

Annular area preserving homeomorphisms which admit no interior compact invariant sets

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ABSTRACT. For any irrational number α , there exists an ergodic area preserving homeomorphism of the closed annulus which is isotopic to the identity, admits no compact invariant set contained in the interior of the annulus, and has the rotation number α .

1. Introduction

Let $\mathbb{A} = S^1 \times [-1, 1]$ be the closed annulus equipped with the standard area. Denote by G^r ($r = 0, 1, \dots, \infty$) the group of the area preserving C^r diffeomorphisms of \mathbb{A} which are isotopic to the identity, and by $\partial_{\pm}\mathbb{A} = S^1 \times \{\pm 1\}$ the upper or lower boundary curves of the annulus. In [He1], Michael Hermann announced that if $f \in G^\infty$ has no periodic points and if the rotation number of f restricted to $\partial_-\mathbb{A}$ is a non Liouville number, then f is C^∞ conjugate to a rigid rotation near $\partial_-\mathbb{A}$. According to [FS], in fact he seems to have shown the following.

If the rotation number of $f|_{\partial_-\mathbb{A}}$ is non Liouville, then there is a family of invariant smooth circles accumulating to $\partial_-\mathbb{A}$.

B. Fayad and M. Saprykina [FS] then showed the optimality of this result by proving that for any Liouville number α , there is $f \in G^\infty$ such that f is weakly mixing and that the rotation number of the two boundary curves is α .

In this paper we consider a somewhat different topological counter part of the Herman phenomenon. A map in G^r is called a *NICIS map* if it admits no compact invariant set contained in the interior of \mathbb{A} .

Let $\tilde{\mathbb{A}} = \mathbb{R} \times [-1, 1]$ be the universal cover of \mathbb{A} and $p : \tilde{\mathbb{A}} \rightarrow \mathbb{R}$ be the projection onto the first factor. Fix once and for all a lift $\tilde{f} : \tilde{\mathbb{A}} \rightarrow \tilde{\mathbb{A}}$ of $f \in G^r$. Then the map $p \circ \tilde{f} - p : \tilde{\mathbb{A}} \rightarrow \mathbb{R}$ is invariant by the covering transformation group and defines a map on \mathbb{A} , which we shall denote by the same letter $p \circ \tilde{f} - p$. For a NICIS map $f \in G^0$, there is an irrational number α such that for any f -invariant probability measure μ on \mathbb{A} , we have

$$\langle p \circ \tilde{f} - p, \mu \rangle = \alpha.$$

See Appendix A. The number α is called the *rotation number* of \tilde{f} and is denoted by $\rho(\tilde{f})$. The main result of the present paper is the following.

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Theorem 1. (1) If α is a Liouville number, then there is a NICIS C^∞ -diffeomorphism f (and its lift \tilde{f}) such that $\rho(\tilde{f}) = \alpha$.

(2) For any irrational number α , there is a NICIS homeomorphism f such that $\rho(\tilde{f}) = \alpha$.

Recall that an irrational number α is called a *Liouville number* if for any $\tau > 0$ and $\varepsilon > 0$, there is a rational number p/q , $(p, q) = 1$, such that $|\alpha - p/q| < \varepsilon/q^\tau$.

The proof of (1), given in Sect. 2, is based on the fast approximation by conjugacy method developed in [FS]. But since the NICIS diffeomorphism is much simpler to handle than the weakly mixing diffeomorphism treated in [FS], our argument becomes vastly easier than in [FS]. However, as will be apparent in Sect. 2, it looks hopeless to prove (2) by this method, and we show it in Sect. 3 by constructing a conjugacy with a Anzai skew product $F : S^1 \times \mathbb{R} \rightarrow S^1 \times \mathbb{R}$ over the rotation $R_\alpha : S^1 \rightarrow S^1$ by α . The obtained homeomorphism is not weakly mixing, although it is ergodic. In Appendix A we show some fundamental properties of NICIS maps and in Appendix B we prepare necessary prerequisites about Anzai skew products.

