

EXISTENCE OF A CAPILLARY SURFACE WITH PRESCRIBED CONTACT ANGLE IN $M \times \mathbb{R}$

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ABSTRACT. We study the prescribed mean curvature equation with a prescribed boundary contact angle condition in $M \times \mathbb{R}$ where M^n is a Riemannian submanifold in \mathbb{R}^{n+1} . The main purpose is to establish a priori gradient estimates for solutions, from which the long time existence of the solution are derived.

1. INTRODUCTION

In this paper we discuss the existence of solution u of the Capillary problem

$$(1.0.1) \quad \begin{aligned} (a) \quad & \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \frac{1}{\sqrt{\sigma}} D_i(v^i) = \Psi(x, u) \quad (x \in \Omega) \quad V = \sqrt{1+|Du|^2} \\ (b) \quad & v \cdot \gamma = \Phi(x, u) \quad (x \in \partial\Omega) \end{aligned}$$

where Ω is a bounded domain in n -dimensional manifold M with Riemannian metric σ , Ψ and Φ are given functions on $M \times \mathbb{R}$ and $\partial\Omega \times \mathbb{R}$ respectively, v is the downward unit normal to the graph of u and γ is the inner normal to $\partial\Omega \times \mathbb{R}$.

Capillary problems arise from the physical phenomena that occurs whenever two different materials are situated adjacent to each other and do not mix. If one (at least) of the materials is a fluid, which forms with another fluid (or gas) a free surface interface, then the interfaces will be referred to as a capillary surface. A great deal of work has been devoted to capillarity phenomena since the initial works of Young and Laplace in the early nineteenth century (see the book of Finn [1] for an account on the subject).

Notice that the capillary problem is the same as the prescribed mean curvature equation with given contact angle in the boundary. Recently the topics of existence of minimal and constant mean curvature surfaces in $M \times \mathbb{R}$, where M is a Riemannian manifold, have gathered great interest. For example; B. Nelli and H. Rosenberg considered minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$; particularly, surfaces which are vertical graphs over domains in \mathbb{H}^2 [8]. Also H. Rosenberg discussed minimal surfaces in $M \times \mathbb{R}$, where M is the 2-sphere (with the constant curvature one metric) or a complete Riemannian surface with a metric of non-negative curvature, or M is the hyperbolic plane [9]. L. Hauswirth, H. Rosenberg, and J. Spruck proved existence of constant mean curvature graphs in $M \times \mathbb{R}$ where $M = \mathbb{H}^2$ or \mathbb{S}^2 the hyperbolic plane of curvature -1 or the 2-sphere of curvature 1 [4]. Also J. Spruck established a priori interior gradient estimates and existence theorems for n -dimensional graphs of constant mean curvature $H > 0$ in $M^n \times \mathbb{R}$ where M^n is simply connected and complete and Ω is a bounded domain in M [12].

In this work we prove that the prescribed mean curvature equation with given contact angle for every given Ψ (not only constant) satisfying certain conditions has a solution. For reaching this goal we follow Korevaar's technique [5] to estimate the gradient of a solution to the nonparametric capillary problem in an n -dimensional Riemannian manifold $M \subset \mathbb{R}^{n+1}$. For smooth Euclidean domains, J. Spruck has used a maximum principle in two dimensions to obtain global gradient estimates [11]. The analogous n -dimensional estimate in Euclidean domains have also been obtained using integral iteration arguments, in [2], [3] and [10]. Also G. Lieberman [7] discusses a closely related maximum principle argument to Korevaar's method for getting a priori gradient bound.

In the present paper the following theorem will be proved.

Theorem 1.0.1. *Let $\Omega \subset M$ be a bounded domain with C^3 boundary $\partial\Omega$. Then there exists a function $u \in C^2(\Omega) \cap C_{loc}^1(\overline{\Omega})$ a (bounded) solution to the capillary problem (1.0.1) in Ω if there exists $K_1 < \infty$ and*

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$K_2 < \infty$ so that

- (i) $|u| < K_1$
- (ii) $n + |\Psi| + |\Psi_x| < K_2$
- (iii) $\Psi_z \geq 0$
- (iv) $1 - |\Phi| \geq K_2^{-1}$
- (v) $\Phi_z \geq 0$
- (vi) $|\Phi|_{C^2} < K_2$

on some open set containing $\bar{\Omega} \times [-K_1, K_1]$.

In the remainder of this section we set our notation. In §2 we describe how gradient bounds follow from the construction of suitable "barrier" comparison surfaces. For later reference we collect the estimate which must hold to obtain a gradient bound. In §3 we derive a priori local and global gradient bounds for nonparametric capillary surfaces above smooth domains. In §4 we use the continuity method to prove existence of solution.

Let $M^n \subset \mathbb{R}^{n+1}$ be a Riemannian Manifold. We rescale M so that $|x - y|_M < 2|x - y|$ for any two points $x, y \in M$.

We consider the (signed) distance function

$$d(x) = \begin{cases} \min_{y \in \partial\Omega} |x - y|_M & \text{if } x \in \Omega; \\ -\min_{y \in \partial\Omega} |x - y|_M & \text{if } x \in M \setminus \Omega. \end{cases}$$

near $\partial\Omega$, and the inner normal $\gamma = \frac{\nabla d}{|\nabla d|}$. There exists a neighborhood of radius $\mu > 0$ of points within (unsigned) distance μ of $\partial\Omega$, on which d is C^3 and γ is C^2 .

Define the Euclidean ball of radius R and center x in M by $B_R(x) = \{y \in M : |x - y| < R\}$. If x is suppressed it is assumed to be zero.

