GEOMETRIC REALIZABILITY OF COVARIANT DERIVATIVE KÄHLER TENSORS FOR ALMOST PSEUDO-HERMITIAN AND ALMOST PARA-HERMITIAN MANIFOLDS

M. BROZOS-VÁZQUEZ, E. GARCÍA-RÍO, P. GILKEY, AND L. HERVELLA

Abstract. The covariant derivative of the Kähler form of an almost pseudo-Hermitian or of an almost para-Hermitian manifold satisfies certain algebraic relations. We show, conversely, that any 3-tensor which satisfies these algebraic relations can be realized geometrically.
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1. Introduction

The paper of Gray and Hervella [17] puts into a unified framework 16 classes of almost Hermitian manifolds and was the work which inspired other classification results like those in [24, 28, 29]. It is important in the mathematical setting and is used in obvious settings when some class of Kähler or Hermitian manifolds is the central focus of investigation. The Gray-Hervella decomposition plays a role in the discussion of nearly Kähler and almost Kähler geometry as well as in the study of conformal equivalences among almost Hermitian structures (see for example [11, 23], [4], and [5, 7], respectively). It is related to the Tricerri-Vanhecke [28] decomposition of the curvature tensor in [12] and it has a prominent role in understanding the influence of the curvature on the underlying structure of the manifold [19]. The Gray-Hervella classification is related to the 64 classes of almost quaternion-Hermitian structures in [21], showing some interactions amongst them. The different classes have been considered for flag manifolds – they essentially reduce to four classes [26], and the 6-dimensional case has been considered in detail in [3]. The different classes of almost Hermitian structures also enter into the discussion of some harmonicity problems [5].

Although most of this work has been in the positive definite setting, the indefinite case also plays a role (see for example [10, 15, 18, 22, 27]). In addition to the pseudo-Hermitian setting, the almost para-Hermitian geometry is of interest both from the mathematical and the physical point of view [1, 2, 8, 9, 16, 25]. Related work of Gadea and Masque [14] classified almost para-Hermitian structures into 32 different classes by considering separately the two natural distributions associated to the almost para-Hermitian structure.

In this paper we put both the almost para-Hermitian and the almost pseudo-Hermitian structures in an unified context by extending the Gray-Hervella decomposition to the pseudo-Riemannian setting. This is done by analyzing the covariant derivative of the corresponding Kähler form and the decomposition of the space of such tensors under the action of a suitable structure group (see Theorem 1.4 for details). Moreover we consider the geometric realizability of all the different classes by perturbing the given structures. In Theorem 1.1, we show that *any* algebraic covariant derivative Kähler tensor can be geometrically realized by perturbing the underlying structure on a given almost para/pseudo-Hermitian background manifold; Theorem 1.2 provides a similar result in the integrable setting. In Theorem 1.6, we restrict to the complex setting and extend results of [17] from the positive definite context to the indefinite context showing any of the 16 classes has at least one geometrical representative.

We establish notation as follows. Let (M,g) be a pseudo-Riemannian manifold of dimension $m=2\bar{m}$. Let $J_\pm$ be endomorphisms of the tangent bundle TM. We say that (M,g,J_+) is an *almost para-Hermitian manifold* if $J_+^2=\operatorname{id}$ and if $J_+^*g=-g$. Similarly, if $J_-^2=-\operatorname{id}$ and if $J_-^*g=g$, then we say that (M,g,J_-) is an *almost pseudo-Hermitian manifold*. The existence of such structures is related to the signature (p,q) of g. If (M,g) admits an almost para-Hermitian structure J_+, then $p=q$. Similarly if (M,g) admits an almost pseudo-Hermitian structure J_-, then both p and q are even. Thus usually we are not dealing with both J_- and J_+ at the same time on (M,g), but we adopt a common notation to keep the exposition in parallel as much as possible.

Let ∇ be the Levi-Civita connection of g. The *associated Kähler form* and the covariant derivative are defined, respectively, by:

$$\begin{aligned}&\Omega_\pm(x,y):=g(x,J_\pm y),\\&\nabla\Omega_\pm(x,y;z)=zg(x,J_\pm y)-g(\nabla_z x,J_\pm y)-g(x,J_\pm\nabla_z y)\,.\end{aligned}$$

We subscript J and Ω to keep track of the signs involved. For example, as we shall see presently in Lemma 3.1, we have:

$$\begin{aligned}&\nabla\Omega_\pm(x,y;z)=-\nabla\Omega_\pm(y,x;z),\\&\nabla\Omega_\pm(x,y;z)=\pm\nabla\Omega_\pm(J_\pm x,J_\pm y;z)\,.\end{aligned}\tag{1.a}$$

It is convenient to work in an algebraic context as well. Let $(V,\langle\cdot,\cdot\rangle)$ be an inner product space and let $J_\pm^0$ be linear maps of V. We say that $(V,\langle\cdot,\cdot\rangle,J_+^0)$ is a *para-Hermitian vector space* if $(J_+^0)^*\langle\cdot,\cdot\rangle=-\langle\cdot,\cdot\rangle$ and if $(J_+^0)^2=\operatorname{id}$. Similarly, $(V,\langle\cdot,\cdot\rangle,J_-^0)$ is said to be a *pseudo-Hermitian vector space* if $(J_-^0)^*\langle\cdot,\cdot\rangle=\langle\cdot,\cdot\rangle$ and if $(J_-^0)^2=-\operatorname{id}$. Again, the existence of such structures imposes restrictions on the signature. Motivated by Equation (1.a), we define:

$$\begin{aligned}\mathfrak{H}_\pm:=\{H_\pm\in\otimes^3V^*:H_\pm(x,y;z)=-H_\pm(y,x;z)\quad\text{and}\\ H_\pm(J_\pm^0x,J_\pm^0y;z)=\pm H_\pm(x,y;z)\ \forall\ x,y,z\}\,.\end{aligned}$$

Let $H_\pm\in\mathfrak{H}_\pm$. We have

$$H_\pm(x,J_\pm^0y;z)=\pm H_\pm(J_\pm^0x,J_\pm^0J_\pm^0y;z)=H_\pm(J_\pm^0x,y;z)\,.\tag{1.b}$$

The following result shows that Equation (1.a) generates the universal symmetries satisfied by $\nabla\Omega_\pm$ and provides a rich family of examples. It is striking that we can fix the metric and only vary the almost (para)-complex structure; in particular, we could take the background structure to be flat.

Theorem 1.1. *Let $(M,g,J_\pm)$ be a background almost para/pseudo-Hermitian manifold and let $P\in M$. Suppose given $H_\pm$ in $\mathfrak{H}_\pm(T_PM,g_P,J_{\pm,P})$. Then there exists a new almost para/pseudo-Hermitian structure $\tilde J_\pm$ on M which agrees with $J_\pm$ at P so that $\nabla\Omega_\pm(M,g,\tilde J_\pm)(P)=H_\pm$.*

We consider the following subspace:

$$U_{3,\pm}:=\{H_\pm\in\mathfrak{H}_\pm:H_\pm(x,y;z)=\mp H_\pm(x,J_\pm^0y;J_\pm^0z)\ \forall\ x,y,z\}\,.$$

If $(M,g,J_\pm)$ is a para/pseudo-Hermitian manifold (i.e. $J_\pm$ is integrable), then $\nabla\Omega_\pm\in U_{3,\pm}$ as we shall see presently in Lemma 3.2. Conversely:

Theorem 1.2. *Let $(M,g,J_\pm)$ be a background para/pseudo-Hermitian manifold and let $P\in M$. Suppose given $H_\pm$ in $U_{3,\pm}(T_PM,g_P,J_{\pm,P})$. Then there exists a new para/pseudo-Hermitian metric $\tilde g$ on M which agrees with g at P so that $\nabla\Omega_\pm(M,\tilde g,J_\pm)(P)=H_\pm$.*

Theorems 1.1 and 1.2 are global results; it is necessary to have a starting background structure as not every manifold admits a para/pseudo-Hermitian structure of a given signature; in general, there are topological restrictions on M for the existence of a (para)-complex structure or for the existence of a metric of signature (p, q). These Theorems give results in the category of compact manifolds. However it is a direct consequence of the Theorems that one can also restrict attention to an open coordinate chart to get purely local results.

These results are based on a decomposition of $\mathfrak{H}_\pm$ which extends the decomposition given in [17] in the positive definite context. Adopt the *Einstein convention* and sum over repeated indices.

