THE GARDNER EQUATION AND THE L^2-STABILITY OF THE N-SOLITON SOLUTION OF THE KORTEWEG-DE VRIES EQUATION

MIGUEL A. ALEJO, CLAUDIO MUÑOZ, AND LUIS VEGA

ABSTRACT. Multi-soliton solutions of the Korteweg-de Vries equation (KdV) are shown to be globally L^2-stable, and asymptotically stable in the sense of Martel-Merle [23]. The proof is surprisingly simple and combines the Gardner transform, which links the Gardner and KdV equations, together with the Martel-Merle-Tsai and Martel-Merle recent results on stability and asymptotic stability in the energy space [28, 27], applied this time to the Gardner equation. As a by-product, the results of Maddocks-Sachs [22], and Merle-Vega [29] are improved in several directions.

1. INTRODUCTION AND MAIN RESULTS

In this paper we consider the nonlinear L^2-stability, and asymptotic stability, of the N-*soliton* of the Korteweg-de Vries (KdV) equation

$$u_t + (u_{xx} + u^2)_x = 0. \tag{1.1}$$

Here $u = u(t,x)$ is a real valued function, and $(t,x) \in \mathbb{R}^2$. This equation arises in Physics as a model of propagation of dispersive long waves, as was pointed out by Russel in 1834 [31]. The exact formulation of the KdV equation comes from Korteweg and de Vries (1895) [19]. This equation was studied in a numerical work by Fermi, Pasta and Ulam, and by Kruskal and Zabusky [13, 20].

From the mathematical point of view, equation (1.1) is an *integrable model* [2, 3, 21], with infinitely many conservation laws. Moreover, since the Cauchy problem associated to (1.1) is locally well posed in $L^2(\mathbb{R})$ (cf. [8]), each solution is indeed global in time thanks to the *Mass* conservation

$$M[u](t) := \frac{1}{2}\int_{\mathbb{R}} u^2(t,x)dx = M[u](0). \tag{1.2}$$

Another important conserved quantity, defined for $H^1(\mathbb{R})$-valued solutions, is given by the *Energy*

$$E[u](t) := \frac{1}{2}\int_{\mathbb{R}} u_x^2(t,x)dx - \frac{1}{3}\int_{\mathbb{R}} u^3(t,x)dx = E[u](0). \tag{1.3}$$

On the other hand, equation (1.1) has solitary wave solutions called *solitons*, namely solutions of the form

$$u(t,x) = Q_c(x-ct), \quad Q_c(s) := cQ(\sqrt{c}s), \quad c>0, \tag{1.4}$$

Date: January, 2011.
2000 *Mathematics Subject Classification.* Primary 35Q51, 35Q53; Secondary 37K10, 37K40.
Key words and phrases. KdV equation, Gardner equation, integrability, multi-soliton, stability, asymptotic stability, Miura transform.

and

$$Q(s) := \frac{3}{1+\cosh(s)}. \tag{1.5}$$

The study of perturbations of solitons or solitary waves lead to the introduction of the concepts of *orbital and asymptotic stability*. In particular, since energy and mass are conserved quantities, it is natural to expect that solitons are stable in the energy space $H^1(\mathbb{R})$. Indeed, H^1-stability of KdV solitons has been considered in [6, 7]. On the other hand, the asymptotic stability has been studied e.g. in [35, 23].

Concerning the more involved case of the sum of $N(\geq 2)$ decoupled solitons, stability and asymptotic stability results are very recent. First of all, let us recall that, as a consequence of the integrability property, KdV allows the existence of solutions behaving, as time goes to infinity, as the sum of N decoupled solitons. These solutions are well-known in the literature and are called N*-solitons*, or generically *multi-solitons* [14]. Indeed, any N-soliton solution has the form $u(t,x) := U^{(N)}(x; c_j, x_j - c_j t)$, where

$$\big\{U^{(N)}(x; c_j, y_j) \,:\, c_j > 0,\, y_j \in \mathbb{R},\, j = 1, \dots, N\big\} \tag{1.6}$$

is the family of explicit N-soliton profiles (see e.g. Maddocks-Sachs [22], §3.1). In particular, this solution describes multiple soliton's collisions, but since solitons for KdV equation interact in a linear fashion, there is no residual appearing after the collisions, even if the equation is nonlinear in nature. This is also a consequence of the integrability property.

In [22], the authors considered the $H^N(\mathbb{R})$-stability of the N-soliton solution of KdV, by using N-conservation laws. Their approach strongly invokes the integrability of the KdV equation, and therefore, in order to enlarge the class of perturbations allowed, a more general method was needed. Precisely, in [28, 27], the authors improved the preceding result by proving stability and asymptotic stability of the *sum of N solitons*, *well decoupled* at the initial time, in the energy space. Their proof also applies for general nonlinearities and not only for the integrable cases, provided they have stable solitons, in the sense of Weinstein [39]. Note that the *well-preparedness* restriction on the initial data is by now necessary since there is no satisfactory collision theory for the non-integrable cases.[1] The Martel-Merle-Tsai approach is based on the construction of N *almost conserved quantities*, related to the mass of each solitary wave, plus the total energy of the solution. Further developments on the H^1-stability theory can be found e.g. in [4].

As far as we know, the unique stability result for KdV solitons, below $H^1(\mathbb{R})$, was proved by Merle and Vega in [29]. Precisely, in this work, the authors prove that solitons of (1.1) are L^2-stable, by using the *Miura transform*

$$M[v] := \frac{3}{\sqrt{2}} v_x - \frac{3}{2} v^2, \tag{1.7}$$

which links solutions of the *defocusing, modified KdV* equation,

$$v_t + (v_{xx} - v^3)_x = 0, \quad v = v(t,x) \in \mathbb{R}, \quad (t,x) \in \mathbb{R}^2, \tag{1.8}$$

[1] It turns out that Martel, Merle and the second author of this paper have succeed to describe the collision of two solitons for gKdV equations in some asymptotic regimes and with general nonlinearities beyond the integrable cases, see e.g. [24, 25, 26, 34].

with solutions of the KdV equation (1.1). In particular, the image of the family of *kink* solutions of (1.8) under the transformation (1.7) is the soliton Q_c above described, modulo a standard Galilean transformation (cf. [29]). Since the kink solution of (1.8) is H^1-stable (see e.g. [41, 29]), after a local inversion argument, the authors concluded the L^2-stability of the KdV soliton. Other applications of the Miura transform are local well and ill-posedness results (cf. [17, 10]). However, the stability property in the case of H^s-perturbations, $s \neq 0, 1$ is by now a very difficult and open problem.

The Merle-Vega's idea has been applied to different models describing several phenomena. A similar Miura transform is available for the KP II equation, a two-dimensional generalization of the KdV equation. In this case, the transform has an additional term which takes into account the second variable y. This property has been studied by Wickerhauser in [40], and used by Kenig and Martel in [15] in order to obtain well-posedness results. Finally, Mizumachi and Tzvetkov have shown the stability of solitary waves of KdV, seen as solutions of KP II, under periodic transversal perturbations [33] (see also Section 4 for some additional remarks on this subject). For instability results, see e.g. [36]. Finally, we recall the L^2-stability result for solitary waves of the cubic NLS proved by Mizumachi and Pelinovsky in [32]. Now the proof introduces a *Bäcklund transform* linking the zero and the solitary wave solutions.

A natural question to consider is the generalization of the Merle-Vega's result to the case of multi-soliton solutions. In [37] (see also [12]), the author states that the Miura transform sends *multi-kink* solutions of (1.8) towards a well defined family of *multi-soliton* solutions of (1.1). However, we have found that multi-kinks are hard to manipulate, due to the continuous interaction of non-local terms (recall that a kink does not belong to $L^2(\mathbb{R})$). Therefore we will follow a different approach.

