Leibniz 2-algebras and twisted Courant algebroids *

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Abstract

In this paper, we give the categorification of Leibniz algebras, which is equivalent to 2-term sh Leibniz algebras. They reveal the algebraic structure of omni-Lie 2-algebras introduced in [22] as well as twisted Courant algebroids by closed 4-forms introduced in [9]. We also prove that Dirac structures of twisted Courant algebroids give rise to 2-term L_∞-algebras and geometric structures behind them are exactly H-twisted Lie algebroids introduced in [7].

1 Introduction

Recently, people have payed more attention to higher categorical structures by reasons in both mathematics and physics. One way to provide higher categorical structures is by categorifying existing mathematical concepts. One of the simplest higher structures is a 2-vector space, which is the categorification of a vector space. If we further put a compatible Lie algebra structure on a 2-vector space, then we obtain a Lie 2-algebra [2, 19]. The Jacobi identity is replaced by a natural transformation, called the Jacobiator, which also satisfies some coherence laws of its own. Recently, the relation among higher categorical structures and multisymplectic structures, Courant algebroids, and Dirac structures are studied in [3, 16, 26].

A 2-vector space is equivalent to a 2-term complex of vector spaces. A Lie 2-algebra is equivalent to a 2-term L_∞-algebra. L_∞-algebras, sometimes called strongly homotopy (sh) Lie algebras, were introduced in [20, 11] as a model for "Lie algebras that satisfy Jacobi identity up to all higher homotopies". The notion of Leibniz algebras was introduced by Loday [13], which is a generalization of Lie algebras. Their crossed modules were also introduced in [14] to study the cohomology of Leibniz algebras. As a model for "Leibniz algebras that satisfy Jacobi identity up to all higher

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homotopies", Ammar and Poncin introduced the notion of strongly homotopy Leibniz algebra, or Lod_∞-algebra in [1], which is further studied by Uchino in [25].

Courant algebroid was introduced in [12] to study the double of Lie bialgebroids. Equivalent definition was given by Roytenberg in [18]. Courant algebroids have been widely studied because of their applications in both mathematics and physics. Roytenberg proved that every Courant algebroid give rise to an L_∞-algebra [18]. The L_∞-algebra associated to the standard Courant algebroid $TM \oplus T^*M$ is a semidirect product of a Lie algebra with a representation up to homotopy [23, 24]. Recently, Hansen and Strobl introduced the notion of twisted Courant algebroids by closed 4-forms in [9], which arise from the study of three dimensional sigma models with Wess-Zumino term. In general, if one studies generalized geometry, this 4-form will arise naturally as background [10]. Moreover, a closed 4-form is also used to construct a bundle 2-gerbe in [4].

In this paper, we introduce the notion of Leibniz 2-algebras, which is equivalent to 2-term strongly homotopy Leibniz algebras. Similar to the case of Lie algebras, we prove that there is a one-to-one correspondence between 2-term dg Leibniz algebras and crossed modules of Leibniz algebras. With the help of an automorphism $\mathfrak{f}$ of the 2-term DGLA $\mathrm{End}(\mathcal{V})$, where $\mathcal{V}$ is a 2-term complex of vector spaces, we construct a Leibniz 2-algebra $(\mathrm{End}(\mathcal{V}) \oplus \mathcal{V}, l_2^{\mathfrak{f}}, l_3^{\mathfrak{f}})$, which essentially comes from omni-Lie 2-algebra introduced in [22]. Every twisted Courant algebroid by a closed 4-form H gives rise to a Leibniz 2-algebra. In particular, Dirac structures of twisted Courant algebroids give rise to 2-term L_∞-algebras. The geometric structure underlying this 2-term L_∞-algebra is H-twisted Lie algebroid introduced by Grützmann in [7]. B-field transformation [8] is an important tool to provide symmetries of exact Courant algebroids. In a B-field transformation, the 2-form need to be closed. Now for exact twisted Courant algebroids, every 2-form (not need to be closed) provides an automorphism of the corresponding Leibniz 2-algebra.

The paper is organized as follows. In Section 2 we prove that there is a one-to-one correspondence between 2-term dg Leibniz algebras and crossed modules of Leibniz algebras (Theorem 2.6). In Section 3 we introduce the notion of Leibniz 2-algebra, which is the categorification of Leibniz algebras. We show that they are equivalent to 2-term sh Leibniz algebras. In Section 4 associated to any automorphism $\mathfrak{f}$ of $\mathrm{End}(\mathcal{V})$, we construct a Leibniz 2-algebra $(\mathrm{End}(\mathcal{V}) \oplus \mathcal{V}, l_2^{\mathfrak{f}}, l_3^{\mathfrak{f}})$. In Section 5 we show that every twisted Courant algebroid gives rise to a Leibniz 2-algebra (Theorem 5.2). Via the B-field transformation, any 2-form provides an automorphism of the Leibniz 2-algebra associated to an exact twisted Courant algebroid (Theorem 5.9). In Section 6 we study Dirac structures of a twisted Courant algebroid, it turns out that a Dirac structure of a twisted Courant algebroid gives rise to a 2-term L_∞-algebra (i.e. a Lie 2-algebra, Theorem 6.2). We also find that the geometric structure underlying this 2-term L_∞-algebra is H-twisted Lie algebroid. At last, we consider the Dirac structure $\mathcal{G}_\pi$, which is the graph of a bi-vector field π, and obtain h-twisted Poisson structure (the 3-form h is not closed) as well as the associated 2-term L_∞-algebra.

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2 Crossed modules of Leibniz algebras and sh Leibniz algebras

A Leibniz algebra $\mathfrak{g}$ is an R-module, where R is a commutative ring, endowed with a linear map $[\cdot,\cdot]_\mathfrak{g} : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathfrak{g}$ satisfying

$$[g_1, [g_2, g_3]_\mathfrak{g}]_\mathfrak{g} = [[g_1, g_2]_\mathfrak{g}, g_3]_\mathfrak{g} + [g_2, [g_1, g_3]_\mathfrak{g}]_\mathfrak{g}, \quad \forall\, g_1, g_2, g_3 \in \mathfrak{g}.$$

This is in fact a left Leibniz algebra. In this paper, we only consider left Leibniz algebras.

Recall that a representation of the Leibniz algebra $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g})$ is an R-module V equipped with, respectively, left and right actions of $\mathfrak{g}$ on V,

$$[\cdot,\cdot] : \mathfrak{g} \otimes V \longrightarrow V, \quad [\cdot,\cdot] : V \otimes \mathfrak{g} \longrightarrow V,$$

such that for any $g_1, g_2 \in \mathfrak{g}$, the following equalities hold:

$$l_{[g_1,g_2]} = [l_{g_1}, l_{g_2}], \quad r_{[g_1,g_2]} = [l_{g_1}, r_{g_2}], \quad r_{g_2} \circ l_{g_1} = -r_{g_2} \circ r_{g_1}, \tag{1}$$

where $l_{g_1} u = [g_1, u]$ and $r_{g_1} u = [u, g_1]$ for any $u \in V$. The Leibniz cohomology of $\mathfrak{g}$ with coefficients in V is the homology of the cochain complex $C^k(\mathfrak{g}, V) = \mathrm{Hom}_R(\otimes^k \mathfrak{g}, V), (k \geq 0)$ with the coboundary operator $\partial : C^k(\mathfrak{g}, V) \longrightarrow C^{k+1}(\mathfrak{g}, V)$ defined by

$$\begin{aligned} \partial c^k(g_1, \cdots, g_{k+1}) &= \sum_{i=1}^{k} (-1)^{i+1} l_{g_i}(c^k(g_1, \cdots, \widehat{g_i}, \cdots, g_{k+1})) + (-1)^{k+1} r_{g_{k+1}}(c^k(g_1, \cdots, g_k)) \\ &\quad + \sum_{1 \leq i < j \leq k+1} (-1)^i c^k(g_1, \cdots, \widehat{g_i}, \cdots, g_{j-1}, [g_i, g_j]_\mathfrak{g}, g_{j+1}, \cdots, g_{k+1}). \end{aligned} \tag{2}$$

The fact that $\partial \circ \partial = 0$ is proved in [14].