Definition 1.3. Let $(V, \langle\cdot,\cdot\rangle, J^0_\pm)$ be a para/pseudo-Hermitian vector space. Let $\varepsilon_{ij} := \langle e_i, e_j\rangle$ for some basis $\{e_i\}$ for V. Let $\phi \in V^*$. Let $H \in \otimes^3 V^*$. Let GL be the general linear group. Set:

(1) $(\tau_1 H)(x) := \varepsilon^{ij} H(x, e_i; e_j)$.
(2) $\sigma_\pm(\phi)(x, y; z) := \phi(J^0_\pm x)\langle y, z\rangle - \phi(J^0_\pm y)\langle x, z\rangle + \phi(x)\langle J^0_\pm y, z\rangle - \phi(y)\langle J^0_\pm x, z\rangle$.
(3) $W_{1,\pm} := \{H_\pm \in \mathfrak{H}_\pm : H_\pm(x, y; z) + H_\pm(x, z; y) = 0\ \forall\ x, y, z\}$.
(4) $W_{2,\pm} := \{H_\pm \in \mathfrak{H}_\pm : H_\pm(x, y; z) + H_\pm(y, z; x) + H_\pm(z, x; y) = 0\ \forall\ x, y, z\}$.
(5) $W_{3,\pm} := U_{3,\pm} \cap \ker(\tau_1)$.
(6) $W_{4,\pm} := \text{Range}(\sigma_\pm)$.
(7) $\mathcal{O} := \{T \in \text{GL} : T^*\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle\}$.
(8) $\mathcal{U}_\pm := \{T \in \mathcal{O} : TJ^0_\pm = J^0_\pm T\}$.
(9) $\mathcal{U}^\star_\pm := \{T \in \mathcal{O} : TJ^0_\pm = TJ^0_\pm \text{ or } TJ^0_\pm = -J^0_\pm T\}$.
(10) $\text{GL}_\pm := \{T \in \text{GL} : TJ^0_\pm = J^0_\pm T\}$.
(11) $\chi(T) := +1$ if $T \in \mathcal{U}_\pm$ and $\chi(T) := -1$ if $T \in \mathcal{U}^\star_\pm - \mathcal{U}_\pm$.

Theorem 1.4. *Let $m \geq 6$. We have a direct sum orthogonal decomposition of $\mathfrak{H}_\pm$ and of $U_{3,\pm}$ into irreducible inequivalent $\mathcal{U}^\star_\pm$ modules in the form:*

$$\mathfrak{H}_\pm = W_{1,\pm} \oplus W_{2,\pm} \oplus W_{3,\pm} \oplus W_{4,\pm} \quad \textit{and} \quad U_{3,\pm} = W_{3,\pm} \oplus W_{4,\pm}\,.$$

One obtains the corresponding decompositions if $m = 4$ by setting $W_{1,\pm} = 0$ and $W_{3,\pm} = 0$. The modules $W_{i,-}$ are also irreducible $\mathcal{U}_-$ modules so the decomposition of [17] of $\mathfrak{H}_-$ as a $\mathcal{U}_-$ module extends without change from the positive definite to the indefinite setting; we omit the additional analysis this requires in the interests of brevity. The modules $W_{i,+}$ are not, however, irreducible $\mathcal{U}_+$ modules and thus the classification of [14] is a more refined one than we consider here as there are 8 factors in the decomposition rather than 4. By using the structure group $\mathcal{U}^\star_+$ instead of $\mathcal{U}_+$, we shall bypass some of the technical difficulties encountered in [14] and this structure group is sufficient for our purposes.

The focus of Theorem 1.1 and of Theorem 1.2 is to show that *every* element of $\mathfrak{H}_\pm$ and of $U_{3,\pm}$ is geometrically realizable in an appropriate context. One can, however, focus instead on the precise nature of the classes involved. We now restrict to the complex setting. Let ξ be a $\mathcal{U}^\star_-$ submodule of $\mathfrak{H}_-$. We say that (M, g, J_-) is a ξ-*manifold* if $\nabla\Omega_-$ belongs to ξ for every point of the manifold and if ξ is minimal with this property. This gives rise to the celebrated 16 classes of almost Hermitian manifolds (in the positive definite setting) [17]:

Theorem 1.5. *Let ξ be a submodule of $\mathfrak{H}_-$. Then there exists an almost Hermitian ξ-manifold.*

We can generalize this to the indefinite setting; we shall suppose $m \geq 10$ to simplify the discussion:

Theorem 1.6. *Suppose given $(2\bar{p}, 2\bar{q})$ with $2\bar{p} + 2\bar{q} \geq 10$. Let ξ be a submodule of $\mathfrak{H}_-$. Then there exists a ξ-manifold of signature $(2\bar{p}, 2\bar{q})$.*

Many of these classes have geometrical meanings which have been extensively investigated. For example:

(1) $\xi = \{0\}$ defines the class of Kähler manifolds.
(2) $\xi = W_{1,-}$ defines the class of nearly Kähler manifolds.
(3) $\xi = W_{2,-}$ defines the class of almost Kähler manifolds.
(4) $\xi = W_{3,-}$ defines the class of Hermitian semi-Kähler manifolds.
(5) $\xi = W_{1,-} \oplus W_{2,-}$ defines the class of quasi-Kähler manifolds.
(6) $\xi = W_{3,-} \oplus W_{4,-} = U_{3,-}$ defines the class of pseudo-Hermitian manifolds.
(7) $\xi = W_{1,-} \oplus W_{2,-} \oplus W_{3,-}$ defines the class of semi-Kähler manifolds.
(8) $\xi = \mathfrak{H}_-$ defines the class of almost pseudo-Hermitian manifolds.

Here is a brief outline to the paper. In Section 2, we review briefly the representation theory we shall need concerning $\mathcal{U}_\pm^\star$ submodules of $\otimes^k V^*$ and obtain an upper bound on the dimension of the space of quadratic invariants for $\mathfrak{H}_\pm$ as a $\mathcal{U}_\pm^\star$ module. In Section 3, we turn to the geometric setting and study $\nabla\Omega_\pm$. In Section 4, we examine matters in the algebraic context and define projectors on the spaces $W_{1,\pm}$, $W_{2,\pm}$, $U_{3,\pm}$, and $W_{4,\pm}$. In Section 5, we fix the metric and vary the almost (para)-complex structure to prove Theorem 1.1 and Theorem 1.4. In Section 6, we assume the (para)-complex structure to be integrable and vary the metric to prove Theorem 1.2. In Section 7, we use results of [17] to establish Theorem 1.6.

2. Representation theory

Let $(V, \langle\cdot,\cdot\rangle, J_\pm)$ be a para/pseudo-Hermitian space. Extend $\langle\cdot,\cdot\rangle$ to $\otimes^k V$ so

$$\langle (v_1 \otimes \cdots \otimes v_k), (w_1 \otimes \cdots \otimes w_k) \rangle := \prod_{i=1}^{k} \langle v_i, w_i \rangle \,. \tag{2.a}$$

Equation (2.a) defines a non-degenerate symmetric bilinear form on $\otimes^k V$. We use $\langle\cdot,\cdot\rangle$ to identify V with V^* and $\otimes^k V$ with $\otimes^k V^*$. If $\theta \in \otimes^k V^*$ and if $u \in \mathcal{U}_\pm^\star$, the pull-back $u^*\theta \in \otimes^k V^*$ is defined by $u^*\theta(v_1, \ldots, v_k) := \theta(uv_1, \ldots, uv_k)$. Pull-back defines a natural action of $\mathcal{U}_\pm^\star$ on $\otimes^k V^*$ which preserves the canonical inner product of Equation (2.a). Let ξ be a $\mathcal{U}_\pm^\star$-invariant subspace of $\otimes^k V^*$; the natural action of $\mathcal{U}_\pm^\star$ on $\otimes^k V^*$ by pull-back makes ξ into a $\mathcal{U}_\pm^\star$ submodule of $\otimes^k V^*$. One has:

Lemma 2.1. *Let $(V, \langle\cdot,\cdot\rangle, J_\pm^0)$ be a para/pseudo-Hermitian vector space. Let ξ be a $\mathcal{U}_\pm^\star$ submodule of $\otimes^k V^*$.*

(1) *$\langle\cdot,\cdot\rangle$ is non-degenerate on ξ.*
(2) *There is an orthogonal direct sum decomposition $\xi = \eta_1 \oplus \cdots \oplus \eta_k$ where the η_i are irreducible $\mathcal{U}_\pm^\star$-modules.*
(3) *If ξ_1 and ξ_2 are inequivalent irreducible $\mathcal{U}_\pm^\star$ submodules of ξ, then $\xi_1 \perp \xi_2$.*
(4) *The multiplicity with which an irreducible representation appears in ξ is independent of the decomposition in (2).*
(5) *If ξ_1 appears with multiplicity 1 in ξ and if η is any $\mathcal{U}_\pm^\star$ submodule of ξ, then either $\xi_1 \subset \eta$ or else $\xi_1 \perp \eta$.*
(6) *If $0 \to \xi_1 \to \xi \to \xi_2 \to 0$ is a short exact sequence of $\mathcal{U}_\pm^\star$-modules, then $\xi \approx \xi_1 \oplus \xi_2$ as a $\mathcal{U}_\pm^\star$-module.*

Proof. We shall establish Assertion (1) as this is the crucial property; the remaining assertions follow from Assertion (1) using essentially the same arguments as those used in the positive definite setting; we refer to [6] for a detailed exposition. For example, it is Assertion (1) which lets us define orthogonal projection; if ξ is invariant under the action of $\mathcal{U}_\pm^\star$, then $\xi \cap \xi^\perp$ is a totally isotropic invariant subspace of $\otimes^k V^*$ and hence $\xi \cap \xi^\perp = \{0\}$. Thus $\otimes^k V^* = \xi \oplus \xi^\perp$ and orthogonal projection on ξ is given by the first factor in this decomposition.