Indeed, in this work we invoke a *Gardner transform* [30, 11], well-known in the mathematical and physical literature since the late sixties, and which links H^1-solutions of the Gardner equation[2]

$$v_t + (v_{xx} + v^2 - \beta v^3)_x = 0, \quad \text{in } \mathbb{R}_t \times \mathbb{R}_x, \quad \beta > 0, \tag{1.9}$$

with L^2-solutions of the KdV equation (1.1). The explicit formula of this transform is given in (1.16). Let us recall that the Gardner equation is also an integrable model [11], with soliton solutions of the form

$$v(t,x) := Q_{c,\beta}(x - ct),$$

and[3]

$$Q_{c,\beta}(s) := \frac{3c}{1 + \rho\cosh(\sqrt{c}s)}, \quad \text{with} \quad \rho := (1 - \frac{9}{2}\beta c)^{1/2}, \quad 0 < c < \frac{2}{9\beta}. \tag{1.10}$$

In particular, in the formal limit $\beta \to 0$, we recover the standard KdV soliton (1.4)-(1.5). On the other hand, the Cauchy problem associated to (1.9) is globally well-posed under initial data in the energy class $H^1(\mathbb{R})$ (cf. [16]), thanks to the mass (1.2) and *energy* conservation laws.

[2] In this part we follow the notation of [34].

[3] See e.g. [9, 34] and references therein for a more detailed description of solitons and integrability for the Gardner equation.

We are interested in the image of the family of solutions (1.10) under the aforementioned, Gardner transform. Surprisingly enough, it turns out that the resulting family is **nothing but** the KdV soliton family (1.4), see (1.17) below. This formally suggests that multi-soliton solutions of the Gardner equation (1.9) are sent towards (or close enough to) multi-soliton solutions of the KdV model (1.1), as is done in [37] for the case of the Miura transform.

In this paper, we profit of this property to improve the H^1-stability and asymptotic stability properties proved by Martel, Merle and Tsai in [28], and Martel and Merle [27], now in the case of L^2-perturbations of the KdV multi-solitons. We first start with the case of an initial datum close enough to the **sum of N decoupled solitons** of the KdV equation. Our result is the following

Theorem 1.1 (L^2-stability of the sum of N solitons of KdV)**.**
Let $N \geq 2$ and $0 < c_1^0 < c_2^0 < \ldots < c_N^0$. There exist parameters $\alpha_0, A_0, L, \gamma > 0$, such that the following holds. Consider $u_0 \in L^2(\mathbb{R})$, and assume that there exist $L > L_0$, $\alpha \in (0, \alpha_0)$ and $x_1^0 < x_2^0 < \ldots < x_N^0$, such that

$$x_j^0 > x_{j-1}^0 + L, \quad \textit{with} \quad j = 2, \ldots, N,$$

and

$$\|u_0 - R_0\|_{L^2(\mathbb{R})} \leq \alpha, \qquad \textit{with} \qquad R_0 := \sum_{j=1}^{N} Q_{c_j^0}(\cdot - x_j^0). \tag{1.11}$$

Then there exist $x_1(t), \ldots x_N(t)$ such that the solution $u(t)$ of the Cauchy problem for the KdV equation (1.1), with initial data u_0, satisfies

(1) Stability.

$$\sup_{t\geq 0} \big\|u(t) - \sum_{j=1}^{N} Q_{c_j^0}(\cdot - x_j(t))\big\|_{L^2(\mathbb{R})} \leq A_0(\alpha + e^{-\gamma_0 L}). \tag{1.12}$$

(2) Asymptotic stability.
There exist $c_j(t) > 0$ and possibly a new set of $x_j(t) \in \mathbb{R}$, $j = 1, \ldots, N$, such that

$$\lim_{t\to+\infty} \big\|u(t) - \sum_{j=1}^{N} Q_{c_j(t)}(\cdot - x_j(t))\big\|_{L^2(x \geq \frac{c_1^0}{10} t)} = 0. \tag{1.13}$$

Moreover, for all $j = 1, \ldots, N$ one has that $\lim_{t\to+\infty} c_j(t) =: c_j^+ > 0$ exists and satisfies

$$\sum_{j=1}^{N} |c_j^+ - c_j^0| \leq K A_0(\alpha + e^{-\gamma_0 L}),$$

for some constant $K > 0$.

Before explaining the main ideas behind the proof of this result, some remarks are in order.

Remarks.
1. Compared with [29], our proof gives an explicit upper bound on the error term (cf. (1.12)). This improvement is related to a *fixed point* argument needed for the proof of an inversion procedure, see Section 2 for more details. For the proof of this result, one requires the parameter $\beta > 0$ in the Gardner equation (1.9) small

enough. However, since the formal limit $\beta \to 0$ in (1.9) is the KdV equation, the Gardner transform (1.16) linking both equations degenerates to the identity and thus does not improve the regularity of the inverse. However, by taking $\alpha > 0$ small, depending on β small, we are able to obtain a still satisfactory bound on the stability (1.12).

2. We do not believe that (1.13) holds in the whole real line $\{x \in \mathbb{R}\}$, e.g. based in the Martel-Merle [23] result. Indeed, they have constructed a solution the KdV equation composed of a big soliton plus an infinite train of small solitons, still satisfying the stability property. This implies that there is no strong convergence in $H^1(\mathbb{R})$ in the general case.

Finally, our last result corresponds to the global L^2-stability and asymptotic stability of the *N-soliton* solution of KdV. It turns out that this result is just a direct corollary of Theorem 1.1 and the uniform continuity of the KdV flow for L^2-data, as it was pointed out in [28], Corollary 1. We include the proof at the end of Section 3, for the sake of completeness.

Corollary 1.2 (L^2-stability and asymptotic stability of the N-soliton of KdV)**.**
Let $\delta > 0$, $N \geq 2$, $0 < c_1^0 < \ldots < c_N^0$ and $x_1^0, \ldots, x_N^0 \in \mathbb{R}$. There exists $\alpha_0 > 0$ such that if $0 < \alpha < \alpha_0$, then the following holds. Let $u(t)$ be a solution of (1.1) such that

$$\|u(0) - U^{(N)}(\cdot; c_j^0, -x_j^0)\|_{L^2(\mathbb{R})} \leq \alpha,$$

with U^N the N-soliton profile described in (1.6). Then there exist $x_j(t)$, $j = 1, \ldots, N$, such that

$$\sup_{t\in\mathbb{R}} \left\|u(t) - U^{(N)}(\cdot; c_j^0, -x_j(t))\right\|_{L^2(\mathbb{R})} \leq \delta. \tag{1.14}$$

Moreover, there exist $c_j^{+\infty} > 0$ such that

$$\lim_{t\to+\infty} \left\|u(t) - U^{(N)}(\cdot; c_j^{+\infty}, -x_j(t))\right\|_{L^2(x > \frac{c_1^0}{10}t)} = 0, \tag{1.15}$$

and $x_j(t)$ are C^1 for all $|t|$ large enough, with $x_j'(t) \to c_j^{+\infty} \sim c_j^0$ as $t \to +\infty$. A similar result holds as $t \to -\infty$, with the obvious modifications.

Remark. Let us emphasize that the proof of this result requires the *existence* and the *explicit form* of the multi-soliton solution of the KdV equation, and therefore the integrable character of the equation. In particular, we do not believe that a similar result is valid for a completely general, non-integrable gKdV equation, unless one considers some perturbative regimes (cf. [24, 26] for some global H^1-stability results in the non-integrable setting.)

Idea of the proofs. Let us explain the main steps of the proofs. We follow the approach introduced in [29]; however, in this opportunity, in order to consider the case of several solitons, we introduce some new ingredients:

1. *The Gardner transform.* First of all, given any $\beta > 0$ and $v(t) \in H^1(\mathbb{R})$, solution of the Gardner equation (1.9), the Gardner transform [11]

$$u(t) = M_\beta[v](t) := [v - \frac{3}{2}\sqrt{2\beta}v_x - \frac{3}{2}\beta v^2](t), \tag{1.16}$$

is an L^2-solution of KdV (in the integral sense).[4] Compared with the original Miura transform (1.7), it has an additional *linear* term which simplifies the proofs. In particular, a direct computation (see Appendix A) shows that for the Gardner soliton solution (1.10), one has

$$\begin{aligned} M_\beta[Q_{c,\beta}](t) &= \big[Q_{c,\beta} - \frac{3}{2}\sqrt{2\beta}Q'_{c,\beta} - \frac{3}{2}\beta Q_{c,\beta}^2\big](x-ct) \\ &= Q_c(x-ct-\delta), \end{aligned} \tag{1.17}$$

with $\delta = \delta(c,\beta) > 0$ provided $\beta > 0$, and Q_c the KdV soliton solution (1.4). In other words, the Gardner transform (1.16) sends the Gardner soliton towards a slightly translated KdV soliton.