The notion of strongly homotopy (sh) Leibniz algebras, or Lod_∞-algebras was first given in [1]. See also [25] for more details.

Definition 2.1. [25] *A sh Leibniz algebra is a graded vector space $L = L_0 \oplus L_1 \oplus \cdots$ equipped with a system $\{l_k |\ 1 \leq k < \infty\}$ of linear maps $l_k : \wedge^k L \longrightarrow L$ with degree $\deg(l_k) = k - 2$, where the exterior powers are interpreted in the graded sense and the following relation is satisfied:*

$$\sum_{i+j=Const} \sum_{k \geq j} \sum_{\sigma} (-1)^{(k+1-j)(j-1)} (-1)^{j(|x_{\sigma(1)}| + \cdots + |x_{\sigma(k-j)}|)} \sum_{\sigma} \mathrm{sgn}(\sigma) \mathrm{Ksgn}(\sigma)$$
$$l_i(x_{\sigma(1)}, \cdots, x_{\sigma(k-j)}, l_j(x_{\sigma(k+1-j)}, \cdots, x_{\sigma(k)}), x_{\sigma(k+1)}, \cdots, x_{\sigma(i+j-1)}) = 0,$$

where the summation is taken over all $(k-j, j-1)$-unshuffles.

In particular, if we concentrate on the 2-term case, we can give explicit formulas for 2-term sh Leibniz algebras as follows:

Lemma 2.2. *A 2-term sh Leibniz algebra $\mathcal{V}$ consists of the following data:*

- *a complex of vector spaces $\mathcal{V} : V_1 \xrightarrow{\mathrm{d}} V_0$,*
- *a bilinear map $l_2 : V_i \times V_j \longrightarrow V_{i+j}$, where $i + j \leq 1$,*
- *a trilinear map $l_3 : V_0 \times V_0 \times V_0 \longrightarrow V_1$,*

such that for any $w, x, y, z \in V_0$ and $m, n \in V_1$, the following equalities are satisfied:

(a) $\mathrm{d}l_2(x, m) = l_2(x, \mathrm{d}m)$,

(b) $\mathrm{d}l_2(m, x) = l_2(\mathrm{d}m, x)$,

(c) $l_2(\mathrm{d}m, n) = l_2(m, \mathrm{d}n)$,

(d) $\mathrm{d}l_3(x,y,z)=l_2(x,l_2(y,z))-l_2(l_2(x,y),z)-l_2(y,l_2(x,z))$,

(e$_1$) $l_3(x,y,\mathrm{d}m)=l_2(x,l_2(y,m))-l_2(l_2(x,y),m)-l_2(y,l_2(x,m))$,

(e$_2$) $l_3(x,\mathrm{d}m,y)=l_2(x,l_2(m,y))-l_2(l_2(x,m),y)-l_2(m,l_2(x,y))$,

(e$_3$) $l_3(\mathrm{d}m,x,y)=l_2(m,l_2(x,y))-l_2(l_2(m,x),y)-l_2(x,l_2(m,y))$,

(f) *the Jacobiator identity:*

$$\begin{array}{l} l_2(w,l_3(x,y,z))-l_2(x,l_3(w,y,z))+l_2(y,l_3(w,x,z))+l_2(l_3(w,x,y),z) \\ -l_3(l_2(w,x),y,z)-l_3(x,l_2(w,y),z)-l_3(x,y,l_2(w,z)) \\ +l_3(w,l_2(x,y),z)+l_3(w,y,l_2(x,z))-l_3(w,x,l_2(y,z))=0. \end{array}$$

We usually denote a 2-term sh Leibniz algebra by $(V_1\stackrel{\mathrm{d}}{\longrightarrow}V_0,l_2,l_3)$, or simply by $\mathcal{V}$.

If $l_3=0$, we obtain the notion of **2-term differential graded (dg) Leibniz algebra**. If the bilinear map l_2 and the trilinear map l_3 are skew-symmetric, then it is a **2-term L_∞-algebra**.

Lemma 2.3. *For a 2-term dg Leibniz algebra $(V_1\stackrel{\mathrm{d}}{\longrightarrow}V_0,l_2,l_3)$, we have*

$$l_2(l_2(x,m),y)+l_2(l_2(m,x),y)=0,\quad \forall\ x,y\in V_0,m\in V_1. \tag{3}$$

Proof. By Condition (e_2) and (e_3) in Definition 2.2, we have

$$\begin{array}{rcl} l_2(l_2(x,m),y)+l_2(l_2(m,x),y) &=& l_2(x,l_2(m,y))-l_2(m,l_2(x,y)) \\ && +l_2(m,l_2(x,y))-l_2(x,l_2(m,y)) \\ &=& 0.\ \blacksquare \end{array}$$

The notion of crossed module of Leibniz algebras was introduced by Loday and Pirashvili in [14]. The more general notion of crossed module of n-Leibniz algebras, which are generalizations of n-Lie algebras, was given by Casas, Khmaladze and Ladra in [5].

Definition 2.4. *A crossed module of Leibniz algebras is a morphism of Leibniz algebras $\mu:\mathfrak{g}\longrightarrow\mathfrak{h}$ together with a representation of $\mathfrak{h}$ (consists of a left action and a right action satisfying the compatibility condition* (1)*) on $\mathfrak{g}$ such that for any $g,g'\in\mathfrak{g},\ h\in\mathfrak{h}$, the following equalities hold:*

$$\mu(l_hg) = [h,\mu(g)]_{\mathfrak{h}},\quad \mu(r_hg)=[\mu(g),h]_{\mathfrak{h}}; \tag{4}$$
$$l_{\mu(g)}g' = [g,g']_{\mathfrak{g}}=r_{\mu(g')}g; \tag{5}$$
$$l_h[g,g']_{\mathfrak{g}} = [l_hg,g']_{\mathfrak{g}}+[g,l_hg']_{\mathfrak{g}}; \tag{6}$$
$$r_h[g,g']_{\mathfrak{g}} = [g,r_hg']_{\mathfrak{g}}-[g',r_hg]; \tag{7}$$
$$[l_hg+r_hg,g']_{\mathfrak{g}} = 0. \tag{8}$$

Remark 2.5. *Loday and Porashvili defined a crossed module of Leibniz algebras to be a morphism of Leibniz algebras $\mu:\mathfrak{g}\longrightarrow\mathfrak{h}$ together with an action of $\mathfrak{h}$ on $\mathfrak{g}$ satisfying* (4) *and* (5) *in [14]. However, to define an action of Leibniz algebra $\mathfrak{h}$ on Leibniz algebra $\mathfrak{g}$, one needs six relations, which is exactly* (1)*,* (6)*,* (7)*, and* (8)*.*

Theorem 2.6. *There is a one-to-one correspondence between 2-term dg Leibniz algebras and crossed modules of Leibniz algebras.*

Proof. Let $V_1 \xrightarrow{\mathrm{d}} V_0$ be a 2-term dg Leibniz algebra, define $\mathfrak{g} = V_1$, $\mathfrak{h} = V_0$, and the following two bracket operations on $\mathfrak{g}$ and $\mathfrak{h}$:

$$\begin{aligned}[m,n]_{\mathfrak{g}} &= l_2(\mathrm{d}m,n) = l_2(m,\mathrm{d}n), \quad \forall\, m,n \in V_1;\\ [u,v]_{\mathfrak{h}} &= l_2(u,v), \quad \forall\, u,v \in V_0.\end{aligned}$$