2. Fast approximation by conjugation method

Define the C^r -distance d_r on G^r by

$$d_r(f, g) = \max\{\|f - g\|_r, \|f^{-1} - g^{-1}\|_r\},$$

where $\|\cdot\|_r$ denotes the C^r -norm. Let us denote the standard S^1 -action on \mathbb{A} by S_t ($t \in S^1$):

$$S_t(x, y) = (x + t, y).$$

For any $n \in \mathbb{N}$, choose an increasing sequence of positive numbers a_n tending to 1. Let

$$U_n = S^1 \times ([-1, -a_n] \cup (a_n, 1])$$

and

$$G_n^r = \{f \in G^r \mid O_f(x) \cap U_n \neq \emptyset, \forall x \in \mathbb{A}\},$$

where $O_f(x)$ denotes the f -orbit of x . Notice that G_n^r is C^0 -open in G^r and that $f \in G^r$ is NICIS if and only if $f \in \bigcap_n G_n^r$.

For the moment let α be an arbitrary irrational number. We shall construct a diffeomorphism $h_n \in G^\infty$ and a rational number $\alpha_n = p_n/q_n$ ($n \in \mathbb{N}$) such that

$$(A) \quad h_n \circ S_{\alpha_n} = S_{\alpha_n} \circ h_n.$$

Denoting

$$H_n = h_1 \circ h_2 \circ \cdots \circ h_n, \quad f_n = H_n \circ S_{\alpha_{n+1}} \circ H_n^{-1},$$

we will also require the following.

(B) The map f_n converges to some $f \in G^\infty$ in the C^∞ -topology and $\alpha_n \rightarrow \alpha$.

(C) For any n , the C^0 -closure of the set $\{f_m \mid m \geq n\}$ is contained in G_n^0 .

Then the limit f is a NICIS C^∞ -diffeomorphism and by the continuity of the rotation number we have $\rho(\tilde{f}) = \alpha$ for some lift \tilde{f} . The condition (A) plays a key role in getting (B) and (C), since it implies the equalities

$$\begin{aligned} f_{n+1} - f_n &= H_{n+1} \circ S_{\alpha_{n+2}} \circ H_{n+1}^{-1} - H_{n+1} \circ S_{\alpha_{n+1}} \circ H_{n+1}^{-1}, \\ f_{n+1}^{-1} - f_n^{-1} &= H_{n+1} \circ S_{-\alpha_{n+2}} \circ H_{n+1}^{-1} - H_{n+1} \circ S_{-\alpha_{n+1}} \circ H_{n+1}^{-1}, \end{aligned}$$

which enable us to estimate $d_r(f_{n+1}, f_n)$ for example by Lemma 2.1 below.

Assume we already defined h_1, \dots, h_{n-1} and $\alpha_1, \dots, \alpha_n = p_n/q_n$. The construction of h_n is carried out in the following way. We identify the cyclic q_n -covering space of \mathbb{A} with \mathbb{A} in a standard way. First we construct $k_n \in G^\infty$ which keeps the vertical line $I_0 = 0 \times [-1, 1]$ invariant, and let h_n be the lift of k_n which keeps I_0 invariant. Then clearly h_n commutes with S_{1/q_n} , and hence with S_{α_n} .

The construction of k_n goes as follows. Fix real numbers b_n and c_n so that $a_n < b_n < c_n < a_{n+1}$. Choose $k_n \in G^\infty$ such that k_n is the identity on the neighbourhood U_{n+1} and on the vertical line I_0 and that k_n maps the horizontal curve $S^1 \times 0$ to the graph of the function $y = c_n \sin 2\pi x$. The Moser lemma ([M]) shows the existence of such an area preserving diffeomorphism.

Notice that the orbit foliation of the S^1 -action S_t is mapped by k_n to a foliation all of whose leaves intersect $S^1 \times ([-1, -c_n] \cup [c_n, 1])$. Clearly this property is inherited to h_n and also to $H_n = h_1 \circ \dots \circ h_{n-1} \circ h_n$ since $h_1 \circ \dots \circ h_{n-1}$ is the identity on U_n . That is, any orbit of the S^1 -action $H_n \circ S_t \circ H_n^{-1}$ ($t \in S^1$) intersects $S^1 \times ([-1, -c_n] \cup [c_n, 1])$.

Next choose a rational number $\alpha_{n+1} = p_{n+1}/q_{n+1}$ which approximates the given α so as to satisfy the following conditions (2.1), (2.2) and (2.3) below.