Suppose first $(V, \langle \cdot, \cdot \rangle, J_-^0)$ is a pseudo-Hermitian vector space of signature (p, q). We prove Assertion (1) for the smaller group $\mathcal{U}_-$; it then follows automatically for the larger group $\mathcal{U}_-^\star$. Use the Gramm-Schmidt process to choose an orthogonal decomposition $V = V_+ \oplus V_-$ which is J_-^0 invariant so V_+ is spacelike and V_- is timelike. Let $T = \pm\,\mathrm{id}$ on $V_\pm$; $T \in \mathcal{U}_-$ since the decomposition is J_-^0 invariant. Let $\{e_1, ..., e_p\}$ be an orthonormal basis for V_- and let $\{e_{p+1}, ..., e_m\}$ be an orthonormal basis for V_+. Let $\{e^1, ..., e^m\}$ be the corresponding orthonormal dual basis for V^*. Then $T^*(e^i) = \langle e^i, e^i \rangle e^i = \pm e^i$. If $I = (i_1, ..., i_k)$ is a multi-index, set $e^I := e^{i_1} \otimes ... \otimes e^{i_k}$. The collection $\{e^I\}$ is an orthonormal basis for $\otimes^k V^*$ with:

$$\begin{aligned} T^* e^I &= T^*(e^{i_1}) \otimes ... \otimes T^*(e^{i_k}) = \langle e^{i_1}, e^{i_1} \rangle e^{i_1} \otimes ... \otimes \langle e^{i_k}, e^{i_k} \rangle e^{i_k} \\ &= \langle e^I, e^I \rangle e^I = \pm e^I\,. \end{aligned}$$

Thus if $T^* w = w$, then w is a spacelike vector in $\otimes^k V^*$ while if $T^* w = -w$, then w is a timelike vector in $\otimes^k V^*$. Let ξ be a non-trivial $\mathcal{U}_-$ invariant subspace of $\otimes^k V^*$. Since $T \in \mathcal{U}_-$, T preserves ξ. Decompose $\xi = \xi_+ \oplus \xi_-$ into the ± 1 eigenspaces of T^*. Since ξ_+ is spacelike and ξ_- is timelike, the metric on ξ is non-degenerate and Assertion (1) follows in this framework.

The argument is a bit different in the para-Hermitian setting. Let $(V, \langle \cdot, \cdot \rangle, J_+^0)$ be a para-Hermitian vector space. Find an orthogonal direct sum decomposition $V = V_+ \oplus V_-$ where V_+ is spacelike, where V_- is timelike, and where $J_+^0 : V_\pm \to V_\mp$. As before, let $T = \pm\,\mathrm{id}$ on $V_\pm$; T does not belong to $\mathcal{U}_+$ but it does belong to $\mathcal{U}_+^\star$. The remainder of the argument now follows as in the complex case; it is necessary to assume ξ is invariant under $\mathcal{U}_+^\star$ and not simply under $\mathcal{U}_+$ – this is the crucial difference. □

Remark 2.2. Lemma 2.1 fails for the group $\mathcal{U}_+$ and it is for this reason that the decomposition of $\mathfrak{H}_+$ has more factors as a $\mathcal{U}_+$ module than as a $\mathcal{U}_+^\star$ module. Let $(V, \langle \cdot, \cdot \rangle, J_+^0)$ be a para-Hermitian vector space. Decompose $V = W_+ \oplus W_-$ into the ± 1 eigenspaces of J_+^0. Then $W_\pm$ are totally isotropic subspaces of V which are invariant under $\mathcal{U}_+$.

Let ξ be a $\mathcal{U}_\pm^\star$ submodule of $\otimes^k V^*$. We say that a symmetric inner product $\theta \in S^2(\xi^*)$ is a *quadratic invariant* if $\theta(\gamma x, \gamma y) = \theta(x, y)$ for all $\gamma \in \mathcal{U}_\pm^\star$ and for all $x, y \in \xi$; let $S^2_{\mathcal{U}_\pm^\star}(\xi)$ be the space of all quadratic invariants. The following is well known – see, for example, the discussion in [6]. The proof follows exactly the same lines as in the positive definite setting given Lemma 2.1 (1).

Lemma 2.3. *Let ξ be a $\mathcal{U}_\pm^\star$ submodule of $\otimes^k V^*$. Suppose that ξ_i are non-trivial $\mathcal{U}_\pm^\star$-modules so that $\xi_1 \oplus \cdots \oplus \xi_\ell$ is a $\mathcal{U}_\pm^\star$ submodule of ξ. Also suppose that $\dim\{S^2_{\mathcal{U}_\pm^\star}(\xi)\} \le \ell$. Then:*

1. *$\xi = \xi_1 \oplus \cdots \oplus \xi_\ell$, $\xi_i \perp \xi_j$ for $i \neq j$, and $\dim\{S^2_{\mathcal{U}_\pm^\star}(\xi)\} = \ell$.*
2. *The modules ξ_i are all irreducible and ξ_i is not isomorphic to ξ_j for $i \neq j$.*

We now examine the space of quadratic invariants for the setting at hand.

Lemma 2.4. $\dim\{S^2_{\mathcal{U}_\pm^\star}(\mathfrak{H}_\pm)\} \le 4$.

Proof. Since the original discussion in [17] was in the positive definite setting, we shall provide full details. Let $(V, \langle \cdot, \cdot \rangle, J_\pm^0)$ be a para/pseudo-Hermitian vector space and let ξ be a G submodule of $\otimes^k V^*$. A spanning set for the space of quadratic invariants if $G = \mathcal{O}$ or if $G = \mathcal{U}_-$ in the positive definite setting is given in [30] and in [13, 20], respectively. The extension to the groups $\mathcal{U}_\pm^\star$ is straightforward (see [6] for example). In brief, if $G=\mathcal{U}_\pm^\star$, everything is given by contraction of indices using the inner product $\langle\cdot,\cdot\rangle$ and the structure $J_\pm^0$ where $J_\pm^0$ must appear an even number of times. The following is a convenient formalism. We identify θ with the corresponding quadratic function $\theta(x):=\theta(x,x)$. We consider 3 distinct orthonormal bases $\{e^1_{i_1},e^2_{i_2},e^3_{i_3}\}$ for V which are indexed by $\{i_1,i_2,i_3\}$, respectively, for $1\le i_1\le m$, $1\le i_2\le m$, and $1\le i_3\le m$. Let

$$\varepsilon_I=\langle e^1_{i_1},e^1_{i_1}\rangle\langle e^2_{i_2},e^2_{i_2}\rangle\langle e^3_{i_3},e^3_{i_3}\rangle=\pm1\,.$$

We consider a string S of 6 symbols grouped into 2 monomials of 3 symbols where each index 1, 2, 3 appears twice and where some of the indices are decorated with $J_\pm^0$. Thus, for example, if $S=(1,2;J_\pm^0 2)(1,3;J_\pm^0 3)$ and if $H_\pm\in\mathfrak{H}_\pm$, then the associated invariant $\mathcal{I}(S)$ is given by:

$$\mathcal{I}(S)(H_\pm):=\sum_{i_1=1}^m\sum_{i_2=1}^m\sum_{i_3=1}^m\varepsilon_I H_\pm(e^1_{i_1},e^2_{i_2};J_\pm^0e^2_{i_2})H_\pm(e^1_{i_1},e^3_{i_3};J_\pm^0e^3_{i_3})\,.$$

The space of quadratic invariants of $\mathfrak{H}_\pm$ is spanned by such invariants. We will stratify the invariants by the number of times $J_\pm^0$ appears; this gives rise to 2 basic cases each of which has 2 subcases.

(1) General remarks.
  (a) We can replace the basis $\{e^1_{i_1}\}$ by $\{J_\pm^0e^1_{i_1}\}$ and thereby replace ε_I by $\mp\varepsilon_I$. Thus $\mathcal{I}\{(\dots,1,\dots,1,\dots)\}=\mp\mathcal{I}\{(\dots,J_\pm^0 1,\dots,J_\pm^0 1,\dots)\}$.
  (b) We need only consider strings where either a given index is undecorated or it is decorated exactly once.
  (c) We may permute the bases. Thus $\mathcal{I}\{(1,2;3)(1,2;3)\}=\mathcal{I}\{(2,3;1)(2,3;1)\}$.
  (d) By Equation (1.a), $\mathcal{I}\{(\mu,\sigma;\star)(\star,\star;\star)\}=-\mathcal{I}\{(\sigma,\mu;\star)(\star,\star;\star)\}$ $=\pm\mathcal{I}\{(J_\pm^0\mu,J_\pm^0\sigma;\star)(\star,\star;\star)\}$.
  (e) By Equation (1.b), $\mathcal{I}\{(\mu,J_\pm^0\sigma;\star)(\star,\star;\star)\}=\mathcal{I}\{(J_\pm^0\mu,\sigma;\star)(\star,\star;\star)\}$.

