\mathcal{F}_1, \mathcal{F}_2$ are graded D-branes on $(S, 0)$. Then $\mathcal{F}_1 \otimes \mathcal{F}_2$ is a graded D-brane on $(S, 0)$. Now, in a small enough neighborhood $S = \mathrm{Spec}(A)$ where $A = A_0[p_1, \ldots, p_r]$ is a graded ring with $\deg(p_i) = 2$ and $\mathcal{F}_1, \mathcal{F}_2$ are finitely generated graded projective modules M, N respectively, endowed with differentials of degree 1. We can view $M \otimes_A N$ as a filtered complex in the following way. Write

$$F^k(M \otimes_A N) = M_{\geq k} \otimes_A N$$

Since N is flat over A we see that

$$F^k(M \otimes_A N)/F^{k+1}(M \otimes_A N) = M_k \otimes_A N$$

So this spectral sequence has the form

$$M_k \otimes_A N_i \Rightarrow H^{i+j}(M \otimes_A N)$$

Of course, M_k is a projective A_0 module and the A module structure factors through $A/A_{>0} \cong A_0$. Observe that since M and N are bounded below, this filtration has the property that $F^k(M \otimes_A N)_i = 0$ if $k \gg i$. This implies that the associated spectral sequence converges.

Now, in our case above $H^\bullet(N)$ is annihilated by $A_{>0}$. Indeed, our function corresponds to a section with smooth zero locus Z. We can write it locally as $F = \sum_{i=1}^r W_i p_i$. For any derivation $\partial \in \operatorname{Der}(A)$, $\partial(F)$ acts trivially on $H^\bullet(N)$. So $\frac{\partial F}{\partial p_i} = W_i$ acts trivially. Moroever, if $\partial \in \operatorname{Der}(A_0)$ we can extend it to a derivation $\partial \in \operatorname{Der}(A)$ by setting $\partial(p_i) = 0$. Then $\partial(F) = \sum_{i=1}^r \partial(W_i)p_i$. In order for Z to be smooth, at each point x on $Z \subset Y$ the vectors $(\partial(W_1), \ldots, \partial(W_r))$ as ∂ ranges over T_xY have to span a space of dimension r. Therefore, the elements $\partial(F) = \sum_{i=1}^r \partial(W_i)p_i$ and $W_j p_i$ have to generate all of A_2 over A_0. Since A_2 generates $A_{>0}$, it follows that $H^\bullet(N)$ is annihilated by $A_{>0}$.

Let $K(p_1, \ldots, p_r)$ be the Koszul complex. Since N is flat over A, $K(p_1, \ldots, p_r) \otimes_A N \to A_0 \otimes_A N$ is a quasi isomrophism. Since $H^\bullet(N)$ is annihilated by $A_{>0}$ the convergent spectral sequence computing $H^\bullet(K(p_1, \ldots, p_r) \otimes_A N)$ collapses to

$$\wedge^\bullet \mathbb{C}^r \otimes_{\mathbb{C}} H^\bullet(N).$$

Therefore, if $N_1 \to N_2$ is a quasi-isomorphism then $A_0 \otimes_A N_1 \to A_0 \otimes_A N_2$ is also a quasi-isomorphism. We conclude that if $H^\bullet(N_1), H^\bullet(N_2)$ are annihilated by $A_{>0}$ and $N_1 \to N_2$ is a quasi-isomorphism then $M \otimes_A N_1 \to M \otimes_A N_2$ is a quasi-isomorphism. This completes the local analysis. If $\mathcal{F}_2 \to \mathcal{F}_3$ is a quasi-isomorphism then $\mathcal{F}_1 \otimes \mathcal{F}_2 \to \mathcal{F}_1 \otimes \mathcal{F}_3$ is a quasi-isomorphism. Therefore $\mathcal{H}(\mathcal{H}om(\mathcal{E}, \mathcal{E}))|_Z = 0$ and since $\mathcal{H}(\mathcal{H}om(\mathcal{E}, \mathcal{E}))$ is coherent and supported on Z this implies that $\mathcal{H}(\mathcal{H}om(\mathcal{E}, \mathcal{E})) = 0$. It follows that $H^0 \operatorname{Hom}(\mathcal{E}, \mathcal{E}) = 0$ and $\mathcal{E} = 0$ in $H^0 \operatorname{DBr}^{\mathrm{gr}}(S, F)$. □

5. Application to projective complete intersections

Now, we will see how we may combine the results of the previous sections with Segal's theorem to derive Orlov's theorem and a generalization. Suppose $X \subset \mathbb{P} = \mathbb{P}^{N-1}$ is a smooth Calabi-Yau complete intersection. Let $W_1, \ldots, W_r \in \mathbb{C}[x_1, \ldots, x_N]$ be homogeneous equations for X with $d_i = \deg(W_i)$. The Calabi-Yau condition is $\sum_{i=1}^r d_i = N$. There are several relevant LG pairs. First, we can combine the W_i into a section s_W of the bundle $\oplus_{i=1}^r \mathcal{O}(d_i)$ on $\mathbb{P}$. This section gives rise to a function W on the total space Y of the bundle $\oplus_{i=1}^r \mathcal{O}(-d_i)$. This function is linear on each fiber of the projection $p : Y \to \mathbb{P}$. Since Y is the total space of a vector bundle it has an action of $\mathbb{C}^\times$. However, as in section 3, we consider the new "doubled" action induced by the squaring endomorphism of $\mathbb{C}^\times$. Let $\mathcal{O}_Y(a) = \pi^*\mathcal{O}(a)$ and note that $\oplus_{i=1}^r \mathcal{O}_Y(-d_i)$ has a tautological section s. The function W can be factored as

$$\mathcal{O}_Y \xrightarrow{s_W} \bigoplus_{i=1}^r \mathcal{O}_Y(d_i) \xrightarrow{\vee s} \mathcal{O}_Y$$

where $\vee s$ denotes contraction with s.