(2.1) *The number q_{n+1} is big enough so that any orbit of the diffeomorphism $H_n \circ S_{1/q_{n+1}} \circ H_n^{-1}$ intersects $S^1 \times ([-1, -b_n] \cup [b_n, 1])$. We also assume*

$$|\alpha - \alpha_{n+1}| < |\alpha - \alpha_n|$$

for any n .

The same intersection property is enjoyed also by the diffeomorphism $f_n = H_n \circ S_{\alpha_{n+1}} \circ H_n^{-1}$, since the both diffeomorphisms, periodic of order q_{n+1} , have the same orbit structure.

Furthermore if $g \in G^0$ satisfies

$$\|g^j - f_n^j\|_0 < b_n - a_n \quad (1 \leq j \leq q_{n+1}),$$

then $g \in G_n^0$, i. e. any orbit of g intersects U_n . Let

$$\varepsilon_n = \min\left\{\frac{b_n - a_n}{2}, \frac{b_{n-1} - a_{n-1}}{2^2}, \dots, \frac{b_1 - a_1}{2^n}\right\}.$$

If we choose diffeomorphisms $f_n = H_n \circ S_{\alpha_{n+1}} \circ H_n^{-1}$ in such a way that

$$\|f_{n+1}^j - f_n^j\|_0 < \varepsilon_n \quad (1 \leq j \leq q_{n+1})$$

for any n , then the sequence $\{f_n\}$ satisfies (C), and if further f_n converges to some f in G^0 , then f belongs to $\cap_n G_n^0$, that is, f is a NICIS map.

Denoting the Lipschitz constant by $L(\cdot)$, we have

$$\|f_{n+1}^j - f_n^j\|_0 = \|H_{n+1} \circ S_{j\alpha_{n+2}} \circ H_{n+1}^{-1} - H_{n+1} \circ S_{j\alpha_{n+1}} \circ H_{n+1}^{-1}\|_0 \leq L(H_{n+1})j|\alpha_{n+2} - \alpha_{n+1}|.$$

Since $L(h_n) \leq L(k_n)q_n$, $1 \leq j \leq q_{n+1}$ and

$$|\alpha_{n+2} - \alpha_{n+1}| < 2|\alpha - \alpha_{n+1}|$$

by (2.1), we get

$$\|f_{n+1}^j - f_n^j\|_0 \leq C_n q_1 \cdots q_n q_{n+1}^2 2|\alpha - \alpha_{n+1}|,$$

where $C_n = L(k_1) \cdots L(k_n)$.

Thus the following condition (2.2) assures $\|f_{n+1}^j - f_n^j\|_0 < \varepsilon_n$.

(2.2) We have

$$\left| \alpha - \frac{p_{n+1}}{q_{n+1}} \right| < \frac{\varepsilon_n}{2C_n q_1 \cdots q_n} \cdot \frac{1}{q_{n+1}^2}.$$

Finally to guarantee the convergence of f_n , we need the following lemma. The statement (1) is just Lemma 5.6 of [FS], and (2) is easy to establish.

Lemma 2.1. (1) For $r > 0$, $\alpha, \beta \in \mathbb{R}$ and for $H \in G^r$, we have

$$d_r(H \circ S_\alpha \circ H^{-1}, H \circ S_\beta \circ H^{-1}) \leq C_2(r) \|H\|_{r+1}^{r+1} |\alpha - \beta|,$$

where $C_2(r)$ is a constant depending only on r .

(2) We have

$$\|h_n\|_r \leq C_3(n, r) q_n^r,$$

where $C_3(n, r)$ is a constant depending only on n and r . □

The lemma implies that

$$d_r(f_{n+1}, f_n) \leq C_4(n, r) q_1^{(r+1)^2} \cdots q_n^{(r+1)^2} q_{n+1}^{(r+1)^2} |\alpha - \alpha_{n+1}|,$$

where $C_4(n, r)$ is a constant depending only on n and r .

Therefore the following (2.3) guarantees that $d_n(f_{n+1}, f_n) \leq 1/2^n$ and that f_n converges in G^∞ .