(2) $J_\pm^0$ does not appear. This gives rise to 3 invariants:
  (a) Each index appears in each variable:
    (i) $\psi_1:=\mathcal{I}\{(1,2;3)(1,2;3)\}$.
    (ii) $\psi_2:=\mathcal{I}\{(1,2;3)(1,3;2)\}$.
  (b) Only one index appears in both variables:
    (i) $\psi_3:=\mathcal{I}\{(1,2;1)(3,2;3)\}$.

(3) $J_\pm^0$ appears twice. This gives rise to another invariant:
  (a) Each index appears in each variable:
    (i) $\psi_4:=\mathcal{I}\{(1,J_\pm^0 2;J_\pm^0 3)(1,2;3)\}$.
    (ii) $\mathcal{I}\{(1,J_\pm^0 2;3)(1,J_\pm^0 3;2)\}=\mathcal{I}\{(J_\pm^0 1,2;3)(J_\pm^0 1,2;3)\}$ $=\mp\mathcal{I}\{(1,2;3)(1,2;3)\}=\mp\psi_1$.
  (b) Only one index appears in both variables:
    (i) $\mathcal{I}\{(J_\pm^0 1,2;1)(J_\pm^0 3,2;3)\}=\mathcal{I}\{(1,J_\pm^0 2;1)(3,J_\pm^0 2;3)\}$ $=\mp\mathcal{I}\{(1,2;1)(3,2;3)\}=\mp\psi_3$.

We have enumerated all the possibilities and constructed 4 invariants. □

3. Geometric analysis

If $(x^1,\dots,x^m)$ is a system of local coordinates on M, let $\partial_{x_i}:=\frac{\partial}{\partial x_i}$.

Lemma 3.1. *Let $(M,g,J_\pm)$ be an almost para/pseudo-Hermitian manifold. Then:*
(1) $\nabla\Omega_\pm\in\mathfrak{H}_\pm$.
(2) $\nabla\Omega_\pm(x,y;z)=g(x,(\nabla_zJ_\pm)y)=g(x,\nabla_zJ_\pm y)-g(x,J_\pm\nabla_zy)$ $=g(x,\nabla_zJ_\pm y)+g(J_\pm x,\nabla_zy)$.

Proof. Since $\Omega_\pm \in C^\infty(\Lambda^2)$, $\nabla\Omega_\pm \in C^\infty(\Lambda^2 \otimes V^*)$. We prove Assertion (1) by studying the action of $J_\pm^*$:

$$\begin{aligned}
&\nabla\Omega_\pm(J_\pm x, J_\pm y; z)\\
&\quad= zg(J_\pm x, J_\pm J_\pm y) - g(\nabla_z J_\pm x, J_\pm J_\pm y) - g(J_\pm x, J_\pm \nabla_z J_\pm y)\\
&\quad= \mp zg(x, J_\pm y) \mp g(\nabla_z J_\pm x, y) \pm g(x, \nabla_z J_\pm y)\\
&\quad= \mp zg(x, J_\pm y) \mp zg(J_\pm x, y) \pm g(J_\pm x, \nabla_z y) \pm zg(x, J_\pm y) \mp g(\nabla_z x, J_\pm y)\\
&\quad= \pm zg(x, J_\pm y) \mp g(x, J_\pm \nabla_z y) \mp g(\nabla_z x, J_\pm y) = \pm\nabla\Omega_\pm(x, y; z).
\end{aligned}$$

We use the fact that $\nabla g = 0$ to prove Assertion (2) by computing:

$$\begin{aligned}
&\nabla_z\Omega_\pm(x, y) = zg(x, J_\pm y) - g(\nabla_z x, J_\pm y) - g(x, J_\pm \nabla_z y)\\
&= zg(x, J_\pm y) - g(\nabla_z x, J_\pm y) - g(x, \nabla_z J_\pm y) + g(x, \nabla_z J_\pm y) - g(x, J_\pm \nabla_z y)\\
&= (\nabla_z g)(x, J_\pm y) + g(x, \nabla_z J_\pm y) - g(x, J_\pm \nabla_z y)\\
&= g(x, \nabla_z J_\pm y) - g(x, J_\pm \nabla_z y) = g(x, \nabla_z J_\pm y) + g(J_\pm x, \nabla_z y).
\end{aligned}$$ □

Let $g(x, y; z) := zg(x, y)$. We continue our study and assume $J_\pm$ is integrable:

Lemma 3.2. *Let $(M, g, J_\pm)$ be a para/pseudo-Hermitian manifold. Then:*

1. $\nabla\Omega_\pm(\partial_{x_i}, \partial_{x_j}; \partial_{x_k}) = \frac{1}{2}\{g(\partial_{x_i}, \partial_{x_k}; J_\pm\partial_{x_j}) - g(\partial_{x_j}, \partial_{x_k}; J_\pm\partial_{x_i})$ $+g(J_\pm\partial_{x_i}, \partial_{x_k}; \partial_{x_j}) - g(J_\pm\partial_{x_j}, \partial_{x_k}; \partial_{x_i})\}$.
2. $\nabla\Omega_\pm(M, g, J_\pm) \in U_{3,\pm}$.
3. $\nabla\Omega_\pm(M, e^{2f}g, J_\pm) = e^{2f}\{\nabla\Omega_\pm(M, g, J_\pm) - \sigma_{\pm,g}(df)\}$.
4. $W_{4,\pm} \subset U_{3,\pm}$.

Proof. Since $J_\pm$ is integrable, we may choose coordinates so $J_\pm\partial_{x_i} \in \{\partial_{x_1}, ..., \partial_{x_m}\}$. Let $x = \partial_{x_i}$, $y = \partial_{x_j}$, and $z = \partial_{x_k}$. We may apply Lemma 3.1 and the Koszul formula for the Christoffel symbols in a coordinate frame to see:

$$\begin{aligned}
\nabla_z\Omega_\pm(x, y) &= g(x, \nabla_z J_\pm y) + g(J_\pm x, \nabla_z y)\\
&= \tfrac{1}{2}\{g(x, z; J_\pm y) + g(x, J_\pm y; z) - g(z, J_\pm y; x)\}\\
&+ \tfrac{1}{2}\{g(J_\pm x, z; y) + g(J_\pm x, y; z) - g(z, y; J_\pm x)\}\,.
\end{aligned}$$

Assertion (1) now follows from the identity:

$$g(x, J_\pm y; z) + g(J_\pm x, y; z) = z\{g(x, J_\pm y) + g(J_\pm x, y)\} = 0\,.$$

We prove Assertion (2) by checking that $\nabla\Omega_\pm$ satisfies the defining relation for $U_{3,\pm}$ in this instance. We use Assertion (1) to compute:

$$\begin{aligned}
&\nabla\Omega_\pm(x, J_\pm y; J_\pm z)\\
=\ & \tfrac{1}{2}\{g(x, J_\pm z; J_\pm J_\pm y) - g(J_\pm y, J_\pm z; J_\pm x)\}\\
+\ & \tfrac{1}{2}\{g(J_\pm x, J_\pm z; J_\pm y) - g(J_\pm J_\pm y, J_\pm z; x)\}\\
=\ & \tfrac{1}{2}\{\pm g(x, J_\pm z; y) \pm g(y, z; J_\pm x) \mp g(x, z; J_\pm y) \pm g(J_\pm y, z; x)\}\\
=\ & \tfrac{1}{2}\{\mp g(J_\pm x, z; y) \pm g(y, z; J_\pm x) \mp g(x, z; J_\pm y) \pm g(J_\pm y, z; x)\}\\
=\ & \mp\nabla_\pm(x, y; z)\,.
\end{aligned}$$

We also use Assertion (1) to prove Assertion (3) by checking:

$$\begin{aligned}
&\nabla\Omega_{\pm,e^{2f}g}(x, y; z)\\
=\ & e^{2f}\{\nabla\Omega_{\pm,g}(x, y; z) + df(J_\pm y)g(x, z) - df(J_\pm x)g(y, z)\\
+\ & df(y)g(J_\pm x, z) - df(x)g(J_\pm y, z)\}\\
=\ & e^{2f}\{\nabla\Omega_{\pm,g}(x, y; z) - \sigma_{\pm,g}(df)(x, y; z)\}\,.
\end{aligned}$$

Let $(V,\langle\cdot,\cdot\rangle,J_\pm^0)$ be a para/pseudo-Hermitian vector space. Let f be a smooth function on V and consider the manifold $(M,g,J_\pm):=(V,e^{2f}\langle\cdot,\cdot\rangle,J_\pm^0)$. We apply Assertion (2) and Assertion (3) to prove Assertion (4) by checking:

$$e^{2f}\sigma_{\pm,\langle\cdot,\cdot\rangle}(df)=-\nabla\Omega_{\pm,e^{2f}\langle\cdot,\cdot\rangle}\in U_{3,\pm}. \qquad \square$$

4. Algebraic considerations

We now turn our attention to purely algebraic considerations. For the remainder of this section, let $(V,\langle\cdot,\cdot\rangle,J_\pm^0)$ be a para/pseudo-Hermitian vector space.

Definition 4.1. Let $H_\pm\in\mathfrak{H}_\pm$.