We now describe Segal's theorem. To begin with we consider $V := \operatorname{Spec}(\mathbb{C}[x_1, \ldots, x_N, p_1, \ldots, p_r])$ with two $\mathbb{C}^\times$ actions. Under the first action, $\deg(x_i) = 1$ and $\deg(p_i) = -d_i$. Under the second, the one which provides the even graded structure, we have $\deg(x_i) = 0$ and $\deg(p_i) = 2$. The function $F = \sum_{i=1}^r p_i W_i$ has degree zero for the first $\mathbb{C}^\times$ action and degree 2 for the second. To distinguish the two actions we set $G = \mathbb{C}^\times$ and call the first action an action of G and the second an action of $\mathbb{C}^\times$. There are two possible open sets of semistable points in V associated to the identity and inversion characters of G. Write V_+ and V_- for the points semistable with respect to the identity and inversion characters, respectively. We see that $[V_+/G] \cong Y$. We will describe $[V_-/G]$ in more detail below. Both semistable sets are $\mathbb{C}^\times$ invariant and hence we obtain three LG pairs $([V/G], F), (Y, W), ([V_-/G], W)$ fitting into a diagram

$$\operatorname{DBr}^{\mathrm{gr}}([V_-/G], W) \xleftarrow{j^*} \operatorname{DBr}^{\mathrm{gr}}([V/G], F) \xrightarrow{j^*} \operatorname{DBr}^{\mathrm{gr}}(Y, W)$$

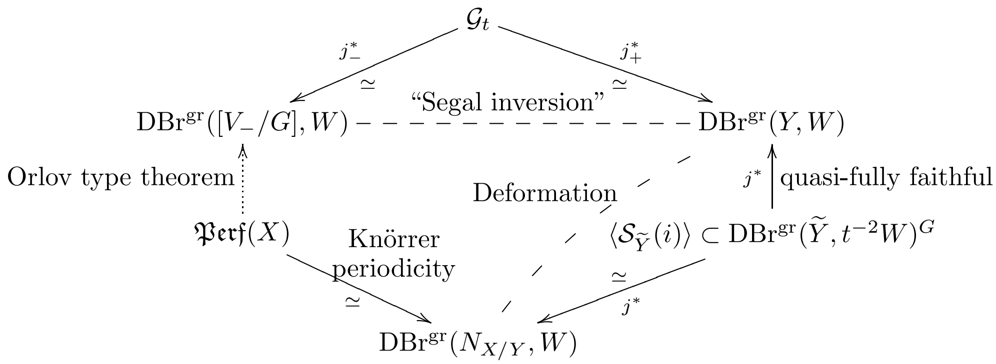
Let $\mathcal{G}_t$ be the full DG subcategory of $\operatorname{DBr}^{\mathrm{gr}}([V/G], F)$ whose objects are graded D-branes $\mathcal{E}$ whose underlying G-equivariant vector bundle is a direct sum of character line bundles in the set $\mathcal{O}_V(t), \ldots, \mathcal{O}_V(t + N - 1)$. We can now formulate Segal's theorem.

Theorem (3.3, [Seg09])**.** *The functors*

$$\mathrm{DBr}^{\mathrm{gr}}([V_-/G], W) \xleftarrow{j^*} \mathcal{G}_t \xrightarrow{j^*} \mathrm{DBr}^{\mathrm{gr}}(Y, W)$$

are quasi-equivalences.

In conclusion, we have the following diagram, where the solid arrows are functors which are quasi-equivalences when labelled by $\simeq$. The dashed lines indicate the "phenomenona" responsible for the various equivalences and comparisons and the dotted arrow on the left represents the fully faithful functor between the homotopy categories of $\mathfrak{Perf}(X)$ and $\mathrm{DBr}^{\mathrm{gr}}([V_-/G], W)$ that one obtains by going around the diagram counter clockwise.



The quasi-equivalences induce triangulated equivalences in the homotopy categories, and thus we obtain the following theorem.