(2.3) We have

$$\left| \alpha - \frac{p_{n+1}}{q_{n+1}} \right| \leq \frac{1}{2^n C_4(n, n) q_1^{(n+1)^2} \cdots q_n^{(n+1)^2}} \cdot \frac{1}{q_{n+1}^{(n+1)^2}}.$$

Now if α is a Liouville number, then it is possible to choose p_n/q_n successively so as to satisfy (2.1), (2.2) and (2.3) above. This shows (1) of Theorem 1.

Notice that if we are working in G^0 , the condition for C^0 -convergence becomes;

(2.3')

$$\left| \alpha - \frac{p_{n+1}}{q_{n+1}} \right| \leq \frac{1}{2^n C_4(n, 0) q_1 \cdots q_n} \cdot \frac{1}{q_{n+1}}.$$

and any irrational number α admits a sequence $\{p_n/q_n\}$ satisfying (2.1) and (2.3'). On the other hand the condition (2.2) cannot be relaxed, and it remains still necessary to require the well approximability of α . Recall an irrational number α is called *well approximable* if for any $\varepsilon > 0$, there is a rational number p/q such that $|\alpha - p/q| < \varepsilon/q^2$.

3. Anzai skew products

The purpose of this section is to give a proof of Theorem 1 (2). Denote by $R_\alpha : S^1 \rightarrow S^1$ the rotation by α , and let φ be a real valued continuous function on S^1 . Define a homeomorphism $F_{\alpha, \varphi}$ of the annulus $S^1 \times \mathbb{R}$ by

$$F_{\alpha, \varphi}(x, y) = (R_\alpha(x), y + \varphi(x)).$$

The homeomorphism $F_{\alpha, \varphi}$ is called a *Anzai skew product* over R_α . Throughout this section we assume

(3.1) The rotation number α is irrational.

(3.2) The function φ satisfies $\int_{S^1} \varphi(x) dx = 0$.

(3.3) The function φ is nonintegrable, i. e. the functional equation $h \circ R_\alpha - h = \varphi$ has no continuous solution $h : S^1 \rightarrow \mathbb{R}$.

(3.4) *The function φ is absolutely continuous.*

Notice that if condition (3.2) is not fulfilled, then the unique ergodicity of R_α implies that all the orbits of $F_{\alpha,\varphi}$ are discrete in $S^1 \times \mathbb{R}$ and there is no $F_{\alpha,\varphi}$ -invariant probability measure on $S^1 \times \mathbb{R}$. Also if there is a solution h in (3.3), then the graph of h as well as its vertical translates will be an invariant curve. On the other hand conditions (3.3) implies that there are no compact invariant set of $F_{\alpha,\varphi}$ (Theorem 14.11, [GH]).

For a point $(x, y) \in S^1 \times \mathbb{R}$, denote by $\lambda(x, y)$ the limit set of (x, y) by the homeomorphism $F_{\alpha,\varphi}$, *i. e.* the union of the α -limit set and the ω -limit set:

$$\lambda(x, y) = \alpha(x, y) \cup \omega(x, y).$$

Following H. Poincaré, define

$$\mathbb{L}_x = \{y \in \mathbb{R} \mid (x, y) \in \lambda(x, 0)\}.$$

Since $F_{\alpha,\varphi}$ commutes with the vertical translations, \mathbb{L}_x forms a closed subsemigroup of \mathbb{R} , and we have

$$\lambda(x, y) \cap (x \times \mathbb{R}) = (x, y + \mathbb{L}_x).$$

Define

$$D = \{x \in S^1 \mid \mathbb{L}_x = \mathbb{R}\},$$

and notice that $x \in D$ if and only if the orbit of (x, y) is dense in $S^1 \times \mathbb{R}$ for some (any) $y \in \mathbb{R}$. Also define

$$P = \{x \in S^1 \mid \mathbb{L}_x = [0, \infty)\}, \quad N = \{x \in S^1 \mid \mathbb{L}_x = (-\infty, 0]\}.$$

The following theorem plays a crucial role in what follows. It was proven in [K] for C^1 -function φ in the situation where the λ -limit set is replaced by the ω -limit set. But this is not enough for our purpose. We shall include the proof in Appendix B.

Theorem 3.1. *Assume that $F_{\alpha,\varphi}$ satisfies (3.1) \sim (3.4). Then all the three subsets D , P and N are nonempty and they constitute a partition of S^1 into Borel sets.*

Here is an outline of the proof of Theorem 1 (2). Given any irrational number α , we shall construct a function $\varphi : S^1 \rightarrow \mathbb{R}$ satisfying (3.2) \sim (3.4). Moreover φ is to satisfy (3.5) and (3.6) below.