We embed $M \subset M \times \mathbb{R}$ in the usual way: $M = \{(x, z) \in M \times \mathbb{R} : z = 0\}$. The capillary tube above $\partial\Omega$ is defined to be $\partial\Omega \times \mathbb{R} = \{(x, z) : x \in \partial\Omega, z \in \mathbb{R}\}$. We often extend functions (or vector fields) defined on $U \subset M$ to $U \times \mathbb{R} \subset M \times \mathbb{R}$ by making them constant in the z -direction. In particular, we so extend d and γ and they represent the distance function and normal vector field associated to $\partial\Omega \times \mathbb{R}$.

For a function u on M define $S = \text{graph}(u)$ to be $\{(x, z) \in M \times \mathbb{R} : z = u(x)\}$.

Let x^1, \dots, x^n be a system of local coordinate for M with corresponding metric σ_{ij} . Subscripts on functions generally denote partial derivatives, e.g. $f_i = \frac{\partial f}{\partial x^i}$, whereas superscripts refer to components of vectors. For total derivative we use D for a function of x and z , that is

$$Df = Df(x, u(x)) = f_x + f_z Du = (D_1 f, \dots, D_n f).$$

The downward unit normal to S is given by

$$v = (v^1, \dots, v^{n+1}) = \frac{1}{\sqrt{1 + |Du|^2}}(u_1, \dots, u_n, -1)$$

where $u_i = \sigma^{ij} D_j u$ and $|Du| = |Du|_M$.

We extend v and u away from S and Ω by making them constant in the vertical direction. Measure the steepness of S by

$$V = (v^{n+1})^{-1} = \sqrt{1 + |Du|^2}$$

2. MAXIMUM PRINCIPLE

In this section we will set up a technique to construct suitable "barrier" comparison surfaces to obtain the gradient estimate in the third section using the maximum principle Lemma (2.2.2).

2.1. Barrier Technique. We construct a family of surfaces $\Sigma(t) = \text{graph}(u^t(x))$ for sufficiently small nonnegative t , with $\Sigma(0) = \Sigma \subset \subset S = \text{graph}(u)$. Denote the interior of Σ by Σ° , and its boundary by $\partial\Sigma$. The $\Sigma(t)$ are constructed by deforming Σ smoothly along a vector field Z . Although one can modify the $\Sigma(t)$ so that they are actually barriers (i.e. lying in a useful way entirely above or below $\text{graph}(u)$), we use them directly as comparison surfaces: for small t , the height separation $s(t)$ between S and $\Sigma(t)$ will be seen to be about $t(Z \cdot v)V$. For suitable Z we can use the contact angle boundary condition to show that $(Z \cdot v)V$

is bounded at any (relatively large) maximum value of $s(t)$ which occurs on the intersection of Σ with the capillary tube. $(Z \cdot \nu)V$ will be bounded by construction on the part of $\partial\Sigma$ that is inside the tube, $\partial\Sigma \cap S^\circ$. Finally, we will be able to use the prescribed mean curvature equation to show that $(Z \cdot \nu)V$ is bounded at any maximum of $s(t)$ which occurs on Σ° . We will therefore conclude a bound for $(Z \cdot \nu)V$ on Σ , i.e. a local gradient estimate.

Proceeding with our construction, we assume there exists an open subset $O \subset M \times \mathbb{R}$ with $\Sigma \subset O$, on which the deformation vector field Z is defined, with $|Z|_{C^2(O)} < \infty$. For $P \in \Sigma$ and small t , define $\tilde{P}(P, t)$ by solving the ODE

$$(2.1.1) \quad \frac{d}{dt} \tilde{P} = Z(\tilde{P})$$

$$(2.1.2) \quad \tilde{P}(P, 0) = P$$

and define the resulting perturbed surface by

$$(2.1.3) \quad \Sigma(t) = \{\tilde{P}(P, t) : P \in \Sigma\}$$

It follows from ODE theory that $\Sigma(t)$ is the graph of a C^2 function $u^t(x)$, with domain nearly that of u^0 . If we make the further requirement that Z be tangential:

$$(2.1.4) \quad Z(Q) \cdot \gamma(Q) = 0 \text{ for all } Q \in \partial\Omega \times \mathbb{R} \cap O$$

(and that $\partial\Omega \times \mathbb{R} \cap O$ is C^1), then the ODE (2.1.1) preserves $\partial\Omega \times \mathbb{R}$, a fact which implies that the domain of u^t is contained in $\bar{\Omega}$, which will be useful later in boundary (contact angle) calculations. We define the quantities to be estimated later. Writing $\tilde{P}(P, t) = (\tilde{x}, u^t(\tilde{x}))$, denote the point in $S = \text{graph}(u)$ directly above (or below) it by $\hat{P}(P, t) = (\tilde{x}, u(\tilde{x}))$. Let $s(P, t)$ be the (signed) vertical distance from $\tilde{P}(P, t)$ to S , $s(P, t) = u(\tilde{x}) - u^t(\tilde{x})$. Let $\nu(P, t)$, $\Pi(P, t)$ and $H(P, t)$ be the normal, tangential plane and mean curvature of $\Sigma(t)$ at $\tilde{P}(P, t)$, respectively. Whenever t is suppressed its value is zero. Hereafter t should be small enough such that \tilde{P} and \hat{P} are in the injectivity ball of P in $M \times \mathbb{R}$.

For a fixed point $P \in \Sigma$, we consider a unitary frame $\{f_1, f_2, \dots, f_{n+1}\}$ of $M \times \mathbb{R}$ with the following properties:

- (1) For each $Q \in \Sigma$, $f_i \in \Pi(Q)$ for $i = 1, \dots, n$ and $f_{n+1} = -\nu(Q)$.
- (2) At P they are orthonormal. Moreover, the vectors f_1, \dots, f_{n-1} are horizontal, that is, they have the last component (in $M \times \mathbb{R}$) equal to 0, and f_n is in the direction of steepest ascent in $\Pi(P)$.