2. *Lifting.* Given an initial data u_0 satisfying (1.11), with $\alpha > 0$ small, we solve the Ricatti equation $u_0 = M_\beta[v_0]$ in $H^1(\mathbb{R})$. In addition, we prove that the function v_0 is actually close in $H^1(\mathbb{R})$ to the sum of N-solitons of the Gardner equation. However, for the proof of this result, we do not follow the Merle-Vega approach, which is mainly based in a minimization procedure. Instead, we solve the Ricatti equation by using a *fixed point* argument in a neighborhood of R_0. It turns out that in order to do this, we need to assume that β, the free parameter of the Gardner equation, is small enough, and therefore we require α smaller, depending on β. In any case, and as a by-product, we obtain explicit bounds on the distance of the solution v_0 and the Gardner multi-soliton solution, that one can see in Theorem 1.1. This is done in Section 2.

3. *Conclusion.* Finally, we invoke the H^1-stability theory developed by Martel-Merle-Tsai and Martel-Merle [28, 27], in the particular case of the Gardner equation. The final conclusion follows directly after a new application of the Gardner transform (1.16). This is done in Section 3. Finally, the global character of the stability and asymptotic stability properties follow after a simple continuity argument applied to the N-soliton solution of the KdV equation. This is done at the end of Section 3.

We recall that the proof of Theorem 1.1 does not use the *full integrable character* of (1.1) and (1.9), but only the Gardner transform linking both equations. However, for the proof of Corollary 1.2, we need to work with the N-soliton solution. In addition, we simplify and improve the proof of [29], since the lifting procedure is easier to prove in the case of localized solutions, and we give an explicit bound in the stability result. It is expected that this method may be applied to others models, see Section 4 for more details.

2. Lifting

Let $u_0 \in L^2(\mathbb{R})$ satisfying (1.11). Let us denote by $z_0 := u_0 - R_0$, such that $\|z_0\|_{L^2(\mathbb{R})} \leq \alpha$. In this section, our objective is to solve the nonlinear Ricatti equation

$$M_\beta[v_0] = u_0 = R_0 + z_0, \tag{2.1}$$

with M_β the Gardner transform given by (1.16). We will do that provided α is small enough. In other words, we want to solve the Gardner transform in a neighborhood of the multi-soliton solution R_0. This is the purpose of the following

[4]See Section 4 for additional information about this transform.

Proposition 2.1 (Local invertibility around R_0).

There exists $\beta_0 > 0$ such that, for all $0 < \beta < \beta_0$, the following holds. There exist $K_0, L_0, \gamma_0, \alpha_0 > 0$ such that for all $0 < \alpha < \alpha_0$, $L > L_0$, and $\|z_0\|_{L^2(\mathbb{R})} \leq \alpha$, there exists a solution $v_0 \in H^1(\mathbb{R})$ of (2.1), such that

$$\big\|v_0 - \sum_{j=1}^N Q_{c_j^0,\beta}(\cdot - x_j^0 - \delta_j)\big\|_{H^1(\mathbb{R})} \leq K_0(\frac{\alpha}{\sqrt{\beta}} + e^{-\gamma_0 L}), \tag{2.2}$$

with

$$\delta_j = \delta_j(c_j^0) := (c_j^0)^{-1/2}\cosh^{-1}(\frac{1}{\rho_j}),^{5} \quad \rho_j := (1 - \frac{9}{2}\beta c_j^0)^{1/2}, \quad j = 1, \ldots, N, \tag{2.3}$$

and $Q_{c,\beta}$ being the soliton solution of the Gardner equation (1.9).

Proof. 1. First of all, in what follows we assume $\beta > 0$ small in such a way that $\beta < \frac{2}{9c_N^0}$ and $Q_{c_j^0,\beta}$ is well defined for all $j = 1, \ldots, N$. Let us consider

$$S_0(x) := \sum_{j=1}^N Q_{c_j^0,\beta}(x - x_j^0 - \delta_j),$$

with δ_j defined in (2.3). Let us recall that

$$M_\beta[Q_{c_j^0,\beta}(x - x_j^0 - \delta_j)] = Q_{c_j^0}(x - x_j^0),$$

(cf. Appendix A). A Taylor expansion shows that $\delta_j = O(\beta)$, independent of c_j^0, as β approaches zero. Therefore, in what follows we may suppose that

$$x_j^0 + \delta_j \geq x_{j-1}^0 + \delta_{j-1} + \frac{9}{10}L, \quad j = 2, \ldots, N, \tag{2.4}$$

by taking β small enough.

2. It is clear that $S_0 \in H^1(\mathbb{R})$ with $\|S_0\|_{H^1(\mathbb{R})} \leq K$, independent of β. Moreover, a direct computation, using (1.17) and (2.4), shows that

$$\begin{aligned} M_\beta[S_0](t) &= \sum_{j=1}^N M_\beta[Q_{c_j^0,\beta}(\cdot - x_j^0 - \delta_j)] \\ &\qquad - \frac{3}{2}\beta \sum_{i\neq j} Q_{c_i^0,\beta}(\cdot - x_i^0 - \delta_i) Q_{c_j^0,\beta}(\cdot - x_j^0 - \delta_j) \\ &= \sum_{j=1}^N Q_{c_j^0}(\cdot - x_j^0) - \frac{3}{2}\beta \sum_{i\neq j} Q_{c_i^0,\beta}(\cdot - x_i^0 - \delta_i) Q_{c_j^0,\beta}(\cdot - x_j^0 - \delta_j) \\ &= \sum_{j=1}^N Q_{c_j^0}(\cdot - x_j^0) + O_{L^2(\mathbb{R})}(\beta e^{-\gamma_0 L}) \\ &= R_0 + O_{L^2(\mathbb{R})}(\beta e^{-\gamma_0 L}), \end{aligned} \tag{2.5}$$

for some $\gamma_0 > 0$, independent of β small.

3. Now we look for a solution $v_0 \in H^1(\mathbb{R})$ of (2.1), of the form $v_0 = S_0 + w_0$, and w_0 small in $H^1(\mathbb{R})$. In other words, w_0 has to solve the nonlinear equation

$$\mathcal{L}[w_0] = (R_0 - M_\beta[S_0]) + z_0 + \frac{3}{2}\beta w_0^2, \tag{2.6}$$

[5] We take the positive inverse.

with

$$\mathcal{L}[w_0] := -\frac{3}{2}\sqrt{2\beta}w_{0,x} + (1-3\beta S_0)w_0. \tag{2.7}$$

We may think $\mathcal{L}$ as a unbounded operator in $L^2(\mathbb{R})$, with dense domain $H^1(\mathbb{R})$. From standard energy estimates, one has that for $\beta > 0$ small enough, any solution $w_0 \in H^1(\mathbb{R})$ of the linear problem

$$\mathcal{L}[w_0] = f, \quad f \in L^2(\mathbb{R}), \tag{2.8}$$

must satisfy

$$\|(w_0)_x\|_{L^2(\mathbb{R})} \leq \frac{K}{\sqrt{\beta}}(\|w_0\|_{L^2(\mathbb{R})} + \|f\|_{L^2(\mathbb{R})}),$$

with $K > 0$ independent of β. On the other hand, to obtain a-priori L^2-bounds, note that from the Young inequality and Plancherel,[6]

$$\|\hat{S}_0 \star \hat{w}_0\|_{L^2(\mathbb{R})} \leq \|\hat{S}_0\|_{L^1(\mathbb{R})}\|\hat{w}_0\|_{L^2(\mathbb{R})}.$$