Theorem 5.1. *There is a fully faithful triangulated functor* $D^b \operatorname{coh}(X) = H^0\,\mathfrak{Perf}(X) \to H^0\,\mathrm{DBr}^{\mathrm{gr}}(Y, W) \simeq H^0\,\mathrm{DBr}^{\mathrm{gr}}([V_-/G], W)$ *whose essential image weakly generates.*

If $r = 1$, $[V_-/G]$ has a simple description and $H^0\,\mathrm{DBr}^{\mathrm{gr}}([V_-/G], W)$ is naturally equivalent to the category of graded matrix factorizations. In this case $V = \operatorname{Spec}(\mathbb{C}[x_1, \dots, x_N, p])$ and $V_- = \operatorname{Spec}(\mathbb{C}[x_1, \dots, x_N, p, p^{-1}])$. The ring $\mathbb{C}[x_1, \dots, x_n, p]$ has two gradings and the degrees are $\deg(x_i) = (1, 0)$ and $\deg(p) = (-N, 2)$. Let $R = \mathbb{C}[x_1, \dots, x_N]$. The finitely generated bigraded projective modules over $R_p = R[p, p^{-1}]$ are direct sums of the modules $R_p(a, b)$, the free R_p module generated by an element of degree (a, b). Note that R_p is also generated by p as a module and therefore $R_p \cong R_p(-N, 2)$ as bigraded modules. The only units of R_p not in degree zero are the powers of p and hence there are no isomorphisms between the modules in the collection $R_p(a, b)$ that do not come from $R_p \cong R_p(-N, 2)$ by shifts and compositions. We see that the bigraded modules $R_p(a, 0), R_p(a, 1)$ are all distinct and that every bigraded projective module is isomorphic to a direct sum of these.

An object of $\mathrm{DBr}^{\mathrm{gr}}([V_-/G], W)$ is a bigraded projective R_p module $\mathcal{E} = \bigoplus_j R_p(a_j, b_j)$ and an endomorphism d of degree $(0, 1)$ satisfying $d^2 = pf$, where f is the defining equation of our hypersurface. Clearly

$$\begin{aligned}
\operatorname{Hom}_{R_p}(R_p(a, 0), R_p(a', 0))_{(0,1)} &= \operatorname{Hom}_{R_p}(R_p(a, 1), R_p(a', 1))_{(0,1)} = 0 \\
\operatorname{Hom}_{R_p}(R_p(a, 1), R_p(a', 0))_{(0,1)} &= (R_p)_{(a'-a,0)} = R_{a'-a} \\
\operatorname{Hom}_{R_p}(R_p(a, 0), R_p(a', 1))_{(0,1)} &= p(R_p)_{(a'-a,0)} = pR_{a'-a}
\end{aligned}$$

It is now clear that a graded D-brane on $([V_-/G], W)$ is the same as a graded matrix factorization of f over $R = \mathbb{C}[x_1, \dots, x_N]$. Given two graded D-branes $\mathcal{E}, \mathcal{F}$ on $([V_-/G], W)$ we write E, F, respectively, for the corresponding R_p modules. Since

$$\operatorname{Hom}_{\mathrm{DBr}^{\mathrm{gr}}}(\mathcal{E}, \mathcal{F}) = \operatorname{Hom}_{R_p}(E, F)_{(0,*)}$$

we see that $H^0 \operatorname{Hom}_{\mathrm{DBr}^{\mathrm{gr}}}(\mathcal{E}, \mathcal{F})$ is given by the space of graded chain maps $E \to F$ modulo nullhomotopic chain maps. Hence $H^0\,\mathrm{DBr}^{\mathrm{gr}}([V_-/G], W)$ is equivalent to the category of graded matrix factorizations and clearly the equivalence is compatible with the triangulated structure.

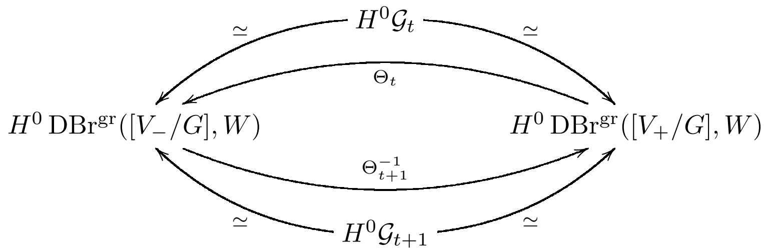
We will now show that the above functor is an equivalence when X is a hypersurface.

Theorem 5.2. *Conjecture 1 holds when X is a hypersurface.*

Proof. We must show that the collection $\mathcal{S}_Y(i)$ generates $\mathrm{DBr}^{\mathrm{gr}}(Y, W)$ when X is is a hypersurface and $Y = K$, the total space of the canonical bundle. First, we write $\mathbb{P} = \mathbb{P}(V)$ and $R = \mathrm{Sym}^\bullet V^\vee$ and consider the objects

$$\mathcal{S}_- = (\wedge^\bullet \Omega^1_R[1], dW \wedge + \frac{1}{N} E \vee)$$

where $E \in \mathrm{Der}(R)$ is the Euler field which looks like $E = \sum_{i=1}^n x_i \partial_i$ in coordinates. This gives an object in $\mathrm{DBr}^{\mathrm{gr}}([V_-/G], W)$. According to [Tu, Theorem 6.8], the category $\mathrm{DBr}^{\mathrm{gr}}([V_-/G], W)$ is generated by the objects $\mathcal{S}_-(i)$ for $i = 0, \ldots, N-1$. Consider the diagram, with the notation as above



