G-FANO THREEFOLDS, I

YURI PROKHOROV

Abstract. We classify Fano threefolds with only terminal singularities whose canonical class is Cartier and divisible by 2, and satisfying an additional assumption that the G-invariant part of the Weil divisor class group is of rank 1 with respect to an action of some group G. In particular, we find a lot of examples of Fano 3-folds with "many" symmetries.

Contents

1. Introduction.

1.1. Let Y be an algebraic variety X over a field $\Bbbk$ and let G be a group. Following works of Yu. I. Manin [Man67] we say that X is a G-variety if the group G acts on $\bar{X} := X \otimes_{\Bbbk} \bar{\Bbbk}$, where $\bar{\Bbbk}$ is the algebraic closure of $\Bbbk$. Moreover, we assume that X, G and $\Bbbk$ satisfy one of the following two conditions.

(a) *Algebraic case.* G is the Galois group of $\bar{\Bbbk}$ over $\Bbbk$ acting on $\bar{X}$ through the second factor. The action of G on X is trivial.

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(b) *Geometric case.* The field $\Bbbk$ is algebraically closed, G is a finite group, and the action of G on X is given by a homomorphism $G \to \operatorname{Aut}_{\Bbbk}(X)$.

A G-morphism (resp. rational G-map) of G-varieties is a $\Bbbk$-morphism (resp. $\Bbbk$-rational map) commuting with the action of G in the geometric case. A projective G-morphism $f : X \to Z$ of normal G-varieties is called *G-Mori fiber space* if X has at worst terminal $G\mathbb{Q}$-factorial singularities, $f_*\mathscr{O}_X = \mathscr{O}_Z$, the relative invariant Picard group $\operatorname{Pic}(X/Z)^G$ is of rank 1, and the anticanonical divisor $-K_X$ is f-ample. In the particular case where $\dim Z = 0$, X is called a *$G\mathbb{Q}$-Fano variety.* If furthermore the canonical divisor is Cartier, then we say that X is G-Fano variety.

Throughout this paper we assume that the ground field has characteristic 0. The following is an easy consequence of the Minimal Model Program [Mor88, 0.3.14] (cf. [Pro09b, 4.2]).

1.2. Proposition. *Let V be a G-variety of dimension ≤ 3. The following are equivalent:*

(i) $\kappa(V) = -\infty$,
(ii) *V is geometrically uniruled,*
(iii) *V is G-birationally isomorphic to a variety X having a structure of G-Mori fiber space.*

Birational classification of G-surfaces is developed very well [Man67], [Isk80b]. In this and subsequent papers we consider G-Mori fiber spaces $X \to Z$, where $\dim X = 3$ and Z is a point, i.e. the case of $G\mathbb{Q}$-Fano threefolds.

1.3. Let X be a G-Fano threefold. It is well-known that $\operatorname{Pic}(X)$ is a finitely generated torsion free abelian group (see, e.g. [IP99, Prop. 2.1.2]). Consider the following composed object:

$$\mathrm{V}(X) = \big(\operatorname{Cl}(X), \operatorname{Pic}(X), K_X, (\ \ ,\ \ ,\ \ )\big),$$

where $\operatorname{Pic}(X)$ is regarded as a sublattice of $\operatorname{Cl}(X)$, $K_X \in \operatorname{Pic}(X)$ is the canonical class of X, and $(\ \ ,\ \ ,\ \ )$ is the intersection form $\operatorname{Pic}(X) \times \operatorname{Pic}(X) \times \operatorname{Cl}(X) \to \mathbb{Z}$. Since the singularities of X are isolated cDV [Rei87], $\operatorname{Pic}(X)$ is a primitive sublattice in $\operatorname{Cl}(X)$, i.e. the quotient $\operatorname{Cl}(X)/\operatorname{Pic}(X)$ is torsion free [Kaw88, 5.1]. Moreover, since $\rho(X)^G = 1$, we have

(1.3.1) $\qquad \operatorname{Cl}(X)^G$ is a subgroup of rank 1 containing K_X.

1.4. In this paper we give a classification of one class of Gorenstein G-Fano threefolds. More precisely, we consider Fano threefolds such that $-K_X = 2S$ for some ample Cartier divisor S. Then X is called a *del*

Pezzo threefold (see 3.1). Smooth del Pezzo threefolds were classified by Iskovskikh [Isk80a], see also [Fuj84], [IP99]. Singular ones were discussed from different points of view in many works [Fuj86], [Shi89], [Fuj90], [CJR08], [JP08]. We are interested basically in group actions on terminal del Pezzo threefolds X and the structure of the lattice $\mathrm{Cl}(X)$.

1.5. Let S be a smooth del Pezzo surface of degree $d := K_S^2$. Then we have $\mathrm{Pic}(S) = \mathbb{Z}^{10-d}$. Define

$$\Delta := \{\alpha \in \mathrm{Pic}(S) \mid \alpha^2 = -2, \quad \alpha \cdot K_S = 0\}.$$

Then Δ is a root system in $(K_S)^\perp \otimes \mathbb{R}$. Depending on d, Δ is of the following type ([Man86]):

d	1	2	3	4	5	6	7	8
Δ	E_8	E_7	E_6	D_5	A_4	$A_1 \times A_2$	$-$	A_1

1.6. Now let X be a del Pezzo threefold. Let $S \in |-\frac{1}{2}K_X|$ be a smooth member [Sho79], [Shi89] and let $\iota : S \hookrightarrow X$ be the natural embedding. Then S is a del Pezzo surface of degree $d = -\frac{1}{8}K_X^3$. It is easy to show that the restriction map $\iota^* : \mathrm{Cl}(X) \to \mathrm{Pic}(S)$ is injective and its cokernel is torsion free (see 3.10.3). Define the following subsets in $\Delta \subset \mathrm{Pic}(S)$:

$$\begin{aligned} \Delta' &:= \{\alpha \in \mathrm{Pic}(S) \mid \alpha^2 = -2,\ \alpha \cdot K_S = \alpha \cdot \iota^* \mathrm{Cl}(X) = 0\}, \\ \Delta'' &:= \{\alpha \in \iota^* \mathrm{Cl}(X) \mid \alpha^2 = -2,\ \alpha \cdot K_S = 0\}. \end{aligned}$$

In other words,

$$\Delta' = \Delta \cap (\iota^* \mathrm{Cl}(X))^\perp, \qquad \Delta'' = \Delta \cap \iota^* \mathrm{Cl}(X).$$

If Δ' (resp. Δ'') is non-empty, then it is a root subsystem in Δ. Assume that X is a G-variety. Then the group G naturally acts on $\iota^* \mathrm{Cl}(X)$ and Δ'' preserving the class of K_S and the intersection form.

Our classification of G-del Pezzo threefolds is by types of root systems Δ' and Δ''.

1.7. Theorem. *Let X be a G-del Pezzo threefold and let $\mathrm{d}(X) := -\frac{1}{8}K_X^3$. There are the following possibilities:*

	r	X	$\bar{X}$	Z	Δ'	Δ''	p	s
				d$(X) = 1$				
1^o	1	V_1	–	pt	E_8	–	0	21 − h
2^o	2	(5.2.6)	–	$\mathbb{P}^1$	D_7	–	0	22 − h
3^o	2	(5.2.1)	–	$\mathbb{P}^2$	A_7	–	0	22
4^o	2	(5.2.12)	V_2	pt	E_7	A_1	2	22 − h, h $\leq$ 10
5^o	3		(5.2.7)	$\mathbb{P}^1$	D_6	$2A_1$	4	23 − h
6^o	3		(5.2.2)	$\mathbb{P}^2$	A_6	A_1	2	23
7^o	3		V_3	pt	E_6	A_2	6	23 − h
8^o	4		(5.2.3)	$\mathbb{P}^2$	A_5	$A_1 \times A_2$	8	24
9^o	4		V_4	pt	D_5	A_3	12	24 − h, h $\leq$ 2
10^o	5		(5.2.8)	$\mathbb{P}^1$	D_4	D_4	24	25 − h
11^o	5		V_5	pt	A_4	A_4	20	25
12^o	6		(5.2.5)	$\mathbb{P}^2$	A_3	D_5	40	26
13^o	7	8.1	V_6	$\mathbb{P}^2$	A_2	E_6	72	27
14^o	8	7.8	$\mathbb{P}^3$	pt	A_1	E_7	126	28
				d$(X) = 2$				
15^o	1	V_2	–	pt	E_7	–	0	10 − h
16^o	2	(5.2.7)	–	$\mathbb{P}^1$	D_6	A_1	0	11 − h
17^o	2	(5.2.2)	–	$\mathbb{P}^2$	A_6	–	0	11
18^o	2	(5.2.13)	V_3	pt	E_6	–	2	11 − h, h $\leq$ 5
19^o	3	4.2.1	–	$(\mathbb{P}^1)^2$	A_5	A_2	0	12
20^o	3		(5.2.3)	$\mathbb{P}^2$	A_5	A_1	2	12
21^o	3		V_4	pt	D_5	A_1	4	12 − h, h $\leq$ 2
22^o	4		(5.2.8)	$\mathbb{P}^1$	D_4	$3A_1$	8	13 − h
23^o	4		V_5	pt	A_4	A_2	6	13
24^o	5		(5.2.5)	$\mathbb{P}^2$	A_3	$A_1 \times A_3$	12	14
25^o	6	8.1	V_6	$\mathbb{P}^2$	A_2	D_5	20	15
26^o	7	7.7	$\mathbb{P}^3$	pt	A_1	D_6	32	16
				d$(X) = 3$				
27^o	1	V_3	–	pt	E_6	–	0	≤ 5
28^o	2	(5.2.3)	–	$\mathbb{P}^2$	A_5	A_1	0	6
29^o	3	9.4	(5.2.8)	$\mathbb{P}^1$	D_4	–	3	6 $\leq$ s $\leq$ 7
30^o	5	8.1	V_6	$\mathbb{P}^2$	A_2	$2A_2$	9	9

	r	X	$\bar{X}$	Z	Δ'	Δ''	p	s
31^o	6	7.6	$\mathbb{P}^3$	pt	A_1	A_5	15	10
				$\mathrm{d}(X) = 4$				
32^o	1	V_4	–	pt	D_5	–	0	≤ 2
33^o	2	(5.2.8)	–	$\mathbb{P}^1$	D_4	–	0	$1 \leq \mathrm{s} \leq 3$
34^o	3	4.2.2	–	$(\mathbb{P}^1)^2$	A_3	$2A_1$	0	4
35^o	4	8.1	V_6	$\mathbb{P}^2$	A_2	$2A_1$	4	5
36^o	5	7.5	$\mathbb{P}^3$	pt	A_1	$A_1 \times A_3$	8	6
				$\mathrm{d}(X) = 5$				
37^o	1	V_5	–	pt	A_4	–	0	0
				$\mathrm{d}(X) = 6$				
38^o	2	V_6	–	$\mathbb{P}^2$	A_2	A_1	0	0
39^o	3	$(\mathbb{P}^1)^3$	–	pt	A_1	A_2	0	0
				$\mathrm{d}(X) = 8$				
40^o	1	$\mathbb{P}^3$	–	pt	–	–	0	0

Here $\bar{X}/Z$ is a primitive birational model of X (see Theorem 3.10) and $\mathrm{h} := h^{1,2}(\hat{X})$, where $\hat{X}$ is the standard resolution of X. For compactness, we denote $(\mathbb{P}^1)^k := \underbrace{\mathbb{P}^1 \times \cdots \times \mathbb{P}^1}_{k}$. For other notation we refer to 2.1.

1.8. Remark. For $\mathrm{d}(X) \leq 2$ any del Pezzo threefold automatically has G-structure (see Remark 3.4.1). So, in this case, 1^o – 26^o is a complete list of del Pezzo threefolds with $\mathrm{d}(X) \leq 2$.

1.9. Remark. Singular three-dimensional cubics (without group action) whose singularities are only *nodes* and their small resolutions were classified in [FW]. There is the following correspondence between our list and the classification in [FW]: $31^o \longleftrightarrow$ J15, $30^o \longleftrightarrow$ J14, $29^o \longleftrightarrow$ J11, $28^o \longleftrightarrow$ J9, $27^o \longleftrightarrow$ J1–J5.

We hope that our result can be useful for applications to the classification of finite subgroups of the Cremona group $\mathrm{Cr}_3(\Bbbk)$ [Pro09b], [Pro09a], and also the the birational classification of rational algebraic threefolds over non-closed fields (cf. [Man67]).

The paper is organized as follows. In Sections 2 and 3 we collect some known results. In Sections 4 and 5 we classify primitive del Pezzo threefolds with $\mathrm{rk}\,\mathrm{Cl}(X) = 3$ and 2, respectively. The results of §5 were known earlier [JP08]. We give a short proof for the convenience of the reader. Section 6 describes root systems Δ' on del Pezzo threefolds. In

Sections 7 and 8 we classify del Pezzo threefolds with $\operatorname{rk}\operatorname{Cl}(X) \geq 8-d$, where d is the (half-canonical) degree of X. Section 9 is devoted to the proof of Theorem 1.7. Finally, in Section 11, we discuss some open questions.

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2. Preliminaries.

2.1. Notation. We work over an algebraically closed field of characteristic 0. Throughout this paper X denotes a del Pezzo threefold with at worst terminal Gorenstein singularities. Thus we can write $-K_X = 2S$, where $S = S_X$ is a ample Cartier divisor of S defined up to linear equivalence. Everywhere below we use the following notation:

$\rho = \rho(X) := \operatorname{rk}\operatorname{Pic}(X)$;
$\operatorname{Cl}(X)$ is the Weil divisor class group;
$\mathrm{r} = \mathrm{r}(X) := \operatorname{rk}\operatorname{Cl}(X)$;
$\mathrm{d} = \mathrm{d}(X) := S^3 = -K_X^3/8$, the degree of X;
$\mathrm{p} = \mathrm{p}(X)$ is the number of planes on X;
$\mathrm{s} = \mathrm{s}(X)$ is the number of singular points of X under an additional assumption that X has at worst nodes;
$V_5 \subset \mathbb{P}^6$ is a *smooth* del Pezzo threefold of degree 5 (see [Isk80a], [IP99]);
$V_6 \subset \mathbb{P}^7$ is a *smooth* del Pezzo threefold of degree 6 with $\rho = 2$, see Theorem 3.5;
V_d, for $d = 1, \dots, 4$, is a del Pezzo threefold of degree d with terminal *factorial* singularities (see Theorem 3.4).

2.2. Terminal singularities (see [Rei87]**).** Let (X, P) be a germ of a three-dimensional terminal singularity. Then (X, P) is isolated, i.e, $\operatorname{Sing}(X) = \{P\}$. The *index* of (X, P) is the minimal positive integer r such that rK_X is Cartier. If $r = 1$, then (X, P) is Gorenstein. In this case (X, P) is analytically isomorphic to a hypersurface singularity of multiplicity 2.

Let X be a threefold with Gorenstein terminal singularities. Then any Weil $\mathbb{Q}$-Cartier divisor is Cartier (see e.g. [Kaw88, Lemma 5.1]). Equivalently, $\operatorname{Cl}(X)$ is a primitive sublattice in $\operatorname{Pic}(X)$.