It is straightforward to see that both $[\cdot,\cdot]_{\mathfrak{g}}$ and $[\cdot,\cdot]_{\mathfrak{h}}$ are Leibniz brackets. Let $\mu = \mathrm{d}$, by Condition (a) in Definition 2.2, we have

$$\mu[m,n]_{\mathfrak{g}} = \mathrm{d}l_2(\mathrm{d}m,n) = l_2(\mathrm{d}m,\mathrm{d}n) = [\mu(m),\mu(n)]_{\mathfrak{h}},$$

which implies that μ is a morphism of Leibniz algebras. Define the representation of $\mathfrak{h}$ on $\mathfrak{g}$ by l_2, i.e.

$$l_u m = l_2(u,m), \quad r_u m = l_2(m,u), \quad \forall\, u \in \mathfrak{h}, m \in \mathfrak{g}.$$

It is well defined. In fact, by Condition (e_1), we have

$$l_{[u,v]_{\mathfrak{h}}} = [l_u, l_v].$$

By (3), we have

$$r_v r_u + r_v l_u = 0.$$

Now by Condition (e_3), we have

$$r_{[u,v]_{\mathfrak{h}}} = [l_u, r_v],$$

which implies that (1) holds.

By Conditions (a)-(c), we have (4) and (5). By Condition (e_1) and (a), we have

$$\begin{aligned}l_u[m,n]_{\mathfrak{g}} &= l_2(u, l_2(\mathrm{d}m,n))\\ &= l_2(l_2(u,\mathrm{d}m),n) + l_2(\mathrm{d}m, l_2(u,n))\\ &= l_2(\mathrm{d}l_2(u,m),n) + l_2(\mathrm{d}m, l_2(u,n))\\ &= [l_u m, n]_{\mathfrak{g}} + [m, l_u n]_{\mathfrak{g}},\end{aligned}$$

which yields (6). By Condition (e_2), we have

$$\begin{aligned}r_u[m,n]_{\mathfrak{g}} &= l_2(l_2(\mathrm{d}m,n),u)\\ &= l_2(\mathrm{d}m, l_2(n,u)) - l_2(n, l_2(\mathrm{d}m,u))\\ &= [m, r_u n]_{\mathfrak{g}} - [n, r_u m]_{\mathfrak{g}},\end{aligned}$$

which implies that (7) holds. By (3), we have

$$[r_u m + l_u m, n]_{\mathfrak{g}} = l_2(l_2(u,m) + l_2(m,u), \mathrm{d}n) = 0.$$

Thus we get (8). Therefore, we obtain a crossed module of Leibniz algebras.

Conversely, a crossed module of Leibniz algebras gives rise to a 2-term dg Leibniz algebra with $\mathrm{d} = \mu$, $V_1 = \mathfrak{g}$ and $V_0 = \mathfrak{h}$, where the brackets are given by:

$$\begin{aligned}l_2(m,n) &\triangleq 0, \qquad \forall\, m,n \in \mathfrak{g};\\ l_2(u,v) &\triangleq [u,v]_{\mathfrak{h}_0}; \quad \forall\, u,v \in \mathfrak{h},\\ l_2(u,m) &\triangleq l_u m;\\ l_2(m,u) &\triangleq r_u m.\end{aligned}$$

The crossed module conditions give various conditions for 2-term dg Leibniz algebras. We omit details. ■

Definition 2.7. *A 2-term sh Leibniz algebra* $(V_1 \xrightarrow{\mathrm{d}} V_0, l_2, l_3)$ *is called skeletal if* $\mathrm{d} = 0$.

Skeletal 2-term sh Leibniz algebras can be classified by the third cohomology of Leibniz algebras.

Proposition 2.8. *There is a one-to-one correspondence between skeletal 2-term sh Leibniz algebras* $(V_1 \xrightarrow{0} V_0, l_2, l_3)$ *and quadruples* $(\mathfrak{g}, V, \rho, \phi)$*, where* $\mathfrak{g}$ *is a Leibniz algebra,* V *is a vector space,* ρ *is a representation of* $\mathfrak{g}$ *on* V*,* ϕ *is* 3*-cocycle on* $\mathfrak{g}$ *with coefficient in* V*.*

Proof. For a skeletal 2-term sh Leibniz algebra $(V_1 \xrightarrow{0} V_0, l_2, l_3)$, by Condition (d) in Definition 2.2, V_0 is a Leibniz algebra. By Condition $(e_1), (e_3)$ in Definition 2.2 and (3) in Lemma 2.3, we get that $l_2 : V_i \times V_j \longrightarrow V_1(i + j = 1)$ gives rise to a representation of Leibniz algebra V_0 on V_1. Now Condition (g) means that $\partial l_3(w, x, y, z) = 0$ by Formula (2).

The converse part is also straightforward, this completes the proof. ■

Definition 2.9. *Let* $\mathcal{V}$ *and* $\mathcal{V}'$ *be* 2*-term sh Leibniz algebras, a morphism* $\mathfrak{f}$ *from* $\mathcal{V}$ *to* $\mathcal{V}'$ *consists of*

- *linear maps* $f_0 : V_0 \longrightarrow V_0'$ *and* $f_1 : V_1 \longrightarrow V_1'$ *commuting with the differential, i.e.*

$$f_0 \circ \mathrm{d} = \mathrm{d}' \circ f_1;$$

- *a bilinear map* $f_2 : V_0 \times V_0 \longrightarrow V_1'$,

such that for all $x, y, z \in L_0,\ m \in L_1$*, we have*

$$\begin{cases} l_2'(f_0(x), f_0(y)) - f_0 l_2(x, y) &= \mathrm{d}' f_2(x, y), \\ l_2'(f_0(x), f_1(m)) - f_1 l_2(x, m) &= f_2(x, \mathrm{d}m), \\ l_2'(f_1(m), f_0(x)) - f_1 l_2(m, x) &= f_2(\mathrm{d}m, x), \end{cases} \tag{9}$$

and

$$\begin{aligned} &f_1(l_3(x, y, z)) + l_2'(f_0(x), f_2(y, z)) - l_2'(f_0(y), f_2(x, z)) - l_2'(f_2(x, y), f_0(z)) \\ &-f_2(l_2(x, y), z) + f_2(x, l_2(y, z)) - f_2(y, l_2(x, z)) - l_3'(f_0(x), f_0(y), f_0(z)) = 0. \end{aligned} \tag{10}$$

In particular, if $\mathcal{V}$ and $\mathcal{V}'$ are 2-term L_∞-algebras and f_2 is skew-symmetric, we recover the definition of morphisms between 2-term L_∞-algebras.

If (f_0, f_1) is an isomorphism of underlying complexes, we say that (f_0, f_1, f_2) is an isomorphism.

It is obvious that 2-term sh Leibniz algebras and morphisms between them form a category.

3 Leibniz 2-algebras

Leibniz 2-algebras are the categorification of Leibniz algebras. Vector spaces can be categorified to 2-vector spaces. A good introduction for this subject is [2]. Let Vect be the category of vector spaces.

Definition 3.1. [2] *A* 2*-vector space is a category in the category* Vect*.*

Thus a 2-vector space C is a category with a vector space of objects C_0 and a vector space of morphisms C_1, such that all the structure maps are linear. Let $s, t : C_1 \longrightarrow C_0$ be the source and target maps respectively. Let $\cdot_{\mathrm{v}}$ be the composition of morphisms.

