

# A SHORT PROOF OF THE DOOB-MEYER THEOREM

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ABSTRACT. Every submartingale  $S$  of class  $D$  has a unique Doob-Meyer decomposition  $S = M + A$ , where  $M$  is a martingale and  $A$  is a predictable increasing process starting at 0.

We provide a short and elementary prove of the Doob-Meyer decomposition theorem. Several previously known arguments are included to keep the paper self-contained.

## 1. INTRODUCTION

Throughout this article we fix a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and a right-continuous complete filtration  $(\mathcal{F}_t)_{0 \leq t \leq T}$ .

An adapted process  $(S_t)_{0 \leq t \leq T}$  is of class  $D$  if the family of random variables  $S_\tau$  where  $\tau$  ranges through all stopping times is uniformly integrable ([Mey62]).

The purpose of this paper is to give a short and elementary proof of the following

**Theorem 1.1** (Doob-Meyer). *Let  $S = (S_t)_{0 \leq t \leq T}$  be a càdlàg submartingale of class  $D$ . Then,  $S$  can be written in a unique way in the form*

$$(1) \quad S = M + A$$

where  $M$  is a martingale and  $A$  is a predictable increasing process starting at 0.

Doob [Doo53] noticed that in discrete time an integrable process  $S = (S_n)_{n=1}^\infty$  can be uniquely represented as the sum of a martingale  $M$  and a predictable process  $A$  starting at 0; in addition, the process  $A$  is increasing iff  $S$  is a submartingale. The continuous time analogue, Theorem 1.1, goes back to Meyer [Mey62, Mey63], who introduced the class  $D$  and proved that every submartingale  $S = (S_t)_{0 \leq t \leq T}$  can be decomposed in the form (1), where  $M$  is a martingale and  $A$  is a *natural* process. The modern formulation is due to Doléans-Dade [DD67, DD68] who obtained that an increasing process is natural iff it is predictable. Further proofs of Theorem 1.1 were given by Rao [Rao69], Bass [Bas96] and Jakubowski [Jak05].

Rao works with the  $\sigma(L^1, L^\infty)$ -topology and applies the Dunford-Pettis compactness criterion to obtain the desired continuous time decomposition as a weak- $L^1$  limit from discrete approximations. To obtain that  $A$  is predictable one then invokes the theorem of Doléans-Dade.

Bass gives a more elementary proof based on the dichotomy between predictable and totally inaccessible stopping times.

Jakubowski proceeds as Rao, but notices that predictability of the process  $A$  can also be obtained through an application of Komlos' Lemma [Kom67].

The proof presented subsequently combines ideas from [Jak05] and [BSV10] to construct the continuous time decomposition using a suitable Komlos-type lemma.

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## 2. PROOF OF THEOREM 1.1

The proof of uniqueness is standard and we have nothing to add here; see for instance [Kal02, Lemma 25.11].

For the remainder of this article we work under the assumptions of Theorem 1.1 and fix  $T = 1$  for simplicity.

Denote by  $\mathcal{D}_n$  and  $\mathcal{D}$  the set of  $n$ -th resp. all dyadic numbers  $j/2^n$  in the interval  $[0, 1]$ . For each  $n$ , we consider the discrete time Doob decomposition of the sampled process  $S^n = (S_t)_{t \in \mathcal{D}_n}$ , that is, we define  $A^n, M^n$  by  $A_0^n := 0$ ,

$$(2) \quad A_t^n - A_{t-1/2^n}^n := \mathbb{E}[S_t - S_{t-1/2^n} | \mathcal{F}_{t-1/2^n}] \text{ and}$$

$$(3) \quad M_t^n := S_t - A_t^n$$

so that  $(M_t^n)_{t \in \mathcal{D}_n}$  is a martingale and  $(A_t^n)_{t \in \mathcal{D}_n}$  is predictable with respect to  $(\mathcal{F}_t)_{t \in \mathcal{D}_n}$ .

The idea of the proof is, of course, to obtain the continuous time decomposition (1) as a limit, or rather, as an accumulation point of the processes  $M^n, A^n, n \geq 1$ .

Clearly, in infinite dimensional spaces a (bounded) sequence need not have a convergent subsequence. As a substitute for the Bolzano-Weierstrass Theorem we establish the Komlos-type Lemma 2.1 in Section 2.1.