Since S_0 is in the Schwartz class, one has $\hat{S}_0 \in L^1(\mathbb{R})$, with uniform bounds. By taking $\beta > 0$ small and the Fourier transform in (2.8), one has

$$(-\frac{3}{2}i\sqrt{2\beta}\xi + 1)\hat{w}_0(\xi) = \hat{f}(\xi) + O_{L^2(\mathbb{R})}(\beta\hat{w}_0).$$

Therefore, using Plancherel,

$$\|w_0\|_{L^2(\mathbb{R})} \leq K\|f\|_{L^2(\mathbb{R})}.$$

In concluding, one has, for some fixed constant $K_0 > 0$,

$$\|w_0\|_{H^1(\mathbb{R})} \leq \frac{K_0}{\sqrt{\beta}}\|f\|_{L^2(\mathbb{R})}, \tag{2.9}$$

for any $w_0 \in H^1(\mathbb{R})$ solution of (2.8). In order to prove the existence and uniqueness of a solution of (2.8), we use a fixed point approach, in the spirit of [40, 15]. Let us introduce the ball

$$\mathcal{B}_0 := \Big\{ w_0 \in H^1(\mathbb{R}) \ \Big| \ \|w_0\|_{H^1(\mathbb{R})} \leq \frac{K_0}{\sqrt{\beta}}\|f\|_{L^2(\mathbb{R})} \Big\},$$

and the complex operator in the Fourier space,

$$T_0[g](\xi) := \frac{3\beta\hat{S}_0 \star g(\xi) + \hat{f}(\xi)}{1 + \frac{3}{2}i\sqrt{2\beta}\xi}.$$

It is clear that problem (2.8) can be written in Fourier variables as the fixed point problem

$$g = T_0[g], \quad g := \hat{w}_0.$$

By simple inspection one can see that T_0 is a contraction on $\mathcal{B}_0$. Indeed, note that for $w_0 \in \mathcal{B}_0$, $g := \hat{w}_0$,

$$\|T_0[g]\|_{L^2(\mathbb{R})} \leq K(\beta\|g\|_{L^2(\mathbb{R})} + \|f\|_{L^2(\mathbb{R})}) \leq \frac{K_0}{2\sqrt{\beta}}\|f\|_{L^2(\mathbb{R})},$$

and

$$\|\xi T_0[g]\|_{L^2(\mathbb{R})} \leq K(\beta\|\xi g\|_{L^2(\mathbb{R})} + \frac{1}{\sqrt{\beta}}\|f\|_{L^2(\mathbb{R})}) \leq \frac{K_0}{2\sqrt{\beta}}\|f\|_{L^2(\mathbb{R})},$$

by taking K_0 larger. The contraction part works easier. The fixed point theorem gives the existence and uniqueness result.

[6] Here $\hat{}$ denotes the Fourier transform.

In what follows, let us denote by $T := \mathcal{L}^{-1} : L^2(\mathbb{R}) \to H^1(\mathbb{R})$ the resolvent operator constructed in step 3.

4. Finally, from (2.6), we want to solve the nonlinear problem

$$w_0 = T[w_0] = \mathcal{L}^{-1}\big[(R_0 - M_\beta[S_0]) + z_0 + \frac{3}{2}\beta w_0^2\big]. \tag{2.10}$$

In order to use, once again, a fixed point argument, let us introduce the ball

$$\mathcal{B} := \Big\{ w_0 \in H^1(\mathbb{R}) \;\Big|\; \|w_0\|_{H^1(\mathbb{R})} \leq 2K_0(\frac{\alpha}{\sqrt{\beta}} + e^{-\gamma_0 L}) \Big\},$$

with $K_0 > 0$ the constant from (2.9), and $\gamma_0 > 0$ given in (2.5). Let $w_0 \in \mathcal{B}$. Note that, from (2.10), (2.5) and (2.9)

$$\begin{aligned} \|T[w_0]\|_{H^1(\mathbb{R})} &\leq \frac{K_0}{\sqrt{\beta}}[\|R_0 - M_\beta[S_0]\|_{L^2(\mathbb{R})} + \alpha + \beta\|w_0^2\|_{L^2(\mathbb{R})}] \\ &\leq \frac{K_0}{\sqrt{\beta}}[K\beta e^{-\gamma_0 L} + \alpha + 4K_0^2\beta(\frac{\alpha}{\sqrt{\beta}} + e^{-\gamma_0 L})^2] \\ &\leq K_0(K\sqrt{\beta} + KK_0\beta e^{-\gamma_0 L} + KK_0\alpha\sqrt{\beta})e^{-\gamma_0 L} \\ &\qquad + K_0\frac{\alpha}{\sqrt{\beta}}(1 + KK_0\alpha). \end{aligned}$$

By taking β_0 small, and then α_0 smaller if necessary, we can ensure that the above conclusions still hold and therefore

$$\|T[w_0]\|_{H^1(\mathbb{R})} \leq \frac{3}{2}K_0(\frac{\alpha}{\sqrt{\beta}} + e^{-\gamma_0 L}).$$

This proves that $T(\mathcal{B}) \subseteq \mathcal{B}$. In the same way, one can prove that T is a contraction. Indeed, we have for $w_1, w_2 \in \mathcal{B}$,

$$\begin{aligned} \|T[w_1] - T[w_2]\|_{H^1(\mathbb{R})} &\leq K_0\beta\|\mathcal{L}^{-1}[w_1^2 - w_2^2]\|_{H^1(\mathbb{R})} \\ &\leq KK_0(\frac{\alpha}{\sqrt{\beta}} + e^{-\gamma_0 L})\beta\|w_1 - w_2\|_{H^1(\mathbb{R})} \\ &< \frac{1}{2}\|w_1 - w_2\|_{H^1(\mathbb{R})}, \end{aligned}$$

provided β_0 is small enough. Therefore, T is a contraction mapping from $\mathcal{B}$ into itself, and there exists a unique fixed point for T. The proof is now complete. □

3. Proof of the Main Theorems

In this section we prove Theorem 1.1 and Corollary 1.2.

3.1. **Proof of Theorem 1.1.**

1. Let us assume the hypotheses mentioned in the statement of Theorem 1.1, in particular (1.11). From Proposition 2.1, by taking α_0 smaller if necessary, there exist $\beta > 0$ small, and $v_0 \in H^1(\mathbb{R})$, solution of the Ricatti equation (2.1), which satisfies (2.2).

Next, we recall the following H^1-stability result valid for the Gardner equation.

Proposition 3.1 (H^1-stability for Gardner solitons, [28, 27])**.**
Let $0 < c_1^0 < c_2^0 < \ldots < c_N^0 < \frac{2}{9\beta}$ *be such that*

$$\partial_c \int_{\mathbb{R}} Q_{c,\beta}^2 \Big|_{c=c_j} > 0, \quad \textit{for all } j = 1, \ldots, N. \qquad (\textit{Weinstein's criterium.}) \tag{3.1}$$

There exists $\tilde{\alpha}_0, \tilde{A}_0, \tilde{L}_0, \tilde{\gamma} > 0$ *such that the following is true. Let* $v_0 \in H^1(\mathbb{R})$*, and assume that there exists* $\tilde{L} > \tilde{L}_0$*,* $\tilde{\alpha} \in (0, \tilde{\alpha}_0)$ *and* $\tilde{x}_1^0 < \tilde{x}_2^0 < \ldots < \tilde{x}_N^0$*, such that*

$$\|v_0 - \sum_{j=1}^N Q_{c_j^0,\beta}(\cdot - \tilde{x}_j^0)\|_{H^1(\mathbb{R})} \leq \tilde{\alpha}, \tag{3.2}$$

$$\tilde{x}_j^0 > \tilde{x}_{j-1}^0 + \tilde{L}, \quad j = 2, \ldots, N. \tag{3.3}$$