(1) $(\pi_{1,\pm}H_\pm)(x,y;z):=\frac{1}{6}\big\{H_\pm(x,y;z)+H_\pm(y,z;x)+H_\pm(z,x;y)$
$\qquad\pm H_\pm(x,J_\pm^0y;J_\pm^0z)\pm H_\pm(y,J_\pm^0z;J_\pm^0x)\pm H_\pm(z,J_\pm^0x;J_\pm^0y)\big\}$.

(2) $(\pi_{2,\pm}H_\pm)(x,y;z):=\frac{1}{6}\big\{2H_\pm(x,y;z)-H_\pm(y,z;x)-H_\pm(z,x;y)$
$\qquad\pm 2H_\pm(x,J_\pm^0y;J_\pm^0z)\mp H_\pm(y,J_\pm^0z;J_\pm^0x)\mp H_\pm(z,J_\pm^0x;J_\pm^0y)\big\}$.

(3) $(\pi_{3,\pm}H_\pm)(x,y;z):=\frac{1}{2}\{H_\pm(x,y;z)\mp H_\pm(x,J_\pm^0y;J_\pm^0z)\}$.

(4) $\pi_{4,\pm}:=\pm\frac{1}{m-2}\sigma_\pm(J_\pm^0)^*\tau_1$.

Lemma 4.2.

(1) $\pi_{1,\pm}$ *is a projection from* $\mathfrak{H}_\pm$ *onto* $W_{1,\pm}$.

(2) $\pi_{2,\pm}$ *is a projection from* $\mathfrak{H}_\pm$ *onto* $W_{2,\pm}$.

(3) $\pi_{3,\pm}$ *is a projection from* $\mathfrak{H}_\pm$ *onto* $U_{3,\pm}$.

(4) $\pi_{4,\pm}$ *is a projection from* $\mathfrak{H}_\pm$ *onto* $W_{4,\pm}$.

Proof. Set:

$$\begin{aligned}(\kappa_\pm^1H_\pm)(x,y;z)&:=\pm H_\pm(x,J_\pm^0y;J_\pm^0z),\\ (\kappa_\pm^2H_\pm)(x,y;z)&:=H_\pm(y,z;x)+H_\pm(z,x;y)\\ &\qquad\pm H_\pm(y,J_\pm^0z;J_\pm^0x)\pm H_\pm(z,J_\pm^0x;J_\pm^0y)\,.\end{aligned}$$

We may use Equation (1.a) and Equation (1.b) to see that $\kappa_\pm^1H_\pm$, and $\kappa_\pm^2H_\pm$ are anti-symmetric in the first two arguments. We show that $\kappa_\pm^1\mathfrak{H}_\pm\subset\mathfrak{H}_\pm$ and that $\kappa_\pm^2\mathfrak{H}_\pm\subset\mathfrak{H}_\pm$ by checking:

$$\begin{aligned}&(\kappa_\pm^1H_\pm)(J_\pm^0x,J_\pm^0y;z)=H_\pm(J_\pm^0x,J_\pm^0J_\pm^0y;J_\pm^0z)=\pm H_\pm(x,J_\pm^0y;J_\pm^0z)\\ &\quad=\pm\kappa_\pm^1H_\pm(x,y;z),\\ &(\kappa_\pm^2H_\pm)(J_\pm^0x,J_\pm^0y;z)=H_\pm(J_\pm^0y,z;J_\pm^0x)+H_\pm(z,J_\pm^0x;J_\pm^0y)\\ &\quad\pm H_\pm(J_\pm^0y,J_\pm^0z;J_\pm^0J_\pm^0x)\pm H_\pm(z,J_\pm^0J_\pm^0x;J_\pm^0J_\pm^0y)\\ &\quad=H_\pm(y,J_\pm^0z;J_\pm^0x)+H_\pm(z,J_\pm^0x;J_\pm^0y)\pm H_\pm(y,z;x)\pm H_\pm(z,x;y)\\ &\quad=\pm(\kappa_\pm^2H_\pm)(x,y;z).\end{aligned}$$

We see $\pi_{1,\pm}\mathfrak{H}_\pm\subset\mathfrak{H}_\pm$, $\pi_{2,\pm}\mathfrak{H}_\pm\subset\mathfrak{H}_\pm$, and $\pi_{3,\pm}\mathfrak{H}_\pm\subset\mathfrak{H}_\pm$ by expressing:

$$\begin{aligned}&\pi_{1,\pm}=\tfrac{1}{6}\{\mathrm{id}+\kappa_\pm^1+\kappa_\pm^2\},\quad \pi_{2,\pm}=\tfrac{1}{6}\{2(\mathrm{id}+\kappa_\pm^1)-\kappa_\pm^2\},\\ &\pi_{3,\pm}=\tfrac{1}{2}\{\mathrm{id}-\kappa_\pm^1\}\,.\end{aligned}$$

Let $H_\pm\in\mathfrak{H}_\pm$. We verify $\pi_{1,\pm}H_\pm\in W_{1,\pm}$, that $\pi_{2,\pm}H_\pm\in W_{2,\pm}$, and that $\pi_{3,\pm}H_\pm\in U_{3,\pm}$ by checking that the defining relations are satisfied in each case:

$$\begin{aligned}&(\pi_{1,\pm}H_\pm)(x,z;y):=\tfrac{1}{6}\big\{H_\pm(x,z;y)+H_\pm(z,y;x)+H_\pm(y,x;z)\\ &\quad\pm H_\pm(x,J_\pm^0z;J_\pm^0y)\pm H_\pm(z,J_\pm^0y;J_\pm^0x)\pm H_\pm(y,J_\pm^0x;J_\pm^0z)\big\}\\ &\quad=-\pi_{1,\pm}H_\pm(x,y;z),\end{aligned}$$

$$\begin{aligned}
&(\pi_{2,\pm}H_\pm)(x,y;z)+(\pi_{2,\pm}H_\pm)(y,z;x)+(\pi_{2,\pm}H_\pm)(z,x;y)\\
&\quad=\tfrac16\big\{2H_\pm(x,y;z)-H_\pm(y,z;x)-H_\pm(z,x;y)\\
&\qquad\pm2H_\pm(x,J_\pm^0y;J_\pm^0z)\mp H_\pm(y,J_\pm^0z;J_\pm^0x)\mp H_\pm(z,J_\pm^0x;J_\pm^0y)\\
&\qquad+2H_\pm(y,z;x)-H_\pm(z,x;y)-H_\pm(x,y;z)\\
&\qquad\pm2H_\pm(y,J_\pm^0z;J_\pm^0x)\mp H_\pm(z,J_\pm^0x;J_\pm^0y)\mp H_\pm(x,J_\pm^0y;J_\pm^0z)\\
&\qquad+2H_\pm(z,x;y)-H_\pm(x,y;z)-H_\pm(y,z;x)\\
&\qquad\pm2H_\pm(z,J_\pm^0x;J_\pm^0y)\mp H_\pm(x,J_\pm^0y;J_\pm^0z)\mp H_\pm(y,J_\pm^0z;J_\pm^0x)\big\}=0,
\end{aligned}$$

$$\begin{aligned}
&(\pi_{3,\pm}H_\pm)(x,J_\pm^0y;J_\pm^0z)\\
&\quad=\tfrac12\{H_\pm(x,J_\pm^0y;J_\pm^0z)\mp H_\pm(x,J_\pm^0J_\pm^0y;J_\pm^0J_\pm^0z)\}\\
&\quad=\tfrac12\{H_\pm(x,J_\pm^0y;J_\pm^0z)\mp H_\pm(x,y;z)\}\\
&\quad=\mp\tfrac12\{H(x,y;z)\mp H(x,J_\pm^0y;J_\pm^0z)\}=\mp(\pi_{3,\pm}H_\pm)(x,y;z).
\end{aligned}$$