Let g be the Riemannian metric equivalent to $\sigma + dz^2$ of $M \times \mathbb{R}$ corresponding to this new frame.

Any vector field X on S can be written as $X = X^\alpha f_\alpha = X^i f_i - \chi f_{n+1}$. (Here and in the sequel we use the summation convention, summing from 1 to $n+1$ if the repeated indices are Greek and from 1 to n if they are Latin.) For any function χ , we can define natural tangent-plane analogs to the gradient ∇ and Δ of $(M \times \mathbb{R}, g)$:

$$(2.1.5) \quad \begin{aligned} \nabla_{\Pi} \chi &= \nabla \chi - (\chi_\nu) \nu \\ \Delta_{\Pi} \chi &= \Delta \chi - \chi_{\nu\nu} \\ (\chi_\nu &= \nabla \chi \cdot \nu). \end{aligned}$$

For $|y|$ small less than the injectivity radius of $M \times \mathbb{R}$ at P , S may be given near P by the exponential mapping $y \rightarrow \exp_P(y^i f_i + U(y) f_{n+1})$, for some C^3 function U . Let $[U_{ij}(y)]$ denote its Hessian matrix, and for $y = 0$ write $U_{ij} U_{ij} = |A|^2$. $[U_{ij}(0)]$ is the matrix for the second fundamental form of S at P , with respect to the $\{f_i\}$ frame on S .

Similarly, we can write a point $\tilde{P}(Q, t) \in \Sigma(t)$ close enough to P as $\tilde{P}(Q, t) = \exp_P(\tilde{y}^i f_i + U^t(\tilde{y}) f_{n+1})$, for t small enough. We use the notation $\tilde{P}(Q, t) = (\tilde{y}, U^t(\tilde{y}))$ and $Q = (y, U(y))$.

Lemma 2.1.1. *Let $P \in \Sigma$ and $t = 0$. Let the vector field Z satisfy (2.1.1) and let $u \in C^3(O)$. Express $Z = Z^i f_i - \zeta f_{n+1}$ in the P -based coordinate system. Then the surfaces $\Sigma(t)$ which result from the vector flow (2.1.1) evolve so that at $t = 0$*

$$\begin{aligned}
(2.1.6) \quad & \frac{\partial}{\partial t} s(P, t) = \zeta V \\
& \frac{\partial}{\partial t} v(P, t) = -\nabla_{\Pi} \zeta \\
& \frac{\partial}{\partial t} H(P, t) = -(2Z_i^k U_{ki} + \Delta_{\Pi} \zeta - \zeta_v H)
\end{aligned}$$

in the strong sense that for $L = s, v, H$ we have

$$(2.1.7) \quad L(P, t) = L(P, 0) + t \frac{\partial L}{\partial t} + o(t)$$

with the error term $o(t)$ uniform for $P \in \Sigma$. In the special case that Z is a normal-perturbation vector field ($Z = \eta v$, with the function $\eta \in C^2(O)$ and v), the evolution formulae are given by

$$\begin{aligned}
(2.1.8) \quad & \frac{\partial}{\partial t} s(P, t) = \eta V \\
& \frac{\partial}{\partial t} v(P, t) = -\nabla_{\Pi} \eta \\
& \frac{\partial}{\partial t} H(P, t) = -(2\eta |A|^2 + \Delta_{\Pi} \eta - \eta_v H)
\end{aligned}$$

Proof. Consider the curve $\alpha(t) = \widehat{P}(P, t)$, the vertical projection onto Σ of the curve $\widetilde{P}(P, t)$ that solves the ODE (2.1.1). Then we can write:

$$\alpha(t) = \exp_{\widetilde{P}(P, t)}(s(P, t)e_{n+1}),$$

and therefore we have the equation:

$$\begin{aligned}
\frac{d\alpha}{dt} &= \frac{d}{dt} \exp_{\widetilde{P}(P, t)}(s(P, t)e_{n+1}) = \\
&= d(\exp_{\widetilde{P}(P, t)})_{s(P, t)e_{n+1}} \left(\frac{\partial \widetilde{P}(P, t)}{\partial t} + \frac{ds(P, t)}{dt} e_{n+1} \right) \\
&= d(\exp_{\widetilde{P}(P, t)})_{s(P, t)e_{n+1}} \left(Z(\widetilde{P}(P, t)) + \frac{ds(P, t)}{dt} e_{n+1} \right)
\end{aligned}$$

At $t = 0$, we have $\widetilde{P}(P, 0) = P$ and $s(P, 0) = 0$, therefore:

$$\frac{d\alpha}{dt}(0) = d(\exp_P)_0(Z(P) + \frac{ds(P, 0)}{dt} e_{n+1}) = Z(P) + \frac{ds(P, 0)}{dt} e_{n+1}$$

Since $\alpha(t)$ is a curve in Σ , we have that $\frac{d\alpha}{dt}(0) \in \Pi(P)$, and therefore its product with the normal vector is 0:

$$0 = \frac{d\alpha}{dt}(0) \cdot v(P) = (Z(P) + \frac{ds(P, 0)}{dt} e_{n+1}) \cdot v(P)$$

and from this and the fact that $e_{n+1} \cdot v(P) = V^{-1}$ we get the result for $\frac{ds}{dt}$.