In order to apply this auxiliary result, we require that the sequence  $(M_1^n)_{n \geq 1}$  is uniformly integrable. This follows from the class  $D$  assumption as shown by [Rao69]. To keep the paper self-contained, we provide a proof in Section 2.2.

Finally, in Section 2.3, we obtain the desired decomposition by passing to a limit of the discrete time versions. As the Komlos-approach guarantees convergence in a strong sense, predictability of the process  $A$  follows rather directly from the predictability of the approximating processes. This idea is taken from [Jak05].

**2.1. Komlos' Lemma.** Following Komlos [Kom67]<sup>1</sup>, it is sometimes possible to obtain an accumulation point of a bounded sequence in an infinite dimensional space if appropriate convex combinations are taken into account.

A particularly simple result of this kind holds true if  $(f_n)_{n \geq 1}$  is a bounded sequence in a Hilbert space. In this case

$$A = \sup_{n \geq 1} \inf \{ \|g\|_2 : g \in \text{conv}\{f_n, f_{n+1}, \dots\} \}$$

is finite and for each  $n$  we may pick some  $g_n \in \text{conv}\{f_n, f_{n+1}, \dots\}$  such that  $\|g_n\|_2 \leq A + 1/n$ . If  $n$  is sufficiently large with respect to  $\varepsilon > 0$ , then  $\|(g_k + g_m)/2\|_2 > A - \varepsilon$  for all  $m, k \geq n$  and hence

$$\|g_k - g_m\|_2^2 = 2\|g_k\|_2^2 + 2\|g_m\|_2^2 - \|g_k + g_m\|_2^2 \leq 4(A + \frac{1}{n})^2 - 4(A - \varepsilon)^2.$$

By completeness,  $(g_n)_{n \geq 1}$  converges in  $\|\cdot\|_2$ .

By a straight forward truncation procedure this Hilbertian Komlos-Lemma yields an  $L^1$ -version which we will need subsequently.<sup>2</sup>

**Lemma 2.1.** *Let  $(f_n)_{n \geq 1}$  be a uniformly integrable sequence of functions on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . Then there exist functions  $g_n \in \text{conv}\{f_n, f_{n+1}, \dots\}$  such that  $(g_n)_{n \geq 1}$  converges in  $\|\cdot\|_{L^1(\Omega)}$ .*

*Proof.* For  $i, n \in \mathbb{N}$  set  $f_n^{(i)} := f_n \mathbb{1}_{\{|f_n| \leq i\}}$  such that  $f_n^{(i)} \in L^2(\Omega)$ .

We claim that there exist for every  $n$  convex weights  $\lambda_n^n, \dots, \lambda_{N_n}^n$  such that the functions  $\lambda_n^n f_n^{(i)} + \dots + \lambda_{N_n}^n f_{N_n}^{(i)}$  converge in  $L^2(\Omega)$  for every  $i \in \mathbb{N}$ .

<sup>1</sup>Indeed, [Kom67] considers Cesaro sums along subsequences rather than arbitrary convex combinations. But for our purposes, the more modest conclusion of Lemma 2.1 is sufficient.

<sup>2</sup>Lemma 2.1 is also a trivial consequence of Komlos' original result [Kom67] or other related results that have been established through the years. Cf. [KS09, Chapter 5.2] for an overview.

To see this, one first uses the Hilbertian lemma to find convex weights  $\lambda_n^n, \dots, \lambda_{N_n}^n$  such that  $(\lambda_n^n f_n^{(1)} + \dots + \lambda_{N_n}^n f_{N_n}^{(1)})_{n \geq 1}$  converges. In the second step, one applies the lemma to the sequence  $(\lambda_n^n f_n^{(2)} + \dots + \lambda_{N_n}^n f_{N_n}^{(2)})_{n \geq 1}$ , to obtain convex weights which work for the first two sequences. Repeating this procedure inductively we obtain sequences of convex weights which work for the first  $m$  sequences. Then a standard diagonalization argument yields the claim.