Then there exists $\tilde{x}_1(t), \ldots \tilde{x}_N(t)$ *such that the solution* $v(t)$ *of the Cauchy problem associated to (1.9), with initial data* v_0*, satisfies*

$$v(t) = S(t) + w(t), \quad S(t) := \sum_{j=1}^N Q_{c_j^0,\beta}(\cdot - \tilde{x}_j(t)),$$

and

$$\sup_{t \geq 0} \Big\{ \|w(t)\|_{H^1(\mathbb{R})} + \sum_{j=1}^N |\tilde{x}_j'(t) - c_j| \Big\} \leq \tilde{A}_0(\tilde{\alpha} + e^{-\tilde{\gamma}\tilde{L}}). \tag{3.4}$$

Proof. Although this proof is not present in the literature, it is a direct consequence of [28] (see also Section 5 in [27].) For the proof of (3.1), note that from (1.10)

$$\partial_c \int_{\mathbb{R}} Q_{c,\beta}^2 = \frac{3}{2} c^{1/2} \int_{\mathbb{R}} Q^2 + O(\beta) > 0, \tag{3.5}$$

for β small. See also [5] for the explicit computation. □

2. Since v_0 satisfies (2.2), by taking $\alpha_0 > 0$ smaller and L_0 larger if necessary, we can apply the above Proposition with

$$\tilde{\alpha} := K_0(\frac{\alpha}{\sqrt{\beta}} + e^{-\gamma_0 L}), \quad \tilde{L} := \frac{9}{10} L, \tag{3.6}$$
$$\tilde{x}_j^0 := x_j^0 + \delta_j, \ j = 2, \ldots, N.$$

Therefore, there exist $\tilde{A}_0 > 0$, parameters $\tilde{x}_j(t) \in \mathbb{R}$ and a solution $v(t)$ of (1.9), defined for all $t \geq 0$, and satisfying

$$\sup_{t \geq 0} \big\| v(t) - \sum_{j=1}^N Q_{c_j^0,\beta}(\cdot - \tilde{x}_j(t)) \big\|_{H^1(\mathbb{R})} \leq \tilde{A}_0(\alpha + e^{-\gamma L}), \tag{3.7}$$

for some $\gamma > 0$ and $\tilde{A}_0 = \tilde{A}_0(\beta)$ (note that $\tilde{L}$ and L are of similar size).

Now we are ready to prove the first part of Theorem 1.1.

3. **L^2-stability.** The final steps of the stability proof are similar to those followed in [29]: Let us define

$$\bar{u}(t) := M_\beta[v](t).$$

with M_β given in (1.16). Note that

(1) The initial datum satisfy

$$\bar{u}(0) = M_\beta[v](0) = M_\beta[v_0] = u_0 = R_0 + z_0.$$

(2) $\bar{u}(t)$ is an L^2-solution of the KdV equation (1.1).
(3) From the definition of $M_\beta[v](t)$ and (3.4), one has

$$\begin{aligned}\bar{u}(t) &= M_\beta[S(t)+w(t)]\\ &= M_\beta[S](t)+M_\beta[w](t)-3\beta S(t)w(t).\end{aligned}$$

Let us consider this last term. From (3.7), one has

$$\|M_\beta[w](t)-3\beta S(t)w(t)\|_{L^2(\mathbb{R})}\leq \tilde{A}_0(\alpha+e^{-\gamma L}),$$

and, similarly to (2.5),

$$\begin{aligned}M_\beta[S](t) &= \sum_{j=1}^N M_\beta[Q_{c_j^0,\beta}(\cdot-\tilde{x}_j(t))]\\ &\qquad -\frac{3}{2}\beta\sum_{i\neq j}Q_{c_i^0,\beta}(\cdot-\tilde{x}_i(t))Q_{c_j^0,\beta}(\cdot-\tilde{x}_j(t))\\ &= \sum_{j=1}^N Q_{c_j^0}(\cdot-\tilde{x}_j(t)-\delta_j)+O_{L^2(\mathbb{R})}(\beta e^{-\gamma\tilde{L}})\\ &=: \sum_{j=1}^N Q_{c_j^0}(\cdot-x_j(t))+O_{L^2(\mathbb{R})}(\beta e^{-\gamma L}),\end{aligned}$$

with $x_j(t):=\tilde{x}_j(t)+\delta_j$. Therefore, the final conclusion follows from the uniqueness of $u(t)$, solution of (1.1) with initial data u_0 [8].

4. **Asymptotic stability in $L^2(\mathbb{R})$.** Finally, in this paragraph we prove that solitons are asymptotically stable in $L^2(\mathbb{R})$, in the sense of Martel-Merle, namely estimate (1.13). For this purpose, we recall the following result proved in [27] (see also Remark 3 in that paper).

Proposition 3.2 (Asymptotic stability in $H^1(\mathbb{R})$, [27]).
Suppose that (3.4) holds. Then, there exist $c_j=c_j(t)\in(0,\frac{2}{9\beta})$, and $\rho_j(t)\in\mathbb{R}$, $j=1,\ldots,N$, such that

$$|c_j(t)-c_j^0|+|c_j(t)-\rho_j'(t)|\leq K\tilde{A}_0(\tilde{\alpha}+e^{-\tilde{\gamma}\tilde{L}}),\tag{3.8}$$

and

$$\lim_{t\to+\infty}\Big\|v(t)-\sum_{j=1}^N Q_{c_j,\beta}(\cdot-\rho_j(t))\Big\|_{H^1(x>\frac{c_1^0}{10}t)}=0.\tag{3.9}$$

Moreover, there exist $c_j^+\in(0,\frac{2}{9\beta})$ such that $c_j(t)\to c_j^+$ as $t\to+\infty$, and $j=1,\ldots,N$.

Let us recall that the last conclusion above is a consequence of the fact that the integral $\int_{\mathbb{R}}Q_{c,\beta}^2$ varies with c (see (3.5)), which is a sufficient condition to obtain the convergence of the scaling parameters.

From the above result we can define

$$\tilde{w}(t):=v(t)-\sum_{j=1}^N Q_{c_j,\beta}(\cdot-\rho_j(t)),$$

such that

$$\lim_{t\to+\infty}\|\tilde{w}(t)\|_{H^1(x\geq\frac{1}{10}c_1^0t)}=0.\tag{3.10}$$

Using the Gardner transform, we know that for $\delta_j(t) := \delta_j(c_j(t))$ (cf. Proposition 2.1),

$$\begin{aligned} u(t) &= M_\beta[v](t) \\ &= M_\beta[\sum_{j=1}^N Q_{c_j,\beta}(\cdot - \rho_j(t))] + M_\beta[\tilde w](t) - 3\beta \tilde w(t) \sum_{j=1}^N Q_{c_j,\beta}(\cdot - \rho_j(t)) \\ &= \sum_{j=1}^N Q_{c_j}(\cdot - \rho_j(t) + \delta_j(t)) - 3\beta \sum_{i\neq j}^N Q_{c_i,\beta}(\cdot - \rho_i(t)) Q_{c_j,\beta}(\cdot - \rho_j(t)) \\ &\qquad + M_\beta[\tilde w](t) - 3\beta \tilde w(t) \sum_{j=1}^N Q_{c_j,\beta}(\cdot - \rho_j(t)). \end{aligned}$$

Now, it is clear from (3.8) that

$$\lim_{t\to+\infty} \big\| \sum_{i\neq j}^N Q_{c_i,\beta}(\cdot - \rho_i(t)) Q_{c_j,\beta}(\cdot - \rho_j(t)) \big\|_{L^2(x > \frac{c_1^0}{10} t)} = 0.$$

On the other hand, from (3.10) one has

$$\lim_{t\to+\infty} \big\| M_\beta[\tilde w](t) - 3\beta \tilde w(t) \sum_{j=1}^N Q_{c_j,\beta}(\cdot - \rho_j(t)) \big\|_{L^2(x > \frac{c_1^0}{10} t)} = 0.$$

Finally, by redefining $x_j(t) := \rho_j(t) - \delta_j(t)$, using (3.8) and an argument similar to Step 1 in the proof of Proposition 2.1, we obtain the final conclusion. The proof is complete.

Remark. It is important to stress that the invertibility property above mentioned in Proposition 2.1 depends on β small, and it should be present in the main result, namely Theorem 1.1. In fact, we have chosen α_0 depending on β such that $\tilde\alpha$ in (3.6) is small enough to apply the stability result for the Gardner equation. Therefore, in (1.12) the dependence in β is hidden under the constant A_0.