To calculate $v(P, t)$ and $H(P, t)$, we consider the curve from $Q = (y, U(y)) \in \Sigma$ to $(\widetilde{y}, U^t(\widetilde{y})) \cong \widetilde{P}(Q, t)$ that solves the ODE (2.1.1). By the Taylor expansion for \exp map we have

$$\begin{aligned}
(2.1.9) \quad & \widetilde{y}^i(t) = y^i + \int_0^t Z^i(\widetilde{P}(Q, a)) ds + o(t) \\
& U^t(\widetilde{y}) = U(y) - \int_0^t \zeta(\widetilde{P}(Q, s)) ds + o(t)
\end{aligned}$$

The first equation in (2.1.9) expresses \widetilde{y} as a function of y . We use it to estimate the Jacobian of the inverse transformation, which expresses y as a function of \widetilde{y} . With this Jacobian we can use the chain rule

and the second equation to estimate the first two derivatives of $U^t(\tilde{y})$, with respect to \tilde{y} , yielding estimates for $v(P, t)$ and $H(P, t)$. We note that the $o(t)$ -error terms below depend at most on second derivatives of Z and third derivatives of U .

$$\begin{aligned}\frac{\partial \tilde{y}^i}{\partial y^m} &= \delta_m^i + t D_m Z^i(y, U(y)) + \int_0^t D_m(Z^k(\tilde{P}(Q, s)) - Z^k(Q)) ds + o(t) \\ \frac{\partial y^k}{\partial \tilde{y}^i} &= \delta_k^i - t D_i Z^k(Q) + o(t) \\ \frac{\partial U^t(\tilde{y})}{\partial \tilde{y}^i} &= U_i(y) - t(D_i \zeta + U_k D_i Z^k) + o(t) \\ \frac{\partial^2 U^t(\tilde{y})}{\partial \tilde{y}^i \partial \tilde{y}^j} &= U_{ij} - t(U_{ik} D_j Z^k + U_{kj} D_i Z^k + D_j D_i \zeta + U_k D_j D_i Z^k) + o(t)\end{aligned}$$

The terms involving DU are zero at the value of \tilde{y} corresponding to $y = 0$, and the derivative estimates there simplify to

$$\begin{aligned}\frac{\partial U^t(\tilde{y})}{\partial \tilde{y}^i} &= -t \zeta_i + o(t) \\ \frac{\partial^2 U^t(\tilde{y})}{\partial \tilde{y}^i \partial \tilde{y}^j} &= U_{ij} - t(U_{ik} Z_j^k + U_{kj} Z_i^k + \zeta_{ij} + \zeta_{n+1} U_{ij}) + o(t)\end{aligned}\tag{2.1.10}$$

Estimating the normal and mean curvature of $\Sigma(t)$ at $\tilde{P}(P, t)$ with the aid of (2.1.10) yields the evolution formulas for v and H .

In the case $Z = \eta v$ is normal perturbation, we have

$$\begin{aligned}Z^k(Q) &= \eta(Q)(v \cdot f_k) \quad (1 \leq k \leq n) \\ \zeta(Q) &= \eta(Q)(v \cdot f_{n+1})\end{aligned}\tag{2.1.11}$$

In computing the derivative of Z at P we use the facts that the gradient of U is zero there and that because $v^{n+1} = -1$ is a minimum value, so is the gradient of v^{n+1} . These computation yield

$$\zeta(P) = \eta, \quad \zeta_i(P) = \eta_i, \quad \zeta_{ii} = \eta_{ii} - \eta U_{ik} U_{ik}\tag{2.1.12}$$

$$\zeta_v(P) = \eta_v, \quad Z_i^k(P) = \eta U_{ik}\tag{2.1.13}$$

Substituting the above expressions into the estimate (2.1.6) yields (2.1.8). □

2.2. Maximum Principle Lemma. The maximality of $s(P, t)$ has two possible geometry consequences: depending on the location of P , we can obtain inequalities which are implicit in comparison principles for surfaces of related mean curvature and contact angle:

Lemma 2.2.1. *Let a positive maximum of $s(Q, t)$ (over Σ) occur at $P \in \Sigma$.*

- (1) *If $P \in (\partial \Sigma \cap \partial S)^\circ$, then $v(\tilde{P}(P, t)) \cdot \gamma(\tilde{P}(P, t)) \geq v(\hat{P}(P, t)) \cdot \gamma(\hat{P}(P, t))$.*
- (2) *If $P \in \Sigma^\circ$, then $H(\tilde{P}(P, t)) \geq H(\hat{P}(P, t))$.*

Proof. These inequalities follow directly from calculus and the ellipticity of the contact angle and mean curvature operators. In both cases the function $s = u - u^t$ has a local maximum at $\tilde{P}(P, t)$. In case (1) it follows that the gradient Ds points in the exterior normal direction, $-\gamma$, implying the contact angle inequality. In case (2) it follows that Ds is zero and D^2s is negative semi-definite, implying the mean curvature inequality. □

Now let $P \in \Sigma^\circ$ be a point where s is maximum, we write $\tilde{P} = (x, \tilde{u}(x))$ and $\hat{P} = (x, u(x))$, so that $s = u(x) - \tilde{u}(x)$. Since it is a maximum, we have that $Ds(x) = 0$, so that $Du(x) = D\tilde{u}(x)$ and therefore the tangent plane to $\Sigma(t)$ at \tilde{P} is parallel to the tangent plane to Σ at \hat{P} . This also implies that, if we write $\tilde{P} = (\tilde{y}, U^t(\tilde{y}))$ and $\hat{P} = (y, U(y))$ in the f_i basis, we have that $DU(y) = DU^t(\tilde{y})$.