(3.5) *There exists a function $h \in L^1(S^1, dx)$ such that $h \circ R_\alpha - h = \varphi$, where dx denotes the Lebesgue measure on S^1 .*

(3.6) *We have $\varphi(-x - \alpha) = \varphi(x)$ for any $x \in S^1$.*

We identify $S^1 \times \mathbb{R}$ with the interior $\overset{\circ}{\mathbb{A}}$ of the closed annulus \mathbb{A} in a standard way by a diffeomorphism which maps a fiber $x \times \mathbb{R}$ onto a fiber $x \times (-1, 1)$. Clearly the map $F_{\alpha,\varphi}$ extends to a homeomorphism of the closed annulus \mathbb{A} in such a way that the restriction to each boundary component is the rotation by α . Using a solution h of (3.5), let us define a measurable map $\Gamma(h) : S^1 \rightarrow \mathbb{A}$ by $\Gamma(h)(x) = (x, h(x))$. Then we get a probability measure $\mu = \Gamma(h)_* dx$ on \mathbb{A} . Condition (3.5) implies that μ is invariant by $F_{\alpha,\varphi}$.

Clearly μ is nonatomic, and $\mu(\partial\mathbb{A}) = 0$. Furthermore once we show that μ is of full support, we can find a homeomorphism g of \mathbb{A} , identity on the boundary, such that $g_*\mu$ is the standard area of \mathbb{A} ([OU]). Then the conjugate $g \circ F_{\alpha,\varphi} \circ g^{-1}$ is a

NICIS homeomorphism with rotation number equal to α . Also it is ergodic since it is metrically isomorphic to the rotation R_α on S^1 .

Given α , the function φ is constructed as follows. For $x \in \mathbb{R}$, $\|x\|$ denotes the distance in $S^1 = \mathbb{R}/\mathbb{Z}$ of the class of x to 0. Choose a sequence $\{q_n\}_{n=1}^\infty$ of positive integers satisfying

$$(3.7) \quad \|q_n \alpha\| < 1/q_n, \quad (3.8) \quad q_{n+1} > 10q_n,$$

and define

$$\varphi_N(x) = \sum_{n=1}^N \frac{1}{ni} ((e^{2\pi i q_n \alpha} - 1)e^{2\pi i q_n x} - (e^{-2\pi i q_n \alpha} - 1)e^{-2\pi i q_n x}).$$

We have by (3.7)

$$|e^{\pm 2\pi i q_n \alpha} - 1| < 2\pi \|q_n \alpha\| < \frac{2\pi}{q_n}.$$

This, together with the assumption (3.8) shows that φ_N converges uniformly to a continuous function φ . To show that φ is absolutely continuous, we have;

$$\begin{aligned} \int_{S^1} |\varphi'(x)| dx &\leq \left(\int_{S^1} |\varphi'(x)|^2 dx \right)^{1/2} \\ &\leq \left(2 \sum_{n=1}^\infty \frac{1}{n^2} |e^{2\pi i q_n \alpha} - 1|^2 (2\pi q_n)^2 \right)^{1/2} \leq 4\sqrt{2}\pi^2 \sum_n \frac{1}{n^2} < \infty. \end{aligned}$$

Since $\varphi' \in L^1(S^1, dx)$, the function φ is absolutely continuous.

The Fourier series of the solution h of the equation $h \circ R_\alpha - h = \varphi$ is given by

$$h(x) = \sum_{n=1}^\infty \frac{1}{ni} (e^{2\pi i q_n x} - e^{-2\pi i q_n x}) = \sum_{n=1}^\infty \frac{1}{2n} \sin 2\pi q_n x.$$

Direct computation shows that $h \in L^2(S^1, dx) \subset L^1(S^1, dx)$. On the other hand h is not continuous. To see this, recall that the continuity of h is equivalent to the uniform convergence of the associated Cesàro sum (Theorem 73 of [HR]). Let

$$C_n = \{x \in S^1 \mid \sin 2\pi q_n x \geq 1/2\}.$$

Then by (3.8) a connected component of C_n contains a connected component of C_{n+1} , and thus one can choose a point x_0 from the intersection of all the C_n 's. Then clearly the Cesàro sum diverges at x_0 .