By uniform integrability,  $\lim_{i \rightarrow \infty} \|f_n^{(i)} - f_n\|_1 = 0$ , uniformly with respect to  $n$ . Hence, once again, uniformly with respect to  $n$ ,

$$\lim_{i \rightarrow \infty} \|(\lambda_n^n f_n^{(i)} + \dots + \lambda_{N_n}^n f_{N_n}^{(i)}) - (\lambda_n^n f_n + \dots + \lambda_{N_n}^n f_{N_n})\|_1 = 0.$$

Thus  $(\lambda_n^n f_n + \dots + \lambda_{N_n}^n f_{N_n})_{n \geq 1}$  is a Cauchy sequence in  $L^1(\Omega)$ .  $\square$

## 2.2. Uniform integrability of the discrete approximations.

**Lemma 2.2.** *The sequence  $(M_1^n)_{n \geq 1}$  is uniformly integrable.*

*Proof.* Subtracting  $\mathbb{E}[S_1 | \mathcal{F}_t]$  from  $S_t$  we may assume that  $S_1 = 0$  and  $S_t \leq 0$  for all  $0 \leq t \leq 1$ . Then  $M_1^n = -A_1^n$ , and for every  $(\mathcal{F}_t)_{t \in \mathcal{D}_n}$ -stopping time  $\tau$

$$(4) \quad S_\tau^n = -\mathbb{E}[A_1^n | \mathcal{F}_\tau] + A_\tau^n.$$

We claim that  $(A_1^n)_{n=1}^\infty$  is uniformly integrable. For  $c > 0$ ,  $n \geq 1$  define

$$\tau_n(c) = \inf \{ (j-1)/2^n : A_{j/2^n}^n > c \} \wedge 1.$$

From  $A_{\tau_n(c)}^n \leq c$  and (4) we obtain  $S_{\tau_n(c)} \leq -\mathbb{E}[A_1^n | \mathcal{F}_{\tau_n(c)}] + c$ . Thus,

$$\int_{\{A_1^n > c\}} A_1^n d\mathbb{P} = \int_{\{\tau_n(c) < 1\}} \mathbb{E}[A_1^n | \mathcal{F}_{\tau_n(c)}] d\mathbb{P} \leq c \mathbb{P}[\tau_n(c) < 1] - \int_{\{\tau_n(c) < 1\}} S_{\tau_n(c)} d\mathbb{P}.$$

Note  $\{\tau_n(c) < 1\} \subseteq \{\tau_n(\frac{c}{2}) < 1\}$ , hence, by (4)

$$\begin{aligned} \int_{\{\tau_n(\frac{c}{2}) < 1\}} -S_{\tau_n(\frac{c}{2})} d\mathbb{P} &= \int_{\{\tau_n(\frac{c}{2}) < 1\}} A_1^n - A_{\tau_n(\frac{c}{2})}^n d\mathbb{P} \\ &\geq \int_{\{\tau_n(c) < 1\}} A_1^n - A_{\tau_n(\frac{c}{2})}^n d\mathbb{P} \geq \frac{c}{2} \mathbb{P}[\tau_n(c) < 1]. \end{aligned}$$

Combining the above inequalities we obtain

$$(5) \quad \int_{\{A_1^n > c\}} A_1^n d\mathbb{P} \leq -2 \int_{\{\tau_n(\frac{c}{2}) < 1\}} S_{\tau_n(\frac{c}{2})} d\mathbb{P} - \int_{\{\tau_n(c) < 1\}} S_{\tau_n(c)} d\mathbb{P}.$$

On the other hand

$$\mathbb{P}[\tau_n(c) < 1] = \mathbb{P}[A_1^n > c] \leq \mathbb{E}[A_1^n] / c = -\mathbb{E}[M_1^n] / c = -\mathbb{E}[S_0] / c,$$

hence, as  $c \rightarrow \infty$ ,  $\mathbb{P}[\tau_n(c) < 1]$  goes to 0, uniformly in  $n$ . As  $S$  is of class  $D$ , (5) implies that the sequence  $(A_1^n)_{n \geq 1}$  is uniformly integrable and hence  $(M_1^n)_{n \geq 1} = (S_1 - A_1^n)_{n \geq 1}$  is uniformly integrable as well.  $\square$