2.2.1. Theorem-Definition ([Kaw88, Corollary 4.5]). *Let X be a threefold with terminal singularities. Then there exists a projective birational morphism $\xi : \hat{X} \to X$ such that*

(i) *$\hat{X}$ is normal and has only terminal $\mathbb{Q}$-factorial singularities;*
(ii) *ξ is a crepant morphism, that is, $K_{\hat{X}} = \xi^* K_X$;*
(iii) *ξ is small, that is, its exceptional locus does not contain any divisors.*

Such a morphism is called $\mathbb{Q}$-factorialization of X. Any two $\mathbb{Q}$-factorializations of X are connected by a sequence of flops.

2.2.2. Theorem [Cut88]. *Let X be a rationally connected threefold with terminal factorial singularities. Assume that $-K_X = 2S$ for some divisor S and $\rho(X) > 1$. Let $f : X \to Z$ be an extremal K-negative contraction. Then one of the following holds:*

(i) *$Z \simeq \mathbb{P}^1$ and f is a quadric bundle, i.e. there is an embedding $X \hookrightarrow \mathbb{P}(\mathscr{E})$, where $\mathscr{E}$ is a rank 4 vector bundle on Z, so that each fiber of f is a quadric in the fiber of $\mathbb{P}(\mathscr{E})/Z$;*
(ii) *X is smooth, Z is a smooth rational surface, and $X = \mathbb{P}(\mathscr{E})$, where $\mathscr{E}$ is a rank 2 vector bundle on Z;*
(iii) *Z is a threefold with terminal factorial singularities and f is blowup of a smooth point on Z.*

2.3. Below we give some basic facts about Fano varieties.

2.3.1. Define the *Fano index* of a (possibly singular) Fano variety X as follows:

$$qF(X) = \sup\{q \mid -K_X \equiv qH,\ q \in \mathbb{Q},\ H \text{ is a Cartier divisor}\}.$$

The following fact is well-known (see e.g. [IP99, Th. 3.1.14]).

2.3.2. Proposition. *Let X be a Fano variety with at worst log terminal singularities.*

(i) *$qF(X) \in \mathbb{Q}$ and $0 < qF(X) \le \dim X + 1$.*
(ii) *If $qF(X) = \dim X + 1$, then $X \simeq \mathbb{P}^{\dim X}$.*
(iii) *If $qF(X) = \dim X$, then $X \simeq Q \subset \mathbb{P}^{\dim X+1}$ is a quadric.*

If additionally X is Gorenstein, then $qF(X)$ is an integer.

2.3.3. Remark. A very interesting invariant of a G-Fano variety is the cone $\overline{\mathrm{NE}}^1(X) \subset \mathrm{Cl}(X) \otimes \mathbb{R}$ of effective divisors. Let $\xi : X' \to X$ be a small $\mathbb{Q}$-factorialization. There are natural identifications $\mathrm{Cl}(X) = \mathrm{Cl}(X')$ and $\overline{\mathrm{NE}}^1(X) = \overline{\mathrm{NE}}^1(X')$. The variety X' is a Mori dream space [HK00]. Hence $\overline{\mathrm{NE}}^1(X)$ is a G-invariant polyhedral cone generated by a finite number of effective divisors.

3. Generalities on del Pezzo threefolds

3.1. Definition. Let X be a projective variety X with at worst terminal Gorenstein singularities.* We say that X is a *del Pezzo variety* (resp. *weak del Pezzo variety*) if its anti-canonical class $-K_X$ is divisible by $\dim X - 1$ and is ample (resp. nef and big).

Note that if X is a del Pezzo variety, then either $X \simeq \mathbb{P}^3$ or $qF(X) = 2$.

3.2. Theorem ([Fuj86]). *Let X be a del Pezzo variety of dimension ≥ 3. Then $\mathrm{d}(X) \leq 8$. Moreover, if $\mathrm{d}(X) = 8$, then $X \simeq \mathbb{P}^3$. If $\mathrm{d}(X) = 7$, then $X \simeq \mathbb{P}(\mathscr{O}_{\mathbb{P}^2} \oplus \mathscr{O}_{\mathbb{P}^2}(1))$. If $\mathrm{d}(X) = 6$, then $\rho(X) = 2$ or 3. If $\mathrm{d}(X) \leq 5$, then $\rho(X) = 1$.*

Sketch of the proof. Assume for simplicity that $\dim X = 3$. By [Nam97] there exists a smoothing X_t, that is, a flat family X_t such that $X_0 \simeq X$ and a general member X_t is a smooth del Pezzo threefold with $\mathrm{d}(X_t) = \mathrm{d}(X)$ and $\rho(X_t) = \rho(X)$. Then assertions about Picard number follows from the classification of smooth del Pezzo threefolds [Fuj84], [Isk80a, ch. 2, Th. 1.1], [IP99]. If $\mathrm{d}(X) = 8$ or 7, then X is smooth by [Fuj86] and then we again can use the classification mentioned above. □

3.3. Lemma. *Let X be a del Pezzo threefold. If X is factorial and singular, then $\rho(X) = 1$.*

Proof. Assume that $\rho(X) > 1$. Let $f_i : X \to Z$ be all extremal contractions. If $\dim Z_i = 2$, then by Theorem 2.2.2 the variety X is smooth, a contradiction. Again by Theorem 2.2.2 each f_i has a two-dimensional fiber F_i. Since f_i are extremal contractions, these fibers F_i do not meet each other. In particular, this implies that $\dim Z_i \neq 1$ for all i. Therefore, all the contractions f_i are birational and F_i are exceptional divisors. Take an effective curve $C \equiv (-nK_X)^2$. There is a decomposition $C \equiv \sum \alpha_i \ell_i$, where ℓ_i are corresponding extremal curves and $\alpha_i \geq 0$. For any j we have

$$0 < F_j \cdot (-nK_X)^2 = F_j \cdot C = F_j \cdot \sum \alpha_i \ell_i = F_j \cdot \alpha_j \ell_j < 0,$$

a contradiction. □

3.4. Theorem ([Isk80a], [Shi89], [Fuj84],[Fuj86], [Fuj90]). *Let X be a del Pezzo variety of dimension $n \geq 3$ and let $S = -\frac{1}{d-1} K_X$.*

(i) $\dim |S| = \mathrm{d}(X) + 1$.

*In papers [Fuj90], [CJR08] authors considered del Pezzo varieties whose singularities are more general than terminal.

(ii) *The linear system* $|S|$ *is base point free (resp. very ample) for* $\mathrm{d}(X) \geq 2$ *(resp.* $\mathrm{d}(X) \geq 3$*). If* $\mathrm{d}(X) \geq 4$*, then the image of* $X_{\mathrm{d}(X)} \subset \mathbb{P}^{\mathrm{d}(X)+n-2}$ *of* X *under the embedding given by* $|S|$ *is an intersection of quadrics.*

(iii) *If* $\mathrm{d}(X) = 1$*, then the linear system* $|S|$ *has a unique base point which is a smooth point of* X*. In this case* $|S|$ *defines a rational map* $X \dashrightarrow \mathbb{P}^{n-1}$ *whose general fiber is an elliptic curve. The variety* X *is isomorphic to a hypersurface of degree* 6 *in the weighted projective space* $\mathbb{P}(1^n, 2, 3)$*.*

(iv) *If* $\mathrm{d}(X) = 2$*, then* $|S|$ *defines a double cover* $X \to \mathbb{P}^n$ *whose branch locus* $B \subset \mathbb{P}^n$ *is a hypersurface of degree* 4 *with at worst isolated singularities. The variety* X *is isomorphic to a hypersurface of degree* 4 *in* $\mathbb{P}(1^{n+1}, 2)$*.*

(v) *If* $\mathrm{d}(X) = 3$*, then* X *is isomorphic to a cubic in* $\mathbb{P}^{n+1}$*.*

(vi) *If* $\mathrm{d}(X) = 4$*, then* X *is isomorphic to a complete intersection of two quadrics in* $\mathbb{P}^{n+2}$*.*

The del Pezzo threefolds with $\mathrm{d}(X) = 1$ and $\mathrm{d}(X) = 2$ have their names: *double Veronese cone* and *quartic double solid*, respectively.

3.4.1. Remark. Let X be a del Pezzo variety of degree 1 (resp. 2). Then there is a finite of degree 2 morphism $\varphi : X \to \mathbb{P}(1^n, 2)$ (resp. $\varphi : X \to \mathbb{P}^n$). The corresponding natural Galois involution $X \to X$ is called *Bertini* (resp. *Geiser*) involution. Therefore, any del Pezzo variety X with $\mathrm{d}(X) \leq 2$ is a G-del Pezzo.

3.5. Theorem ([Fuj84]**,** [IP99]**).** *Let* X *be a smooth del Pezzo threefold with* $\mathrm{d}(X) = 6$*. Then either* $X \simeq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ *or* $X \simeq V_6 \subset \mathbb{P}^7$*, where* V_6 *is unique up to isomorphism and can be realized by one of the following ways:*

(i) *a divisor of bidegree* $(1, 1)$ *in* $\mathbb{P}^2 \times \mathbb{P}^2$*,*

(ii) $\mathbb{P}_{\mathbb{P}^2}(T_{\mathbb{P}^2})$*, where* $T_{\mathbb{P}^2}$ *is the tangent bundle on* $\mathbb{P}^2$*,*

(iii) *the variety of full flags* $\mathrm{F}(\mathbb{P}^2)$ *on* $\mathbb{P}^2$*.*

3.6. Del Pezzo threefolds of degree 1 **(**[Isk80a]**,** [Shi89]**,** [Fuj84]**,** [Fuj86]**,** [Fuj90]**).** Let X be a del Pezzo threefold of degree 1. The half-anticanonical map $\phi : X \dashrightarrow \mathbb{P}^2$ has a unique base point P which must be a smooth point of X and the anticanonical map $\psi : X \to \mathbb{P}^6$ is a finite morphism of degree 2 whose image is the Veronese cone $V_4 \simeq \mathbb{P}(1^3, 2)$. Here $\psi(P) = O := (0, 0, 0, 1)$ and $\psi^{-1}(O) = P$. The branch divisor $B \subset \mathbb{P}(1^3, 2)$ is a surface of weighted degree 6 such that

$B \not\ni O$. Thus we have the following diagram.

$$\begin{array}{ccc} & X & \\ {}^{\psi}\swarrow & & \searrow^{\phi} \\ \mathbb{P}(1^3, 2) & \overset{p}{\dashrightarrow} & \mathbb{P}^2 \end{array}$$

where p is the projection from O.

3.7. Proposition (see [IP99]**).** *Let X be a smooth del Pezzo threefold. Then the Hodge number $h^{1,2}(X)$ is given by the following table:*

$\mathrm{d}(X)$	1	2	3	4	5	6	7	8
$h^{1,2}(X)$	21	10	5	2	0	0	0	0

3.8. Definition. Let X be a weak del Pezzo threefold and let $S = -\frac{1}{2}K_X$. An irreducible surface $\Pi \subset X$ is called a *plane* if $S^2 \cdot \Pi = 1$ and in case $\mathrm{d}(X) = 1$ the base point of $|S|$ does not lie on Π.

3.8.1. Lemma. *Let X be a del Pezzo threefold. If $\Pi \subset X$ is a plane, then $\Pi \simeq \mathbb{P}^2$ and $\mathscr{O}_\Pi(S) = \mathscr{O}_{\mathbb{P}^2}(1)$.*

Proof. The statement is obvious if $\mathrm{d}(X) \geq 3$ because the divisor S is very ample in this case. If $\mathrm{d}(X) = 2$, then $|S|$ defines a double cover $\varphi : X \to \mathbb{P}^3$ so that $\varphi(\Pi)$ is a projective plane on $\mathbb{P}^3$. Thus $\varphi|_\Pi : \pi \to \varphi(\Pi) \simeq \mathbb{P}^2$ is a finite birational morphism, so it is an isomorphism. Finally if $\mathrm{d}(X) = 1$, then $|S|$ defines a rational map $\varphi : X \dashrightarrow \mathbb{P}^2$ so that its restriction to Π is a morphism which must be finite and birational. As above we get $\Pi \simeq \mathbb{P}^2$. □

3.8.2. Lemma. *If $\Pi \subset X$ is a plane, then there is a $\mathbb{Q}$-factorialization $\xi : \hat{X} \to X$ such that for the proper transform $\hat{\Pi}$ of Π we have $\hat{\Pi} \simeq \mathbb{P}^2$ and $\mathscr{O}_{\hat{\Pi}}(\hat{\Pi}) \simeq \mathscr{O}_{\mathbb{P}^2}(-1)$. Therefore, $\hat{\Pi}$ is contractible, i.e. there is a birational contraction $\hat{X} \to X'$ of $\hat{\Pi}$ to a smooth point. Conversely, if $\xi : \hat{X} \to X$ is a $\mathbb{Q}$-factorialization and $\hat{\Pi} \subset \hat{X}$ is an irreducible surface such that $\hat{\Pi} \simeq \mathbb{P}^2$ and $\mathscr{O}_{\hat{\Pi}}(\hat{\Pi}) \simeq \mathscr{O}_{\mathbb{P}^2}(-1)$, then $f(\hat{\Pi})$ is a plane on X.*

Proof. Let $\Pi \subset X$ be a plane. Take a $\mathbb{Q}$-factorialization $\xi : \hat{X} \to X$ so that $\hat{\Pi}$ is f-nef. One can do it by performing flops over X. Assume that $\hat{\Pi}$ is nef. Then by the base point free theorem the linear system $|n\hat{\Pi}|$ is base point free for $n \gg 0$. Hence $|n\Pi|$ has no fixed components. Since X has at worst isolated singularities, by adjunction we have

$$K_\Pi = (-2S + \Pi)|_\Pi \geq -2S|_\Pi,$$

a contradiction.

Thus $\hat{\Pi}$ is not nef. Then there is a K-negative extremal ray R such that $\hat{\Pi} \cdot R < 0$. Since $K_{\hat{X}}$ is divisible by 2, from the classification of extremal rays (Theorem 2.2.2) we see that $\hat{\Pi}$ is contractible to a smooth point, $\hat{\Pi} \simeq \mathbb{P}^2$ and $\mathscr{O}_{\hat{\Pi}}(\hat{\Pi}) \simeq \mathscr{O}_{\mathbb{P}^2}(-1)$.