The proof that the function φ satisfies the conditions (3.2) \sim (3.5) is now complete, while the condition (3.6) can be verified by a direct computation.

Finally let us show that the measure μ has full support. Notice that condition (3.6) implies that

$$J \circ F_{\alpha, \varphi} \circ J = F_{\alpha, \varphi}^{-1},$$

where J is the involution of $S^1 \times \mathbb{R}$ defined by

$$J(x, y) = (-x, -y).$$

Since our definition of \mathbb{L}_x is invariant by time reversion, this shows that $-P = N$ on S^1 . Therefore the measures of P and N must be the same. But since R_α is ergodic, this implies that D is of full measure in S^1 .

Suppose for contradiction that the support S of the measure μ is not the whole $S^1 \times \mathbb{R}$. Then for any $x \in D$ and $y \in \mathbb{R}$, the point (x, y) cannot belong to S , for if it did, then we would have $S = S^1 \times \mathbb{R}$ since the orbit of (x, y) is dense in $S^1 \times \mathbb{R}$.

Therefore for the projection $q : S^1 \times \mathbb{R} \rightarrow S^1$, we have $q_*\mu(D) = 0$. But $q_*\mu$ is just the Lebesgue measure dx , contradicting the fact that D is of full measure. This completes the proof of Theorem 1 (2).

4. Appendix A: Properties of NICIS homeomorphisms

Let f be an area preserving homeomorphism of the annulus \mathbb{A} , isotopic to the identity. Fix once and for all a lift \tilde{f} of f to the universal covering space $\tilde{\mathbb{A}}$ of \mathbb{A} . As before the function $p \circ \tilde{f} - p$, defined on $\tilde{\mathbb{A}}$ and invariant by the deck transformation group, is considered to be a function defined on \mathbb{A} .

Denote by $\mathcal{P}_f(\mathbb{A})$ the space of the probability measures on \mathbb{A} invariant by f . Define

$$\mathcal{R}(\tilde{f}) = \{ \langle p \circ \tilde{f} - p, \mu \rangle \mid \mu \in \mathcal{P}_f(\mathbb{A}) \}$$

and call it the *rotation set of \tilde{f}* . The rotation set is a compact interval (maybe one point) of \mathbb{R} .

Proposition 4.1. *If f is a NICIS homeomorphism, then the rotation set $\mathcal{R}(\tilde{f})$ consists of a single irrational number α .*

PROOF. It is shown in [F1] that if $\mathcal{R}(\tilde{f})$ is not a singleton, then there is a periodic orbit of f in the interior $\overset{\circ}{\mathbb{A}}$ of \mathbb{A} , contradicting the condition of a NICIS homeomorphism. Assume that the rotation set $\mathcal{R}(\tilde{f})$ consists of a rational number α . In the ergodic decomposition of the Lebesgue measure, there must exist an ergodic component ν whose support has nonempty intersection with $\overset{\circ}{\mathbb{A}}$. Since

$$\langle p \circ \tilde{f} - p, \nu \rangle = \alpha,$$

again there must be a periodic orbit in $\overset{\circ}{\mathbb{A}}$, by Proposition 2.3 of [FH2]. \square

Next we shall show some topological properties of the orbits of NICIS homeomorphisms.

Proposition 4.2. *A NICIS homeomorphism f admit dense orbits.*

PROOF. We shall show that any f -invariant open subset $U \subset \overset{\circ}{\mathbb{A}}$ is dense in $\overset{\circ}{\mathbb{A}}$. By the Baire property of $\overset{\circ}{\mathbb{A}}$, this will imply the existence of dense orbits. In fact, choose a countable open basis U_j of $\overset{\circ}{\mathbb{A}}$. The set $\mathcal{O}_f(U_j)$ of points whose orbits intersect U_j will then be an open dense subset of $\overset{\circ}{\mathbb{A}}$, and a point in the nonempty intersection $\cap_j \mathcal{O}_f(U_j)$ will have a dense orbit.