Write $\xi(t)$ for the Σ -secant vector such that $\hat{P}(P, t) = \exp_P(\xi(t))$, and recall that $\alpha(t) = \hat{P}(P, t)$. Using the calculations above for $\alpha(t)$, we can compute:

$$\frac{d}{dt}\Big|_{t=0} \exp_P(\xi(t)) = \frac{d\alpha(0)}{dt} = Z(P) + \frac{ds(P, 0)}{dt} e_{n+1},$$

which is a vector in $\Pi(P)$. On the other hand,

$$\frac{d}{dt}\Big|_{t=0} \exp_P(\xi(t)) = d(\exp_P)_0\left(\frac{d\xi}{dt}\Big|_{t=0}\right) = \frac{d\xi}{dt}\Big|_{t=0} = \dot{\xi}(0),$$

so that $\xi(t) = \xi(0) + t\dot{\xi}(0) + o(t) = t(Z(P) + \frac{ds(P, 0)}{dt} e_{n+1}) + o(t)$. We can then write $y = t\dot{\xi}(0) + o(t)$ and $U(y) = o(t)$.

Now we compute the Taylor expansion of $U_i(y)$ at 0:

$$(2.2.1) \quad U_i(y) = U_i(0) + DU_i(0) \cdot y + O(|y|^2) = DU_i(0) \cdot t\dot{\xi}(0) + o(t) = t(\dot{\xi}(0)^k U_{ik}) + o(t)$$

where we have used that $DU(0) = 0$. On the other hand, by (2.1.6) we have that $U_i^t(\tilde{y}) = -t\zeta_i + o(t)$. By equating $DU(y)$ and $DU^t(\tilde{y})$ and dividing by t , we obtain:

$$(2.2.2) \quad \dot{\xi}(0)^k U_{ik} = -\zeta_i + o(1)$$

We know that $Z = Z^m f_m + \zeta v$, for $m = 1..n$. On the other hand, by the definition of the basis (using the fact that f_1, \dots, f_{n-1} are horizontal and f_n is in the direction of Du), we can see that $e_{n+1} \cdot f_n = \frac{|Du|}{V}$ (in fact, $f_n = (\frac{1}{V|Du|} Du + \frac{|Du|}{V} e_{n+1})$), so that we have

$$(2.2.3) \quad \dot{\xi}(0)^k = Z^k(P) + \zeta(P)|Du|\delta_{kn}$$

That implies:

$$(2.2.4) \quad \zeta|Du|U_{ni} = -\zeta_i - Z^k U_{ki} + o(1) \text{ for } 1 \leq i \leq n$$

Now we will prove the key Lemma to get the gradient bound in the next section.

Lemma 2.2.2. *Let $u \in C^3(\overline{\Omega})$ solve the capillary problem in Ω . Consider $\Sigma \subset\subset S = \text{graph}(u)$, $\Sigma \subset O$, and a C^2 tangential deformation vector field $Z = \eta v + X$. Assume that $\partial\Sigma$ is the union of $\overline{\partial\Sigma \cap S^\circ}$ and $(\partial\Sigma \cap \partial S)^\circ$. Fixing any point $P \in \Sigma$, denote the decomposition of X with respect to the tilted basis $\{f_\alpha\}$ at P by $X = X^i f_i - \chi f_{n+1}$. Then we can conclude the gradient estimate*

$$(Z \cdot v)V = (\eta + \chi)V \leq M$$

on Σ , for some $0 < M < \infty$, if we can verify the following three inequalities, for some $\delta > 0$:

- (1) $(Z \cdot v)V \leq M - \delta$, ($P \in \overline{\partial\Sigma \cap S^\circ}$).
- (2) $\nabla_\Pi(\eta + \chi) \cdot \gamma > \delta - \frac{\partial}{\partial t} \Phi(\tilde{P}(P, t)) + v \cdot \frac{\partial}{\partial t} \gamma(\tilde{P}(P, t))$, (at $t = 0$ for $P \in (\partial\Sigma \cap \partial S)^\circ$).
- (3) $\eta|A|^2 + 2X_i^k U_{ki} + \Delta_\Pi(\eta + \chi) - (\eta + \chi)_v H + \dot{\xi}(0) \cdot \nabla \Psi > \delta$, (for $P \in \Sigma^\circ$).

Proof. For small $t > 0$, we consider the maximum on Σ of the function $s(Q, t)$, and let that maximum occur at $P \in \Sigma$. If $s(P, t) \leq Mt$ for small t , since $s = t(\eta + \chi)V + o(t)$, that is enough to prove the gradient estimate. Consider the following cases for P :

- (1) If $P \in \overline{\partial\Sigma \cap S^\circ}$, then inequality (1) shows that the gradient estimate holds.

- (2) If $P \in (\partial\Sigma \cap \partial S)^\circ$: then using Capillary equation we have $\Phi(P, 0) = \gamma(P, 0) \cdot \nu(P, 0)$ Also we have $\frac{\partial v(P, t)}{\partial t} = -\nabla_{\Pi}(\eta + \chi)$ Thus inequality (2) will imply at $t = 0$:

$$(2.2.5) \quad \begin{aligned} & -\frac{\partial v}{\partial t}(P, t) \cdot \gamma > \delta - \frac{\partial}{\partial t} \Phi(\tilde{P}(P, t)) + v \cdot \frac{\partial}{\partial t} \gamma(\tilde{P}(P, t)) \\ & \frac{\partial}{\partial t} \Phi(\tilde{P}(P, t)) > \delta + \frac{\partial}{\partial t} (v \cdot \gamma(P, t)) \end{aligned}$$