The converse statement is obvious. □

3.8.3. Lemma. *Let X be a del Pezzo threefold and let $S \in |-\frac{1}{2}K_X|$ be a smooth member. Let $l \in \operatorname{Pic}(S)$ be an element such that $l^2 = l \cdot K_S = -1$ (the class of a line $L \subset S$). Assume that $l \in \iota^* \operatorname{Cl}(X)$, where $\iota : S \hookrightarrow X$ is the embedding. Then there exists a unique plane $\Pi \subset X$ such that $\iota^*\Pi = l$ (i.e., $\Pi \cap S = L$).*

Proof. Denote by Π any divisor whose class coincides with $\iota^* l$. Let $\xi : \hat{X} \to X$ be a $\mathbb{Q}$-factorialization as in Lemma 3.8.2, let $\hat{S} := \xi^{-1}(S)$, and let $\hat{\Pi}$ be the proper transform of Π. By Shokurov's adjunction theorem the pair $(\hat{X}, \hat{\Pi})$ is purely log terminal (PLT). Hence, by the Kawamata-Viehweg vanishing [Fuk97]

$$H^1(\hat{X}, \mathscr{O}_{\hat{X}}(\hat{\Pi} - \hat{S})) = H^1(\hat{X}, \mathscr{O}_{\hat{X}}(\hat{S} + K_{\hat{X}} + \hat{\Pi})) = 0.$$

Then one can see from the exact sequence

$$0 \longrightarrow \mathscr{O}_{\hat{X}}(\hat{\Pi} - \hat{S}) \longrightarrow \mathscr{O}_{\hat{X}}(\hat{\Pi}) \longrightarrow \mathscr{O}_{\hat{S}}(\iota^* l) \longrightarrow 0$$

that $H^0(\hat{X}, \mathscr{O}_{\hat{X}}(\hat{\Pi})) \neq 0$, so we may assume that both $\hat{\Pi}$ and Π are effective. Since $S^2 \cdot \Pi = 1$, Π is a plane. Finally, if there is another plane Π' such that $\iota^*\Pi' = l$, then $\Pi \sim \Pi'$ and $\mathscr{O}_{\hat{\Pi}}(\hat{\Pi}) = \mathscr{O}_{\hat{\Pi}}(\hat{\Pi}')$ is positive, a contradiction. □

3.8.4. Definition. We say that a del Pezzo threefold X is *imprimitive* if it contains at least one plane. Otherwise we say that X is *primitive*.

The following two theorems are easy consequences of [CJR08, Prop. 2.8].

3.9. Theorem. *Let X be a primitive weak del Pezzo threefold with at worst terminal Gorenstein singularities. Let $\xi : \hat{X} \to X$ be a $\mathbb{Q}$-factorialization. Then there exists a K-negative Mori contraction $f : \hat{X} \to Z$ such that one of the following holds:*

- (i) *Z is a point, $\rho(\hat{X}) = 1$, X is factorial, and ξ is an isomorphism;*
- (ii) *$Z \simeq \mathbb{P}^2$, $\rho(\hat{X}) = 2$, and f is a $\mathbb{P}^1$-bundle, i.e. $\hat{X}$ is smooth and $\hat{X} = \mathbb{P}_{\mathbb{P}^2}(\mathscr{E})$, where $\mathscr{E}$ is a rank-2 vector bundle on $\mathbb{P}^2$;*
- (iii) *$Z \simeq \mathbb{P}^1$, $\rho(\hat{X}) = 2$, and f is a quadric bundle, i.e. any fiber of f is an irreducible quadric in $\mathbb{P}^3$;*
- (iv) *$Z \simeq \mathbb{P}^1 \times \mathbb{P}^1$, $\rho(\hat{X}) = 3$, and f is a $\mathbb{P}^1$-bundle.*

Proof. Almost all the statements are proved in [CJR08, Prop. 2.8]. We have to show only that $Z \not\simeq \mathbb{F}_2$. Indeed, if $\dim Z = 2$, then for a general member $\bar{S} \in |-\frac{1}{2}K_{\bar{X}}|$, the restriction $f|_{\bar{S}} : \bar{S} \to Z$ is birational. Hence Z is a del Pezzo surface. $\square$

3.10. Theorem. *Let X be an imprimitive del Pezzo threefold with at worst terminal Gorenstein singularities. Then there exists a diagram*

(3.10.1)
$$\begin{array}{ccccc} & \hat{X} & \xrightarrow{\sigma} & \bar{X} & \\ {\scriptstyle\xi}\swarrow & & & & \searrow{\scriptstyle f} \\ X & & & & Z \end{array}$$

where

(i) ξ *is a $\mathbb{Q}$-factorialization;*
(ii) $\hat{X}$ *is an weak del Pezzo threefold with at worst terminal factorial singularities;*
(iii) σ *is a blowup in smooth distinct points $P_1, \dots, P_n \in \bar{X}$;*
(iv) $\mathrm{d}(X) = \mathrm{d}(\hat{X}) = \mathrm{d}(\bar{X}) + n$*;*
(v) $\bar{X}$ *is a primitive weak del Pezzo threefold with $\rho(\bar{X}) \le 2$, thus $\bar{X}$ is described by* (i)-(iii) *of Theorem* 3.9.

3.10.2. Corollary. *Let X be a del Pezzo threefold. Then* $\mathrm{r}(X) + \mathrm{d}(X) \le 9$.

Proof. We have $9 \ge \rho(\bar{X}) + \mathrm{d}(\bar{X}) = \rho(\hat{X}) + \mathrm{d}(\hat{X}) = \mathrm{r}(X) + \mathrm{d}(X)$. $\square$

3.10.3. Corollary. *Let X be a weak del Pezzo threefold and let $S \in |-\frac{1}{2}K_X|$ be a smooth element. Then the restriction map* $\mathrm{Cl}(X) \to \mathrm{Pic}(S)$ *is injective and its cokernel is torsion free.*

Proof. Clearly the assertion is invariant under taking small modifications. In view of construction (3.10.1), it is sufficient to prove that the restriction map $\mathrm{Cl}(\bar{X}) \to \mathrm{Pic}(\bar{S})$ is injective and its cokernel is torsion free, where $\bar{S} = \sigma(S)$. Thus we may assume that X is a primitive factorial weak del Pezzo threefold. The assertion is obvious if $\rho(X) = 1$. Assume that $\rho(X) = 2$. Then $\rho(Z) = 1$. Let Θ be the ample generator of $\mathrm{Pic}(Z)$. The group $\mathrm{Cl}(X)$ is generated by $f^*\Theta$ and the class of S. Recall that Z is either $\mathbb{P}^1$ or $\mathbb{P}^2$. Hence $f^*\Theta|_S$ is either a conic or the pull-back of a line on $\mathbb{P}^2$, respectively. It is easy to see that $f^*\Theta|_S$ and $-K_S \sim S|_S$ generate a rank 2 primitive sublattice in $\mathrm{Pic}(S)$. The case $\rho(X) = 3$ can be treated similarly. $\square$

4. Primitive del Pezzo threefolds with $\mathrm{r}(X) = 3$

4.1. Lemma. *Let X be a primitive del Pezzo threefold with $\mathrm{r}(X) = 3$ and let $\mathcal{F} = |F|$ be a complete one-dimensional linear system (pencil) of Weil divisors without fixed components. There is a small $\mathbb{Q}$-factorialization $\xi : \hat{X} \to X$ such that the proper transform $\hat{\mathcal{F}}$ of $\mathcal{F}$ on $\hat{X}$ is base point free and defines a fibration $f : \hat{X} \to \mathbb{P}^1$. Moreover, f factors through a (not unique) $\mathbb{P}^1$-bundle contraction*

$$f : \hat{X} \xrightarrow{f_1} \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{f_2} \mathbb{P}^1 \tag{4.1.1}$$

Proof. Take a $\mathbb{Q}$-factorialization $\xi : \hat{X} \to X$ so that $\hat{\mathcal{F}}$ is ξ-nef (one can get it by performing flops over X). Then $\hat{\mathcal{F}}$ is nef. Indeed, otherwise there is a K-negative extremal ray R such that $\hat{\mathcal{F}} \cdot R < 0$. Since $\hat{\mathcal{F}}$ has no fixed components, R must define a flipping contraction. On the other hand, K_X is Cartier, a contradiction [Mor88, Th. 6.2]. Thus $\hat{\mathcal{F}}$ is nef. Then $\hat{\mathcal{F}}$ defines a contraction to a (rational) curve by the base point free theorem. Further, since $\mathrm{r}(X) = 3$, we have $\rho(\hat{X}) = 3$. Running the MMP over $\mathbb{P}^1$ we obtain f_1. □

4.1.2. Remark-definition. In notation of (4.1.1), another ruling on $\mathbb{P}^1 \times \mathbb{P}^1$ defines another pencil $\mathcal{F}'$ on X. In this situation, we say that pencils $\mathcal{F}$ and $\mathcal{F}'$ are *conjugate*. Thus there is one-to-one correspondence between

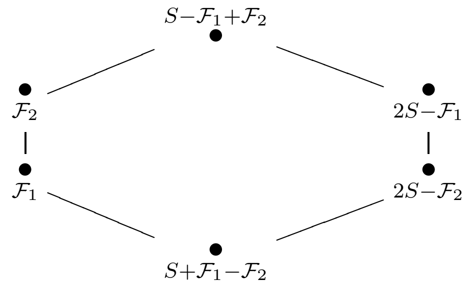
- (i) the set of pairs of conjugate pencils $\mathcal{F}$, $\mathcal{F}'$ and
- (ii) the set of $\mathbb{Q}$-factorializations $X' \to X$ together with a structure of $\mathbb{P}^1$-bundle $f' : \hat{X}' \to \mathbb{P}^1 \times \mathbb{P}^1$.

4.1.3. Corollary. *The cone of effective divisors $\overline{\mathrm{NE}}^1(X)$ is generated by classes of pencils $\mathcal{F}$ as in Lemma 4.1.*

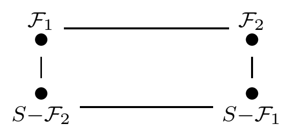
Proof. Let $\xi : \hat{X} \to X$ be a small $\mathbb{Q}$-factorialization. The cone $\overline{\mathrm{NE}}^1(\hat{X})$ is generated by a finite number of effective divisors D_i (see e.g. [HK00] and Remark 2.3.3). Running D_i-MMP on $\hat{X}$, after a number of flops, we get a $\mathbb{P}^1$-bundle over $\mathbb{P}^1 \times \mathbb{P}^1$ (because X is primitive). This shows that D_i must coincide with some $\mathcal{F}$. □

4.2. Theorem. *Let X be a primitive del Pezzo threefold with $\mathrm{r}(X) = 3$. Let $\{\mathcal{F}_i\}$ be the set of all pencils as in Lemma 4.1. Then there are the following possibilities for $\{\mathcal{F}_i\}$, where we draw the graph for $\{\mathcal{F}_i\}$ so that every two elements are connected by an edge if and only if they are conjugate.*

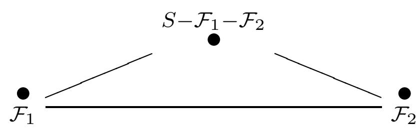
4.2.1. $\mathrm{d}(X) = 2$



4.2.2. $\mathrm{d}(X) = 4$



4.2.3. $\mathrm{d}(X) = 6$



In the last case $X \simeq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Proof. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two conjugate pencils and let $\xi : \hat{X} \to X$ be the corresponding small $\mathbb{Q}$-factorialization. Clearly, we have

$$\mathcal{F}_1^2 \equiv \mathcal{F}_2^2 \equiv 0,\ \mathcal{F}_1 \cdot \mathcal{F}_2 \cdot S = 1,\ S^2 \cdot \mathcal{F}_1 = S^2 \cdot \mathcal{F}_2 = 2.$$

For any j, write $\mathcal{F}_j \sim aS + b_1\mathcal{F}_1 + b_2\mathcal{F}_2$, where $a \geq 0$. Then

$$\begin{aligned} 0 = \mathcal{F}_j^2 \cdot S &= a^2 d + 4a(b_1 + b_2) + 2b_1b_2, \\ 2 = \mathcal{F}_j \cdot S^2 &= ad + 2(b_1 + b_2), \end{aligned} \tag{4.2.4}$$

where $d := \mathrm{d}(X)$. Therefore,

$$\begin{aligned} b_1 + b_2 &= \tfrac{1}{2}(2 - ad), \\ b_1b_2 &= \tfrac{1}{2}a(ad - 4). \end{aligned}$$

Since this system has an integer solution in b_1, b_2, the discriminant

$$\frac{1}{4}(2 - ad)^2 - 2a(ad - 4) = \frac{1}{4}\bigl(4 - a(8 - d)(ad - 4)\bigr)$$

must be a square and ad must be even. Assuming $a > 0$ (i.e. $\mathcal{F}_j \neq \mathcal{F}_1$, $\mathcal{F}_2$), we get $ad = 8$, 6, 4, or 2. Hence, up to permutation of b_1 and b_2, there are the following solutions with $a > 0$:

$d = 1,\quad (a, b_1, b_2) = (4, -1, 0),\ (4, 0, -1);$
$d = 2,\quad (a, b_1, b_2) = (1, -1, 1),\ (1, 1, -1),\ (2, -1, 0),\ (2, 0, -1);$

$d = 4, \quad (a, b_1, b_2) = (1, -1, 0),\ (1, 0, -1);$
$d = 6, \quad (a, b_1, b_2) = (1, -1, -1).$

Note that if $\mathcal{F}_j$ and $\mathcal{F}_k$ are conjugate, then $\mathcal{F}_j \cdot \mathcal{F}_k \cdot S = 1$. From this one can see that for each $\mathcal{F}_j$ there are exactly two divisors in $\{\mathcal{F}_i\}$ that conjugate to $\mathcal{F}_j$. Moreover, if $d \neq 1$, then conjugacy relations are given by graphs in 4.2.1, 4.2.2, 4.2.3. In the case $d = 1$ we get the following (disconnected) graph:

Hence there are only two extremal K-negative contractions on $\hat{X}$. On the other hand, the cone $\overline{\mathrm{NE}}^1(\hat{X})$ has at least three extremal rays, a contradiction. □

4.3. Remark. Let X be a primitive del Pezzo threefold with $\mathrm{r}(X) = 3$ and $\mathrm{d}(X) = 2$ or 4. Let $\xi : \hat{X} \to X$ be a $\mathbb{Q}$-factorialization. Then $\hat{X} \simeq \mathbb{P}(\mathscr{E})$, where $\mathscr{E}$ is a stable rank two vector bundle on $Z = \mathbb{P}^1 \times \mathbb{P}^1$ with $c_1(\mathscr{E}) = 0$, $c_2(\mathscr{E}) = 6 - \mathrm{d}(\mathscr{E})$.

4.3.1. Example. If $\mathrm{d}(X) = 2$, an example of such $\mathscr{E}$ can be obtained as a restriction of the null-correlation bundle $\mathscr{N}$ from $\mathbb{P}^3$ to Z, where $Z \subset \mathbb{P}^3$ is the Segre embedding. Recall that the null-correlation bundle is defined by the exact sequence

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^3} \longrightarrow \Omega_{\mathbb{P}^3}(2) \longrightarrow \mathscr{N}(1) \longrightarrow 0.$$

Its projectivization $Y := \mathbb{P}(\mathscr{N})$ is a Fano fourfold of index 2 [SW90]. This Y has also a structure of $\mathbb{P}^1$-bundle over a smooth three-dimensional quadric. Let $\hat{X} = \mathbb{P}(\mathscr{E}) = \pi^{-1}(Z)$, where $\pi : Y \to \mathbb{P}^3$ is the natural projection. Then $\hat{X}$ is a weak del Pezzo threefold of type 4.2.2.

Examples of del Pezzo threefolds of type 4.2.1 can be constructed similarly by restricting to $Z \subset \mathbb{P}^3$ rank two stable vector bundles $\mathscr{F}$ with $c_1 = 0$, $c_2 = 2$ [Har78, §9].