3.2. **The case of negative times.** One may concern whether the preceding result, valid for positive times, can be extended as in [22], for negative times, or even better, for all time. We have a first, positive answer for this question. Indeed, by using a continuity argument inside the *interaction region* and the explicit multi-soliton solution of the KdV equation, one has the following

Proposition 3.3 (L^2-stability for negative times).

Let $\delta > 0$ fixed. Under the hypotheses of Theorem 1.1, by taking α_0 smaller and L_0 larger if necessary, there exist $\tilde T \geq 0$ and $x_j(t) \in \mathbb{R}$, $j = 1, \ldots, N$, defined for all $|t| \geq \tilde T$, and such that

$$\sup_{t\leq -\tilde T} \big\| u(t) - \sum_{j=1}^N Q_{c_j^0}(\cdot - x_j(t)) \big\|_{L^2(\mathbb{R})} \leq \delta. \tag{3.11}$$

Moreover, as above, the asymptotic stability result (1.13) can be extended as $t \to -\infty$, with the obvious modifications.

Proof of Proposition 3.3.

We use the notation introduced in [24]. Let $\delta > 0$ fixed, and let

$$T := T(x_1^0, \dots, x_N^0; c_1^0, \dots, c_N^0) < 0$$

be the first *interaction time* among the solitons. In particular, for $t \le T$, solitons are well ordered and separated (in terms of their mutual distance L), but with the inverse order compared with case of positive times. Note that this definition depends only on the set $(c_j^0, x_j^0)_{j=1,\dots,N}$. By taking L_0 larger if necessary, one has from (1.11) and the explicit form of $U^{(N)}$,

$$\|u_0 - U^{(N)}(\cdot; c_j^0, -x_j^0)\|_{L^2(\mathbb{R})} \le 2\alpha.$$

Let $\tilde u(t,x) := U^{(N)}(x; c_j^0, -(x_j^0 + c_j^0 t))$ be the N-soliton solution associated to the initial datum $U^{(N)}(x; c_j^0, -x_j^0)$, [22]. From the uniform, continuous dependence on the initial datum in $L^2(\mathbb{R})$ of the KdV equation (cf. [8]), one has that

$$\|u(t) - \tilde u(t)\|_{L^2(\mathbb{R})} \le \delta,$$

for all $t \in [T, 0]$, provided α_0 is chosen small enough. However, from the definition of T and a computation one has that

$$\big\|\tilde u(T) - \sum_{j=1}^N Q_{c_j^0}(\cdot - x_j^+ - c_j^0 T)\big\|_{L^2(\mathbb{R})} \le K e^{-\gamma L},$$

for some $\gamma > 0$, and where $x_j^+ = x_j^+((c_i^0), (x_i^0))$ are the shifts induced by the elastic collision [14]. Note that by definition of T, each soliton is well ordered and separated for $t \le T$ (in the inverse sense compared with $t \ge 0$), and therefore $x_j^+ + c_j^0 T + L \le x_{j-1}^+ + c_{j-1}^0 T$, for all $j = 2, \dots, N$. Therefore, by taking α_0 smaller and L_0 larger if necessary, we can apply Theorem 1.1 backwards in time (just note that $u(-t, -x)$ is also a solution of KdV) to conclude the proof.

□

Remark. We could have used alternatively the H^1-local well-posedness theory given in [18] for the Gardner equation, and then the Gardner transform to obtain a similar result as above.

3.3. **Proof of Corollary 1.2.**

We follow the proof of Corollary 1 in [28]. The proof is also similar to the proof of Proposition 3.3. First note that the N-soliton behaves as the sum of N-soliton as the distance among each soliton diverges. Indeed,

$$\lim_{\inf(y_{j+1} - y_j) \to +\infty} \big\|U^{(N)}(\cdot; c_j^0, -y_j) - \sum_{j=1}^N Q_{c_j^0}(\cdot - y_j)\big\|_{L^2(\mathbb{R})} = 0. \tag{3.12}$$

Let $\delta > 0$ be a small fixed number. For γ_0, A_0, L_0 and α_0 as in the statement of Theorem 1.1, let $\alpha_1 < \alpha_0$, $L > L_0$ be such that $A_0(\alpha_1 + e^{-\gamma_0 L}) < \frac{1}{2}\delta$ and

$$\big\|U^{(N)}(\cdot; c_j^0, -y_j) - \sum_{j=1}^N Q_{c_j^0}(\cdot - y_j)\big\|_{L^2(\mathbb{R})} \le \frac{1}{2}\delta, \tag{3.13}$$

for $y_{j+1} - y_j > L$. We may suppose $A_0 \ge 1$.

Now, let $\tilde{u}(t,x) := U^{(N)}(x; c_j^0, -(x_j^0 + c_j^0 t))$ be the N-soliton solution of (1.1) with initial datum $U^{(N)}(\cdot; c_j^0, -x_j^0)$. Let $\tilde{T} = \tilde{T}(\alpha_1, L) > 0$ be such that, for all $t \geq \tilde{T}$,

$$\big\|\tilde{u}(t) - \sum_{j=1}^N Q_{c_j^0}(\cdot - (x_j^0 + c_j^0 t))\big\|_{L^2(\mathbb{R})} \leq \frac{1}{2}\alpha_1, \tag{3.14}$$

and for all j, $x_{j+1}^0 + c_{j+1}^0 \tilde{T} \geq x_j^0 + c_j^0 \tilde{T} + 2L$.

Therefore, by the uniform continuous dependence of the solution of (1.1) with respect to the initial datum in $L^2(\mathbb{R})$ (see [8]), there exists $\alpha > 0$ such that if $\|u(0) - \tilde{u}(0)\|_{L^2(\mathbb{R})} \leq \alpha$, then for all $t \in [0, \tilde{T}]$, one has $\|u(t) - \tilde{u}(t)\|_{L^2(\mathbb{R})} \leq \frac{1}{2}\alpha_1 < \delta$. In particular, by (3.14),

$$\big\|u(\tilde{T}) - \sum_{j=1}^N Q_{c_j^0}(\cdot - (x_j^0 + c_j^0 \tilde{T}))\big\|_{L^2(\mathbb{R})} \leq \alpha_1 < \alpha_0.$$

Thus, by Theorem 1.1, there exist $x_j(t)$, such that for all $t \geq \tilde{T}$,

$$\big\|u(t) - \sum_{j=1}^N Q_{c_j^0}(\cdot - x_j(t))\big\|_{L^2(\mathbb{R})} \leq A_0(\alpha_1 + e^{-\gamma_0 L}) < \frac{1}{2}\delta.$$

Moreover, $x_{j+1}(t) > x_j(t) + L$. Together with (3.13), this gives the stability result for positive times $t \geq \tilde{T}$.

Next, taking α smaller if necessary, we can use a similar argument to that in the proof of Proposition 3.3 to extend the stability result backwards in time, until before the first collision time. Finally, we extend the result for all negative times by using again Theorem 1.1.

Finally, the asymptotic stability result follows from (3.12) and the second part of Theorem 1.1. The proof is complete.

4. Final remarks

In this last section we would like to stress some points which are of independent interest.

1. On the relationship between the Miura and Gardner transforms. There is a second way to see the transform (1.16), which uses the standard Miura transform M, considered in (1.7). This way is probably not new in the literature, so we recall it by completeness purposes.

Indeed, let $v \in H^1(\mathbb{R})$ be a solution of (1.9). Then, the auxiliary function

$$\tilde{v}(t,x) := \frac{1}{3\sqrt{\beta}} - \sqrt{\beta}\, v\big(t, x + \frac{t}{3\beta}\big)$$

solves the mKdV equation (1.8). Note that $\tilde{v}$ is a L^∞-function with nonzero limits at infinity. Next, $M[\tilde{v}]$ is a solution of (1.1), with nonzero limit at infinity. Using the fact that (1.1) is *Galilean invariant*, one has that

$$u(t,x) = \frac{1}{6\beta} + M[v]\big(t, x - \frac{t}{3\beta}\big),$$

is also an $L^2(\mathbb{R})$-solution of KdV. Finally, one can easily check that the composition of these applications gives (1.16). See e.g. [5] for further applications of this transform.