Theorem [Seg09, Theorem 3.13] states that for any object $\mathcal{E}$ of $H^0\,\mathrm{DBr}^{\mathrm{gr}}([V_+/G], W)$ there is an arrow $\epsilon_\mathcal{E} : \mathcal{S}(t) \otimes H^0(\mathrm{Hom}(\mathcal{S}(t), \mathcal{E})) \to \mathcal{E}$ and an isomorphism $\mathrm{cone}(\epsilon_\mathcal{E}) \to \Theta_{t+1}^{-1}\Theta_t(\mathcal{E})$. Hence $\Theta_{t+1}^{-1}\Theta_t(\mathcal{E}) \in \langle \mathcal{S}(t), \mathcal{E} \rangle$, where the angle brackets indicate the smallest full triangulated subcategory of $\mathrm{DBr}^{\mathrm{gr}}(Y, W)$ containing $\mathcal{E}$ and $\mathcal{S}(t)$. Since $\mathcal{S}_-(t + r) = \Theta_{t+r}(\mathcal{S}(t + r))$ this implies that

$$\Theta_{t+r}^{-1}\mathcal{S}_-(t) \cong \Theta_{t+r}^{-1}\Theta_{t+r-1}\Theta_{t+r-1}^{-1} \cdots \Theta_{t+1}^{-1}\Theta_t\mathcal{S}(t) \in \langle \mathcal{S}(t), \ldots, \mathcal{S}(t + r - 1) \rangle.$$

Now, since Θ_N^{-1} is a triangulated equivalence, the objects $\Theta_N^{-1}\mathcal{S}_-, \ldots, \Theta_N^{-1}\mathcal{S}_-(N-1)$ generate $\mathrm{DBr}^{\mathrm{gr}}(Y, W)$. However, as we just observed the objects $\Theta_N^{-1}\mathcal{S}_-, \ldots, \Theta_N^{-1}\mathcal{S}_-(N-1)$ are in the subcategory of $\mathrm{DBr}^{\mathrm{gr}}(Y, W)$ generated by $\mathcal{S}, \ldots, \mathcal{S}(N-1)$. Therefore $\mathcal{S}, \ldots, \mathcal{S}(N-1)$ generate $\mathrm{DBr}^{\mathrm{gr}}(Y, W)$. □

We obtain Orlov's theorem as a corollary:

Corollary 5.3. *When X is a smooth Calabi-Yau hypersurface, there is a triangulated equivalence of categories between $D^b\,\mathrm{coh}(X)$ and the triangulated category of graded matrix factorizations of the defining equation.*

When $r > 1$ it is more difficult to reduce $\mathrm{DBr}^{\mathrm{gr}}(Y, W)$ to an algebraic model, however it is possible to obtain a result in this direction. We begin with the stack $[V/G]$. Recall that $V = \mathrm{Spec}(\mathbb{C}[x_1, \ldots, x_N, p_1, \ldots, p_r])$ and $G = \mathbb{C}^\times$ acts by $\deg(x_i) = 1$ and $\deg(p_j) = -d_i$. The $\mathbb{C}^\times$ action giving the LG structure is $\deg(x_i) = 0$ and $\deg(p_j) = 2$. Graded D-branes on this stack are just pairs of bigraded free modules over A with homomorphisms in either direction squaring to $F = \sum_{i=1}^r p_i W_i$. Note that $A = \bigoplus_{a \in \mathbb{Z}} \Gamma(Y, \mathcal{O}(a))$ and this graded structure corresponds to the G action. Each of these spaces has a second grading, coming from the $\mathbb{C}^\times$ action.

Segal proved that the pullback functor

$$\mathrm{DBr}^{\mathrm{gr}}([V/G], F) \to \mathrm{DBr}^{\mathrm{gr}}(Y, W)$$

is quasi-essentially surjective. With a little more work we can show that it is full and describe a relationship between $H^0\,\mathrm{DBr}^{\mathrm{gr}}([V/G], F)$ and $H^0\,\mathrm{DBr}^{\mathrm{gr}}(Y, W) \simeq D^b\,\mathrm{coh}(X)$. First, recall that if $\mathcal{E}, \mathcal{F}$ are any graded D-branes on (Y, W) there is a natural convergent spectral sequence

$$H^i(Y, \mathcal{H}om(\mathcal{E}, \mathcal{F}))_j \Rightarrow H^{i+j}\check{C}^\bullet(Y, \mathcal{H}om(\mathcal{E}, \mathcal{F}))_\bullet$$

If $\mathcal{E}$ and $\mathcal{F}$ are pullbacks of graded D-branes on $[V/G]$ then their underlying equivariant sheaves are sums of character line bundles $\mathcal{O}(a)$. Now, the projection $p : Y \to \mathbb{P}^{N-1}$ is affine and therefore

$$H^i(Y, \mathcal{O}(a)) = H^i(\mathbb{P}^{N-1}, p_*\mathcal{O}(a)) = \bigoplus_{n_1, \ldots, n_r \geq 0} H^i(\mathbb{P}^{N-1}, \mathcal{O}(a + n_1 d_r + \cdots + n_r d_r))$$