Since Z is tangential then we have $\hat{x} \in \partial\Omega$, so using Taylor expansion for $v \cdot \gamma$ and Φ also by knowing $\tilde{P} + se_{n+1} = \hat{P}$ then we have:

$$(2.2.6) \quad \begin{aligned} v \cdot \gamma(P, t) &= v \cdot \gamma(P, 0) + t \frac{\partial}{\partial t} (v \cdot \gamma(P, t))|_{t=0} + o(t) \\ v \cdot \gamma(P, t) - v \cdot \gamma(P, 0) &< -\delta t + t \frac{\partial}{\partial t} \Phi(\tilde{P}(P, t))|_{t=0} + o(t) \\ v \cdot \gamma(\hat{P}(P, 0)) - v \cdot \gamma(P, 0) &< -\delta t + t \frac{\partial}{\partial t} \Phi(\tilde{P}(P, t)) + o(t) \\ \frac{\partial}{\partial t} \Phi(\hat{P}(P, t))|_{t=0} &< C + \frac{\partial}{\partial t} \Phi(\tilde{P}(P, t))|_{t=0} + o(t) \\ (\eta + \chi)V &= \frac{\partial}{\partial t} s|_{t=0} < M \end{aligned}$$

- (3) If $P \in \Sigma^\circ$, using Capillary equation we have $H(P, 0) = \Psi(P, 0)$ and from (3), (2.1.6) and (2.1.8) we have:

$$(2.2.7) \quad -\frac{\partial H}{\partial t}(P, t)|_{t=0} + \dot{\xi}(0) \cdot \nabla \Psi > \delta$$

By Taylor expansion for H and Ψ also (2.2.1) we have:

$$(2.2.8) \quad \begin{aligned} H(P, 0) - H(P, t) + o(t) + t\dot{\xi}(0) \cdot \nabla \Psi &> t\delta \\ H(P, 0) - H(\hat{P}, t) + t\dot{\xi}(0) \cdot \nabla \Psi + o(t) &> t\delta \\ \Psi(P, 0) - \Psi(\hat{P}, t) + t\dot{\xi}(0) \cdot \nabla \Psi + o(t) &> t\delta \\ -\dot{\xi}(0) \cdot \nabla \Psi - \frac{\partial}{\partial t} (\tilde{u}^t + s(P, t))|_{t=0} \Psi_z + \dot{\xi}(0) \cdot \nabla \Psi + Ct + o(t) &> \delta \\ (\eta + \chi)V \Psi_z + Z \cdot e_{n+1} \Psi_z + Ct + o(t) &< \delta \end{aligned}$$

Therefore we have $(\eta + \chi)V < M$. □

3. GRADIENT BOUNDS IN SMOOTH DOMAINS

We prove three a priori gradient estimates for solutions to the capillary problem: local interior and boundary estimates when there is positive gravity, and global estimates when there is not.

3.1. Local gradient bound. we will prove local interior and boundary gradient estimates assuming positive gravity using Lemma (2.2.2).

Theorem 3.1.1. *Let $u \in C^3(\Omega)$ solve the prescribed mean curvature equation, with positive gravity $\Psi_Z > k > 0$. If R be less than injectivity radius of $M \times \mathbb{R}$ and $\overline{B_R}(0) \subset \Omega$, then there exists a finite $M = M(R, K_1, K_2, k)$ so that*

$$V(x) \leq M \frac{R^2}{R^2 - |x|^2}$$

for all $x \in B_R$.

Proof. In the formalism of §2, define the subset $\Sigma \subset\subset S = \text{graph}(u)$ and the deformation vector field Z by

$$(3.1.1) \quad \Sigma = S \cap (\overline{B_R} \times \mathbb{R}), \quad Z = \eta \nu, \quad \eta(x, z) = 1 - \frac{|x|^2}{R^2}$$

we only need to find $0 < M < \infty$ with which to satisfy condition (3) in Lemma (2.2.2) in order to get interior estimate ($\Sigma \subset S^o$ and $\eta = 0$ on $\partial\Sigma$). Fixing a point $P \in \Sigma^o$, using (2.2.3) for the vector $\dot{\xi}(0)$ (2.2.3) and the definition of f_n we have

$$(3.1.2) \quad \begin{aligned} \dot{\xi}(0) \cdot \nabla \Psi &= ((Z^k + \eta |Du| \delta_n^k) f_k) \cdot (\Psi_{x_k} e_k + \Psi_z e_{n+1}) \\ \dot{\xi}(0) \cdot \nabla \Psi &\geq \eta k \frac{|Du|^2}{V} - C \end{aligned}$$

Thus by definition of η there is a constant C so that

$$\eta |A|^2 + \Delta_{\Pi} \eta - \eta_v H + \dot{\xi}(0) \cdot \nabla \Psi > \eta k \frac{|Du|^2}{V} - C$$

so that condition (3) in Lemma (2.2.2) can be verified for sufficiently large M . \square

Remark 3.1.2. Let $u \in C^3(\Omega)$ solve the prescribed mean curvature equation with positive gravity k . Let Ω satisfy a uniform interior sphere condition of radius $R > 0$ (i.e. each $P \in \Omega$ is contained in a sub ball of Ω having radius at least R). Then it follows immediately from pervious theorem and the definition of the distance function d , that there is an M so that $V(x) \leq \frac{M}{d(x)}$ for any $x \in \Omega$.

Now we will prove an a priori boundary gradient estimate when there is positive gravity.