Another way to show existence of del Pezzo threefolds of types 4.2.2 and 4.2.1 is by writing down explicit equations:

4.3.2. Example. Let $X \subset \mathbb{P}^5$ is given by the equations

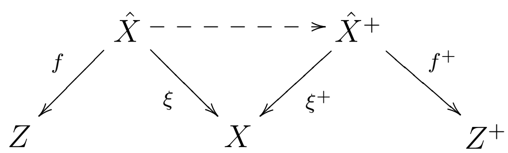
$$\begin{cases} x_1x_3 - x_2x_4 + a_{3,4}x_3x_5 + a_{3,6}x_3x_6 + a_{4,5}x_4x_5 + a_{4,6}x_4x_6 = 0 \\ x_1x_5 - x_2x_6 + b_{3,4}x_3x_5 + b_{3,6}x_3x_6 + b_{4,5}x_4x_5 + b_{4,6}x_4x_6 = 0 \end{cases}$$

where $a_{i,j}$, $b_{i,j}$ are sufficiently general constants. Then X is a del Pezzo threefold having exactly 4 nodes. By Corollary 10.6.2 $\mathrm{r}(X) \geq 3$. On the other hand, by results of 7.5 and §8 below $\mathrm{r}(X) = 3$. Finally, two quadrics $x_5 = x_6 = x_1x_3 - x_2x_4 = 0$ and $x_3 = x_4 = x_1x_5 - x_2x_6 = 0$ determine two conjugate pencils. Therefore, X is of type 4.2.2.

5. Del Pezzo threefolds with $\mathrm{r}(X) = 2$

The results of this section are contained in [JP08]. We give a short self-contained proof for the convenience of the reader.

5.1. Let X be a del Pezzo threefold with $\mathrm{r}(X) = 2$. There exists the following diagram:



where ξ, ξ^+ are small $\mathbb{Q}$-factorializations, $\hat{X} \dashrightarrow \hat{X}^+$ is a flop, and f, f^+ are K-negative extremal contractions. We may assume that $\dim Z \ge \dim Z^+$. Let $S = -\frac{1}{2}K_X$ and let $\hat{S} = h^*S$. Let M (resp. M^+) be the ample generator of $\mathrm{Pic}(X)$ (resp. $\mathrm{Pic}(X^+)$). Put $L := f^*M$ and $L^+ := f^{+*}M^+$. Let L' be the proper transform of L^+ on $\hat{X}$. If f is birational, then $E \subset \hat{X}$ denotes the f-exceptional divisor. Similarly, if f^+ is birational, then $E' \subset \hat{X}$ is the proper transform of f^+-exceptional divisor. Only one such a solution has $a = 0$. Hence the case $d = 1$ is impossible and for $d = 6$ we have $X \simeq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

5.1.1. Remark. If in the above notation $\mathrm{d}(X) \le 2$, then by 3.4.1 there is a natural (Bertini or Geiser) involution $\tau : X \to X$. In this case, we can take $\hat{X}^+ \simeq \hat{X}$ and $\xi^+ = \tau \circ \xi$. Therefore, $Z \simeq Z^+$ and f^+ has the same type as f.

The following theorem was proved (in much stronger form) in [JP08]. For convenience of the reader we provide a short proof.

5.2. Theorem. *In the above notation there are the following possibilities.*

	f	f^+	d	$\mathrm{Pic}(\hat{X})$	s
(5.2.1)	$\mathbb{P}^1$-*bundle*	$\mathbb{P}^1$-*bundle*	1	$L + L' \sim 6\hat{S}$	22
(5.2.2)			2	$L + L' \sim 3\hat{S}$	11
(5.2.3)			3	$L + L' \sim 2\hat{S}$	6
(5.2.4)			6	$L + L' \sim \hat{S}$	0
(5.2.5)	$\mathbb{P}^1$-*bundle*	*quadric bundle*	5	$L + L' \sim \hat{S}$	1

	f	f^+	d	$\mathrm{Pic}(\hat{X})$	s
(5.2.6)	*quadric bundle*	*quadric bundle*	1	$L + L' \sim 4\hat{S}$	≤ 22
(5.2.7)			2	$L + L' \sim 2\hat{S}$	≤ 11
(5.2.8)			4	$L + L' \sim \hat{S}$	≤ 3
(5.2.9)	*birational*	$\mathbb{P}^1$*-bundle*	4	$E + L' \sim \hat{S}$	3
(5.2.10)			7	$E + 2L' \sim \hat{S}$	0
(5.2.11)	*birational*	*quadric bundle*	3	$E + L' \sim \hat{S}$	4, 5, 6
(5.2.12)	*birational*	*birational*	1	$E + E' \sim 2\hat{S}$	$12 \leq \mathrm{s} \leq 22$
(5.2.13)			2	$E + E' \sim \hat{S}$	$6 \leq \mathrm{s} \leq 11$

Here in the 5th column we indicate relations between L, L', E, and E' in $\mathrm{Pic}(\hat{X})$*.*

Proof. First we consider the case where X is primitive, i.e. both f and f^+ are of fiber type. Write $L' \sim a\hat{S} + bL$. Clearly, $a > 0$. Since L' is not ample, $b \leq 0$. Since L' and $\hat{S}$ generate $\mathrm{Pic}(\hat{X})$, we have $b = -1$. Let $n := \dim Z$ and $n' := \dim Z^+$. Further,

$$\hat{S}^3 = d, \qquad \hat{S}^2 \cdot L = n + 1, \qquad \hat{S} \cdot L^2 = n - 1, \qquad L^3 = 0$$

and similarly

$$\hat{S}^2 \cdot L' = n' + 1, \qquad \hat{S} \cdot L'^2 = n' - 1.$$

This gives us

$$n' + 1 = \hat{S}^2 \cdot L' = ad - (n + 1), \quad ad = n + n' + 2.$$

On the other hand, by Remark 5.1.1 $d \geq 3$ whenever $n \neq n'$. This gives us the possibilities (5.2.1) – (5.2.8) in our table.

Assume that f is birational. If $Z \simeq \mathbb{P}^3$, then we get the case 5.2.10. Thus we may assume that E and $S \sim L - E$ generate $\mathrm{Pic}(\hat{X})$. Assume that f^+ is of fiber type. As above, $L' \sim a\hat{S} - E$ and $n' + 1 = \hat{S}^2 \cdot L' = ad - 1$. So, $ad = n' + 2 \leq 4$. On the other hand, by Remark 5.1.1 $d \geq 3$. Hence $a = 1$ and $d = n' + 2$. This gives us (5.2.9) and (5.2.11).

Finally assume that both f and f^+ are birational. Since $\mathrm{Pic}(\hat{X}) = \mathbb{Z} \cdot \hat{S} \oplus \mathbb{Z} \cdot E = \mathbb{Z} \cdot \hat{S} \oplus \mathbb{Z} \cdot E'$ and $\dim |E'| = 0$, we can write $E' \sim a\hat{S} - E$. Hence,

$$1 = E' \cdot \hat{S}^2 = (a\hat{S} - E) \cdot \hat{S}^2 = ad - 1, \qquad ad = 2.$$

We get cases (5.2.12) and (5.2.13). □

5.3. Corollary. *Let X be a del Pezzo threefold with $\mathrm{r}(X) = 1$. Assume that X is singular. Then $\mathrm{d}(X) \le 4$. If $\mathrm{d}(X) = 4$, then every singular point $P \in X$ is rs-nondegenerate (see 10.1). Moreover, $\lambda(X, P) = \nu(X, P)$ and $\sum_P \lambda(X, P) \le 2$.*

Proof. Let $P \in X$ be a sufficiently general point. Let $\sigma : \tilde{X} \to X$ be the blowup of P, let $E := \sigma^{-1}(P)$, and let $\tilde{S}$ be the proper transform of $S = -\frac{1}{2}K_X$. Write $-K_{\tilde{X}} = 2\sigma^* S - 2E = 2\tilde{S}$. Since the linear system $|\tilde{S}|$ is base point free and big, $\tilde{X}$ is a weak del Pezzo threefold with at worst factorial terminal singularities, $\rho(\tilde{X}) = 2$, and $\mathrm{d}(\tilde{X}) = \mathrm{d}(X) - 1$. If $\mathrm{d}(X) \ge 5$, then by Theorem 5.2 we have only one possibility 5.2.9. But then both $\tilde{X}$ and X are smooth. If $\mathrm{d}(X) = 4$, then we have case 5.2.11. In this case any singularity $\tilde{P} \in \tilde{X}$ is analytically isomorphic to the hypersurface singularity given by $x_1x_2 + x_3^2 + x_4^n = 0$. Then $\lambda(\tilde{X}, \tilde{P}) = \nu(\tilde{X}, \tilde{P}) = \lfloor n/2 \rfloor$. The last inequality follows by Proposition 10.6 □

5.4. By [JP08] all the cases in the table do occur†. Below we give explicit examples of some del Pezzo threefolds with $\mathrm{r}(X) = 2$.

5.4.1. Case 5.2.3. $X = X_3 \subset \mathbb{P}^4$ is given by an equation of the form

$$(x_1x_4 - x_2x_3)\ell_1 + (x_2^2 - x_1x_3)\ell_2 + (x_3^2 - x_1x_3)\ell_3 = 0,$$

where $\ell_i(x_1, \dots, x_5)$ are linear forms.

5.4.2. Case 5.2.5 (cf. 7.4 and 8.3.) Let Y be the blowup of $\mathbb{P}^1 \times \mathbb{P}^2$ along a smooth curve C of bidegree $(2, 1)$. Then Y is a Fano threefold with $-K_Y^3 = 38$ and $\rho(Y) = 3$ [MM82]. Let $S \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a (unique) effective divisor of bidegree $(0, 1)$ containing C and let $\tilde{S}$ be the proper transform of S on Y. Then $\tilde{S} \simeq S \simeq \mathbb{P}^1 \times \mathbb{P}^2$ and $\mathscr{O}_{\tilde{S}}(\tilde{S})$ is of type $(-1, -1)$. Therefore, there exists a contraction $\varphi : Y \to X$, where $\varphi(\tilde{S})$ is a node. Here X is a quintic del Pezzo threefold as in 5.2.5.

5.4.3. Case 5.2.7. $X \subset \mathbb{P}(1^4, 2)$ is given by the equation

$$x_5^2 = (x_1x_2 - x_3x_4)^2 + (x_1x_2 - x_3x_4)q_1(x_1, \dots, x_4) + q_2(x_1, \dots, x_4)^2,$$

where q_1 and q_2 are sufficiently general quadratic forms.

5.4.4. Case 5.2.8. $X \subset \mathbb{P}^5$ is given by the equations

$$x_1x_2 + x_3x_4 + x_5^2 + x_6l_1(x_1, \dots x_6) = x_1x_3 + x_6l_2(x_1, \dots x_6) = 0,$$

where l_i are linear forms. It is easy to see that X contains two singular quadrics $Q_1 = \{x_6 = x_1 = x_3x_4 + x_5^2 = 0\}$ and $Q_2 = \{x_6 = x_3 = x_1x_2 + x_5^2 = 0\}$. They generate two pencils. Hence X is of type 5.2.8. For general choice of l_i the variety X has exactly one node.

†There is a typographical error in [JP08, Th. 3.6]: the case $c_2(\mathcal{F}) = 6$ exists.

5.4.5. Case 5.2.9. $X \subset \mathbb{P}^5$ is given by the equations

$$x_3x_4 - x_5^2 + x_6l_1(x_1, \dots x_6) = x_1x_4 - x_2x_5 + x_6l_2(x_1, \dots x_6) = 0,$$

where l_i are sufficiently general linear forms. Its singular locus consists of three points

$$\{x_3 = x_4 = x_5 = x_6 = l_1 = 0\},\ \{x_2 = x_4 = x_5 = x_6 = x_3l_2 - x_1l_1 = 0\}$$

and X contains the plane $\{x_4 = x_5 = x_6 = 0\}$.

5.4.6. Case 5.2.11. Let $X \subset \mathbb{P}^4$ be given by the following equation:

$$x_1u(x_1, x_2, x_3, x_4, x_5) + x_2v(x_1, x_2, x_3, x_4, x_5) = 0,$$

where u and v are quadratic forms. This cubic contains the plane $\Pi := \{x_1 = x_2 = 0\}$ and for sufficiently general u and v the singular locus consists of four nodes. The projection from Π gives us a quadric bundle structure on $\hat{X}$ (which is blowup of Π). For some special choices of u and v the cubic X can have one or two extra (factorial) singular points (see [FW]) and $\mathrm{r}(X) = 2$.

5.4.7. Case 5.2.12. $X \subset \mathbb{P}(1^3, 2, 3)$ is given by the equation

$$x_5^2 = x_4^3 + x_4^2\phi_2 + x_4\phi_4 + \phi_3^2,$$

where $\phi_i(x_1, x_2, x_3)$ are sufficiently general homogeneous forms of degree i.

5.4.8. Case 5.2.13. $X \subset \mathbb{P}(1^4, 2)$ is given by the equation

$$x_5^2 = x_1\phi_3(x_1, \dots, x_4) + q(x_1, \dots, x_4)^2,$$

where ϕ_3 and q are sufficiently general homogeneous forms of degree 3 and 2, respectively.

6. Root systems

6.1. Let X be a del Pezzo threefold of degree $d = \mathrm{d}(X)$. In this section we study the image of the restriction map $\iota^* : \mathrm{Cl}(X) \to \mathrm{Pic}(S)$, where $S \in |-\frac{1}{2}K_X|$ is a smooth member contained into the smooth locus of X and $\iota : S \hookrightarrow X$ is an embedding. Define Δ and Δ' as in 1.5. If X is imprimitive, we apply construction (3.10.1) with all corresponding notation. In the primitive case, to unify notation, we put $\sigma = \mathrm{id}$.

Note that S does not pass through singular points of X. Thus we may identify S and $\hat{S} = \xi^{-1}(S)$. Let $\bar{S} := \sigma(S)$. Then $\bar{S}$ is a smooth del Pezzo surface, $\bar{S} \in |-\frac{1}{2}K_{\bar{X}}|$ and $\sigma_S : S \to \bar{S}$ is a blowup of $\mathrm{r}(X) - \mathrm{r}(\bar{X})$ distinct points. Define $\bar{\Delta}$ and $\bar{\Delta}'$ for $\bar{S}$ as in 1.5.

6.2. Theorem. (i) *In the above notation the image* $\iota^* \operatorname{Cl}(X)$ *is the orthogonal complement to* Δ'*. In particular,*

$$\operatorname{rk}\Delta' + \operatorname{rk}\operatorname{Cl}(X) + \operatorname{d}(X) = 10. \tag{6.2.1}$$

(ii) *We have* $\Delta' = \sigma_S^* \bar{\Delta}'$*.*

(iii) *According to possibilities for* Z *we have the following cases:*

(a) *If* Z *is a point (i.e.* $\rho(\bar{X}) = 1$*), then* $\bar{\Delta}' = \bar{\Delta}$*. Here* $\bar{\Delta}'$ *is of type* E_8*,* E_7*,* E_6*,* D_5*,* A_4*,* A_1 *in cases* $\operatorname{d}(\bar{X}) = 1, 2, 3, 4, 5,$ *and* 8*, respectively.*

(b) *If* $Z \simeq \mathbb{P}^2$*, then* $\bar{\Delta}' = \{\alpha \in \bar{\Delta} \mid \alpha \cdot f^* K_Z = 0\}$*. Here* $\bar{\Delta}'$ *is of type* A_m *(recall that* $\operatorname{d}(\bar{X}) = 1, 2, 3, 5,$ *or* 6*).*

(c) *If* $Z \simeq \mathbb{P}^1$*, then* $\bar{\Delta}' = \{\alpha \in \bar{\Delta} \mid \alpha \cdot C = 0\}$*, where* C *is a conic on* $\bar{S}$*. Here* $\bar{\Delta}'$ *is of type* D_m *(recall that* $\operatorname{d}(\bar{X}) = 1, 2,$ *or* 4*).*‡

(d) *If* $X \simeq (\mathbb{P}^1)^3$*, then* Δ' *is the subsystem* A_1 *in* $\Delta \simeq A_1 \times A_2$*.*

(e) *If* X *is of type* 4.2.1 *or* 4.2.2*, then* Δ' *is of type* A_5 *or* A_3*, respectively.*

Proof.