Let $H_{1,\pm}\in W_{1,\pm}$, let $H_{2,\pm}\in W_{2,\pm}$, and let $H_{3,\pm}\in U_{3,\pm}$. We complete the proof of Assertion (1), of Assertion (2), and of Assertion (3) by verifying:

$$\begin{aligned}
&(\pi_{1,\pm}H_{1,\pm})(x,y;z)=\tfrac16\big\{H_{1,\pm}(x,y;z)+H_{1,\pm}(y,z;x)+H_{1,\pm}(z,x;y)\\
&\qquad\pm H_{1,\pm}(x,J_\pm^0y;J_\pm^0z)\pm H_{1,\pm}(y,J_\pm^0z;J_\pm^0x)\pm H_{1,\pm}(z,J_\pm^0x;J_\pm^0y)\big\}\\
&\quad=\tfrac16\big\{H_{1,\pm}(x,y;z)-H_{1,\pm}(y,x;z)-H_{1,\pm}(x,z;y)\\
&\qquad\mp H_{1,\pm}(J_\pm^0z,J_\pm^0y;x)\mp H_{1,\pm}(J_\pm^0x,J_\pm^0z;y)\mp H_{1,\pm}(J_\pm^0y,J_\pm^0x;z)\big\}\\
&\quad=\tfrac16\big\{H_{1,\pm}(x,y;z)+H_{1,\pm}(x,y;z)+H_{1,\pm}(x,y;z)\\
&\qquad-H_{1,\pm}(z,y;x)-H_{1,\pm}(x,z;y)-H_{1,\pm}(y,x;z)\big\}=H_{1,\pm}(x,y;z),
\end{aligned}$$

$$\begin{aligned}
&(\pi_{2,\pm}H_{2,\pm})(x,y;z)=\tfrac16\big\{2H_{2,\pm}(x,y;z)-H_{2,\pm}(y,z;x)-H_{2,\pm}(z,x;y)\\
&\qquad\pm2H_{2,\pm}(x,J_\pm^0y;J_\pm^0z)\mp H_{2,\pm}(y,J_\pm^0z;J_\pm^0x)\mp H_{2,\pm}(z,J_\pm^0x;J_\pm^0y)\big\}\\
&\quad=\tfrac16\big\{3H_{2,\pm}(x,y;z)\pm H_{2,\pm}(J_\pm^0x,y;J_\pm^0z)\pm H_{2,\pm}(x,J_\pm^0y;J_\pm^0z)\\
&\qquad\mp H_{2,\pm}(y,J_\pm^0z;J_\pm^0x)\mp H_{2,\pm}(J_\pm^0z,x;J_\pm^0y)\big\}\\
&\quad=\tfrac16\big\{3H_{2,\pm}(x,y;z)\mp H_{2,\pm}(J_\pm^0z,J_\pm^0x;y)\mp H_{2,\pm}(y,J_\pm^0z;J_\pm^0x)\\
&\qquad\mp H_{2,\pm}(J_\pm^0y,J_\pm^0z;x)\mp H_{2,\pm}(J_\pm^0z,x;J_\pm^0y)\\
&\qquad\mp H_{2,\pm}(y,J_\pm^0z;J_\pm^0x)\mp H_{2,\pm}(J_\pm^0z,x;J_\pm^0y)\big\}\\
&\quad=\tfrac16\big\{3H_{2,\pm}(x,y;z)-H_{2,\pm}(z,x;y)-H_{2,\pm}(y,z;x)\\
&\qquad\mp2H_{2,\pm}(J_\pm^0y,z;J_\pm^0x)\mp2H_{2,\pm}(z;J_\pm^0x;J_\pm^0y)\big\}\\
&\quad=\tfrac16\{4H_{2,\pm}(x,y;z)\pm2H_{2,\pm}(J_0^\pm x,J_0^\pm y;z)\}=H_{2,\pm}(x,y;z),
\end{aligned}$$

$$\begin{aligned}
&(\pi_{3,\pm}H_{3,\pm})(x,y;z)=\tfrac12\{H_{3,\pm}(x,y;z)\mp H_{3,\pm}(x,J_\pm^0y;J_\pm^0z\}\\
&\quad=\tfrac12\{H_{3,\pm}(x,y;z)+H_{3,\pm}(x,y;z)\}=H_{3,\pm}(x,y;z).
\end{aligned}$$

We now turn to the final assertion. We compute:

$$\begin{aligned}
\tau_1(\sigma_\pm(\phi))(x)&=\varepsilon^{ij}\{\phi(J_\pm^0x)\langle e_i,e_j\rangle-\phi(J_\pm^0e_i)\langle x,e_j\rangle\}\\
&\quad+\varepsilon^{ij}\{\phi(x)\langle J_\pm^0e_i,e_j\rangle-\phi(e_i)\langle J_\pm^0x,e_j\rangle\}\\
&=m\phi(J_\pm^0x)-\phi(J_\pm^0x)+\mathrm{Trace}(J_\pm^0)\phi(x)-\phi(J_\pm^0x)\\
&=(m-2)((J_\pm^0)^*\phi)(x)+\mathrm{Trace}(J_\pm^0)\phi(x)\,.
\end{aligned}$$

Since $\mathrm{Trace}(J_\pm^0)=0$, we have $\tau_1\sigma_\pm=(m-2)(J_\pm^0)^*$. It is immediate that $\pi_{4,\pm}$ takes values in $W_{4,\pm}$. We complete the proof by checking:

$$\pi_{4,\pm}\sigma_\pm\phi = \pm\tfrac{1}{m-2}(\sigma_\pm(J_\pm^0)^*\tau_1)(\sigma_\pm\phi) = \pm\sigma_\pm(J_\pm^0)^*(J_\pm^0)^*\phi = \sigma_\pm\phi. \qquad \square$$

We examine these modules further:

Lemma 4.3.

(1) $W_{1,\pm} + W_{2,\pm} \subset \ker \pi_{3,\pm}$.

(2) $W_{1,\pm} \cap W_{2,\pm} = \{0\}$.

(3) $W_{1,\pm} \oplus W_{2,\pm} \oplus W_{3,\pm} \oplus W_{4,\pm}$ *is a* $\mathcal{U}_\pm^\star$ *submodule of* $\mathfrak{H}_\pm$.

Proof. Suppose first that $H_{1,\pm} \in W_{1,\pm}$. Then

$$\begin{aligned}
\pi_{3,\pm}H_{1,\pm}(x,y;z) &= \tfrac{1}{2}\{H_{1,\pm}(x,y;z) \mp H_{1,\pm}(x,J_\pm^0 y;J_\pm^0 z)\}\\
&= \tfrac{1}{2}\{H_{1,\pm}(x,y;z) \mp H_{1,\pm}(J_\pm^0 x,y;J_\pm^0 z)\}\\
&= \tfrac{1}{2}\{H_{1,\pm}(x,y;z) \pm H_{1,\pm}(J_\pm^0 x,J_\pm^0 z;y)\}\\
&= \tfrac{1}{2}\{H_{1,\pm}(x,y;z) + H_{1,\pm}(x,z;y)\} = 0.
\end{aligned}$$

Next suppose that $H_{2,\pm} \in W_{2,\pm}$. We have

$$\begin{aligned}
\pi_{3,\pm}H_{2,\pm}(x,y;z) &= \tfrac{1}{2}\{H_{2,\pm}(x,y;z) \mp H_{2,\pm}(x,J_\pm^0 y;J_\pm^0 z)\}\\
&= \tfrac{1}{2}\{H_{2,\pm}(x,y;z) \pm H_{2,\pm}(J_\pm^0 y,J_\pm^0 z;x) \pm H_{2,\pm}(J_\pm^0 z,x;J_\pm^0 y)\}\\
&= \tfrac{1}{2}\{H_{2,\pm}(x,y;z) + H_{2,\pm}(y,z;x) \pm H_{2,\pm}(z,J_\pm^0 x;J_\pm^0 y)\}\\
&= \tfrac{1}{2}\{H_{2,\pm}(x,y;z) + H_{2,\pm}(y,z;x) \mp H_{2,\pm}(J_\pm^0 x,J_\pm^0 y;z)\\
&\qquad \mp H_{2,\pm}(J_\pm^0 y,z;J_\pm^0 x)\}\\
&= \tfrac{1}{2}\{H_{2,\pm}(y,z;x) \mp H_{2,\pm}(J_\pm^0 y,z;J_\pm^0 x)\}\\
&= -\tfrac{1}{2}\{H_{2,\pm}(z,y;x) \mp H_{2,\pm}(z,J_\pm^0 y;J_\pm^0 x)\} = -\pi_{3,\pm}H_{2,\pm}(z,y;x).
\end{aligned}$$

This shows that

$$\begin{aligned}
\pi_{3,\pm}H_{2,\pm}(x,y;z) &= -\pi_{3,\pm}H_{2,\pm}(y,x;z) = \pi_{3,\pm}H_{2,\pm}(z,x;y)\\
&= -\pi_{3,\pm}H_{2,\pm}(x,z;y)\,.
\end{aligned}$$

Consequently $H_{1,\pm} := \pi_{3,\pm}H_{2,\pm} \in W_{1,\pm}$. Thus:

$$\pi_{3,\pm}H_{2,\pm} = \pi_{3,\pm}\pi_{3,\pm}H_{2,\pm} = \pi_{3,\pm}H_{1,\pm} = 0\,.$$

Let $H_\pm \in W_{1,\pm} \cap W_{2,\pm}$. We establish Assertion (2) by checking:

$$\begin{aligned}
0 &= H_\pm(x,y;z) + H_\pm(y,z;x) + H_\pm(z,x;y)\\
&= H_\pm(x,y;z) - H_\pm(y,x;z) - H_\pm(x,z;y)\\
&= 3H_\pm(x,y;z)\,.
\end{aligned}$$

If $\pi_\pm : \mathfrak{H}_\pm \to \mathfrak{H}_\pm$ satisfies $\pi_\pm^2 = \pi_\pm$, then Lemma 2.1 shows

$$\mathfrak{H}_\pm = \ker(\pi_\pm) \oplus \operatorname{Range}(\pi_\pm)\,.$$

By Lemma 4.2, we can apply this observation to $\pi_{3,\pm}$ and to $\pi_{4,\pm}$. By Assertion (1) and by Assertion (2),

$$W_{1,\pm} \cap W_{2,\pm} = \{0\} \quad \text{so} \quad W_{1,\pm} \oplus W_{2,\pm} \subset \ker(\pi_{3,\pm})\,.$$