Therefore $H^i(Y^*, \mathcal{O}(a)) = 0$ unless $i = 0$ or $i = N-1$. As we remarked above $H^0(Y^*, \mathcal{O}(a)) = A_a$. On the other hand

$$\bigoplus_{n_1,\ldots,n_r \geq 0} H^{N-1}(\mathbb{P}^{r-1}, \mathcal{O}(a + n_1 d_r + \cdots + n_r d_r)) \cong \bigoplus_{n_1,\ldots,n_N \geq 0} H^0(\mathbb{P}^{r-1}, \mathcal{O}(-(N + a + n_1 d_1 + \cdots + n_r d_r)))^\vee$$

by Serre duality. If $a > 0$ then this is zero and if $a < 0$ then it is finite dimensional since there are finitely many choices of $n_1, \ldots, n_r$ such that $-a \geq N + n_1 + \cdots + n_r$. For $a \geq N$ we may write

$$H^{N-1}(Y, \mathcal{O}(-a)) = \left(\bigoplus_{\substack{n_1,\ldots,n_r \geq 0 \\ n_1 d_1 + \cdots + n_r d_r \leq a - N}} H^0(\mathbb{P}^{N-1}, \mathcal{O}(a - N - n_1 d_1 - \cdots - n_r d_r)) \right)^\vee .$$

We can understand the $A_0 = \Gamma(Y, \mathcal{O}_Y)$ module

$$\bigoplus_{\substack{n_1,\ldots,n_r \geq 0 \\ n_1 d_1 + \cdots + n_r d_r \leq a - N}} H^0(\mathbb{P}^{N-1}, \mathcal{O}(a - N - n_1 d_1 - \cdots - n_r d_r))$$

as follows. Given a bigraded A module M we form the bigraded A module $M[p_1^{-1}, \ldots, p_r^{-1}] = M \otimes_A A[p_1^{-1}, \ldots, p_r^{-1}]$ where $\deg(p_i^{-1}) = (d_i, -2)$. Let $D(M) = \operatorname{coker}((p_i)M^\vee \hookrightarrow M^\vee[p_i^{-1}])$. Then we see that

$$D(A(a))_N^\vee \cong H^{N-1}(Y, \mathcal{O}(a))$$

where $\vee$ denotes the linear dual.

So if $\mathcal{E}, \mathcal{F}$ are graded D-branes we have

$$H^{N-1}(Y, \mathcal{H}om(\mathcal{E}, \mathcal{F})) \cong D(\operatorname{Hom}(\mathcal{E}, \mathcal{F}))_N^\vee$$

Now the last nontrivial differential in the spectral sequence above has the form

$$H^* H^{N-1}(\mathcal{H}om(\mathcal{E}, \mathcal{F})) \to H^* H^0(\mathcal{H}om(\mathcal{E}, \mathcal{F}))$$

By comparing $\mathcal{E}$ and $\mathcal{F}$ to graded D-branes whose underlying vector bundles lie in the range $\mathcal{O}, \ldots, \mathcal{O}(N-1)$ we can show that this differential has to be injective. Suppose that $\mathcal{E}' \to \mathcal{E}$ and $\mathcal{F}' \to \mathcal{F}$ become isomorphisms in the homotopy category and $\mathcal{E}', \mathcal{F}'$ belong to $j^*\mathcal{G}_0$. Then we have a diagram of quasi-isomorphisms

$$\operatorname{Hom}_{\mathrm{DBr}^{\mathrm{gr}}}(\mathcal{E}', \mathcal{F}') \longrightarrow \operatorname{Hom}_{\mathrm{DBr}^{\mathrm{gr}}}(\mathcal{E}', \mathcal{F}) \longleftarrow \operatorname{Hom}_{\mathrm{DBr}^{\mathrm{gr}}}(\mathcal{E}, \mathcal{F})$$

of filtered complexes. These induce morphisms of the associated spectral sequence. At the last nontrivial page, the diagram becomes a commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & H^* H^{N-1}(\mathcal{H}om(\mathcal{E}', \mathcal{F})) & \longleftarrow & H^* H^{N-1}(\mathcal{H}om(\mathcal{E}, \mathcal{F})) \\ \downarrow & & \downarrow & & \downarrow \\ H^* H^0(\mathcal{H}om(\mathcal{E}', \mathcal{F}')) & \longrightarrow & H^* H^0(\mathcal{H}om(\mathcal{E}', \mathcal{F})) & \longleftarrow & H^* H^0(\mathcal{H}om(\mathcal{E}, \mathcal{F})) \end{array}$$

of quasi-isomorphisms. This implies that the vertical arrows are injective and that the restriction functor $H^0 \, \mathrm{DBr}^{\mathrm{gr}}([V/G], W) \to H^0 \, \mathrm{DBr}^{\mathrm{gr}}(Y, W)$ is full.

Theorem 5.4. *There is a functorial short exact sequence for any graded D-branes E, F on $([V/G], F)$,*

$$0 \to H^m(D(\operatorname{Hom}(F, E)))_N)^\vee \to H^m \operatorname{Hom}(E, F)_0 \to H^m \, \mathrm{DBr}^{\mathrm{gr}}(Y, W)(j^*\mathcal{E}, j^*\mathcal{F}) \to 0$$

The second arrow is $H^m(j^*)$. However, while the first arrow is functorial, it is inexplicit. An explicit construction for this arrow would lead to a completely explicit description of $D^b \operatorname{coh}(X)$ in the same spirit as Orlov's theorem.