2. The KP II model.[7] It is also interesting to stress that, modulo a constant transformation on the scaling, the KdV soliton $Q_c(x-ct)$, defined in (1.5), and seen as a two-variable function of x and y, is a non-localized solution of the KP II equation

$$\begin{aligned} &u_t + (u_{xx} + 3u^2)_x + 3v_y = 0, \quad \text{in} \quad \mathbb{R}_t \times \mathbb{R}^2_{x,y}, \\ &u = u(t,x,y), \quad v := \partial_x^{-1} u_y. \end{aligned} \tag{4.1}$$

In [33], Mizumachi and Tzvetkov follow the idea of Merle and Vega and perform a Miura transform to show that Q_c is stable under small perturbations in the space $L^2(\mathbb{R}_x \times \mathbb{T}_y)$, where here $\mathbb{T}_y$ is the one-dimensional torus in the y-variable. The pivot equation is now the integrable model called *modified* KP II equation, which is given by

$$u_t + (u_{xx} - 2u^3)_x + 3v_y + 6u_x v = 0, \quad \text{in} \quad \mathbb{R}_t \times \mathbb{R}^2_{x,y},$$

with v defined in (4.1). Note in addition that, after a standard scaling modification, the one-variable kink solution of the mKdV equation (1.8) is also an admissible solution of this last equation (seen as a two-variable function).

We believe that, using the methods developed in this paper, the stability – under periodic transversal perturbations– of the KdV multi-soliton $U^{(N)}$, seen as a solution of (4.1), constant in the y-variable, can be handled via a Gardner transform pointing this time to the *integrable*, Gardner generalization of KP II, namely

$$\begin{aligned} &\tilde{u}_t + (\tilde{u}_{xx} + 3\tilde{u}^2 - 2\tilde{u}^3)_x + 3\tilde{v}_y + 6\tilde{u}_x \tilde{v} = 0, \quad \text{in} \quad \mathbb{R}_t \times \mathbb{R}^2_{x,y}, \\ &\tilde{u} = \tilde{u}(t,x,y), \quad \tilde{v} := \partial_x^{-1} \tilde{u}_y, \end{aligned}$$

for which a scaled version of the Gardner soliton (1.10), seen as a constant function in the y-variable, is a simple solution. The Gardner-KP II transform is given in this case by the simple formula

$$\tilde{M}[\tilde{u}] := \tilde{u} + \tilde{u}_x + \tilde{v} - \tilde{u}^2.$$

Appendix A. Proof of (1.17)

In this small paragraph we prove, for the sake of completeness, that the Gardner transform sends solitons of the Gardner equation towards translated solitons of KdV, namely the identity (1.17). Let us recall that $\rho = (1 - \frac{9}{2}\beta c)^{1/2}$. Indeed, note

[7] In this paragraph we follow the notation of [33].

that by (1.16) and (1.10),

$$\begin{aligned}
&M_\beta[Q_{c,\beta}](t,x) = \\
&= \big[Q_{c,\beta} - \frac{3}{2}\sqrt{2\beta}Q'_{c,\beta} - \frac{3}{2}\beta Q^2_{c,\beta}\big](x-ct) \\
&= \Big[\frac{3c}{1+\rho\cosh(\sqrt{c}s)} + \frac{9}{2}\sqrt{2\beta}c^{3/2}\rho\frac{\sinh(\sqrt{c}s)}{(1+\rho\cosh(\sqrt{c}s))^2} \\
&\qquad - \frac{27\beta c^2}{2(1+\rho\cosh(\sqrt{c}s))^2}\Big]\Big|_{s=x-ct} \\
&= \frac{3c\rho}{(1+\rho\cosh(\sqrt{c}s))^2}\big[\rho + \cosh(\sqrt{c}s) + \frac{3}{2}\sqrt{2\beta c}\sinh(\sqrt{c}s)\big]\Big|_{s=x-ct}.
\end{aligned}$$

Now, let us note that for $\beta > 0$ one has $\rho < 1$ and therefore $\delta := \frac{1}{\sqrt{c}}\cosh^{-1}(\frac{1}{\rho}) > 0$ is a well defined quantity, provided we take e.g. the positive inverse of cosh. Note that the shift in the KdV soliton is *always* present since $\beta > 0$. Moreover, with this choice one has

$$\cosh(\sqrt{c}\delta) = \frac{1}{\rho}, \qquad \sinh(\sqrt{c}\delta) = \frac{1}{\rho}\sqrt{1-\rho^2} = \frac{3}{2\rho}\sqrt{2\beta c} > 0. \tag{A.1}$$

We replace these identities above, to obtain

$$\begin{aligned}
&M_\beta[Q_{c,\beta}](t,x) = \\
&= \frac{3c}{\cosh^2(\sqrt{c}\delta)\big[1+\frac{\cosh(\sqrt{c}s)}{\cosh(\sqrt{c}\delta)}\big]^2} \times \\
&\qquad \times\big[1 + \cosh(\sqrt{c}\delta)\cosh(\sqrt{c}s) + \sinh(\sqrt{c}\delta)\sinh(\sqrt{c}s)\big]\Big|_{s=x-ct} \\
&= 3c\frac{(1 + \cosh(\sqrt{c}\delta)\cosh(\sqrt{c}s) + \sinh(\sqrt{c}\delta)\sinh(\sqrt{c}s))}{1+\sinh^2(\sqrt{c}\delta) + \cosh^2(\sqrt{c}s) + 2\cosh(\sqrt{c}\delta)\cosh(\sqrt{c}s)}\Big|_{s=x-ct}.
\end{aligned} \tag{A.2}$$

Note that

$$\begin{aligned}
&1 + \cosh(\sqrt{c}\delta)\cosh(\sqrt{c}(x-ct)) + \sinh(\sqrt{c}\delta)\sinh(\sqrt{c}(x-ct)) \\
&\qquad = 1 + \cosh(\sqrt{c}(x-ct+\delta)) > 0,
\end{aligned} \tag{A.3}$$

and

$$\begin{aligned}
&(1 + \cosh(\sqrt{c}\delta)\cosh(\sqrt{c}(x-ct)) + \sinh(\sqrt{c}\delta)\sinh(\sqrt{c}(x-ct))) \times \\
&\qquad \times(1 + \cosh(\sqrt{c}\delta)\cosh(\sqrt{c}(x-ct)) - \sinh(\sqrt{c}\delta)\sinh(\sqrt{c}(x-ct))) = \\
&= (1 + \cosh(\sqrt{c}\delta)\cosh(\sqrt{c}(x-ct)))^2 - \sinh^2(\sqrt{c}\delta)\sinh^2(\sqrt{c}(x-ct))) \\
&= 1 + 2\cosh(\sqrt{c}\delta)\cosh(\sqrt{c}(x-ct)) + \cosh^2(\sqrt{c}\delta)\cosh^2(\sqrt{c}(x-ct)) \\
&\qquad - \sinh^2(\sqrt{c}\delta)\sinh^2(\sqrt{c}(x-ct))) \\
&= 1 + 2\cosh(\sqrt{c}\delta)\cosh(\sqrt{c}(x-ct)) + \cosh^2(\sqrt{c}(x-ct)) + \sinh^2(\sqrt{c}\delta),
\end{aligned}$$

which is the denominator in (A.2). Therefore, from (A.3) we can simplify this term to obtain

$$\begin{aligned} &M_\beta[Q_{c,\beta}](t,x) = \\ &= \frac{3c}{1+\cosh(\sqrt{c}\delta)\cosh(\sqrt{c}(x-ct)) - \sinh(\sqrt{c}\delta)\sinh(\sqrt{c}(x-ct))} \\ &= \frac{3c}{1+\cosh(\sqrt{c}(x-ct-\delta))} \\ &= Q_c(x-ct-\delta), \end{aligned}$$

as desired (cf. (1.4)-(1.5)). An a-posteriori analysis shows that the final result is independent of the sign chosen for δ, provided $\sinh(\sqrt{c}\delta)$ is chosen negative in (A.1).