It is well known that the 2-category of 2-vector spaces is equivalent to the 2-category of 2-term complexes of vector spaces. Roughly speaking, given a 2-vector space C, $\mathrm{Ker}(s) \xrightarrow{t} C_0$ is a 2-term complex. Conversely, any 2-term complex of vector spaces $\mathcal{V} : V_1 \xrightarrow{\mathrm{d}} V_0$ gives rise to a 2-vector space of which the set of objects is V_0, the set of morphisms is $V_0 \oplus V_1$, the source map s is given by $s(v+m) = v$, and the target map t is given by $t(v+m) = v + \mathrm{d}m$, where $v \in V_0,\ m \in V_1$. We denote the 2-vector space associated to the 2-term complex of vector spaces $\mathcal{V} : V_1 \xrightarrow{\mathrm{d}} V_0$ by $\mathbb{V}$:

$$\mathbb{V} = \begin{array}{c} \mathbb{V}_1 := V_0 \oplus V_1 \\ \mathrm{s}\downarrow\downarrow\mathrm{t} \\ \mathbb{V}_0 := V_0. \end{array} \tag{11}$$

In this paper, we always assume that a 2-vector space is of the above form.

Definition 3.2. *A Leibniz 2-algebra is a 2-vector space $\mathbb{V}$ endowed with a bilinear functor (bracket) $[\![\cdot,\cdot]\!] : \mathbb{V} \times \mathbb{V} \longrightarrow \mathbb{V}$ and a natural isomorphism $J_{x,y,z}$ for every $x, y, z \in \mathbb{V}_0$,*

$$J_{x,y,z} : [\![[\![x,y]\!],z]\!] \longrightarrow [\![x,[\![y,z]\!]]\!] - [\![y,[\![x,z]\!]]\!], \tag{12}$$

such that the following Jacobiator identity is satisfied:

$$\begin{aligned} &J_{[\![w,x]\!],y,z}(J_{w,x,[\![y,z]\!]} - [\![y, J_{w,x,z}]\!]) = \\ &[\![J_{w,x,y}, z]\!]\,(J_{w,[\![x,y]\!],z} - J_{x,[\![w,y]\!],z})([\![w, J_{x,y,z}]\!] - J_{x,y,[\![w,z]\!]} - [\![x, J_{w,y,z}]\!] + J_{w,y,[\![x,z]\!]}), \end{aligned} \tag{13}$$

or, in terms of a diagram,

$$\begin{array}{ccc} & [\![[\![[\![w,x]\!],y]\!],z]\!] & \\ {}^{[\![J_{w,x,y},z]\!]}\swarrow & & \searrow^{J_{[\![w,x]\!],y,z}} \\ [\![[\![w,[\![x,y]\!]]\!],z]\!] - [\![[\![x,[\![w,y]\!]]\!],z]\!] & & [\![[\![w,x]\!],[\![y,z]\!]]\!] - [\![y,[\![[\![w,x]\!],z]\!]]\!] \\ \downarrow{\scriptstyle J_{w,[\![x,y]\!],z} - J_{x,[\![w,y]\!],z}} & & \downarrow{\scriptstyle J_{w,x,[\![y,z]\!]} - [\![y,J_{w,x,z}]\!]} \\ P & \xrightarrow{[\![w,J_{x,y,z}]\!] - J_{x,y,[\![w,z]\!]} - [\![x,J_{w,y,z}]\!] + J_{w,y,[\![x,z]\!]}} & Q \end{array}$$

where P and Q are given by

$$\begin{aligned} P &= [\![w,[\![[\![x,y]\!],z]\!]]\!] - [\![[\![x,y]\!],[\![w,z]\!]]\!] - [\![x\,[\![[\![w,y]\!],z]\!]]\!] + [\![[\![w,y]\!],[\![x,z]\!]]\!], \\ Q &= [\![w,[\![x,[\![y,z]\!]]\!]]\!] - [\![x,[\![w,[\![y,z]\!]]\!]]\!] - [\![y,[\![w,[\![x,z]\!]]\!]]\!] + [\![y,[\![x,[\![w,z]\!]]\!]]\!]. \end{aligned}$$

In particular, if the Jacobiator is trivial, we call a **strict Leibniz 2-algebra**; if the bilinear functor $[\![\cdot,\cdot]\!]$ and the trilinear natural isomorphism J are skew-symmetric, we recover the notion of **semistrict Lie 2-algebras** [2].

Definition 3.3. *Let $\mathbb{V}$ and $\mathbb{V}'$ be two Leibniz 2-algebras, a morphism from $\mathbb{V}$ to $\mathbb{V}'$ consists of*

- *a linear functor F from the underlying 2-vector space of $\mathbb{V}$ to that of $\mathbb{V}'$,*

- *a skewsymmetric natural transformation*

$$F_2(x,y) : F_0(l_2(x,y)) \longrightarrow l_2'(F_0(x), F_0(y)),$$

such that

$$\begin{aligned} & (F_1 J_{x,y,z})(F_2(x, \llbracket y,z \rrbracket) - F_2(y, \llbracket x,z \rrbracket))(\llbracket F_0(x), F_2(y,z) \rrbracket - \llbracket F_0(y), F_2(x,z) \rrbracket) \\ = \ & F_2(\llbracket x,y \rrbracket, z)(\llbracket F_2(x,y), F_0(z) \rrbracket)(J_{F_0(x),F_0(y),F_0(z)}), \end{aligned}$$

or in terms of diagram,

$$\begin{array}{ccc} F_0 \llbracket \llbracket x,y \rrbracket, z \rrbracket & \xrightarrow{\quad F_1 J_{x,y,z} \quad} & F_0(\llbracket x, \llbracket y,z \rrbracket \rrbracket - \llbracket y, \llbracket x,z \rrbracket \rrbracket) \\ \Big\downarrow {\scriptstyle F_2(\llbracket x,y \rrbracket, z)} & & \Big\downarrow {\scriptstyle F_2(x, \llbracket y,z \rrbracket) - F_2(y, \llbracket x,z \rrbracket)} \\ \llbracket F_0 \llbracket x,y \rrbracket, F_0(z) \rrbracket & & \llbracket F_0(x), F_0 \llbracket y,z \rrbracket \rrbracket - \llbracket F_0(y), F_0 \llbracket x,z \rrbracket \rrbracket \\ \Big\downarrow {\scriptstyle \llbracket F_2(x,y), F_0(z) \rrbracket} & & \Big\downarrow {\scriptstyle \llbracket F_0(x), F_2(y,z) \rrbracket - \llbracket F_0(y), F_2(x,z) \rrbracket} \\ \llbracket \llbracket F_0(x), F_0(y) \rrbracket, F_0(z) \rrbracket & \xrightarrow{J_{F_0(x),F_0(y),F_0(z)}} & \llbracket F_0(x), \llbracket F_0(y), F_0(z) \rrbracket \rrbracket - \llbracket F_0(y), \llbracket F_0(x), F_0(z) \rrbracket \rrbracket . \end{array}$$

It is obvious that Leibniz algebras and morphisms between them form a category. In the case of semistrict Lie 2-algebras, it is well known that the category of semistrict Lie 2-algebras and the category of 2-term L_∞-algebras are equivalent [2, Theorem 4.3.6]. Similarly, we have

Theorem 3.4. *The category of Leibniz* 2*-algebras and the category of* 2*-term sh Leibniz algebras are equivalent.*

Proof. We only give a sketch on how to construct a Leibniz 2-algebra from a 2-term sh Leibniz algebra and how to construct a 2-term sh Leibniz algebra from a Leibniz 2-algebra. The other proof is similar to Theorem 4.3.6 in [2]. We omit details.