By Lemma 4.2, we have $U_{3,\pm} = \operatorname{Range}(\pi_{3,\pm})$. Consequently

$$W_{1,\pm} \oplus W_{2,\pm} \oplus U_{3,\pm}$$

is a submodule of $\mathfrak{H}_\pm$. By Lemma 4.2,

$$W_{4,\pm} = \operatorname{Range}(\pi_{4,\pm}) \subset U_{3,\pm}.$$

Since $W_{4,\pm} = \pi_{4,\pm}U_{3,\pm}$, $W_{3,\pm} \oplus W_{4,\pm}$ is a $\mathcal{U}_\pm^\star$ submodule of $U_{3,\pm}$. $\square$

5. Varying the almost (para)-complex structure

Fix a background almost para/pseudo-Hermitian manifold $(M,g,J_\pm)$ and a point P of M for the remainder of Section 5. Let $\mathcal{O}(M)$ be the fiber bundle whose fibre over a point Q of M is the associated structure group $\mathcal{O}(T_QM,g_Q)$. The Lie algebra $\mathfrak{o}$ of $\mathcal{O}$ is the vector space of all matrices which are skew-adjoint with respect to the inner product. Let $\vartheta\in\mathfrak{o}_P\otimes T_P^*$M. Let Θ be a smooth section to $\mathcal{O}(M)$ so that $\Theta(P)=\text{id}$, so that $\Theta=\text{id}$ off a neighborhood of P, and so that $d\Theta=\vartheta$. Let:

$$J_\pm^\Theta:=\Theta^{-1}J_\pm\Theta\,.$$

Since Θ takes values in $\mathcal{O}$, $(M,g,J_\pm^\Theta)$ is an almost para/pseudo-Hermitian manifold as well. Define:

$$\Xi_\pm(\vartheta)(x,y;z):=g(x,(-\vartheta(z)J_\pm+J_\pm\vartheta(z))y)(P)\,.$$

Lemma 5.1. *Adopt the notation established above.*

1. $\left\{\nabla\Omega_\pm(M,g,J_\pm^\Theta)(x,y;z)-\nabla\Omega_\pm(M,g,J_\pm)(x,y;z)\right\}(P)$ $=\Xi_\pm(d\vartheta)(x,y;z)$.
2. $\Xi_\pm$ *is a* $\mathcal{U}_\pm^\star$ *module morphism from* $\mathfrak{o}\otimes V^*\otimes\chi$ *to* $\mathfrak{H}_\pm$.
3. *If* $m\geq 6$*, then* $\pi_{1,\pm}\{\Xi_\pm(\mathfrak{o})\}\neq\{0\}$ *and* $\pi_{3,\pm}\{\Xi_\pm(\mathfrak{o})\}\cap W_{3,\pm}\neq\{0\}$.
4. $\pi_{2,\pm}\{\Xi_\pm(\mathfrak{o})\}\neq\{0\}$ *and* $\pi_{4,\pm}\{\Xi_\pm(\mathfrak{o})\}\neq\{0\}$.

Proof. Since $\Theta(P)=\text{id}$, $(J_\pm^\Theta-J_\pm)(P)=0$. We use Lemma 3.1 to prove Assertion (1) by computing:

$$\begin{aligned}&\Omega_\pm(M,g,J_\pm^\Theta)(P)-\Omega_\pm(M,g,J_\pm)(P)\\&=g(x,\{\nabla_z(J_\pm^\Theta-J_\pm)-(J_\pm^\Theta-J_\pm)\nabla_z\}y)(P)\\&=g(x,\{z(J_\pm^\Theta-J_\pm)\}y)(P)=g(x,\{z(\Theta^{-1}J_\pm\Theta-J_\pm)\}y)(P)\\&=g(x,\{-z(\Theta)J_\pm+J_\pm z(\Theta)\}y)(P).\end{aligned}$$

Assertion (2) is an immediate consequence of Assertion (1). The proof of Assertions (3) and (4) is a purely algebraic computation. Introduce an orthonormal basis $\{e_1,\dots,e_{\bar m},f_1,\dots,f_{\bar m}\}$ for V so

$$J_\pm:e_i\to f_i\quad\text{and}\quad J_\pm:f_i\to\pm e_i\,.$$

We set $\varepsilon_i:=\langle e_i,e_i\rangle$. Define $\vartheta_0\in\mathfrak{o}$ by setting:

$$\vartheta_0e_i=\left\{\begin{array}{rcl}\varepsilon_2e_2&\text{if}&i=1\\-\varepsilon_1e_1&\text{if}&i=2\\0&\text{if}&i>2\end{array}\right\}\quad\text{and}\quad\vartheta_0f_i=\left\{\begin{array}{rcl}0&\text{if}&i=1\\0&\text{if}&i=2\\0&\text{if}&i>2\end{array}\right\}.$$

Suppose first that $m\geq 6$. We set $\vartheta=\vartheta_0\otimes e^3$. Choose $\alpha\in C^\infty(M)$ to be compactly supported near P with $d\alpha(P)=dx^3$. If $\varepsilon_1=\varepsilon_2$, then the corresponding Θ may be taken to be:

$$\Theta\partial_{x_i}=\left\{\begin{array}{rcl}\cos(\alpha)e_1+\varepsilon_2\sin(\alpha)e_2&\text{if}&i=1\\-\varepsilon_2\sin(\alpha)e_1+\cos(\alpha)e_2&\text{if}&i=2\\e_i&\text{if}&i\geq 3\end{array}\right\}\quad\text{and}\quad\Theta\partial_{y_i}=\partial_{y_i}\ \forall\ i\,,$$

whereas if $\varepsilon_1=-\varepsilon_2$, then Θ may be taken to be:

$$\Theta\partial_{x_i}=\left\{\begin{array}{rcl}\cosh(\alpha)e_1+\varepsilon_2\sinh(\alpha)e_2&\text{if}&i=1\\\varepsilon_2\sinh(\alpha)e_1+\cosh(\alpha)e_2&\text{if}&i=2\\e_i&\text{if}&i\geq 3\end{array}\right\}\quad\text{and}\quad\Theta\partial_{y_i}=\partial_{y_i}\ \forall\ i\,.$$

Set $H_\pm:=\Xi_\pm(\vartheta_0\otimes e^3)$. The non-zero components of $H_\pm$ are determined by:

$$H_\pm(f_2,e_1;e_3)=\mp1\quad\text{and}\quad H_\pm(f_1,e_2;e_3)=\pm1\,.$$

Clearly $\tau_1 H_\pm = 0$; thus $\pi_{3,\pm}H_\pm \in W_{3,\pm}$. We prove Assertion (3) by computing:

$$\pi_{1,\pm}H_\pm(f_2,e_1;e_3) = \mp\tfrac{1}{6} \quad\text{and}\quad \pi_{3,\pm}H_\pm(f_2,e_1;e_3) = \mp\tfrac{1}{2}\,.$$

Next we clear the previous notation and let $H_\pm = \Xi_\pm(\vartheta_0 \otimes e^2)$; here we need to have $d\alpha(P) = dx^2$. The non-zero components of $H_\pm$ are determined by:

$$H_\pm(f_2,e_1;e_2) = \mp 1 \quad\text{and}\quad H_\pm(f_1,e_2;e_2) = \pm 1\,.$$

Since $\tau_1(H_\pm) = \pm\varepsilon_2$, the component of $H_\pm$ in $W_{4,\pm}$ is non-zero. We complete the proof of Assertion (4) by checking:

$$\begin{aligned}(\pi_{2,\pm}H_\pm)(f_2,f_1;f_2) &:= \tfrac{1}{6}\big\{2H_\pm(f_2,f_1;f_2) - H_\pm(f_1,f_2;f_2) - H_\pm(f_2,f_2;f_1)\\ &\quad \pm 2H_\pm(f_2,J^0_\pm f_1;J^0_\pm f_2) \mp H_\pm(f_1,J^0_\pm f_2;J^0_\pm f_2) \mp H_\pm(f_2,J^0_\pm f_2;J^0_\pm f_1)\big\}\\ &= \tfrac{1}{6}\{0-0-0-2-1+0\} = -\tfrac{1}{2}.\end{aligned}$$
□

Proof of Theorem 1.4. Let $m \geq 6$. By Lemma 5.1, $W_{i,\pm}$ are non-trivial modules for $1 \leq i \leq 4$. By Lemma 4.3, $W_{1,\pm} \oplus W_{2,\pm} \oplus W_{3,\pm} \oplus W_{4,\pm}$ is a $\mathcal{U}^\star_\pm$ submodule of $\mathfrak{H}_\pm$. By Lemma 2.4, $\dim\{S^2_{\mathcal{U}^\star_\pm}(\mathfrak{H}_\pm)\} \leq 4$. Theorem 1.4 now follows from Lemma 2.1 and from Lemma 2.3. □