6.3. Assume that X is primitive. Then $\hat{X} = \bar{X}$ and $\sigma = \mathrm{id}$. All the statements are obvious if $\mathrm{r}(X) = 1$. We assume that $\mathrm{r}(X) \geq 2$. Let $f : X \to Z$ be an extremal K-negative contraction. Let $S \in |-\frac{1}{2}K_X|$ be a smooth member. Denote by $\delta : S \to Z$ the restriction of f to S. Since $f : X \to Z$ is an extremal contraction, the image of $\iota^* : \operatorname{Pic}(X) \to \operatorname{Pic}(S)$ is generated by $\delta^* \operatorname{Pic}(Z)$ and $-K_S = -\frac{1}{2}K_X|_S$. Clearly, $f : S \to Z$ is surjective. Fix a standard basis in $\operatorname{Pic}(S)$ [Dol, ch. 8]:

$$\mathbf{h}, \quad \mathbf{e}_1, \ldots, \mathbf{e}_n,$$

where $n = 9 - d$ and

$$\mathbf{h}^2 = 1, \quad \mathbf{e}_i^2 = -1, \quad \mathbf{e}_i \cdot \mathbf{e}_j = 0 \quad \text{for } i \neq j.$$

Since $\iota^* \operatorname{Pic}(X)$ is generated by $\delta^* \operatorname{Pic}(Z)$ and $-K_S$, we have

$$\Delta' = \{\alpha \in \Delta \mid \alpha \cdot \delta^* \operatorname{Pic}(Z) = 0\}.$$

6.3.1. Case $Z \simeq \mathbb{P}^2$ and f is a $\mathbb{P}^1$-bundle. Then $f : S \to \mathbb{P}^2$ is the blowup of $n = 9 - d$ points and we can choose the basis $\mathbf{h}, \mathbf{e}_1, \ldots, \mathbf{e}_n$ so that $\mathbf{h} = f^* \mathscr{O}_{\mathbb{P}^2}(1)$ and $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are f-exceptional. In this case, $\iota^* \operatorname{Pic}(X)$ is generated by $\mathbf{h}$ and K_S. Hence $\Delta' = \{\alpha \in \Delta \mid \alpha \cdot \mathbf{h} = 0\}$. Then $\Delta' = \{\mathbf{e}_i - \mathbf{e}_j \mid i \neq j\}$ is a root subsystem of rank $n-1$ generated by $\mathbf{e}_1 - \mathbf{e}_2, \ldots, \mathbf{e}_{n-1} - \mathbf{e}_n$. Thus Δ' is of type A_{n-1}.

‡Cases (b) and (c) overlap for X with $\operatorname{d}(X) = 5$.

6.3.2. Case $Z \simeq \mathbb{P}^1$, i.e. f is a quadric bundle. Then $n \geq 4$ and $\delta : S \to \mathbb{P}^1$ is a conic bundle. Let C be a fiber. By changing the basis $\mathbf{h}$, $\mathbf{e}_1, \dots, \mathbf{e}_n$ we may assume that $C \sim \mathbf{h} - \mathbf{e}_1$. Then $\Delta' = \{\alpha \in \Delta \mid \alpha \cdot C = 0\}$, i.e. Δ' consists of the following elements:

- $\mathbf{e}_i - \mathbf{e}_j, \quad i, j > 1,\ i \neq j.$
- $\pm(\mathbf{h} - \mathbf{e}_1 - \mathbf{e}_i - \mathbf{e}_j), \quad i, j > 1,\ i \neq j.$

Simple roots can be taken as follows:

$$\mathbf{h} - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3, \quad \mathbf{e}_2 - \mathbf{e}_3, \dots, \mathbf{e}_{n-1} - \mathbf{e}_n.$$

Hence Δ' is of type D_{n-1} if $n \geq 5$ and A_3 if $n = 4$.

6.3.3. Case $Z \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and f is a $\mathbb{P}^1$-bundle. Let $\ell_i := F_i|_S$. Then we may assume that $\ell_1 \sim \mathbf{h} - \mathbf{e}_1$, $\ell_2 \sim \mathbf{h} - \mathbf{e}_2$. Δ' consists of the following elements:

- $\mathbf{e}_i - \mathbf{e}_j, \quad i, j > 2,\ i \neq j.$
- $\pm(\mathbf{h} - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_i), \quad i > 2.$

Simple roots can be taken as follows:

$$\mathbf{h} - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3, \quad \mathbf{e}_3 - \mathbf{e}_4, \dots, \mathbf{e}_{n-1} - \mathbf{e}_n.$$

Thus Δ' is of type A_{n-2}.

This proves our theorem in the case where X is primitive.

6.4. Now consider the case where X is imprimitive. Obviously, the statement of (iii) follows from 6.3. There is a birational contraction $\sigma : \hat{X} \to \bar{X}$, where $\bar{X}$ is primitive and σ is a composition of blowups of smooth points. Let $E_1, \dots, E_l$ be σ-exceptional divisors and let $\mathbf{e}_i = E_i \cap S$ for $i = 1, \dots, l$, $l = \mathrm{r}(X) - \mathrm{r}(\bar{X})$. By the above, the statement of our theorem holds for $\bar{X}$ with root system $\bar{\Delta}' \subset \bar{\Delta} \subset \mathrm{Pic}(\bar{S})$. We have a commutative diagram

$$\begin{array}{ccccccc}
\mathrm{Pic}(\bar{S}) & \xhookrightarrow{\sigma_S^*} & \mathrm{Pic}(S) & \simeq & \mathrm{Pic}(\bar{S}) & \oplus & \sum_{i=1}^{l} \mathbf{e}_i \cdot \mathbb{Z} \\
\bar{\iota}^* \uparrow & & \iota^* \uparrow & & \bar{\iota}^* \uparrow & & \Vert \\
\mathrm{Cl}(\bar{X}) & \xhookrightarrow{\sigma^*} & \mathrm{Cl}(X) & \simeq & \mathrm{Cl}(\bar{X}) & \oplus & \sum_{i=1}^{l} E_i \cdot \mathbb{Z}
\end{array}$$

Now it is easy to see that $\iota^* \mathrm{Cl}(X)^{\perp} \subset \sigma_S^* \mathrm{Pic}(\bar{S})$. Therefore,

$$\sigma_S^* \bar{\Delta}' \subset \Delta \cap \iota^* \mathrm{Cl}(X)^{\perp} \subset \Delta \cap \sigma_S^* \mathrm{Pic}(\bar{S}).$$

On the other hand, $\sigma_S^* \bar{\Delta}' \supset \Delta \cap \sigma_S^* \mathrm{Pic}(\bar{S})$. Hence, $\sigma_S^* \bar{\Delta}' = \Delta \cap \iota^* \mathrm{Cl}(X)^{\perp}$. This proves (ii). As a consequence we have that the left hand side of (6.2.1) is preserved under birational contractions σ. By 6.3 the equality (6.2.1) holds for primitive del Pezzo threefolds. Thus (6.2.1) holds for imprimitive ones as well. This proves (i).

□

7. Del Pezzo threefolds with maximal $\mathrm{r}(X)$

Recall that $\mathrm{r}(X) + \mathrm{d}(X) \le 9$ by Corollary 3.10.2. In this section we study del Pezzo threefolds with $\mathrm{r}(X) + \mathrm{d}(X) = 9$.

We say that points $P_1, \dots, P_n \in \mathbb{P}^3$ are *in general position* if no three of them lie on one line and no four of them lie on one plane.

7.1. Theorem. *Let X be a del Pezzo threefold with $\mathrm{r}(X) + \mathrm{d}(X) = 9$. Assume that $X \not\simeq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Then*

- (i) *X can be obtained by applying construction (3.10.1) to $\mathbb{P}^3 \simeq V_8 \subset \mathbb{P}^9$ where σ is the blowup of $n := \mathrm{r}(X) - 1$ points $P_1, \dots, P_n \in V_8$ in general position.*
- (ii) *Singular points of X are images of proper transforms of*
  - (a) *lines passing through P_i and P_j, $i \neq j$,*
  - (b) *twisted cubics passing through six distinct points $P_{i_1}, \dots, P_{i_6}$ (see Claim 7.1.2 below).*
- (iii) *If all the singularities of X are nodes, then $\mathrm{s}(X) = 28$, 16, 10, 6, 3, 1 in cases $\mathrm{d}(X) = 1$, 2, 3, 4, 5, 6, respectively.*
- (iv) *If $\mathrm{d}(X) \ge 2$, then all the singularities of X are nodes.*

Conversely, assume that X is a del Pezzo threefold whose singularities are at worst nodes and assume that $\mathrm{s}(X) = 28$, 16, 10, 6, 3, 1 in cases $\mathrm{d}(X) = 1$, 2, 3, 4, 5, 6, respectively. Then $\mathrm{d}(X) + \mathrm{r}(X) = 9$.

Note that in the case $\mathrm{d}(X) = 1$ the statement of (iv) is wrong: one can easily construct X having only 27 singular points, where one of them is not a node.

7.1.1. Corollary. *Let X be a del Pezzo threefold with $\mathrm{r}(X)+\mathrm{d}(X) = 9$. If $\mathrm{d}(X) \ge 3$ and $\mathrm{d}(X) \neq 6$, then X is unique up to isomorphism. If $\mathrm{d}(X) = 2$ (resp. $\mathrm{d}(X) = 1$), then X belongs to a 3-dimensional (resp. 6-dimensional) family. There are exactly two isomorphism classes of Pezzo threefolds with $\mathrm{d}(X) = 6$, $\mathrm{r}(X) = 3$.*

Proof. (i) If X is primitive, then either $X \simeq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ or $X \simeq \mathbb{P}^3$ by Theorems 3.2, 4.2, and 5.2. Thus we assume that X is imprimitive and $\mathrm{d}(X) \le 7$. We use notation of Theorem 3.10. Run construction (3.10.1) in such a way that n is maximal possible. On the last step we get a primitive weak del Pezzo threefold $\bar{X}$ with $\rho(\bar{X}) = 9 - \mathrm{d}(\bar{X})$. Moreover, if $\rho(\bar{X}) = 3$, then $n = 0$, $\rho(\hat{X}) = \mathrm{r}(X) = 3$, and $\mathrm{d}(X) = 6$. By Theorem 4.2 we have $X \simeq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. On the other hand, by Theorem 5.2 $\rho(\bar{X}) \neq 2$. Hence $\rho(\bar{X}) = 1$, $\mathrm{d}(\bar{X}) = 8$, and then $\bar{X} \simeq \mathbb{P}^3$.

It remains to show that the centers $P_1, \dots, P_n$ of the blowup $\hat{X} \to \bar{X} \simeq \mathbb{P}^3$ are in general position. Indeed, if distinct points P_i, P_j, P_k

lie on a line $L \subset \mathbb{P}^3$, then for its proper transform L' on $\hat{X}$ we have $-K_{\hat{X}} \cdot L' = -K_{\mathbb{P}^3} \cdot L - 3 \cdot 2 < 0$, a contradiction. Similarly, if four distinct points P_i, P_j, P_k, P_l lie on a plane $D \subset \mathbb{P}^3$, then then for its proper transform $\hat{D}$ on $\hat{X}$ we have $K_{\hat{X}}^2 \cdot \hat{D} = K_{\mathbb{P}^3}^2 \cdot D - 4 \cdot 4 = 0$. Hence $\hat{D}$ is contracted by the anticanonical map, a contradiction. This proves (i).

(ii) Let $P \in X$ be a singular point. Then $\xi^{-1}(P)$ is a curve. Let $\hat{C} \subset \xi^{-1}(P)$ be a component and let $\bar{C} := \sigma(\hat{C}) \subset \bar{X}$. There are two members $\hat{S}'$, $\hat{S}'' \in |-\frac{1}{2}K_{\hat{X}_0}|$ such that $C \subsetneq \hat{S}' \cap \hat{S}''$. Then $\bar{C} \subsetneq \bar{S}' \cap \bar{S}''$, where $\bar{S}'$, $\bar{S}'' \subset \hat{X}_0 = \mathbb{P}^3$ are proper transforms of $\hat{S}'$ and $\hat{S}''$. Therefore, $\deg \bar{C} \leq 3$ and $\bar{C}$ is not a plane cubic. If $\deg \bar{C} = 2$, then $\bar{C}$ is a conic and it must contain four distinct points from $P_1, \dots, P_n$. This contradicts our assumption that $P_1, \dots, P_n$ are in general position. Therefore, $\bar{C}$ is either a line or a twisted cubic. This proves (ii).

(iii) follows by Corollary 10.6.2.

(iv) If $\mathrm{d}(X) \geq 3$, then X is unique up to isomorphism and the statement (iv) can be checked directly (see 7.3-7.6 below). Let $\mathrm{d}(X) = 2$ the ξ-exceptional set consists of proper transforms of lines $L_{i,j}$ passing through pairs of distinct points P_i, P_j and one twisted cubic C passing through $P_1, \dots, P_6$. Moreover, the lines $L_{i,j}$ meet C transversally. By blowing the points $P_1, \dots, P_6$ up we get these curves disjointed. Thus ξ is a small resolution whose exceptional set is a disjointed union of 16 smooth rational curves.

The last assertion follows by Corollary 10.6.2. □

7.1.2. Claim. *Let $P_1, \dots, P_6 \in \mathbb{P}^3$ be a points in general position. Then there exists a twisted cubic curve $C = C_3 \subset \mathbb{P}^3$ containing $P_1, \dots, P_6$. This curve is unique.*

Proof. Let Q be a quadratic cone with vertex at P_1 containing $P_2, \dots, P_6$. Then $Q \simeq \mathbb{P}(1,1,2)$ and $\dim |\mathscr{O}_{\mathbb{P}(1,1,2)}(3)| = 5$. Hence there exists a member $C \in |\mathscr{O}_{\mathbb{P}(1,1,2)}(3)|$ passing through $P_2, \dots, P_6$. Clearly, C passes also through the vertex P_1. Then $C \subset Q \subset \mathbb{P}^3$ is a curve of degree 3 and it is irreducible because $P_1, \dots, P_6$ are in general position.