Let $(V_1 \xrightarrow{\mathrm{d}} V_0, l_2, l_3)$ be a 2-term sh Leibniz algebra, we introduce a bilinear functor $\llbracket \cdot, \cdot \rrbracket$ on the 2-vector space $\mathbb{V}$ given by (11) by

$$\llbracket x+m, y+n \rrbracket = l_2(x,y) + l_2(x,n) + l_2(m,y) + l_2(m, \mathrm{d}n).$$

It is straightforward to see that it does not satisfy the Leibniz rule and the Jacobiator is given by

$$J_{x,y,z} = \llbracket \llbracket x,y \rrbracket, z \rrbracket + l_3(x,y,z).$$

By Condition (f), it is not hard to see that (13) is satisfied. Thus from a 2-term sh Leibniz algebra, we can obtain a Leibniz 2-algebra.

Conversely, given a Leibniz 2-algebra $\mathbb{V}$, we define l_2 and l_3 on the 2-term complex $V_1 \xrightarrow{\mathrm{d}} V_0$ by

- $l_2(x,y) = \llbracket x,y \rrbracket, \quad \forall\, x,y \in V_0.$

- $l_2(x,m)=[\![x,m]\!]\,,\ l_2(m,x)=[\![m,x]\!]\,,\quad \forall\ x\in V_0,\ m\in V_1.$
- $l_2(m,n)=0,\quad \forall\ m,n\in V_1.$
- $l_3(x,y,z)=Pr_1 J_{x,y,z},\quad \forall\ x,y,z\in V_0$, where $Pr_1:\mathbb{V}_1=V_0\oplus V_1\longrightarrow V_1$ is the projection.

Then one can verify that $(V_1\stackrel{\mathrm{d}}{\longrightarrow}V_0,l_2,l_3)$ is a 2-term dg Leibniz algebra. ■

4 Omni-Lie 2-algebras

From now on, when we say a Leibniz 2-algebra, we mean a 2-term sh Leibniz algebra. In this section, we provide an example of Leibniz 2-algebras which comes from omni-Lie 2-algebras [22], which is the categorification of Weinstein's omni-Lie algebras.

Let $\mathcal{V}:V_1\stackrel{\mathrm{d}}{\longrightarrow}V_0$ be a complex of vector spaces. Define $\mathrm{End}^0_{\mathrm{d}}(\mathcal{V})$ by

$$\mathrm{End}^0_{\mathrm{d}}(\mathcal{V})\triangleq\{(A_0,A_1)\in\mathfrak{gl}(V_0)\oplus\mathfrak{gl}(V_1)|A_0\circ\mathrm{d}=\mathrm{d}\circ A_1\},$$

and define $\mathrm{End}^1(\mathcal{V})\triangleq\mathrm{End}(V_0,V_1)$. There is a differential $\delta:\mathrm{End}^1(\mathcal{V})\longrightarrow\mathrm{End}^0_{\mathrm{d}}(\mathcal{V})$ given by

$$\delta(\phi)\triangleq\phi\circ\mathrm{d}+\mathrm{d}\circ\phi,\quad\forall\ \phi\in\mathrm{End}^1(\mathcal{V}),$$

and a bracket operation $[\cdot,\cdot]$ given by the graded commutator. More precisely, for any $A=(A_0,A_1),B=(B_0,B_1)\in\mathrm{End}^0_{\mathrm{d}}(\mathcal{V})$ and $\phi\in\mathrm{End}^1(\mathcal{V})$, $[\cdot,\cdot]$ is given by

$$[A,B]=A\circ B-B\circ A=(A_0\circ B_0-B_0\circ A_0,A_1\circ B_1-B_1\circ A_1),$$

and

$$[A,\phi]=A\circ\phi-\phi\circ A=A_1\circ\phi-\phi\circ A_0. \tag{14}$$

These two operations make $\mathrm{End}^1(\mathcal{V})\stackrel{\delta}{\to}\mathrm{End}^0_{\mathrm{d}}(\mathcal{V})$ into a 2-term DGLA (proved in [24]), which we denote by $\mathrm{End}(\mathcal{V})$. It plays the same role as $\mathfrak{gl}(V)$ for a vector space V.

Let $\mathfrak{f}=(f_0,f_1,f_2)$ be an automorphism of the 2-term DGLA $\mathrm{End}(\mathcal{V})$. On the complex

$$\mathrm{End}(\mathcal{V})\oplus\mathcal{V}=\mathrm{End}^1(\mathcal{V})\oplus V_1\stackrel{\delta+\mathrm{d}}{\longrightarrow}\mathrm{End}^0_{\mathrm{d}}(\mathcal{V})\oplus V_0,$$

define bilinear map $l_2^{\mathfrak{f}}$ by

$$\begin{cases} l_2^{\mathfrak{f}}(A+u,B+v) &=& [A,B]+f_0(A)(v), & \text{in degree-0},\\ l_2^{\mathfrak{f}}(A+u,\phi+m) &=& [A,\phi]+f_0(A)(m), & \text{in degree-1},\\ l_2^{\mathfrak{f}}(\phi+m,A+u) &=& [\phi,A]+f_1(\phi)(u), & \text{in degree-1},\end{cases} \tag{15}$$

and define trilinear map $l_3^{\mathfrak{f}}$ by

$$l_3^{\mathfrak{f}}(A+u,B+v,C+w)=f_2(A,B)(w).$$

Proposition 4.1. *Let $\mathfrak{f}=(f_0,f_1,f_2)$ be an automorphism of $\mathrm{End}(\mathcal{V})$, then $(\mathrm{End}(\mathcal{V})\oplus\mathcal{V},l_2^{\mathfrak{f}},l_3^{\mathfrak{f}})$ is a Leibniz 2-algebra.*

Proof. We check that all the conditions in Definition 2.2 are satisfied. By the fact that $\delta[A,\phi] = [A,\delta(\phi)]$ and $f_0(A)$ commutes with d, we have

$$\begin{aligned} l_2^{\mathfrak{f}}(A+u,(\delta+\mathrm{d})(\phi+m)) &= l_2^{\mathfrak{f}}(A+u,\delta(\phi)+\mathrm{d}m) \\ &= [A,\delta(\phi)]+f_0(A)(\mathrm{d}m) \\ &= \delta[A,\phi]+\mathrm{d}f_0(A)(m) \\ &= (\delta+\mathrm{d})l_2^{\mathfrak{f}}(A+u,\phi+m), \end{aligned}$$

which implies that (a) holds. Similarly, (b) follows from equalities $[\delta(\phi),A]=\delta[\phi,A]$ and $f_0\circ\delta = \delta\circ f_1$. By the definition of δ, it is not hard to see that

$$[\delta(\phi),\psi]=[\phi,\delta(\psi)], \quad f_0(\delta(\phi))(n)=\delta(f_1(\phi))(n)=f_1(\phi)(\mathrm{d}n).$$

Thus we obtain (c):

$$[(\delta+\mathrm{d})(\phi+m),\psi+n]=[\phi+m,(\delta+\mathrm{d})(\psi+n)].$$

It is straightforward to deduce that

$$\begin{aligned} & l_2^{\mathfrak{f}}(A+u,l_2^{\mathfrak{f}}(B+v,C+w))-l_2^{\mathfrak{f}}(l_2^{\mathfrak{f}}(A+u,B+v),C+w)-l_2^{\mathfrak{f}}(B+v,l_2^{\mathfrak{f}}(A+u,C+w)) \\ = \; & [f_0(A),f_0(B)](w)-f_0([A,B])(w) \\ = \; & \mathrm{d}\circ f_2(A,B)(w). \end{aligned}$$