Suppose U is an f -invariant open subset of $\overset{\circ}{\mathbb{A}}$. Assume U admits an inessential connected component V , *i. e.* a component for which the inclusion $V \rightarrow \overset{\circ}{\mathbb{A}}$ induces the trivial map on the fundamental group. Then V is left invariant by some power of f , say f^m , since f is area preserving. Let \hat{V} be the union of all the closed discs in $\overset{\circ}{\mathbb{A}}$ whose boundary is contained in V . Then f^m leaves the open disc \hat{V} invariant, and admits a fixed point in \hat{V} by the Brouwer plane fixed point theorem, since all the points of \hat{V} are nonwandering w. r. t. f^m . This contradiction shows that any component of U must be essential.

Assume for contradiction that U is not dense in $\mathring{\mathbb{A}}$. By the same argument applied to the interior of the complement of U , we obtain two disjoint essential f -invariant open subset of $\mathring{\mathbb{A}}$. This implies the existence of interior compact invariant set. A contradiction. \square

Next proposition due to [LY] and [F2] shows that NICIS homeomorphisms admit nondense orbits as well. Let us denote $\partial_{\pm}\mathbb{A} = S^1 \times \{\pm 1\}$.

Proposition 4.3. *Given any neighbourhood U of $\partial_+\mathbb{A}$ (or of $\partial_-\mathbb{A}$), a NICIS homeomorphism f admits an orbit contained in $U \cap \mathring{\mathbb{A}}$.*

5. Appendix B: Proof of Theorem 3.1

Throughout this section we assume that the skew product $F_{\alpha, \varphi}$ satisfies the conditions (3.1) ~ (3.4).

Computation shows that for any integer $n > 0$,

$$F_{\alpha, \varphi}^n(x, y) = (R_{\alpha}^n(x), y + \varphi_n(x)), \quad \text{where } \varphi_n(x) = \sum_{i=0}^{n-1} \varphi(R_{\alpha}^i(x)).$$

A positive integer q is called a *closest return time* for α if $\|j\alpha\| > \|q\alpha\|$ for any integer j such that $0 < j < q$, where $\|\cdot\|$ denotes the distance to 0 of the projected image in \mathbb{R}/\mathbb{Z} . The following theorem can be found in Sect. 1.2 of [GLL].

Theorem 5.1. (Improved Denjoy-Koksma theorem) *If φ satisfies (3.2) and (3.4), and q_n is a sequence of closest return times for the irrational number α , then we have*

$$\sup |\varphi_{q_n}| \rightarrow 0.$$

It follows at once that for any $x \in S^1$, the closed subsemigroup \mathbb{L}_x is nonempty. We also need the following lemma, the proof of which is left to the reader.

Lemma 5.2. *For integers $0 \leq n_1 < n_2 < n_3$ assume that n_3 is the smallest positive integer j such that $j\alpha$ lies in the smaller interval bounded by $n_1\alpha$ and $n_2\alpha$. Then $n_3 - n_2$ is a closest return time for α .* \square

The following proposition plays a key role in the proof of Theorem 3.1.

Proposition 5.3. *For any $x \in S^1$ and for any $0 < a < b$, we have*

$$\mathbb{L}_x \cap ([-b, -a] \cup [a, b]) \neq \emptyset.$$

PROOF. It is no loss of generality to assume that the point x in Proposition 5.3 is 0. For contradiction assume that the limit set $\lambda(0, 0) = \alpha(0, 0) \cup \omega(0, 0)$ is disjoint from the set $0 \times ([-b, -a] \cup [a, b])$. Thus one can choose a small positive number ε such that the union of two rectangles

$$Y' = \{(x, y) \mid |x| \leq \varepsilon, a \leq |y| \leq b\}$$

is disjoint from the orbit of $(0, 0)$.

CLAIM. *For even smaller $\delta > 0$, the orbit of $(0, 0)$ is disjoint from $\{|x| \leq \delta, |y| \geq a\}$.*

The claim clearly implies that the orbit of $(0, 0)$ is bounded, which contradicts the nonintegrability condition (3.3) by Theorem 14.11 of [GH].

Let us start the proof of the claim. We are going to show this only for the positive orbit, the other case being similar.