Acknowdlegments. We would like to thank Yvan Martel and Frank Merle for several and useful comments on a first version of this paper, and Y. Martel for the link with the KP II equation. We also thank the referee for some important remarks which helped to improve the quality of this paper.

References

[1] M. Ablowitz, and P. Clarkson, *Solitons, nonlinear evolution equations and inverse scattering*, London Mathematical Society Lecture Note Series, 149. Cambridge University Press, Cambridge, 1991.

[2] M. Ablowitz, D. Kaup, A. Newell, and H. Segur, *The inverse scattering transform-Fourier analysis for nonlinear problems*, Studies in Appl. Math. 53 (1974), no. 4, 249–315.

[3] M. Ablowitz, and H. Segur, *Solitons and the inverse scattering transform*, SIAM Studies in Applied Mathematics, 4. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, Pa., 1981. x+425 pp.

[4] J. Albert, J. Bona, and N. Nguyen, *On the stability of KdV multisolitons*, Differential Integral Equations **20** (2007), no. 8, 841–878.

[5] Alejo, Miguel A., *The geometric modified Korteweg-de Vries(mKdV) equation*, Unpublished PhD thesis, University of the Basque Country (2010).

[6] T.B. Benjamin, *The stability of solitary waves*, Proc. Roy. Soc. London A **328**, (1972) 153–183.

[7] J.L. Bona, P. Souganidis and W. Strauss, *Stability and instability of solitary waves of Korteweg-de Vries type*, Proc. Roy. Soc. London **411** (1987), 395–412.

[8] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. II. The KdV equation*, Geom. Funct. Anal. **3** (1993), no. 3, 209-262.

[9] K.W. Chow, R.H.J Grimshaw, and E. Ding, *Interactions of breathers and solitons in the extended Korteweg-de Vries equation*, Wave Motion **43** (2005) 158–166.

[10] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T.Tao, *Sharp global well-posedness for KdV and modified KdV on* $\mathbb{R}$ *and* $\mathbb{T}$, J. Amer. Math. Soc. **16** (2003), no. 3, 705–749 (electronic).

[11] C.S. Gardner, M.D. Kruskal, and R. Miura, *Korteweg-de Vries equation and generalizations. II. Existence of conservation laws and constants of motion*, J. Math. Phys. **9**, no. 8 (1968), 1204–1209.

[12] F. Gesztesy, and B. Simon, *Constructing solutions of the mKdV-equation*, J. Funct. Anal. **89** (1990), no. 1, 53–60.

[13] E. Fermi, J. Pasta and S. Ulam, *Studies of nonlinear problems I*, Los Alamos Report LA1940 (1955); reproduced in Nonlinear Wave Motion, A.C. Newell, ed., Am. Math. Soc., Providence, R. I., 1974, pp. 143–156.

[14] R. Hirota, *Exact solution of the Korteweg-de Vries equation for multiple collisions of solitons*, Phys. Rev. Lett., **27** (1971), 1192–1194.

[15] C.E. Kenig, and Y. Martel, *Global well-posedness in the energy space for a modified KP II equation via the Miura transform*, Trans. Amer. Math. Soc. **358** no. 6, pp. 2447–2488.

[16] C.E. Kenig, G. Ponce and L. Vega, *Well-posedness and scattering results for the generalized Korteweg–de Vries equation via the contraction principle*, Comm. Pure Appl. Math. **46**, (1993) 527–620.

[17] C.E. Kenig, G. Ponce and L. Vega, *On the ill-posedness of some canonical dispersive equations*, Duke Math. J. **106** (2001), no. 3, 617–633.

[18] C.E. Kenig, G. Ponce and L. Vega, *A bilinear estimate with applications to the KdV equation*, J. Amer. Math. Soc. **9** (1996), no. 2, 573–603.

[19] D.J. Korteweg and G. de Vries, *On the change of form of long waves advancing in a rectangular canal, and on a new type of stationary waves*, Philos. Mag. Ser. 5, **39** (1895), 422–443.

[20] M.D. Kruskal and N.J. Zabusky, *Interaction of "solitons" in a collisionless plasma and recurrence of initial states*, Phys. Rev. Lett. **15** (1965), 240–243.

[21] P. D. Lax, *Integrals of nonlinear equations of evolution and solitary waves*, Comm. Pure Appl. Math. **21**, (1968) 467–490.

[22] J.H. Maddocks, and R.L. Sachs, *On the stability of KdV multi-solitons*, Comm. Pure Appl. Math. **46**, 867–901 (1993).

[23] Y. Martel and F. Merle, *Asymptotic stability of solitons of the subcritical gKdV equations revisited*, Nonlinearity **18** (2005) 55–80.

[24] Y. Martel and F. Merle, *Description of two soliton collision for the quartic gKdV equation*, preprint arXiv:0709.2672 (2007).

[25] Y. Martel and F. Merle, *Stability of two soliton collision for nonintegrable gKdV equations*, Comm. Math. Phys. **286** (2009), 39–79.

[26] Y. Martel and F. Merle, *Inelastic interaction of nearly equal solitons for the quartic gKdV equation*, to appear in Inventiones Mathematicae.

[27] Y. Martel, and F. Merle, *Asymptotic stability of solitons of the gKdV equations with general nonlinearity*, Math. Ann. **341** (2008), no. 2, 391–427.

[28] Y. Martel, F. Merle and T. P. Tsai, *Stability and asymptotic stability in the energy space of the sum of N solitons for subcritical gKdV equations*, Comm. Math. Phys. **231** (2002) 347–373.

[29] F. Merle; and L. Vega, *L^2 stability of solitons for KdV equation*, Int. Math. Res. Not. 2003, no. 13, 735–753.

[30] R.M. Miura, *Korteweg-de Vries equation and generalizations. I. A remarkable explicit nonlinear transformation*, J. Math. Phys. **9**, no. 8 (1968), 1202–1204.

[31] R.M. Miura, *The Korteweg–de Vries equation: a survey of results*, SIAM Review **18**, (1976) 412–459.

[32] T. Mizumachi, and D. Pelinovsky, *Bäcklund transformation and L^2-stability of NLS solitons*, preprint.

[33] T. Mizumachi, and N. Tzvetkov, *Stability of the line soliton of the KP–II equation under periodic transverse perturbations*, preprint.

[34] C. Muñoz, *On the inelastic 2-soliton collision for gKdV equations with general nonlinearity*, Int. Math. Research Notices (2010) **2010** (9): 1624–1719.

[35] R.L. Pego, and M.I. Weinstein, *Asymptotic stability of solitary waves*, Commun. Math. Phys. **164**, 305–349 (1994).

[36] F. Rousset, and N. Tzvetkov, *Transverse nonlinear instability for two-dimensional dispersive models*, Ann. Inst. H. Poincaré Anal. Non Linéaire **26** (2009), no. 2, 477–496.

[37] B. Thaller, *The Dirac equation*, Texts and Monographs in Physics. Springer-Verlag, Berlin, 1992. xviii+357 pp.

[38] A.-M. Wazwaz, *New soliton solutions for the Gardner equation*, Comm. Nonlinear Sc. Num. Sim. **12**, (2007) 1395–1404.

[39] M.I. Weinstein, *Lyapunov stability of ground states of nonlinear dispersive evolution equations*, Comm. Pure. Appl. Math. **39**, (1986) 51—68.

[40] M.V. Wickerhauser, *Inverse scattering for the heat operator and evolutions in* $2+1$ *variables,* Comm. Math. Phys. **108** (1987), 67–89.

[41] P. E. Zhidkov, *Korteweg-de Vries and Nonlinear Schrödinger Equations: Qualitative Theory*, Lecture Notes in Mathematics, vol. 1756, Springer-Verlag, Berlin, 2001.

Departamento de Matemáticas, Facultad de Ciencia y Tecnología, Universidad del País Vasco, Bilbao, España

E-mail address: miguelangel.alejo@ehu.es

E-mail address: Claudio.Munoz@math.uvsq.fr

E-mail address: luis.vega@ehu.es