6. Localization

In this section we will formulate and prove a precise version of the statement that for an LG pair (S,F) the category $\mathrm{DBr}^{\mathrm{gr}}(S,F)$ only depends on a formal neighborhood of the singular locus of the zero locus of F, when this singular locus is itself nonsingular and quasi-projective. To make this precise we need a notion of graded D-brane that makes sense on a formal neighborhood of the zero locus of F.

Let (S,F) be an LG pair with S nonsingular and quasi-projective with an equivariant ample line bundle $\mathcal{L}$. Recall that the *Jacobi ideal (sheaf)* $J(F)$ of F is defined to be the image of the map $\mathcal{T}_S \to \mathcal{O}_S$ given by contraction with dF, where $\mathcal{T}_S$ is the tangent sheaf. The *Tyurina ideal (sheaf)* is defined to be $\tau(F) := J(F) + F\cdot\mathcal{O}_S$. The Tyurina ideal sheaf defines the scheme theoretical singular locus of the zero locus of F. Let Z be the reduced subscheme associated to $\tau(F)$. Observe that $\tau(F)$ is $\mathbb{C}^\times$-equivariant and hence Z is invariant. We assume for the rest of the section that Z is nonsingular.

We consider the subschemes $Z^{(n)}$ defined by $\tau(F)^n$. All of these schemes have an action $\mathbb{C}^\times$ so that the closed immersions $Z^{(n)} \to S$ are equivariant. Let $\widehat{Z}$ be the formal completion of S along Z, where we choose $\tau(F)$ for the ideal of definition.

Definition 6.1. An equivariant structure on a coherent sheaf $\mathcal{E}$ on $\widehat{Z}$ is, for each n, an equivariant structure on $\mathcal{E}|_{Z^{(n)}}$ such that the equivariant structure on $\mathcal{E}|_{Z^{(n)}}$ is obtained by restriction from the equivariant structure on $\mathcal{E}|_{Z^{(n+1)}}$.

View F as a function on $\widehat{Z}$. Now we can formulate the correct notion of a graded D-brane on $(\widehat{Z},F)$.

Definition 6.2. A *graded D-brane on* $\widehat{Z}$ *controlled by* $\mathcal{L}$ is an equivariant vector bundle $\mathcal{E}$ on $\widehat{Z}$ with an endomorphism $d_{\mathcal{E}}$ of degree one such that $d_{\mathcal{E}}^2 = F\cdot \mathrm{id}_{\mathcal{E}}$ and such that for some $m \gg 0$ the natural map $\mathcal{O}_{\widehat{Z}} \otimes \Gamma(\widehat{Z}, \mathcal{E}\otimes\mathcal{L}^{\otimes m}) \to \mathcal{E}\otimes\mathcal{L}^{\otimes m}$ is surjective.

It remains to construct a DG category. Let $U' \subset S$ be an invariant open affine and set $U = U' \cap Z$. Then we define the graded ring

$$\mathcal{O}^{gr}_{\widehat{Z}}(U) = \bigoplus_{k\in\mathbb{Z}} \varprojlim{}_n (\mathcal{O}_S(U')/\tau(F)^n)_k$$

If $\mathcal{E}$ is an equivariant sheaf on $\widehat{Z}$ we can define the $\mathcal{O}^{gr}_{\widehat{Z}}(U)$ module

$$\mathcal{E}^{gr}(U) = \bigoplus_{k\in\mathbb{Z}} \varprojlim{}_n (\mathcal{E}(U')/\tau(F)^n\mathcal{E}(U'))_k.$$

Suppose that $\mathcal{E}$ and $\mathcal{F}$ are two equivariant vector bundles on $\widehat{Z}$. There is a natural graded $\mathcal{O}^{gr}_{\widehat{Z}}(U)$-module structure on the space of continuous homomorphisms

$$\mathrm{Hom}_{gr}(\mathcal{E},\mathcal{F})(U) := \mathrm{Hom}_{cont}(\mathcal{E}^{gr}(U), \mathcal{F}^{gr}(U))$$

There is an alternate description

$$\mathrm{Hom}_{gr}(\mathcal{E},\mathcal{F})(U) = \bigoplus_{k\in\mathbb{Z}} \varprojlim{}_n \left(\mathcal{H}om(\mathcal{E},\mathcal{F})(U')/\tau(F)^n\mathcal{H}om(\mathcal{E},\mathcal{F})(U')\right)_k$$

The endomorphisms of $\mathcal{E}$ and $\mathcal{F}$ induce a differential on $\mathrm{Hom}_{gr}(\mathcal{E},\mathcal{F})(U)$ making it into a complex of $\mathbb{C}$ vector spaces and a DG $\mathcal{O}^{gr}_{\widehat{Z}}(U)$-module. Observe that the formation of $\mathrm{Hom}_{gr}(\mathcal{E},\mathcal{F})(U)$ is compatible with composition in the sense that there is canonical morphism

$$\mathrm{Hom}_{gr}(\mathcal{E}_2,\mathcal{E}_3)(U) \otimes_{\mathcal{O}^{gr}_{\widehat{Z}}(U)} \mathrm{Hom}_{gr}(\mathcal{E}_1,\mathcal{E}_2)(U) \to \mathrm{Hom}_{gr}(\mathcal{E}_1,\mathcal{E}_3)(U) \tag{4}$$

of DG $\mathcal{O}^{gr}_{\widehat{Z}}(U)$-modules.