Thus we arrive at (d). (e_1) follows from

$$\begin{aligned} & l_2^{\mathfrak{f}}(A+u,l_2^{\mathfrak{f}}(B+v,\phi+m))-l_2^{\mathfrak{f}}(l_2^{\mathfrak{f}}(A+u,B+v),\phi+m)-l_2^{\mathfrak{f}}(B+v,l_2^{\mathfrak{f}}(A+u,\phi+m)) \\ = \; & [f_0(A),f_0(B)](m)-f_0([A,B])(m) \\ = \; & f_2(A,B)(\mathrm{d}m) \\ = \; & l_3^{\mathfrak{f}}(A+u,B+v,(\delta+\mathrm{d})(\phi+m)). \end{aligned}$$

Similarly, (e_2) follows from the fact that

$$[f_0(A),f_1(\phi)]-f_1[A,\phi]=f_2(A,\delta(\phi)),$$

and (e_3) follows from the fact that

$$[f_1(\phi),f_0(A)]-f_1[\phi,A]=f_2(\delta(\phi),A).$$

Now we are left to show that $l_3^{\mathfrak{f}}$ satisfies the Jacobiator identity. This essentially follows from the fact that $\mathfrak{f}$ is an automorphism of $\mathrm{End}(\mathcal{V})$. More precisely,

$$\begin{aligned}
& l_2^{\mathfrak{f}}(D+x, l_3^{\mathfrak{f}}(A+u,B+v,C+w)) - l_2^{\mathfrak{f}}(A+u, l_3^{\mathfrak{f}}(D+x,B+v,C+w)) \\
& +l_2^{\mathfrak{f}}(B+v, l_3^{\mathfrak{f}}(D+x,A+u,C+w)) + l_2^{\mathfrak{f}}(l_3^{\mathfrak{f}}(D+x,A+u,B+v),C+w) \\
& -l_3^{\mathfrak{f}}(l_2^{\mathfrak{f}}(D+x,A+u),B+v,C+w) - l_3^{\mathfrak{f}}(A+u, l_2^{\mathfrak{f}}(D+x,B+v),C+w) \\
& -l_3^{\mathfrak{f}}(A+u,B+v,l_2^{\mathfrak{f}}(D+x,C+w)) + l_3^{\mathfrak{f}}(D+x, l_2^{\mathfrak{f}}(A+u,B+v),C+w) \\
& +l_3^{\mathfrak{f}}(D+x,B+v,l_2^{\mathfrak{f}}(A+u,C+w)) - l_3^{\mathfrak{f}}(D+x,A+u,l_2^{\mathfrak{f}}(B+v,C+w)) \\
= \; & f_0(D)f_2(A,B)(w) - f_0(A)f_2(D,B)(w) + f_0(B)f_2(D,A)(w) \\
& -f_2([D,A],B)(w) - f_2(A,[D,B])(w) - f_2(A,B)f_0(D)(w) \\
& +f_2(D,[A,B])(w) + f_2(D,B)f_0(A)(w) - f_2(D,A)f_0(B)(w) \\
= \; & \Big([f_0(D), f_2(A,B)] - [f_0(A), f_2(D,B)] + [f_0(B), f_2(D,A)] \\
& -f_2([D,A],B) + f_2([D,B],A) - f_2([A,B],D)\Big)(w) \\
= \; & 0.
\end{aligned}$$

The last equality follows from the fact that $\mathfrak{f}$ is an automorphism of $\mathrm{End}(\mathcal{V})$. This finishes the proof of $(\mathrm{End}(\mathcal{V}) \oplus \mathcal{V}, l_2^{\mathfrak{f}}, l_3^{\mathfrak{f}})$ being a Leibniz 2-algebra. ■

5 Twisted Courant algebroids

Hansena and Strobl introduced twisted Courant algebroids by closed 4-forms in [9], which arise naturally from the study of three dimensional sigma models with Wess-Zumino term.

Definition 5.1. [9] *A twisted Courant algebroid by a closed* 4*-form* H *is a vector bundle* $E \longrightarrow M$*, together with a fiber metric* $\langle \cdot, \cdot \rangle$ *(so we can identify* E *with* E^**), a bundle map* $\rho : E \longrightarrow TM$ *(called the anchor), a bilinear bracket operation (Dorfman bracket)* $\{\cdot, \cdot\}$ *on* $\Gamma(E)$*, and a closed* 4*-form* H *such that for any* $e_1, e_2, e_3 \in \Gamma(E)$*, we have*

$$\{e,e\} = \frac{1}{2}\rho^* d \langle e, e \rangle ; \tag{16}$$

$$\rho(e_1) \langle e_2, e_3 \rangle = \langle \{e_1, e_2\}, e_3 \rangle + \langle e_2, \{e_1, e_3\} \rangle ; \tag{17}$$

$$\rho^*(i_{\rho(e_1)\wedge\rho(e_2)\wedge\rho(e_3)} H) = \{e_1, \{e_2, e_3\}\} - \{\{e_1, e_2\}, e_3\} - \{e_2, \{e_1, e_3\}\}. \tag{18}$$

We will denote a twisted Courant algebroid by $(E, \langle \cdot, \cdot \rangle, \{\cdot, \cdot\}, \rho, H)$, which is exactly the Courant algebroid ([12, 17]) if $H = 0$.

Roytenberg proved that every Courant algebroid gives rise to a 2-term L_∞-algebra in [18]. Now every twisted Courant algebroid by a closed 4-form gives rise to a Leibniz 2-algebra.

Theorem 5.2. *Every twisted Courant algebroid by a closed* 4*-form* $(E, \langle \cdot, \cdot \rangle, \{\cdot, \cdot\}, \rho, H)$ *gives rise to a Leibniz* 2*-algebra, whose degree-*1 *part is* $\Omega^1(M)$*, degree-*0 *part is* $\Gamma(E)$*, differential is* $\rho^* : \Omega^1(M) \longrightarrow \Gamma(E)$*, the bilinear bracket operation* l_2 *is given by*

$$\left\{ \begin{array}{rcl} l_2(e_1, e_2) & \triangleq & \{e_1, e_2\}, \quad \forall\, e_1, e_2 \in \Gamma(E), \\ l_2(e, \xi) & \triangleq & L_{\rho(e)}\xi, \quad \forall\, e \in \Gamma(E),\ \xi \in \Omega^1(M), \\ l_2(\xi, e) & \triangleq & -i_{\rho(e)} d\xi, \end{array} \right. \tag{19}$$

and the trilinear map l_3^H is given by

$$l_3^H(e_1,e_2,e_3) \triangleq i_{\rho(e_1)\wedge\rho(e_2)\wedge\rho(e_3)}H. \tag{20}$$

To prove this theorem, we need the following two lemmas.

Lemma 5.3. *Let $(E, \langle\cdot,\cdot\rangle, \{\cdot,\cdot\}, \rho, H)$ be a twisted Courant algebroid by a closed 4-form H, then for any $f \in C^\infty(M)$, we have*

$$\begin{aligned}
\{e_1, fe_2\} &= f\{e_1,e_2\} + \rho(e_1)(f)e_2; && (21)\\
\{fe_2, e_1\} &= f\{e_2,e_1\} - \rho(e_1)(f)e_2 + \langle e_1,e_2\rangle \rho^* df; && (22)\\
J_{e_1,e_2,fe_3} &= fJ_{e_1,e_2,e_3} + \big([\rho(e_1),\rho(e_2)] - \rho\{e_1,e_2\}\big)(f)e_3. && (23)
\end{aligned}$$

Proof. By (17), we have

$$\rho(e_1)\langle fe_2, e_3\rangle = \langle \{e_1, fe_2\}, e_3\rangle + \langle fe_2, \{e_1,e_3\}\rangle .$$

On the other hand, we have

$$\begin{aligned}
\rho(e_1)\langle fe_2, e_3\rangle &= \rho(e_1)(f\langle e_2,e_3\rangle)\\
&= \rho(e_1)(f)\langle e_2,e_3\rangle + f\rho(e_1)\langle e_2,e_3\rangle\\
&= \langle \rho(e_1)(f)e_2, e_3\rangle + \langle f\{e_1,e_2\}, e_3\rangle + \langle fe_2, \{e_1,e_3\}\rangle .
\end{aligned}$$