Assume that there are two twisted cubics C, C' passing through $P_1, \dots, P_6$. Since C is an intersection of three quadrics, there exists a quadric Q containing both C and C'. If Q is singular, then $\mathrm{Cl}(Q) \simeq \mathbb{Z}$. Hence, on Q we have $C \cdot C' = 9/2$. On the other hand, $C \cap C' \supset \{P_1, \dots, P_6\}$, a contradiction. If Q is smooth, then $Q \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and we may assume that C is of bidegree $(1,2)$. Then $C \cdot C' < 2 \deg C' = 6$. Again we have a contradiction. □

By Theorem 6.2 we have the following.

7.1.3. Corollary. *Let X be a del Pezzo threefold with* $\mathrm{r}(X) = 9 - \mathrm{d}(X)$ *and* $\mathrm{d}(X) \le 5$. *Then the image of* $\iota^* : \mathrm{Cl}(X) \to \mathrm{Pic}(S)$ *is a sublattice orthogonal to some root* $\alpha \in \Delta$, *i.e.* $\Delta' = \{\pm\alpha\}$. *Moreover,* Δ'' *is of type* E_7, D_6, A_5, $A_1 \times A_3$, A_2 *in cases* $\mathrm{d}(X) = 1, 2, 3, 4, 5$, *respectively.*

7.1.4. Corollary. *Let X be a del Pezzo threefold with* $\mathrm{r}(X) = 9 - \mathrm{d}(X)$ *and* $\mathrm{d}(X) \le 4$.

(i) *If* $\mathrm{d}(X) \ne 2$, *then the image of the natural map* $G \to \mathrm{Aut}(\Delta'')$ *is contained into the Weyl group* $\mathrm{W}(\Delta'')$.

(ii) *If* $\mathrm{d}(X) \le 3$ *and* $\Bbbk$ *is algebraically closed* (*i.e. we are in the geometric case*), *then the map* $G \to \mathrm{Aut}(\Delta'')$ *is an embedding.*

Proof. (i) Similar to [Man67, Ch. 4, 26.5]. If $\mathrm{d}(X) = 1$, then Δ'' is of type E_7 and $\mathrm{Aut}(\Delta'') = \mathrm{W}(\Delta'')$ [Ser87]. For $\mathrm{d}(X) = 3$ and 4 the group $\mathrm{Aut}(\Delta'')$ is a direct product of $\mathrm{W}(\Delta'')$ and $\pm\,\mathrm{id}$. If the image of G is not contained in $\mathrm{W}(\Delta'')$, then the element $\tau := -\,\mathrm{id}$ can be expressed as gw, where $g \in G$ and $w \in \mathrm{W}(\Delta'')$. Note that any reflection $s \in \mathrm{W}(\Delta'')$ can be extended to an element $\mathrm{Aut}(\iota^* \mathrm{Cl}(X))$. Hence, the action of τ can be extended to an action on $\iota^* \mathrm{Cl}(X)$ so that $\tau(K_S) = gw(K_S) = K_S$. Let E be a plane on X and let $\mathbf{e}$ be the class $\iota^*(E)$. Then

$$\tau(\mathbf{e}) = \tau\big(\tfrac{1}{d}K_S + \mathbf{e}\big) - \tfrac{1}{d}\tau(K_S) = -\big(\tfrac{1}{d}K_S + \mathbf{e}\big) - \tfrac{1}{d}K_S = -\tfrac{2}{d}K_S - \mathbf{e}.$$

In particular, $2/d$ must be integral, a contradiction.

(ii) Let G_0 be the kernel of the map $G \to \mathrm{Aut}(\Delta'')$. Then G_0 acts trivially on $\mathrm{Cl}(X)$. In particular, the diagram (3.10.1) is G_0-equivariant. Thus G_0 acts on $\bar{X} = \mathbb{P}^3$ so that there are ≥ 5 fixed points in general position, the images of σ-exceptional divisors. Then G_0 must be trivial. □

7.2. Theorem. *Let X be a del Pezzo threefold with* $\mathrm{r}(X) + \mathrm{d}(X) = 9$. *Assume that* $X \not\simeq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. *Let* $\Pi \subset X$ *is a plane, let* $\hat{\Pi} \subset \hat{X}$ *be its proper transform, and let* $\bar{\Pi} = \sigma(\hat{\Pi}) \subset \bar{X} = \mathbb{P}^3$. *Then* $\bar{\Pi}$ *is of one of the following types:*

(i) $\bar{\Pi}$ *is one of the points* P_i, $\hat{\Pi}$ *is* σ*-exceptional;*

(ii) $\bar{\Pi}$ *is a plane passing through three of the points* P_i*;*

(iii) $\bar{\Pi}$ *is quadratic cone passing through six of the points* P_i *so that one of them is the vertex of the cone;*

(iv) (*only for* $\mathrm{d}(X) = 1$) $\bar{\Pi}$ *is cubic surface passing through all the points* P_i *so that four of them are double points;*

(v) (*only for* $\mathrm{d}(X) = 1$) $\bar{\Pi}$ *is quartic surface passing through all the points* P_i *so that all of them are double points and one of them is a triple point.*

The number of planes on X is given by the following table:

d(X)	7	6	5	4	3	2	1
p(X)	1	2	4	8	15	32	126

Proof. It is easy to see that all the subvarieties Π described in (i)-(v) are planes. So the number of planes is at least the number indicated in the table. On the other hand, for any plane $\Pi \subset X$, the intersection $\Pi \cap S$ is a line whose class in $\mathrm{Pic}(S)$ is orthogonal to the root $\alpha \in \mathrm{Pic}(S)$ (see Corollary 7.1.3). Define $\mathcal{E} := \{e \in \mathrm{Pic}(S) \mid e^2 = K_S{\cdot}e = -1,\ e{\cdot}\Delta' = 0\}$. Thus the number of planes is at most $|\mathcal{E}|$.

Let $\mathbf{h}$, $\mathbf{e}_1, \dots, \mathbf{e}_{9-d}$ be a standard basis of $\mathrm{Pic}(S)$. Since cases $n \le 3$ are trivial, we may assume that $n \ge 4$. Then the Weil group $\mathrm{W}(\Delta)$ transitively acts on Δ [Dol, 8.2.14] and we can take it so that $\alpha = \mathbf{e}_1 - \mathbf{e}_2$. Now it is easy to compute $\mathcal{E}$ (cf. [Dol]). For example, for $d = 6$ we have $\mathcal{E} = \{\mathbf{e}_3, \mathbf{h} - \mathbf{e}_1 - \mathbf{e}_2\}$, and for $d = 5$ we have $\mathcal{E} = \{\mathbf{e}_4, \mathbf{e}_4, \mathbf{h} - \mathbf{e}_1 - \mathbf{e}_2, \mathbf{h} - \mathbf{e}_3 - \mathbf{e}_4\}$. Other cases are similar. For $d = 1$ we also can observe that $\mathcal{E} = \Delta'' + K_S$ and apply Corollary 7.1.3. □

Below we describe del Pezzo threefolds X with $\mathrm{r}(X) + \mathrm{d}(X) = 9$ explicitly and give examples. These threefolds were studied extensively in classical literature (see, e.g., [SR85, ch VIII, §2]). We assume that X is singular (otherwise $X \simeq \mathbb{P}^3$, V_7, or $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$).

7.3. Sextic del Pezzo threefold. Let $X \subset \mathbb{P}^2 \times \mathbb{P}^2$ be given by the equation $x_1y_1 + x_2y_2 = 0$. Then X is a del Pezzo threefold with $\mathrm{d}(X) = 6$ and $\mathrm{r}(X) = 3$. The singular locus consists of one node.

7.4. Quintic del Pezzo threefold (cf. [Tod30]**).** Let $X \subset \mathrm{Gr}(2,5)$ be an intersection of three general Schubert subvarieties of codimension one. Then X is a del Pezzo threefold with $\mathrm{d}(X) = 5$ and $\mathrm{r}(X) = 4$. The singular locus consists of three nodes.

7.5. Quartic del Pezzo threefold. Let $X \subset \mathbb{P}^5$ be an intersection of two quadrics having 6 isolated singular points. Then in some coordinate system X can be given by the equations

$$x_1^2 - x_2^2 = x_3^2 - x_4^2 = x_5^2 - x_6^2. \tag{7.5.1}$$

In [SR85, ch VIII, 2.31] this variety is called the *tetrahedral quartic threefold*. By Corollary 10.6.2 $\mathrm{r}(X) = 5$. The variety X contains 8 planes

$$\Pi_{\epsilon_1,\epsilon_2,\epsilon_3} = \{x_1 + \epsilon_1 x_2 = x_3 + \epsilon_2 x_4 = x_5 + \epsilon_3 x_6 = 0\},$$

where $\epsilon_i = \pm 1$. Clearly,

$$\dim \Pi_{\epsilon_1,\epsilon_2,\epsilon_3} \cap \Pi_{\epsilon_1',\epsilon_2',\epsilon_3'} = -1 + \frac{1}{2}\sum |\epsilon_i + \epsilon_i'|.$$

Therefore, for each plane $\Pi = \Pi_{\epsilon_1,\epsilon_2,\epsilon_3}$ there is exactly 3 planes Π' such that $\Pi \cap \Pi'$ is a point and exactly 3 planes Π' such that $\Pi \cap \Pi'$ is a line. Note that there are two 4-tuples of planes such that planes in each tuple meet each other only by subsets of dimension ≤ 0:

$$\{\Pi_{+++}, \Pi_{+--}, \Pi_{-+-}, \Pi_{--+}\}, \quad \{\Pi_{---}, \Pi_{-++}, \Pi_{+-+}, \Pi_{++-}\}.$$

The involution

$$\tau : (x_1, x_2, x_3, x_4, x_5, x_6) \longmapsto (x_1, -x_2, x_3, -x_4, x_5, -x_6)$$

interchanges these 4-tuples. Hence τ induces a birational (cubo-cubic) involution on $\mathbb{P}^3$. In [Hud27, Ch. XIV, §14, P. 301] it is denoted by T_{tet}. Note however that $\mathrm{Cl}(X)^\tau \not\simeq \mathbb{Z}$, i.e. X is not τ-minimal. X is minimal with respect to the whole automorphism group.

7.6. Segre cubic. If $\mathrm{d}(X) = 3$, then X is can be given by

$$X = X_3^{\mathrm{s}} = \left\{ \sum_{i=1}^{6} x_i = \sum_{i=1}^{6} x_i^3 = 0 \right\} \subset \mathbb{P}^4 \subset \mathbb{P}^5. \tag{7.6.1}$$

This cubic satisfies many remarkable properties (see [SR85, ch VIII, 2.32]) and is called the *Segre cubic*. For example, any cubic hypersurface in $\mathbb{P}^4$ has at most ten isolated singular points, this bound is sharp and achieved exactly for the Segre cubic (up to projective isomorphism). The symmetric group $\mathfrak{S}_6$ acts on X_3^{s} in the standard way. Moreover, it is easy to show that $\mathrm{Aut}(X_3^{\mathrm{s}}) = \mathfrak{S}_6$, so the natural map $\mathrm{Aut}(X_3^{\mathrm{s}}) \to \mathrm{W}(\Delta'')$ is an isomorphism.

7.7. Quartic double solid. Let X be a del Pezzo threefold of degree 2. Let $\phi : X \to \mathbb{P}^3$ be the map the half-anticanonical map. Then ϕ is a double cover whose branch locus $B \subset \mathbb{P}^3$ is a quartic having 16 singular points. It is well-known that such a quartic must be a Kummer surface, so the singularities of B and X are at worst nodes [Hud05], [Nik75] (see also [Jes16]). The singular points of X correspond to 15 lines L_{ij} passing through pairs of points P_i, P_j and one twisted cubic passing through all points $P_1, \dots, P_6$. The threefold X contains 32 planes [SR85, ch VIII, 2.33]. For each such a plane Π the image $\pi(\Pi)$ is a plane touching B along a conic.

7.7.1. Example. Let $S \subset \mathbb{P}^3$ be a surface given by the equation $x_0^4 + x_1^4 + x_2^4 + x_3^4 - 4x_0x_1x_2x_3 = 0$. Then the singular locus of S consists of 16 isolated points which can be obtained from $(\sqrt{-1}, \sqrt{-1}, -1, 1)$, $(-1, -1, 1, 1)$, and $(1, 1, 1, 1)$ by permutation of coordinates. It is easy to check that these points are simple nodes. Let $X \to \mathbb{P}^3$ be a double cover branched along S.

7.8. Double Veronese cone. We use notation of 3.6. Assume for simplicity that the singularities of X are at worst nodes. Then B is a surface having exactly 28 points of type A_1. Conversely if $B \subset \mathbb{P}(1^3, 2)$ is a surface of degree 6 whose singularities are exactly 28 points of type A_1, then the double cover of $\mathbb{P}(1^3, 2)$ with branch divisor B is a del Pezzo threefold with $\mathrm{d}(X) = 1$ and $\mathrm{r}(X) = 8$. See [DO88] for more detailed treatment and more references.

7.8.1. Example. Let $C \subset \mathbb{P}^2$ is given by the equation $f = x_1^4 + x_2^4 + x_3^4$. Then the dual curve C^* is given by $f^* = (x_1^4 + x_2^4 + x_3^4)^3 - 27x_1^4x_2^4x_3^4$. It is easy to check that the discriminant of the polynomial $h(t) = t^3 - (x_1^4 + x_2^4 + x_3^4)t + 2x_1^2x_2^2x_3^2$ is equal to $4f^*$. The last polynomial defines a surface $B \subset \mathbb{P}(1^3, 2)$ of degree 6 having 28 singular points $(\pm 1, \pm 1, \pm 1, 1)$, $(\sqrt{-1}, \sqrt{-1}, \pm 1, 1)$, $(\sqrt{-1}, \pm 1, \sqrt{-1}, 1)$, $(\pm 1, \sqrt{-1}, \sqrt{-1}, 1)$, $(0, 1, \sqrt[4]{-1}, 0)$, $(1, 0, \sqrt[4]{-1}, 0)$, $(\sqrt[4]{-1}, 1, 0, 0)$.

7.9. Corollary. *Let X be a del Pezzo threefold such that $\mathrm{d}(X) + \mathrm{r}(X) = 9$ and $\mathrm{d}(X) \neq 5$, 6, 7. Then X is a G-del Pezzo threefold with respect to some group G.*

8. Del Pezzo threefolds with $\mathrm{r}(X) = 8 - \mathrm{d}(X)$

Let, as above, $V_6 \subset \mathbb{P}^7$ be a smooth del Pezzo threefold of degree 6 and let $f_i : V_6 \to \mathbb{P}^2$, $i = 1$, 2 be $\mathbb{P}^1$-bundles. We say that points $P_1, \dots, P_n \in V_6$ are in *general position* if so are the points $f_i(P_1), \dots, f_i(P_n) \in \mathbb{P}^2$ for $i = 1$ and 2.