Fix a $\mathbb{C}^\times$-invariant affine open cover $\{U_\alpha\}$ of S.

Definition 6.3. The category $\mathrm{DBr}^{\mathrm{gr}}(\widehat{Z},F,\mathcal{L})$ of graded D-branes on $(\widehat{Z},F)$ controlled by $\mathcal{L}$ is the DG category whose objects are graded D-branes on $(\widehat{Z},F)$ controlled by $\mathcal{L}$. The complex of morphisms between $\mathcal{E}$ and $\mathcal{F}$ is the total complex of the bicomplex

$$\check{C}^\bullet(\widehat{Z}, \{U_\alpha\cap Z\}, \mathrm{Hom}_{gr}(\mathcal{E},\mathcal{F}))_\bullet.$$

Composition is induced by (4)

Write $j : \widehat{Z} \to S$ for the natural morphism of locally ringed spaces. If $\mathcal{E}$ is a graded D-brane on (S, F) then $j^*\mathcal{E}$ is a graded D-brane controlled by $\mathcal{L}$. Moreover, if $\mathcal{E}, \mathcal{F}$ are two graded D-branes on (S, F) and U is an invariant open affine, there is a map

$$j^*\mathcal{H}om(\mathcal{E}, \mathcal{F})(U) \to \mathrm{Hom}_{gr}(j^*\mathcal{E}, j^*\mathcal{F})(U \cap Z)$$

of graded $\mathcal{O}_S(U)$ modules that intertwines the natural differentials. This is compatible with compositions and defines a functor $j^* : \mathrm{DBr}^{\mathrm{gr}}(S, F) \to \mathrm{DBr}^{\mathrm{gr}}(\widehat{Z}, F, \mathcal{L})$.

Theorem 6.4. *The completion functor* $j^* : \mathrm{DBr}^{\mathrm{gr}}(S, F) \to \mathrm{DBr}^{\mathrm{gr}}(\widehat{Z}, F, \mathcal{L})$ *is a quasi-equivalence.*

Proof. We must verify that j^* is quasi-fully faithful and quasi-essentially surjective. To prove that j^* is quasi-fully faithful we will check that

$$j^* : \mathcal{H}om(\mathcal{E}, \mathcal{F})(U_\alpha) \to \mathrm{Hom}_{gr}(j^*\mathcal{E}, j^*\mathcal{F})(U_\alpha \cap Z)$$

is a quasi-isomorphism. Since j^* is compatible with the filtrations by Čech degrees it induces a map of spectral sequences. When the above map is a quasi-isomorphism for each α the map of spectral sequences becomes an isomorphism at the first page and hence j^* is a quasi-isomorphism. An exact sequence of graded modules is exact in each homogeneous degree. Moreover, the inverse systems appearing in the definition of the graded completion satisfy the Mittag-Leffler condition. Hence, graded completion is exact. It follows that

$$\bigoplus_{k \in \mathbb{Z}} \varprojlim_n \mathcal{H}(\mathcal{H}om(\mathcal{E}, \mathcal{F})(U'))/\tau^n(F)\mathcal{H}(\mathcal{H}om(\mathcal{E}, \mathcal{F}))(U')_k \cong H^* \mathrm{Hom}_{gr}(j^*\mathcal{E}, j^*\mathcal{F})(U)$$

However, $\tau^n(F)\mathcal{H}(\mathcal{H}om(\mathcal{E}, \mathcal{F}))(U') = 0$ for all $n \geq 1$ and therefore

$$\mathcal{H}(\mathrm{Hom}(\mathcal{E}, \mathcal{F}))(U') \to H^* \mathrm{Hom}_{gr}(j^*\mathcal{E}, j^*\mathcal{F})(U)$$

is an isomorphism.

Now we must show that j^* is quasi-essentially surjective. We will deduce this from Theorem 3.10 of [Orl09b], which we view as a local statement. The theorem says that if B is a graded ring of finite homological dimension and W is a homogeneous element then

$$\mathrm{coker} : H^0 \, \mathrm{DBr}^{\mathrm{gr}}(B, W) \to D^{gr}_{Sg}(B/WB)$$

is a triangulated equivalence. This means that the assignment $E \mapsto \mathrm{coker}(d_{\mathcal{E}^+})$ descends to a functor $H^0 \, \mathrm{DBr}^{\mathrm{gr}}(B, W) \to D^{gr}_{Sg}(B/WB)$.