Thus we have

$$\langle \rho(e_1)(f)e_2, e_3\rangle + \langle f\{e_1,e_2\}, e_3\rangle = \langle \{e_1, fe_2\}, e_3\rangle .$$

Since the fiber metric is nondegenerate, we have

$$\{e_1, fe_2\} = f\{e_1,e_2\} + \rho(e_1)(f)e_2.$$

By (16), first we have

$$\{e_2,e_1\} + \{e_1,e_2\} = \rho^* d\langle e_1,e_2\rangle .$$

Therefore, we have

$$\{fe_2,e_1\} + \{e_1,fe_2\} = \rho^* d\langle e_1, fe_2\rangle = f\rho^* d\langle e_1,e_2\rangle + \langle e_1,e_2\rangle \rho^* df,$$

which implies that

$$\begin{aligned}
\{fe_2,e_1\} &= -\{e_1,fe_2\} + f(\{e_2,e_1\} + \{e_1,e_2\}) + \langle e_1,e_2\rangle \rho^* df\\
&= f\{e_2,e_1\} - \rho(e_1)(f)e_2 + \langle e_1,e_2\rangle \rho^* df.
\end{aligned}$$

By (21), we have

$$\begin{aligned}
J_{e_1,e_2,fe_3} &= \{e_1,\{e_2,fe_3\}\} - \{\{e_1,e_2\},fe_3\} - \{e_2,\{e_1,fe_3\}\}\\
&= \{e_1, f\{e_2,e_3\} + \rho(e_2)(f)e_3\} - f\{\{e_1,e_2\},e_3\} - \rho\{e_1,e_2\}(f)e_3\\
&\quad -\{e_2, f\{e_1,e_3\} + \rho(e_1)(f)e_3\}\\
&= f\{e_1,\{e_2,e_3\}\} + \rho(e_1)(f)\{e_2,e_3\} + \rho(e_2)(f)\{e_1,e_3\} + \rho(e_1)\rho(e_2)(f)e_3\\
&\quad -f\{e_2,\{e_1,e_3\}\} - \rho(e_2)(f)\{e_1,e_3\} - \rho(e_1)(f)\{e_2,e_3\} - \rho(e_2)\rho(e_1)(f)e_3\\
&\quad -f\{\{e_1,e_2\},e_3\} - \rho\{e_1,e_2\}(f)e_3\\
&= fJ_{e_1,e_2,e_3} + \big([\rho(e_1),\rho(e_2)] - \rho\{e_1,e_2\}\big)(f)e_3.
\end{aligned}$$

The proof is completed. ■

Lemma 5.4. *Let* $(E, \langle\cdot,\cdot\rangle, \{\cdot,\cdot\}, \rho, H)$ *be a twisted Courant algebroid by a closed 4-form* H*, then we have*

$$\begin{aligned}
\rho\{e_1, e_2\} &= [\rho(e_1), \rho(e_2)], &(24)\\
\rho\circ\rho^* &= 0, &(25)\\
\{\rho^*(\xi), e\} &= \rho^*(-i_{\rho(e)}d\xi), &(26)\\
\{e, \rho^*(\xi)\} &= \rho^*(L_{\rho(e)}\xi). &(27)
\end{aligned}$$

Proof. By (18), we have $J_{e_1,e_2,fe_3} = fJ_{e_1,e_2,e_3}$. Now (24) is a consequence of (23).
By (24), we have

$$\begin{aligned}
\rho\{fe_2, e_1\} &= [f\rho(e_2), \rho(e_1)]\\
&= f[\rho(e_2), \rho(e_1)] - \rho(e_1)(f)\rho(e_2).
\end{aligned}$$

By (22), we have

$$\rho\{fe_2, e_1\} = f[\rho(e_2), \rho(e_1)] - \rho(e_1)(f)\rho(e_2) + \langle e_1, e_2\rangle\, \rho\circ\rho^* df.$$

Thus we have $\langle e_1, e_2\rangle\, \rho\circ\rho^* df = 0$, which implies (25).
It is not hard to deduce that

$$\{\rho^*(df), e\} = 0, \quad \{e, \rho^*(df)\} = \rho^*(d\rho(e)(f)).$$

Thus for any $g \in C^\infty(M)$, we have

$$\begin{aligned}
\{\rho^*(fdg), e\} &= \{f\rho^*(dg), e\}\\
&= f\{\rho^*(dg), e\} - \rho(e)(f)\rho^*(dg) + \langle\rho^*(dg), e\rangle\, \rho^*(df)\\
&= -\rho(e)(f)\rho^*(dg) + \rho(e)(g)\rho^*(df)\\
&= \rho^*(-i_{\rho(e)}df\wedge dg)\\
&= \rho^*(-i_{\rho(e)}d(fdg)),
\end{aligned}$$

and

$$\begin{aligned}
\{e, \rho^*(fdg)\} &= \{e, f\rho^*(dg)\}\\
&= f\{e, \rho^*(dg)\} + \rho(e)(f)\rho^*(dg)\\
&= f\rho^*(d\rho(e)(g)) + \rho(e)(f)\rho^*(dg)\\
&= \rho^*(L_{\rho(e)}fdg),
\end{aligned}$$

which implies that for any $\xi \in \Omega^1(M)$, we have

$$\{\rho^*(\xi), e\} = \rho^*(-i_{\rho(e)}d(\xi)),$$

and

$$\{e, \rho^*(\xi)\} = \rho^*(L_{\rho(e)}\xi). \blacksquare$$

The proof of Theorem 5.2: We need to show that all the axioms in Definition 2.2 hold. By (26) and (27), it is not hard to see that (a) and (b) hold. (c) follows from the fact that

$$l_2(\rho^*(\xi), \eta) = l_2(\xi, \rho^*(\eta)) = 0.$$

By the definition of twisted Courant algebroids and l_3^H, (d) is obvious. By the definition of l_3^H and (25), we have

$$l_3^H(\rho^*(\xi), e_1, e_2) = l_3^H(e_1, \rho^*(\xi), e_2) = l_3^H(e_1, e_2, \rho^*(\xi)) = 0.$$

On the other hand, we have

$$\begin{aligned} & l_2(e_1, l_2(e_2, \xi)) - l_2(l_2(e_1, e_2), \xi) - l_2(e_2, l_2(e_1, \xi)) \\ = \; & L_{\rho(e_1)} L_{\rho(e_2)} \xi - L_{\rho\{e_1, e_2\}} \xi - L_{\rho(e_2)} L_{\rho(e_1)} \xi \\ = \; & [L_{\rho(e_1)}, L_{\rho(e_2)}] \xi - L_{[\rho(e_1), \rho(e_2)]} \xi \\ = \; & 0, \end{aligned}$$

which implies that (e_1) holds. Similarly, it is straightforward to see that (e_2) and (e_3) follow from the formula

$$i_{[\rho(e_1), \rho(e_2)]} d\xi = L_{\rho(e_1)} i_{\rho(e_2)} d\xi - i_{\rho(e_2)} L_{\rho(e_1)} d\xi.$$

At last, we need to show that the Jacobiator identity holds. Note that l_3^H is skew-symmetric, the Jacobiator identity is equivalent to that

$$\begin{aligned} & l_2(e_1, l_3^H(e_2, e_3, e_4)) - l_2(e_2, l_3^H(e_1, e_3, e_4)) + l_2(e_3, l_3^H(e_1, e_2, e_4)) + l_2(l_3^H(e_1, e_2, e_3), e_4) \\ & + \sum_{i<j} (-1)^{i+j} l_3^H(l_2(e_i, e_j), e_1, \cdots, \widehat{e_i}, \cdots, \widehat{e_j}, \cdots, e_4) = 0. \end{aligned}$$