Choose positive integers n_- and n_+ such that

$$(5.1) \quad -\varepsilon < n_- \alpha < 0 < n_+ \alpha < \varepsilon,$$

$$(5.2) \quad n_{\pm} \text{ is a closest return time for } \alpha, \text{ and}$$

$$(5.3) \quad \text{if } q \text{ is a closest return time for } \alpha \text{ such that } q \geq \min\{n_-, n_+\}, \text{ then}$$

$$\sup|\varphi_q| < b - a.$$

Put

$$Y = \{(x, y) \mid n_- \alpha < x < n_+ \alpha, a \leq |y| \leq b\}.$$

Let us show that whenever a positive iterate of $(0, 0)$ by $F_{\alpha, \varphi}$ lies in the strip $(n_- \alpha, n_+ \alpha) \times \mathbb{R}$, it is contained in $(n_- \alpha, n_+ \alpha) \times (-a, a)$, that is, $n_- \alpha < n \alpha < n_+ \alpha$ implies that $|\varphi_n(0)| < a$. Assume to fix the idea that $n \alpha \in (0, n_+ \alpha)$. Define a sequence $n_1, n_2, \dots, n_k = n$ as follows. First of all let $n_1 = n_+$, and let n_2 be the smallest positive integer j such that $j \alpha$ lies in the interval $I_1 = (0, n_+ \alpha)$. If $n_2 = n$, the definition is over. If not, $n_2 \alpha$ divides the interval I_1 into two subintervals, one of which, say I_2 , contains $n \alpha$. Let n_3 be the smallest positive integer j such that $j \alpha$ lies in the interval I_2 . Proceeding in this way, we end up with some n_k eventually matching the given n .

Now by Lemma 5.2, all the integers $n_1, n_2 - n_1, \dots, n_k - n_{k-1}$ are closest return times for α . Moreover since $\|(n_j - n_{j-1})\alpha\| < \|n_+ \alpha\|$, we have $n_j - n_{j-1} \geq n_+$ ($2 \leq j \leq k$) and (5.3) implies that $\sup|\varphi_{n_j - n_{j-1}}| \leq b - a$ as well as $\sup|\varphi_{n_1}| \leq b - a$.

To begin with, we have $|\varphi_{n_1}(0)| \leq b - a < b$. But since the point $F_{\alpha, \varphi}^{n_1}(0, 0) = (R_{\alpha}^{n_1}(0), \varphi_{n_1}(0))$ does not lie in Y , we have in fact $|\varphi_{n_1}(0)| < a$. Furthermore we obtain

$$|\varphi_{n_2}(0)| = |\varphi_{n_2 - n_1}(R_{\alpha}^{n_1}(0)) + \varphi_{n_1}(0)| \leq |\varphi_{n_2 - n_1}(R_{\alpha}^{n_1}(0))| + |\varphi_{n_1}(0)| \leq (b - a) + a.$$

Again since $F_{\alpha, \varphi}^{n_2}(0, 0)$ does not lie in Y , we actually have $|\varphi_{n_2}(0)| < a$. Proceeding this way one gets $|\varphi_n(0)| < a$.

The proof of the claim and of Proposition 5.3 is now complete. \square

Let us finish the proof of Theorem 3.1. Proposition 5.3 shows that for any $x \in S^1$, the set \mathbb{L}_x is unbounded. If the closed subsemigroup \mathbb{L}_x contains both positive and negative numbers, then $\mathbb{L}_x = \mathbb{R}$ or $\mathbb{L}_x = \mathbb{Z} \cdot c$ for some $c > 0$. Proposition 5.3 shows that $\mathbb{L}_x = \mathbb{R}$ in this case. Also the same proposition shows that if \mathbb{L}_x consists of nonnegative numbers (resp. nonpositive numbers), then $\mathbb{L}_x = [0, \infty)$ (resp. $\mathbb{L}_x = (-\infty, 0]$). This shows that S^1 is the union of D , P and N .

What is left is to show that each of the sets D , P and N is nonempty. The existence of dense orbits of $F_{\alpha, \varphi}$ is shown in [**Hed**] under the assumption (3.1) \sim (3.3), which implies that D is nonempty. In fact the essential point is to show that any point of $S^1 \times \mathbb{R}$ is nonwandering, the rest being exactly the same as the proof of Proposition 4.2. Also in [**B**] the existence of the $F_{\alpha, \varphi}$ -orbits which are bounded below (or above) is shown under the same assumption. This, together with Proposition 5.3, shows that P and N are nonempty.

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