Theorem 3.1.3. *For Ω as in Capillary problem let $u \in C^3(\overline{\Omega})$ solve the capillary problem with positive gravity k . Then for $r > 0$ and $y \in M$, $B_{3r}(y)$, there exists an $M = M(r, k, K_1, K_2, \partial\Omega \cap B_{3r}(y))$, such that $V(x) \leq M$ for each $x \in B_r(y) \cap \overline{\Omega}$.*

Proof. Without loss of generality we may assume $y = 0$. Modify the distance function d outside the μ -neighborhood of $\partial\Omega \cap B_{2r}$ on which it is C^3 . Make it a C^3 function on all of B_{2r} in such a way that this modified d always has magnitude less than the actual (non-negative) distance to $\partial\Omega$, and so that its gradient is bounded in norm by 1. Extend γ to the gradient of the d in $B_{2r}(y)$, making it a C^2 function in the entire ball. It follows from remark (3.1.2) that we have the preliminary estimate

$$(3.1.3) \quad V(x) \leq Cd^{-1} \text{ in } \Omega \cap B_{2r}$$

In analogy with Theorem (3.1.1), we define

$$(3.1.4) \quad \Sigma = S \cap \overline{B_{2r}}, \quad w(x, z) = 4r^2 - |x|^2.$$

Let $0 < \epsilon < 1$ and $N > 0$, we define the vector field $Z = \eta v + X$ by

$$(3.1.5) \quad \eta = \epsilon w + Nd, \quad X = -\epsilon \Phi(w\gamma - d\nabla w)$$

and now we want to show the three conditions of Lemma (2.2.2) hold for sufficiently large M . We estimate the terms of condition (2) in Lemma (2.2.2), for $P \in (\partial\Sigma \cap \partial S)^o$, since $1 - |\Phi|^2$ is bounded above zero we have:

$$(3.1.6) \quad \begin{aligned} \nabla_{\Pi}(\epsilon w + Nd + \chi) \cdot \gamma + \frac{\partial}{\partial t} \Phi(\tilde{P}(P, t)) - v \cdot \frac{\partial}{\partial t} \gamma(\tilde{P}(P, t)) &> \\ \nabla_{\Pi}(Nd) \cdot \gamma - \epsilon \nabla_{\Pi}(\Phi w \gamma \cdot v) \cdot \gamma + D\Phi \cdot Z - v \cdot (D(\nabla d)Z) - C &> \\ \nabla_{\Pi}(Nd) \cdot \gamma - C = N(\gamma - (\gamma \cdot v)v) \cdot \gamma - C = N(|1 - |\Phi|^2) - C. \end{aligned}$$

This implies that we can satisfy condition (2) in Lemma (2.2.2) for large N , independently of M . For such N now we will show that condition (1) in Lemma (2.2.2) can be verified for large M : since $w = 0$ on $\overline{\partial\Sigma \cap S^o}$ and because of the estimate (3.1.3), we have

$$(3.1.7) \quad (\eta + \chi)V = (\epsilon w + Nd + \chi)V \leq (Nd + Cd)Cd^{-1} \leq C \quad (x \in \overline{\partial\Sigma \cap S^o}).$$

From (3.1.5) in the $\{f_\alpha\}$ coordinate we have $|(X^\alpha)_\beta + (X^\beta)_\alpha| \leq C(\epsilon w + d)$, so from symmetry of $[U_{ij}]$ we get:

$$(3.1.8) \quad \eta|A|^2 + 2X_i^k U_{ki} \geq \eta|A|^2 - C\eta|A| \geq -C$$

From this inequality and an inequality analogous to (3.1.2) we estimate

$$(3.1.9) \quad \eta|A|^2 + 2X_i^k U_{ki} + \Delta_\Pi(\eta + \chi) - (\eta + \chi)_v H + \dot{\xi}(0) \cdot \nabla \Psi > C$$

Taken together, (3.1.6), (3.1.7) and (3.1.9) show that there is a large M verify the three conditions of Lemma (2.2.2), hence ζV is bounded above by M on Σ .

$$(3.1.10) \quad \zeta = \epsilon w + Nd + X \cdot v \geq \epsilon w(1 - |\Phi|) + (N - \epsilon C)d \geq \epsilon w(1 - |\Phi|) > \frac{\epsilon}{C}$$

Thus we have Theorem (3.1.3) for $x \in B_r$ and ϵ sufficiently small. \square

3.2. Global gradient estimate. For proving existence of solutions using the continuity method we need to obtain an a priori gradient estimate. Notice that we will get a global gradient bound without assuming positive gravity.

Theorem 3.2.1. *Let $u \in \overline{C^3}(\overline{\Omega})$ solve the capillary problem. Then there is an $M = M(K_1, K_2, \partial\Omega)$ such that $V(x) \leq M$ for all $x \in \overline{\Omega}$.*

Proof. Recall the neighborhood of radius μ about $\partial\Omega$ on which d is C^3 . Extend d to be a C^3 function in all of $\overline{\Omega}$, with $|\nabla d| \leq 1$, and extend γ as ∇d . For the positive parameter N construct an increasing C^2 function $f: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $f(0) = 1$, $f'(0) = N$ and $f(t) = 2$ for $t \geq \mu$.

Introducing another positive parameter L , we define Σ and $Z = \eta v + X$ by

$$(3.2.1) \quad \Sigma = S = \text{graph}(u), \quad \eta = f(d)e^{Lz}, \quad X = -\Phi e^{Lz}\gamma$$