**2.3. The limiting procedure.** For each  $n$ , extend  $M^n$  to a (càdlàg) martingale on  $[0, 1]$  by setting  $M_t^n := \mathbb{E}[M_1^n | \mathcal{F}_t]$ . By Lemma 2.1 and Lemma 2.2 there exist  $M \in L^1(\Omega)$  and for each  $n$  convex weights  $\lambda_n^n, \dots, \lambda_{N_n}^n$  such that with

$$(6) \quad \mathcal{M}^n := \lambda_n^n M^n + \dots + \lambda_{N_n}^n M^{N_n}$$

we have  $\mathcal{M}_1^n \rightarrow M$  in  $L^1(\Omega)$ . Then, by Jensen's inequality,  $\mathcal{M}_t^n \rightarrow M_t := \mathbb{E}[M | \mathcal{F}_t]$  for all  $t \in [0, 1]$ . For each  $n \geq 1$  we extend  $A^n$  to  $[0, 1]$  by

$$(7) \quad A^n := \sum_{t \in \mathcal{D}_n} A_t^n \mathbb{1}_{(t-1/2^n, t]}$$

$$(8) \quad \text{and set } \mathcal{A}^n := \lambda_n^n A^n + \dots + \lambda_{N_n}^n A^{N_n},$$

where we use the same convex weights as in (6). Then the càdlàg process

$$(A_t)_{0 \leq t \leq 1} := (S_t)_{0 \leq t \leq 1} - (M_t)_{0 \leq t \leq 1}$$

satisfies for every  $t \in \mathcal{D}$

$$\mathcal{A}_t^n = (S_t - \mathcal{M}_t^n) \rightarrow (S_t - M_t) = A_t \quad \text{in } L^1(\Omega).$$

Passing to a subsequence which we denote again by  $n$ , we obtain that convergence holds also almost surely. Consequently,  $A$  is almost surely increasing on  $\mathcal{D}$  and, by right continuity, also on  $[0, 1]$ .

As the processes  $A^n$  and  $\mathcal{A}^n$  are left-continuous and adapted, they are predictable. To obtain that  $A$  is predictable, we show that for a.e.  $\omega$  and every  $t \in [0, 1]$

$$(9) \quad \limsup_n \mathcal{A}_t^n(\omega) = A_t(\omega).$$

If  $f_n, f : [0, 1] \rightarrow \mathbb{R}$  are increasing functions such that  $f$  is right continuous and  $\lim_n f_n(t) = f(t)$  for  $t \in \mathcal{D}$ , then

$$(10) \quad \limsup_n f_n(t) \leq f(t) \text{ for all } t \in [0, 1] \text{ and}$$

$$(11) \quad \lim_n f_n(t) = f(t) \text{ if } f \text{ is continuous at } t.$$

Consequently, (9) can only be violated at discontinuity points of  $A$ . As  $A$  is càdlàg, every path of  $A$  can have only finitely many jumps larger than  $1/k$  for  $k \in \mathbb{N}$ . It follows that the points of discontinuity of  $A$  can be exhausted by a countable sequence of stopping times, and therefore it is sufficient to prove  $\limsup_n \mathcal{A}_\tau^n = A_\tau$  for every stopping time  $\tau$ .

By (10),  $\limsup_n \mathcal{A}_\tau^n \leq A_\tau$  and as  $\mathcal{A}_\tau^n \leq \mathcal{A}_1^n \rightarrow A_1$  in  $L^1(\Omega)$  we deduce from Fatou's Lemma that

$$\liminf_n \mathbb{E}[A_\tau^n] \leq \limsup_n \mathbb{E}[\mathcal{A}_\tau^n] \leq \mathbb{E}[\limsup_n \mathcal{A}_\tau^n] \leq \mathbb{E}[A_\tau].$$

Therefore it suffices to prove  $\lim_n \mathbb{E}[A_\tau^n] = \mathbb{E}[A_\tau]$ . For  $n \geq 1$  set

$$\sigma_n := \inf\{t \in \mathcal{D}_n : t \geq \tau\}.$$

Then  $A_\tau^n = A_{\sigma_n}^n$  and  $\sigma_n \downarrow \tau$ . Using that  $S$  is of class  $D$ , we obtain

$$\mathbb{E}[A_\tau^n] = \mathbb{E}[A_{\sigma_n}^n] = \mathbb{E}[S_{\sigma_n}] - \mathbb{E}[M_0] \rightarrow \mathbb{E}[S_\tau] - \mathbb{E}[M_0] = \mathbb{E}[A_\tau].$$

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