8.1. Theorem. *Let X be a del Pezzo threefold with $\mathrm{r}(X) + \mathrm{d}(X) = 8$. Then*

(i) *X can be obtained by applying construction (3.10.1) to $V_6 \subset \mathbb{P}^7$ where σ is the blowup of from $n := 6 - \mathrm{d}(X)$ points $P_1, \dots, P_n \in V_6$ in general position.*

(ii) *Singular points of X are images of proper transforms of*
  (a) *curves of bidegree $(0, 1)$ and $(1, 0)$ passing through one of the points P_i;*
  (b) *curves of bidegree $(1, 1)$ passing through two of the points P_i;*
  (c) *curves of bidegree $(2, 2)$ passing through four of the points P_i;*
  (d) *(only for $\mathrm{d}(X) = 1$) curves of bidegree $(2, 3)$ and $(3, 2)$ passing through all the points P_i.*

(iii) *If all the singularities of X are nodes, then $\mathrm{s}(X) = 27$, 15, 9, 5, 2, 0 in cases $\mathrm{d}(X) = 1$, 2, 3, 4, 5, 6, respectively.*

(iv) *If $\mathrm{d}(X) \geq 2$, then all the singularities of X are nodes.*

Conversely, assume that X is a del Pezzo threefold whose singularities are at worst nodes and assume that $\mathrm{s}(X) = 27,\ 15,\ 9,\ 5,\ 2,\ 0$ *in cases* $\mathrm{d}(X) = 1,\ 2,\ 3,\ 4,\ 5,\ 6$, *respectively. Then* $\mathrm{d}(X) + \mathrm{r}(X) = 8$.

Proof. Run construction (3.10.1) so that n is maximal possible. On the last step we get a primitive weak del Pezzo threefold $\bar{X}$ with $\rho(\bar{X}) = 8 - \mathrm{d}(\bar{X}) < 8$. Moreover, if $\rho(\bar{X}) = 3$, then $n = 0$, $\rho(\hat{X}) = \mathrm{r}(X) = 3$, and $\mathrm{d}(X) = 5$. This is impossible by Theorem 4.2. Therefore, $\rho(\bar{X}) = 2$ and $\mathrm{d}(\bar{X}) = 6$. By Theorem 5.2 we have only one possibility: $\bar{X} \simeq V_6$. □

8.1.1. Corollary. *Let X be a del Pezzo threefold with* $\mathrm{r}(X)+\mathrm{d}(X) = 8$. *If* $\mathrm{d}(X) \geq 5$, *then X is unique up to isomorphism. There are exactly two isomorphism classes of del Pezzo threefolds with* $\mathrm{d}(X) = \mathrm{r}(X) = 4$.

Proof. Indeed, in the case $\mathrm{d}(X) = 4$ two non-isomorphic del Pezzo threefolds X are obtained by blowing up a couple of points corresponding to flags $(L_1, P_1), (L_2, P_2) \in \mathrm{F}(\mathbb{P}^2) = V_6$ such that either $L_1 \cap L_2 \neq P_i$ or $L_1 \cap L_2 = P_i$. □

Similar to Theorem 7.2 one can prove the following.

8.2. Theorem. *Let X be a del Pezzo threefold with* $\mathrm{r}(X) + \mathrm{d}(X) = 8$. *Let $\Pi \subset X$ is a plane, let $\hat{\Pi} \subset \hat{X}$ be its proper transform, and let $\bar{\Pi} = \sigma(\hat{\Pi}) \subset \bar{X} = V_6$. Then $\bar{\Pi}$ is of one of the following types:*

- (i) *$\bar{\Pi}$ is one of the points P_i, $\hat{\Pi}$ is σ-exceptional;*
- (ii) *$f_j(\bar{\Pi})$ is a line for $j = 1$ or 2, and $\bar{\Pi}$ contains two of the points P_i;*
- (iii) *$\bar{\Pi}$ is an element of $|-\frac{1}{2}K_{V_6}|$ passing through four of the points P_i so that one of them is a double point;*
- (iv) *(only for* $\mathrm{d}(X) = 1$*) $f_j(\bar{\Pi})$ is a conic for $j = 1$ or 2, and $\bar{\Pi}$ contains all the points P_i;*
- (v) *(only for* $\mathrm{d}(X) = 1$*) $\bar{\Pi}$ is an element of $|-K_{V_6}|$ passing through all of the points P_i so that all of them are double points and one of them is triple;*
- (vi) *(only for* $\mathrm{d}(X) = 1$*) $\bar{\Pi}$ is an element of $|-K_{V_6} - f_j^*\mathscr{O}_{\mathbb{P}^2}(1)|$, where $j = 1$ or 2, passing through all of the points P_i so that three of them are double points.*

The number of planes on X is given by the following table:

$\mathrm{d}(X)$	6	5	4	3	2	1
$\mathrm{p}(X)$	0	1	4	9	20	72

8.2.1. Corollary. *Let X be a del Pezzo threefold with* $\mathrm{r}(X) = 8 - \mathrm{d}(X)$ *and* $\mathrm{d}(X) \leq 5$. *Then in some standard basis of* $\mathrm{Pic}(S)$ *the image of* ι^* :

$\mathrm{Cl}(X) \to \mathrm{Pic}(S)$ *is a sublattice orthogonal to roots* $\mathbf{e}_1 - \mathbf{e}_2$, $\mathbf{e}_2 - \mathbf{e}_3 \in \Delta$, *i.e.* $\Delta' = \{\pm\mathbf{e}_1 \mp \mathbf{e}_2,\ \pm\mathbf{e}_2 \mp \mathbf{e}_3,\ \pm(\mathbf{h} - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3)\}$. *Moreover,* Δ'' *is of type* E_6, D_5, $2A_2$, $2A_1$ *in cases* $\mathrm{d}(X) = 1, 2, 3, 4$, *respectively.*

8.2.2. Corollary. *Let* X *be a del Pezzo threefold with* $\mathrm{r}(X) = 8 - \mathrm{d}(X)$ *and* $\mathrm{d}(X) \le 2$. *If* $\Bbbk$ *is algebraically closed (i.e. we are in the geometric case), then the map* $G \to \mathrm{Aut}(\Delta'')$ *is an embedding.*

Now we give some examples.

8.3. Quintic del Pezzo threefold (cf. [Tod30]**).** Let $X \subset \mathrm{Gr}(2,5)$ be an intersection of two general Schubert subvarieties of codimension one and one general hyperplane section. Then X is a del Pezzo threefold with $\mathrm{d}(X) = 5$ and $\mathrm{r}(X) = 3$. The singular locus consists of two nodes.

8.4. Quartic del Pezzo threefold. Let $X \subset \mathbb{P}^5$ be given by the equations

$$x_1^2 + x_1x_3 + x_2x_5 = x_1x_3 + x_3^2 + x_4x_6 = 0.$$

Then X is a del Pezzo threefold of degree 4 containing exactly 5 nodes. By Corollary 10.6.2 $\mathrm{r}(X) \ge 4$. On the other hand, X is not of type 7.5 because $\mathrm{s}(X) < 6$. Hence $\mathrm{r}(X) = 4$.

8.5. Cubic hypersurface. Let $X \subset \mathbb{P}^5$ be given by the equation

$$x_1x_2\ell(x_1 \dots, x_5) + (x_3x_4 + x_1x_2)x_5 = 0,$$

where ℓ is a sufficiently general linear form. Then X is a cubic del Pezzo threefold with $\mathrm{s}(X) = 9$, $\mathrm{r}(X) = 5$, and $\mathrm{p}(X) = 9$ (cf. [FW, J14]).

8.6. Quartic double solid. Let Y be a hypersurface in $\mathbb{P}^4$ given by $\{s_1 = 4s_4 - s_2^2 = 0\} \subset \mathbb{P}^5$, where $s_k = \sum x_i^k$. This famous hypersurface is called *Igusa quartic*. The singular locus of Y consists of 15 lines. Consider a general hyperplane section $B := Y \cap \mathbb{P}^3$. Then B is a quartic having 15 nodes (cf. [Jes16]). Let $X \to \mathbb{P}^3$ be a double cover with branch divisor B. Then X is a del Pezzo threefold of degree 2 with $\mathrm{s}(X) = 15$ and $\mathrm{r}(X) = 6$.

8.7. Corollary (cf. [Tod30]**,** [Fuj86]**).** *Let* X *be a del Pezzo threefold of degree* 5. *Then the singularities of* X *are at worst nodes and one of the following holds:*

(i) $X \simeq V_5$, *a smooth del Pezzo quintic threefold;*
(ii) $\mathrm{s}(X) = 1$, $\mathrm{r}(X) = 2$, $\mathrm{p}(X) = 0$, *and* X *is of type* 5.4.2*;*
(iii) $\mathrm{s}(X) = 2$, $\mathrm{r}(X) = 3$, $\mathrm{p}(X) = 1$, *and* X *is of type* 7.4*;*
(iv) $\mathrm{s}(X) = 3$, $\mathrm{r}(X) = 4$, $\mathrm{p}(X) = 4$, *and* X *is of type* 8.3.

Proof. Assertions (iii) or (iv) follows by the results of this and previous sections. If $\mathrm{r}(X) = 2$, then we have case (ii) by Theorem 5.2. Finally, if X is factorial, then it is smooth by Corollary 5.3. □

9. G-del Pezzo threefolds

9.1. In this section we prove Theorem 1.7. We use notation of 6.1. Furthermore we assume that X is a G-del Pezzo threefold. Thus $\mathrm{Cl}(X)^G \simeq \mathbb{Z}$. By Theorem 5.2 we may assume that $\mathrm{r}(X) \geq 3$.

9.2. Lemma. *In the above notation, if* $\mathrm{d}(X) \geq 5$*, then* $X \simeq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Proof. Assume that $X \not\simeq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Then X is singular and $\mathrm{d}(X) = 6$ or 5 by Theorems 3.2 and 3.5.

Consider the case $\mathrm{d}(X) = 6$. Since $\mathrm{r}(X) \geq 3$, our X is described in 7.3. Then X contains exactly two planes Π_1, Π_2 and the divisor $\Pi_1 + \Pi_2$ is G-invariant. Hence $\Pi_1 + \Pi_2 \sim aS$ for some positive integer a. Comparing degrees we get $2 = 6a$, a contradiction.

Now let $\mathrm{d}(X) = 5$. By Lemma 3.3 we may assume that X is not factorial. In this situation, X is imprimitive. The same arguments as above show that the number of planes on X in any G-orbit must be divisible by 5. This contradicts Corollary 8.7. □

9.3. From now on we assume that $\mathrm{d}(X) \leq 4$. By Theorem 4.2 we may assume that X is imprimitive. Let $S \in |-\frac{1}{2}K_X|$ be a general member. Let $n := \mathrm{rk}\,\Delta = 9 - \mathrm{d}(X)$.

9.3.1. Lemma. *If in the above notation* $\mathrm{d}(X) \leq 4$*, then* X *contains at least two planes* Π_1*,* Π_2 *such that* $\dim \Pi_1 \cap \Pi_2 \leq 0$.

Proof. Since X is imprimitive, it contains at least one plane Π_1. Let $\Pi_1, \dots, \Pi_l$ be its orbit. Since $\mathrm{Cl}(X)^G = \mathbb{Z} \cdot S$, $k \geq 4$. If $\dim \Pi_i \cap \Pi_j \geq 1$ for all i, j, the linear span of $\Pi_1, \dots, \Pi_k$ is three-dimensional and so X cannot be an intersection of quadrics. □

First we consider the case $\Delta'' = \emptyset$.

9.4. Proposition. *If in the above notation* $\Delta'' = \emptyset$*, then* $\mathrm{d}(X) = 3$*,* $\mathrm{r}(X) = 3$*,* $\mathrm{p}(X) = 3$*, and* X *is a projection of a del Pezzo threefold* $Y = Y_4 \subset \mathbb{P}^5$ *of type* (5.2.8) *from a point. If moreover the singularities of* X *are at worst nodes, then by* [FW]*,* X *is of type* J11 *or* J12*, and* $6 \leq \mathrm{s}(X) \leq 7$.

Proof. Let $\mathbf{e}_{n-m+1}, \dots, \mathbf{e}_n$ correspond to blowups σ. If $m > 1$, then $\mathbf{e}_{n-1} - \mathbf{e}_n \in \Delta''$. Thus, $m = 1$, $\mathrm{d}(\bar{X}) = \mathrm{d}(X) + 1$, and we may assume that every two planes on X meet each other by a subset of dimension 1. Therefore, $\mathrm{r}(\bar{X}) = 2$ and $\mathrm{r}(X) = 3$. By Lemma 9.3.1 $\mathrm{d}(X) \leq 3$. Therefore, $\mathrm{d}(\bar{X}) \leq 4$, and $8 \geq n \geq 6$.

Consider the case where $f : \bar{X} \to Z = \mathbb{P}^2$ is a $\mathbb{P}^1$-bundle. Then by Theorem 6.2 we may assume that the vectors $\mathbf{e}_1 - \mathbf{e}_2, \dots, \mathbf{e}_{n-2} - \mathbf{e}_{n-1}$ form a basis of Δ'. We have:

$n = 6$, $\mathrm{d}(X) = 3 \Longrightarrow 2\mathbf{h} - \sum \mathbf{e}_i \in \Delta''$,
$n = 7$, $\mathrm{d}(X) = 2 \Longrightarrow 2\mathbf{h} + \mathbf{e}_7 - \sum \mathbf{e}_i \in \Delta''$,
$n = 8$, $\mathrm{d}(X) = 1 \Longrightarrow 3\mathbf{h} - \mathbf{e}_8 - \sum \mathbf{e}_i \in \Delta''$.

Thus, in all cases we have $\Delta'' \neq \emptyset$, a contradiction.

Consider the case where $f : \bar{X} \to Z = \mathbb{P}^1$ is a quadric bundle. Then $\mathrm{d}(X) = \mathrm{d}(\bar{X}) - 1 = 1$ or 3 by Theorem 5.2. Again by Theorem 6.2 we may assume that vectors

$$\mathbf{h} - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3, \quad \mathbf{e}_2 - \mathbf{e}_3, \dots, \mathbf{e}_{n-2} - \mathbf{e}_{n-1}$$

form a basis of Δ'. If $n = 8$, then $3\mathbf{h} - \mathbf{e}_8 - \sum \mathbf{e}_i \in \Delta''$, a contradiction. Therefore, $\mathrm{d}(X) = 3$ and $\bar{X}$ is of type (5.2.8). Thus X is a cubic in $\mathbb{P}^4$. Since for any two planes $\Pi_i, \Pi_j \subset X$ we have $\dim \Pi_i \cap \Pi_j \geq 1$, all the planes on X are contained in one hyperplane. Hence $\mathrm{p}(X) = 3$. By Proposition 10.6 $\mathrm{s}(X) \leq 7 - h^{1,2}(\hat{X})$. If the singularities of X are at worst nodes, then X is of type J11 or J12 by [FW]. □

9.4.1. Example. Consider the cubic $X \subset \mathbb{P}^4$ given by the equation

$$x_1x_2x_3 + x_0\left(\lambda x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2\right) = 0.$$

Then X has 6 (resp. 7) nodes if $\lambda \neq 0$ (resp. $\lambda = 0$). It is easy to see that X contains at least 3 planes, so $\mathrm{r}(X) \geq 3$. By [FW] we have $\mathrm{r}(X) = 3$ and $\mathrm{p}(X) = 3$. The symmetric group $\mathfrak{S}_3$ acts on X by permutations of x_1, x_2, x_3 so that X is a G-Fano threefold.

9.5. Now we assume that $\Delta'' \neq \emptyset$. Then Δ'' is a G-invariant root subsystem in Δ. By the results of §7 and §8 we may assume that $\mathrm{r}(X) \leq 7 - \mathrm{d}(X)$. Further, by Lemma 9.3.1 $\mathrm{d}(X) \leq 3$.