Consider a graded D-brane $\mathcal{E}$ on $(\widehat{Z}, F)$ controlled by $\mathcal{L}$. Write $V(F)$ for the subscheme defined by F and (m) for tensoring with $\mathcal{L}^{\otimes m}$. Let $\widehat{\alpha} = \mathrm{coker}(d^+_{\mathcal{E}})$ and let α be a coherent equivariant sheaf on $V(F)$ such that $j^*\alpha = \widehat{\alpha}$. Suppose that

$$0 \to Q_k \to Q_{k-1} \to \cdots \to Q_0 \to \alpha \to 0$$

is an exact sequence of coherent sheaves on $V(F)$ such that Q_i is locally free and equivariant for $i < k$. Take $m \gg 0$ such that $Q_i(m)$ and $\mathcal{E}(m)$ are globally generated. Choose an equivariant map

$$\mathcal{O}_{\widehat{Z}}(-m) \otimes_{\mathbb{C}} \Gamma(S, Q_1(m)) \to \mathcal{E}_+$$

such that each square of

$$\begin{array}{ccccccccc}
\mathcal{O}_{\widehat{Z}}(-m) \otimes_{\mathbb{C}} \Gamma(S, Q_k(m)) & \longrightarrow & \mathcal{O}_{\widehat{Z}}(-m) \otimes_{\mathbb{C}} \Gamma(S, Q_k(m)) & \longrightarrow & \cdots & \longrightarrow & \mathcal{O}_{\widehat{Z}}(-m) \otimes_{\mathbb{C}} \Gamma(S, Q_k(m)) & \longrightarrow & \mathcal{E}_+ \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
j^*Q_k & \longrightarrow & j^*Q_{k-1} & \longrightarrow & \cdots & \longrightarrow & j^*Q_1 & \longrightarrow & \alpha
\end{array}$$

commutes and

$$\mathcal{O}_{\widehat{Z}}(-m) \otimes_{\mathbb{C}} \Gamma(S, Q_2(m)) \to \mathcal{O}_{\widehat{Z}}(-m) \otimes_{\mathbb{C}} \Gamma(S, Q_1(m)) \to \mathcal{E}_+$$

is zero. Define $P_i = \ker(\mathcal{O}_S(-m) \otimes_{\mathbb{C}} \Gamma(S, Q_i(m)) \to Q_i)$. If $i < k$ then P_i is an equivariant vector bundle. Moreover, if $k \geq \dim(S) - 1$ then P_k is also locally free. Note that P_k fits into an exact sequence

$$0 \to P_k \to \mathcal{O}_S(-m) \otimes_{\mathbb{C}} \Gamma(S, Q_k(m)) \to Q_{k-1} \to \ldots \to Q_0 \to \alpha \to 0.$$

Now, the two term resolutions

$$0 \to P_i \to \mathcal{O}_S(-m) \otimes_{\mathbb{C}} \Gamma(S, Q_i(m)) \to Q_i \to 0$$

imply that for any quasicoherent sheaf β on S, $\mathcal{E}xt^m(Q_i, \beta) = 0$ if $m > 1$. Therefore

$$\mathcal{E}xt^1(P_k, \beta) \cong \mathcal{E}xt^{k+2}(\alpha, \beta) = 0$$

and P_k is locally free.

Suppose that $\phi : P \to Q(1)$ is an injective equivariant map of equivariant vector bundles on S with the property that $FQ \subset \phi(P)$. Over an invariant affine open, P and Q are graded projective modules. We can define a map in the opposite direction $\psi : Q \to P(1)$ by $\psi(q) = \phi^{-1}(Wq)$. This new map is equivariant and by construction $\phi \circ \psi = W \cdot \mathrm{id}_Q$ and $\psi \circ \phi = W \cdot \mathrm{id}_P$. It is the unique such map and hence all of the local maps patch together to give a map $\psi : Q \to P(1)$. So for each i there is a unique equivariant arrow $\mathcal{O}_S(-m) \otimes_{\mathbb{C}} \Gamma(S, Q_i(m)) \to P_i$ which gives $\mathcal{O}_S(-m) \otimes_{\mathbb{C}} \Gamma(S, Q_i(m)) \oplus P_i[1]$ the structure of a graded D-brane which we denote M_i. Observe that for $1 \leq i < k$, M_i is contractible.

Let $C_{k-1} = \mathrm{cone}(M_k \to M_{k-1})$. Since $M_{i+2} \to M_{i+1} \to M_i$ is the zero map we can inductively define $C_i = \mathrm{cone}(C_{i+1} \to M_i)$. Since M_i is contractible if $i < k$, the natural map $C_i \to C_{i+1}[1]$ is an isomorphism in the homotopy category. Now there is a map $j^*C_1 \to \mathcal{E}$. Consider the cone $C = \mathrm{cone}(j^*C_1 \to \mathcal{E})$.

Let $U' \subset S$ be an invariant affine open set and $U = U' \cap Z$. Then since the functor

$$\mathrm{coker} : H^0 \, \mathrm{DBr}^{\mathrm{gr}}(\mathcal{O}^{gr}_{\widehat{Z}}(U), F) \to D^{gr}_{Sg}(\mathcal{O}^{gr}_{\widehat{Z}}(U)/(F))$$

is triangulated, it follows from the construction of C as an iterated cone that $\mathrm{coker}(C)$ is isomorphic to the acyclic complex

$$0 \to Q_k \to Q_1 \to \cdots \to Q_1 \to \alpha \to 0$$

which is itself isomorphic to zero. Since coker is fully faithful, this implies that $C(U)$ is itself contractible. Now since C is locally contractible it is zero in the homotopy category $H^0 \, \mathrm{DBr}^{\mathrm{gr}}(\widehat{Z}, F)$. This means that j^*C_1 is isomorphic to $\mathcal{E}$ in the homotopy category. Hence j^* is quasi-essentially surjective. □

Remark 6.5. Orlov [Orl09a] has obtained a similar theorem in the case of categories of singularities.

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