Since Φ is bounded by construction we have

$$(3.2.2) \quad K_2^{-1}e^{Lz} \leq \zeta \leq Ce^{Lz}$$

We seek to verify conditions (2), (3) in Lemma (2.2.2) for sufficiently large M . We estimate:

$$(3.2.3) \quad \begin{aligned} & \nabla_\Pi(\eta + \chi) \cdot \gamma + \frac{\partial}{\partial t} \Phi(\tilde{P}(P, t)) - v \cdot \gamma(\tilde{P}(P, t)) \geq \\ & e^{Lz}(\nabla_\Pi f \cdot \gamma) + Lf e^{Lz}(\nabla_\Pi z \cdot \gamma) - \nabla_\Pi(|\Phi|^2 e^{Lz}) \cdot \gamma + D\Phi(\tilde{P}(P, t))Z - v \cdot (D\nabla d(\tilde{P}(P, t))Z) \geq \\ & e^{Lz}(N(1 - |\Phi|^2) - C - CLV^{-1}) \end{aligned}$$

whenever $P \in (\partial S \cap \partial\Sigma)^\circ$. If ζV is sufficiently large (depending on L), then the second inequality of (3.2.2) and $|u| < K_1$ imply that $CL\frac{e^{Lz}}{V}$ in (3.2.3) is small. Hence we fix N large enough to verify condition (2) in Lemma (2.2.2) for large M (depending on L). To verify Lemma (2.2.2) we make the preliminary estimate

$$(3.2.4) \quad \eta|A|^2 + 2X_i^k U_{ki} \geq -Ce^{Lz}$$

whenever ζV is large enough, because we have $|X_i^k| \leq Ce^{Lz}$ for $i < n$ and $|X_n^k| \leq CLe^{Lz}$, also $|U_{kn}| \leq (\zeta V)^{-1}C(1 + |A|)$ for t small and $V > 1$. Using $(\zeta V)^{-1}$ to compensate for the L in estimating $X_n^k U_{kn}$, and then applying Cauchy-Schwartz, we will get (3.2.4) for ζV sufficiently large (depending on L). Using (3.2.4) and the fact that $|\nabla_\Pi z|^2 = 1 - V^{-2}$, we have

$$(3.2.5) \quad \begin{aligned} & \eta|A|^2 + 2X_i^k U_{ki} + \Delta_\Pi(\eta + \chi) - (\eta + \chi)_v H + \dot{\xi}(0) \cdot \nabla \Psi \geq \\ & \Delta_\Pi \eta - e^{Lz}(C + CLV^{-1}) \geq e^{Lz}(L^2(1 - V^{-2}) - C - CL) \end{aligned}$$

whenever $P \in \Sigma \cap S^\circ$ and ζV is large enough. Now fixing L large enough, we use (3.2.5) and sufficiently large M to verify condition (3) in Lemma (2.2.2). Since condition (1) in Lemma (2.2.2) is true, all three

conditions of Lemma (2.2.2) can be verified for a fixed N and L . We get ζV is uniformly bounded on Σ , using the first inequality of (3.2.2) we conclude the uniform bound for V on $\overline{\Omega}$. \square

4. EXISTENCE OF SOLUTION

Let $A = D^i(a_i(p))$, $a_i = p^i \cdot (1 + |p|^2)^{-\frac{1}{2}}$, We can rewrite the Capillary problem such that

$$(4.0.6) \quad Au = \Psi(x, u) \text{ in } \Omega$$

$$(4.0.7) \quad v \cdot \gamma = \Phi(x, u) \text{ on } \partial\Omega$$

In view of the a priori estimate which we have just established the existence of a solution will be proved by the method of continuity.

Theorem 4.0.2. *Let Ω be a bounded domain in M with $\partial\Omega$ is C^3 , then the boundary value Capillary problem that we have defined in section §1 has a unique solution $u \in C^{3,\alpha}(\overline{\Omega})$, where $0 < \alpha < 1$.*

Proof. Let τ be a real number with $0 \leq \tau \leq 1$, and consider the boundary value problem

$$(4.0.8) \quad Au_\tau = \tau\Psi(x, u_\tau)$$

$$(4.0.9) \quad v \cdot \gamma = \tau\Phi(x, u_\tau)$$

Let $T = \{\tau : \text{there exists a solution } u_\tau \in C^3(\overline{\Omega})\}$. T is not empty because $0 \in T$. We should show that it is both open and closed. We obtain an a priori estimate of $|u_\tau|_\Omega$ for any $\tau \in T$ independent of τ . Furthermore, let us remark that any solution $u_\tau \in C^3(\overline{\Omega})$ is of class $C^{3,\alpha}(\overline{\Omega})$ with some fixed α , $0 < \alpha < 1$ such that the norm of u_τ in $C^{2,\alpha}(\overline{\Omega})$ is bounded independent of τ . To prove this, we deduce from Theorem (3.2.1) that $|Du_\tau|_\Omega \leq M$ uniformly. We choose a smooth vector field \tilde{a}_i such that $\frac{\partial \tilde{a}_i}{\partial p^j}$ is uniformly elliptic, such that $\tilde{a}_i(p) = a_i(p)$ for $|p| \leq 3M$. From Theorem 2.2 in chapter 10 of [6] we conclude that the problem

$$(4.0.10) \quad \tilde{A}\tilde{u}_\tau = \tau\Psi(x, u_\tau) \text{ in } \Omega$$

$$(4.0.11) \quad v \cdot \gamma = \tau\Phi(x, u_\tau) \text{ on } \partial\Omega$$

has a solution $\tilde{u}_\tau \in C^{3,\alpha}(\overline{\Omega})$ for any τ . Moreover we have $\tilde{A}u_\tau = Au_\tau$. Hence we obtain from uniqueness of the solution $\tilde{u}_\tau = u_\tau$. Thus we can conclude that $|u_\tau|_{3,\alpha,\Omega}$ is uniformly bounded.

$$(4.0.12) \quad |u_\tau|_{3,\alpha,\Omega} \leq M_1$$

where the constant is determined by known quantities. From the estimate (4.0.12) it follows that T is closed. Now, let $\tau_0 \in T$. Then, we consider the boundary value problem (4.0.10) as before. Since $|D\tilde{u}_\tau|_\Omega$ depends continuously on τ , it turns out that $|D\tilde{u}_\tau| \leq M_1$ for $|\tau - \tau_0| < \epsilon$. This yields $\tilde{u}_\tau = u_\tau$ for those τ 's. That means T is open. \square

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