Proof of Theorem 1.1. Let $(M,g,J_\pm)$ be an almost para/pseudo-Hermitian manifold of dimension $m \geq 6$ (the case $m = 4$ is analogous). We consider variations $(M,g,J^\Theta_\pm)$. Subtracting $\nabla\Omega_\pm(M,g,J_\pm)(P)$ has no effect on the question of surjectivity. Every $\vartheta \in \mathfrak{o}\otimes T^*M$ can be written in the form $\vartheta = d\Theta(P)$ for some admissible Θ. Thus it suffices to show $\Xi_\pm(\mathfrak{o}) = \mathfrak{H}_\pm$. By Lemma 5.1, $\Xi(\mathfrak{o})$ is not perpendicular to $W_{\pm,i}$ for $1 \leq i \leq 4$. By Theorem 1.4, $W_{\pm,i}$ is an irreducible submodule of $\mathfrak{H}_\pm$ which occurs with multiplicity 1. Thus by Lemma 2.1, $W_{\pm,i} \subset \Xi(\mathfrak{o})$ for $1 \leq i \leq 4$. Theorem 1.4 now shows $\mathfrak{H}_\pm \subset \Xi(\mathfrak{o})$ as desired. □

6. Varying the metric

Let $(M,g,J_\pm)$ be a para/pseudo-Hermitian manifold. Fix P in M and let

$$(V,\langle\cdot,\cdot\rangle,J^0_\pm) := (T_PM, g_P, J_{\pm,P})\,.$$

Let $\mathfrak{gl}_\pm$ be the Lie algebra of $\mathrm{GL}_\pm$ at P. Given $\tilde\vartheta \in \mathfrak{gl}\otimes V^*$, we may find a smooth map $\tilde\Theta$ from a neighborhood of P in M to $\mathrm{GL}_\pm$ so that $\tilde\Theta(P) = \mathrm{id}$, so that $\tilde\Theta = \mathrm{id}$ away from a neighborhood of P, and so that $d\tilde\Theta(P) = \tilde\vartheta$. We define a new pseudo-Riemannian metric $g^{\tilde\Theta}$ which agrees with g at P and which agrees with g away from a neighborhood of P by setting:

$$g^{\tilde\Theta}(x,y) = (\tilde\Theta x, \tilde\Theta y)\,.$$

Since $\Theta J_\pm = J_\pm\Theta$, $g^{\tilde\Theta}$ is a para/pseudo-Hermitian metric. Set:

$$\tilde\Xi_\pm(\tilde\vartheta) := \left\{\nabla\Omega_\pm(V,g^{\tilde\Theta},J_\pm) - \nabla\Omega_\pm(V,g,J^0_\pm)\right\}(P)\,.$$

We may then use Lemma 3.2 to see that $\tilde\Xi_\pm(\tilde\vartheta) \in W_{\pm,3}$ is independent of the choice of $\tilde\Theta$ and defines a $\mathcal{U}^\star_\pm$ module morphism from $\mathfrak{gl}_\pm \otimes V^* \otimes \chi$ to $\mathfrak{H}_\pm$ by computing:

$$\begin{aligned}&\tilde\Xi_\pm(\tilde\vartheta)(x,y;z)\\ =\;& \tfrac{1}{2}\Big\{\langle\tilde\vartheta(J_\pm y)x,z\rangle + \langle x,\tilde\vartheta(J_\pm y)z\rangle + \langle\tilde\vartheta(y)J_\pm x,z\rangle + \langle J_\pm x,\tilde\vartheta(y)z\rangle\\ &\quad -\langle\tilde\vartheta(J_\pm x)y,z\rangle + \langle y,\tilde\vartheta(J_\pm x)z\rangle + \langle\tilde\vartheta(x)J_\pm y,z\rangle + \langle J_\pm y,\tilde\vartheta(x)z\rangle\Big\}\,.\end{aligned}$$

Thus to prove Theorem 1.2, it suffices to show that $\Xi_\pm$ is surjective. Since we have subtracted the effect of the background metric, we may take the flat metric $g = \langle\cdot,\cdot\rangle$. As in Section 5, we introduce a normalized orthonormal basis

$\{e_1,\dots,e_{\bar m},f_1,\dots,f_{\bar m}\}$ for V. Let α be a smooth function which is compactly supported near $P=0$ with $\alpha(0)=0$ and $d\alpha(0)=dx^1$. Set:

$$\tilde\Theta e_i=\left\{\begin{array}{lll} e^\alpha e_i & \text{if} & i=1,2\\ e_i & \text{if} & i\geq 3\end{array}\right\}\quad\text{and}\quad \tilde\Theta f_i=\left\{\begin{array}{lll} e^\alpha f_i & \text{if} & i=1,2\\ f_i & \text{if} & i\geq 3\end{array}\right\}.$$

Let $\tilde\vartheta=d\tilde\Theta(0)=\tilde\vartheta_0\otimes dx^1$ where $\tilde\vartheta_0$ is orthogonal projection on $\text{Span}\{e_1,e_2,f_1,f_2\}$:

$$\tilde\vartheta_0 e_i=\left\{\begin{array}{lll} e_i & \text{if} & i=1,2\\ 0 & \text{if} & i\geq 3\end{array}\right\}\quad\text{and}\quad \tilde\vartheta_0 f_i=\left\{\begin{array}{lll} f_i & \text{if} & i=1,2\\ 0 & \text{if} & i\geq 3\end{array}\right\}.$$

The associated metric takes the form:

$$\begin{aligned} g_\pm^{\tilde\Theta} &= e^{2\alpha}\varepsilon_1(e^1\otimes e^1\mp f^1\otimes f^1)+e^{2\alpha}\varepsilon_2(e^2\otimes e^2\mp f^2\otimes f^2)\\ &+\sum_{i\geq 3}\varepsilon_i(e^i\otimes e^i\mp f^i\otimes f^i)\,. \end{aligned}$$

Set $H_\pm:=\nabla\Omega_\pm(0)=\tilde\Xi_\pm(\vartheta)$. We use Lemma 3.1 to see $\tau_1(H_\pm)=2e^1$ and thus $H_\pm$ has a non-trivial component in $W_{\pm,4}$. Since $H_\pm(e_1,e_3;f_3)=0$ and $\sigma_\pm(e^1)(e_1,e_3;f_3)\neq 0$, $H_\pm$ also has a non-zero component in $W_{3,\pm}$. Theorem 1.2 now follows. □

7. The 16 classes of almost pseudo-Hermitian manifolds

Proof of Theorem 1.6. If (M,g,J_-) is a ξ-manifold, then $(M,-g,J_-)$ also is a ξ-manifold. Thus by replacing g by $-g$ if need be, we may assume without loss of generality that $p\leq q$ and consequently, as $m\geq 10$, that $6\leq q$ to establish Theorem 1.6. We shall use product structures. The projections $\pi_{i,-}$ for $i=1,2,3$ and the map τ_1 are compatible with Cartesian product; the splitting σ_- is not. This causes a small amount of additional technical fuss.

Suppose first that $W_4\not\subset\xi$. By Theorem 1.5 we may choose a ξ-manifold $(M_1,g_1,J_{1,-})$ of Riemannian signature $(0,q)$. Let $(M_2,g_2,J_{2,-})$ be a flat Kähler torus of signature $(p,0)$. Let

$$M=M_1\times\mathbb{T}^{(p,0)},\qquad g:=g_1+g_2,\quad J_-=J_{1,-}\oplus J_{2,-}\,. \tag{7.a}$$

Then (M,g,J_-) is an almost pseudo-Hermitian manifold of signature (p,q). We have $\nabla\Omega_g=\nabla\Omega_{g_1}$ and $\tau_1(\nabla\Omega_g)=\tau_1(\nabla\Omega_{g_1})=0$. Thus $\pi_{3,-}\nabla\Omega_g$ is projection on $W_{-,3}$; this would not be the case if τ_1 was non-zero and this fact played an important role in the analysis of Section 6. Since $\pi_{i,-}\nabla\Omega_g=\pi_{i,-}\nabla\Omega_{g_1}$, it now follows that (M,g,J_-) is a ξ manifold in this special case.

Next we suppose that $\xi=\eta\oplus W_{-,4}$. Let (M,g,J_-) be an η-manifold of signature (p,q). We make a conformal change of metric and set $\tilde g:=e^{2f}g$; it then follows from Lemma 3.2 that

$$\nabla\Omega_{\tilde g}=e^{2f}\nabla\Omega_g-e^{2f}\sigma_{-,g}(df)$$

where we use the original metric to define the splitting $\sigma_{-,g}$. This has a non-trivial $W_{4,-}$ component and the components $W_{i,-}$ for $1\leq i\leq 3$ are not affected. □

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MB-V: Department of Mathematics, University of A Coruña, Spain
E-mail address: miguel.brozos.vazquez@udc.es

EG-R: Faculty of Mathematics, University of Santiago de Compostela, Spain
E-mail address: eduardo.garcia.rio@usc.es

PG: Mathematics Department, University of Oregon, Eugene OR 97403, USA
E-mail address: gilkey@uoregon.edu

LH: Faculty of Mathematics, University of Santiago de Compostela, Spain
E-mail address: luismaria.hervella@usc.es