9.6. Consider the case $\mathrm{d}(X) = 3$. There are only the following possibilities:

9.6.1. $\mathrm{d}(\bar{X}) = 4$, $\mathrm{r}(\bar{X}) = 2$, $\bar{X}$ is of type (5.2.8), $\mathrm{r}(X) = 3$. Then Δ' is described in 6.3.2: it is of type D_4 and generated by $\mathbf{h} - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3$, $\mathbf{e}_2 - \mathbf{e}_3$, $\mathbf{e}_3 - \mathbf{e}_4$, $\mathbf{e}_4 - \mathbf{e}_5$. Any root $\alpha \in \Delta$ has the form $\alpha = \pm(\mathbf{e}_i - \mathbf{e}_j)$, $\pm(\mathbf{h} - \mathbf{e}_i - \mathbf{e}_j - \mathbf{e}_k)$ or $\pm(2\mathbf{h} - \sum \mathbf{e}_i)$ (see e.g., [Man86, ch. 4, 3.7]). Since $\iota^* \mathrm{Cl}(X) = \Delta'^{\perp}$, we get $\Delta'' = \emptyset$, a contradiction.

9.6.2. $\mathrm{d}(\bar{X}) = 5$, $\mathrm{r}(\bar{X}) = 1$, $\bar{X} = V_5$, $\mathrm{r}(X) = 3$. Similarly, Δ' is of type A_4 and generated by $\mathbf{h} - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3$, $\mathbf{e}_1 - \mathbf{e}_2$, $\mathbf{e}_2 - \mathbf{e}_3$, $\mathbf{e}_3 - \mathbf{e}_4$. In this case, $\Delta'' = \{\pm(\mathbf{e}_5 - \mathbf{e}_6)\}$. It is easy to see that the group G permutes elements $\mathbf{e}_5$, $\mathbf{e}_6 \in \iota^* \mathrm{Cl}(X)$. But then the class of $\mathbf{e}_5 + \mathbf{e}_6$ must be G-invariant, so it is proportional to $-K_S$, a contradiction.

9.6.3. $\mathrm{d}(\bar{X}) = 5$, $\mathrm{r}(\bar{X}) = 2$, $\bar{X}$ is of type (5.2.5), $\mathrm{r}(X) = 4$. Similarly, Δ' is of type A_3 and generated by $\mathbf{e}_1 - \mathbf{e}_2$, $\mathbf{e}_2 - \mathbf{e}_3$, $\mathbf{e}_3 - \mathbf{e}_4$. Then

$\Delta'' = \{\pm(2\mathbf{h} - \sum \mathbf{e}_i),\ \pm(\mathbf{e}_5 - \mathbf{e}_6)\}$. There is a unique element (class of a line on S) $\mathbf{x} \in (\Delta' + \Delta'')^\perp$ such that $\mathbf{x}^2 = K_X \cdot \mathbf{x} = -1$:

$$\mathbf{x} = \mathbf{h} - \mathbf{e}_5 - \mathbf{e}_6.$$

But then $x \in \iota^* \operatorname{Cl}(X)$ and x must be G-invariant, a contradiction.

9.7. Finally we consider cases $\operatorname{d}(X) \le 2$. According to Remark 3.4.1 any del Pezzo threefold with $\operatorname{d}(X) \le 2$ is automatically G-del Pezzo. Thus all the possibilities for $\bar{X}$ with $2 \le \operatorname{d}(\bar{X}) \le 5$ and $\operatorname{r}(\bar{X}) \le 2$ do occur (recall that $3 \le \operatorname{r}(X) \le 7 - \operatorname{d}(X)$):

- $\bar{X} = V_5 \Longrightarrow \operatorname{d}(X) \le 2$, $\Delta' \simeq A_4$;
- $\bar{X} = V_4 \Longrightarrow \operatorname{d}(X) \le 2$, $\Delta' \simeq D_5$;
- $\bar{X} = V_3 \Longrightarrow \operatorname{d}(X) \le 2$, $\Delta' \simeq E_6$;
- $\bar{X}$ is of type (5.2.2) $\Longrightarrow \operatorname{d}(X) = 1$, $\Delta' \simeq A_6$;
- $\bar{X}$ is of type (5.2.3) $\Longrightarrow \operatorname{d}(X) \le 2$, $\Delta' \simeq A_5$;
- $\bar{X}$ is of type (5.2.5) $\Longrightarrow \operatorname{d}(X) \le 2$, $\Delta' \simeq A_3$;
- $\bar{X}$ is of type (5.2.7) $\Longrightarrow \operatorname{d}(X) = 1$, $\Delta' \simeq D_6$;
- $\bar{X}$ is of type (5.2.8) $\Longrightarrow \operatorname{d}(X) \le 2$, $\Delta' \simeq D_4$.

The number of planes can be found by using Lemma 3.8.3 and direct computations.

9.7.1. Example. Let $X \subset \mathbb{P}(1^4, 2)$ is given by the equation

$$y^2 = x_1x_2x_3x_4 + \lambda(x_1^2 + x_2^2 + x_3^2 + x_4^2)^2$$

where λ is a constant. Then X has exactly 12 nodes and contains 8 planes. By Corollary 10.6.2 $\operatorname{r}(X) \ge 3$. Further, by our classification X is of type 22^o.

More examples of del Pezzo therefolds with $\operatorname{d}(X) = 2$ can be constructed similarly by writing down explicit equations (cf. [Jes16]).

10. Appendix: number of singular points of Fano threefolds

10.1. Definition. Let $V \ni P$ be a threefold terminal Gorenstein (=isolated cDV) singularity. We say that $V \ni P$ is *r-nondegenerate* (resolution nondegenerate) if there is a resolution

$$\sigma : V_m \overset{\sigma_m}{\to} \cdots \overset{\sigma_2}{\to} V_1 \overset{\sigma_1}{\to} V = V_0,$$

where each σ_i is a blowup of a singular *point* $P_{i-1} \in V_{i-1}$. Such a resolution σ is called *standard*. In this situation, all varieties V_i also have only isolated cDV singularities. If furthermore each σ_i-exceptional divisor $E_i \subset V_i$ is irreducible, then we say that $V \ni P$ is *rs-nondegenerate* (strongly resolution nondegenerate).

Denote $\lambda(V,P) := m$ and let $\nu(V,P)$ be the number of σ-exceptional divisors. Thus $\lambda(V,P) \le \nu(V,P)$ and the equality holds if and only if $V \ni P$ is rs-nondegenerate.

10.2. Remark. Let $V \ni P$ be a threefold terminal Gorenstein point and let $\sigma_1 : V_1 \to V$ be the blowup of P. Since $V \ni P$ is a hypersurface singularity, we have an (analytic) embedding

$$\begin{array}{ccc} V_1 & \xrightarrow{\sigma_1} & V \\ \downarrow & & \downarrow \\ \tilde{\mathbb{C}}^4 & \xrightarrow{\tilde{\sigma}_1} & \mathbb{C}^4 \end{array}$$

where $\tilde{\sigma}_1 : \tilde{\mathbb{C}}^4 \to \mathbb{C}^4$ is the blowup of the origin. Let $D := \tilde{\sigma}_1^{-1}(P)$ be the exceptional divisor. Then $D \simeq \mathbb{P}^3$. Since $V \ni P$ is a singularity of multiplicity 2, we have one of the following cases:

(i) $D \cap V_1$ is a smooth quadric,
(ii) $D \cap V_1$ is a quadratic cone,
(iii) $D \cap V_1$ is a couple of planes,
(iv) $D \cap V_1$ is a double plane.

In cases (i) and (ii) V_1 is either smooth or have terminal singularity. Moreover, the above arguments show that $2\lambda(V,P) \ge \nu(V,P)$.

10.3. Proposition. *Let $(V \ni 0) \subset \mathbb{C}^4$ be a singularity given by $t^2 = \phi(x,y,z)$, where $\phi = 0$ is an equation of a Du Val singularity. Then $V \ni 0$ is r-nondegenerate. Moreover, if $\phi = 0$ defines a singularity of type A_n, then $V \ni 0$ is rs-nondegenerate.*

Proof. Direct computation. □

10.3.1. Corollary. *Let X be a del Pezzo threefold with $\mathrm{d}(X) \le 2$. Assume that the branch divisor B of the double cover $\varphi : X \to \mathbb{P}(1^3, 2)$ (resp. $\varphi : X \to \mathbb{P}^3$) has only Du Val singularities (see 3.4.1). Then the singularities of X are r-nondegenerate. If moreover B has only singularities of type A, then the singularities of X are rs-nondegenerate.*

10.4. Let W be a smooth projective fourfold and let $V \subset W$ be an effective divisor. Define

$$\beta(W,V) := c_3(W)\cdot V - c_2(W)\cdot V^2 + c_1(W)\cdot V^3 - V^4.$$

If V is smooth then $\beta(W,V)$ coincides with $\deg c_3(V) = \mathrm{Eu}(V)$, the topological Euler number of V.

10.5. Lemma. *In the above notation let $P \in V$ be a singular point, let $\sigma : \tilde{W} \to W$ be the blowup of P, and let $\tilde{V} \subset \tilde{W}$ be the proper transform of V. Then $\beta(\tilde{W},\tilde{V}) = \beta(W,V) + 4$.*

Proof. Let $R = \sigma^{-1}(P)$ be the exceptional divisor in $\tilde{W}$, and let $E = R \cap \tilde{V}$ be the exceptional divisor in $\tilde{V}$. We have

$$\tilde{V} \sim \sigma^* V - 2E, \quad c_3(\tilde{W}) = \sigma^* c_3(W) + 2E^3, \\ c_2(\tilde{W}) = \sigma^* c_2(W) + 2E^2, \quad c_1(\tilde{W}) = \sigma^* c_1(W) - 3E.$$

Using the equality $c(\tilde{V}) = c(\tilde{W}) \cdot c(N_{\tilde{V}/\tilde{W}})^{-1}$, we get

$$\beta(\tilde{W}, \tilde{V}) = c_3(\tilde{W}) \cdot \tilde{V} - c_2(\tilde{W}) \cdot \tilde{V}^2 + c_1(\tilde{W}) \cdot \tilde{V}^3 - \tilde{V}^4 = \\ = (\sigma^* c_3(W) + 2E^3) \cdot (\sigma^* V - 2E) - (\sigma^* c_2(W) + 2E^2) \cdot (\sigma^* V - 2E)^2 + \\ + (\sigma^* c_1(W) - 3E) \cdot (\sigma^* V - 2E)^3 - (\sigma^* V - 2E)^4 = \beta(W, V) - 4E^4.$$

□

10.6. Proposition. *Let X be a Gorenstein Fano threefold whose singularities are r-nondegenerate terminal points. Assume that*

(*) *X can be embedded into a smooth fourfold so that a general member $X' \in |X|$ is smooth.*

Then

$$\sum_{P \in X}{}' \lambda(X, P) \le \sum_{P \in X} \big(2\lambda(X, P) - \nu(X, P)\big) = \\ = \mathrm{r}(X) - \rho(X) + h^{1,2}(X') - h^{1,2}(\hat{X}),$$

where $\hat{X} \to X$ is the standard resolution and the first sum runs through all rs-nondegenerate points $P \in X$.

Proof. Put $\lambda := \sum_{P \in X} \lambda(X, P)$. Thus

$$2 + 2\rho(\hat{X}) - 2h^{1,2}(\hat{X}) = \mathrm{Eu}(\hat{X}) = \beta(\hat{Y}, \hat{X}) = \beta(\hat{Y}, \hat{X}) + 4\lambda = \\ = \mathrm{Eu}(X') + 4\lambda = 2 + 2\rho(X') - 2h^{1,2}(X') + 4\lambda.$$

Since $\rho(\hat{X}) = \mathrm{r}(X) + \sum \nu(X, P)$, this gives the disired inequality. □

10.6.1. Remark. The condition (*) is automatically satisfied if X is a del Pezzo threefold (see Theorem 3.4).

10.6.2. Corollary. *In the notation of* 10.6 *assume additionally that the singularities are rs-nondegenerate. Then*

$$|\operatorname{Sing}(X)| \le \mathrm{r}(X) - \rho(X) + h^{1,2}(X') - h^{1,2}(\hat{X}).$$

The equality holds, if all the singularities are nodes.

11. Concluding remarks and open questions

We would like to propose the following open questions.

11.1. Give a complete birational classification of del Pezzo threefolds over $\mathbb{C}$. Non-trivial cases only are factorial del Pezzo threefolds of degree ≤ 3. All other cases can be reduced to the above ones by using construction 3.10.1 (or birationally trivial). It is well-known that a three-dimensional cubic hypersurface with at worst cDV singularities is rational if and only if it is singular [CG72]. A general smooth (and, in some cases, factorial) del Pezzo threefold of degree ≤ 2 is not rational [AM72], [Bea77], [Tyu79].

11.2. Give a complete birational classification of del Pezzo threefolds over non-closed fields. Here is one example.

11.2.1. Theorem. *Let X be a smooth del Pezzo thereefold of degree 5 over a field $\Bbbk$. Then X is $\Bbbk$-rational.*

Proof. Denote $\bar{X} := X \otimes \bar{\Bbbk}$. Let $\Gamma := \Gamma(X)$ be the Hilbert scheme parameterizing the family of lines on X. It is known that $\bar{\Gamma} := \Gamma \otimes \bar{\Bbbk} \simeq \mathbb{P}^2_{\bar{\Bbbk}}$ (see [Isk80a, Prop. 1.6, ch. 3], [FN89]). Moreover, lines with normal bundle $N_{l/X} \simeq \mathscr{O}(-1) \oplus \mathscr{O}(1)$ are parametrized by some conic $C \subset \Gamma$ [FN89]. The conic C contains a point of degree ≤ 2. Therefore, there is a line $\ell \subset \Gamma$ defined over $\Bbbk$. Let H_ℓ be the union of all lines $L \subset X$ whose class is contained in $\ell \subset \Gamma$. Then H_ℓ in an element of $|-\frac{1}{2}K_X|$ defined over $\Bbbk$ [Isk80a, Proof of Prop. 1.6, ch. 3]. In particular, $\operatorname{Pic}(X) = \mathbb{Z} \cdot \frac{1}{2}K_X$ and the linear system $|-\frac{1}{2}K_X|$ defines an embedding $X \subset \mathbb{P}^6_{\Bbbk}$. A general pencil of hyperplane sections defines a structure of del Pezzo fibration of degree 5 on X. By [Man86, Ch. 4] the variety X is $\Bbbk$-rational. □

11.3. Describe automorphism groups of del Pezzo threefolds over an algebraically closed fields. Which of them are birationally rigid (cf. [CS09], [CS10])? These questions are very usefull for applications to the classification of finite subgroups of Cremona group $\mathrm{Cr}_3(\Bbbk)$ [Pro09b], [Pro09a].

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Department of Algebra, Faculty of Mathematics, Moscow State University, Moscow, 119 991, RUSSIA

Laboratory of Algebraic Geometry, SU-HSE, 7 Vavilova Str., Moscow, 117312, RUSSIA

E-mail address: `prokhoro@gmail.com`