Let the left hand side act on an arbitrary vector field $X \in \mathfrak{X}(M)$, we get

$$\begin{aligned} & \rho(e_1) H(\rho(e_2), \rho(e_3), \rho(e_4), X) - H(\rho(e_2), \rho(e_3), \rho(e_4), [\rho(e_1), X]) \\ & - \rho(e_2) H(\rho(e_1), \rho(e_3), \rho(e_4), X) + H(\rho(e_1), \rho(e_3), \rho(e_4), [\rho(e_2), X]) \\ & + \rho(e_3) H(\rho(e_1), \rho(e_2), \rho(e_4), X) - H(\rho(e_1), \rho(e_2), \rho(e_4), [\rho(e_3), X]) \\ & - d\big(H(\rho(e_1), \rho(e_2), \rho(e_3))\big)(\rho(e_4), X) \\ & + \sum_{i<j} (-1)^{i+j} H([\rho(e_i), \rho(e_j)], \rho(e_1), \cdots, \widehat{e_i}, \cdots, \widehat{e_j}, \cdots, \rho(e_4), X), \end{aligned}$$

which is exactly

$$dH(\rho(e_1), \rho(e_2), \rho(e_3), \rho(e_4), X).$$

Since H is closed 4-form, thus $dH = 0$. Therefore, l_3^H satisfies the Jacobiator identity. This finishes the proof of $(\Omega^1(M) \xrightarrow{\rho^*} \Gamma(E), l_2, l_3^H)$ being a Leibniz 2-algebra. ■

A twisted Courant algebroid by a closed 4-form H $(E, \langle\cdot,\cdot\rangle, \{\cdot,\cdot\}, \rho, H)$ is said to be **exact** if we have the following exact sequence

$$0 \longrightarrow T^*M \xrightarrow{\rho^*} E \xrightarrow{\rho} TM \longrightarrow 0. \tag{28}$$

By choosing an isotropic splitting $s : TM \longrightarrow E$, as vector bundles, we have

$$E \cong \mathcal{T} \triangleq TM \oplus T^*M.$$

We can transfer the twisted Courant algebroid structure to $TM \oplus T^*M$. For any $X + \xi, Y + \eta \in \mathfrak{X}(M) \oplus \Omega(M)$, we have

$$\rho(X + \xi) = \rho(s(X) + \rho^*(\xi)) = X, \tag{29}$$

$$\langle X+\xi, Y+\eta\rangle = \langle s(X)+\rho^*(\xi), s(Y)+\rho^*(\eta)\rangle = \xi(Y)+\eta(X), \tag{30}$$

and

$$\begin{aligned}\{X+\xi, Y+\eta\} &= \{s(X)+\rho^*(\xi), s(Y)+\rho^*(\eta)\}\\ &= \{s(X), s(Y)\}+\{s(X), \rho^*(\eta)\}+\{\rho^*(\xi), s(Y)\}.\end{aligned}$$

By (24) and (25), we have

$$\rho\{s(X), \rho^*(\eta)\} = 0,$$

which implies that $\{s(X), \rho^*(\eta)\} \in \Omega^1(M)$. For any $Z \in \mathfrak{X}(M)$, by (17), we have

$$\begin{aligned}\{s(X), \rho^*(\eta)\}(Z) &= \langle\{s(X), \rho^*(\eta)\}, s(Z)\rangle\\ &= X\langle\rho^*(\eta), s(Z)\rangle - \langle\rho^*(\eta), \{s(X), s(Z)\}\rangle\\ &= X\langle\eta, Z\rangle - \eta([X,Z])\\ &= L_X\eta(Z),\end{aligned}$$

which implies that

$$\{X, \eta\} = L_X\eta. \tag{31}$$

Similarly, we have

$$\begin{aligned}\{\rho^*(\xi), s(Y)\}(Z) &= \langle\{\rho^*(\xi), s(Y)\}, s(Z)\rangle\\ &= \langle -\{s(Y), \rho^*(\xi)\}+\rho^* d\langle\rho^*(\xi), s(Y)\rangle, s(Z)\rangle\\ &= -L_Y\xi(Z)+d(\xi(Y))(Z)\\ &= -(i_Y d\xi)(Z),\end{aligned}$$

which implies that

$$\{\xi, Y\} = -i_Y d\xi. \tag{32}$$

By (24), we can assume that $\{s(X), s(Y)\} - s[X,Y] = h(X,Y)$ for some $h : \mathfrak{X}(M)\times\mathfrak{X}(M) \longrightarrow \Omega^1(M)$. Thus we have

$$\{X, Y\} = [X,Y]+h(X,Y). \tag{33}$$

It is not hard to deduce that $h \in \Omega^3(M)$. To summarize, we have

Theorem 5.5. *For any exact twisted Courant algebroid by a closed 4-form H $(E, \langle\cdot,\cdot\rangle, \{\cdot,\cdot\}, \rho, H)$, as a vector bundle, we have $E \cong \mathcal{T} \triangleq TM\oplus T^*M$. Transfer the twisted Courant algebroid structure to $TM\oplus T^*M$, the anchor ρ and the fiber metric $\langle\cdot,\cdot\rangle$ are given by* (29) *and* (30) *respectively. The bracket $\{\cdot,\cdot\}$ is given by*

$$\{X+\xi, Y+\eta\} = [X,Y]+L_X\eta - i_Y d\xi + h(X,Y), \tag{34}$$

for some 3-form $h \in \Omega^3(M)$. We will denote this bracket operation by $\{\cdot,\cdot\}_h$.

Consequently, any exact twisted Courant algebroid by a closed 4-form H $(E, \langle\cdot,\cdot\rangle, \{\cdot,\cdot\}, \rho, H)$ is isomorphic to the twisted Courant algebroid $(TM\oplus T^*M, \langle\cdot,\cdot\rangle, \{\cdot,\cdot\}_h, \rho, dh)$, i.e. the closed 4-form in an exact twisted Courant algebroid must be exact.

It is well known that if we change different splittings, the 3-form changed by an exact one. For any $B \in \Omega^2(M)$, define $e^B : \mathcal{T} \longrightarrow \mathcal{T}$ by

$$e^B(X+\xi) = X + \xi + i_X B, \quad \forall\ X+\xi \in \Gamma(\mathcal{T}).$$

It is straightforward to deduce that

$$e^B\{X+\xi, Y+\eta\}_{h+dB} = \{e^B(X+\xi), e^B(Y+\eta)\}_h. \tag{35}$$

Thus we have

Proposition 5.6. *Let $(E, \langle\cdot,\cdot\rangle, \{\cdot,\cdot\}, \rho, H)$ be an exact twisted Courant algebroid. If we choose different splitting, we obtain two isomorphic exact twisted Courant algebroid $(\mathcal{T}, \langle\cdot,\cdot\rangle, \{\cdot,\cdot\}_{h+dB}, \rho, dh)$ and $(\mathcal{T}, \langle\cdot,\cdot\rangle, \{\cdot,\cdot\}_h, \rho, dh)$. The isomorphism is given by e^B. In particular, if $dB = 0$, e^B is an automorphism of the exact twisted Courant algebroid $(\mathcal{T}, \langle\cdot,\cdot\rangle, \{\cdot,\cdot\}_h, \rho, dh)$.*

For exact twisted Courant algebroids, since $\rho : \mathcal{T} \longrightarrow TM$ is the projection, $\rho^* : T^*M \longrightarrow \mathcal{T}$ is the inclusion map. Thus by Theorem 5.2, we obtain the following Leibniz 2-algebra.

Corollary 5.7. *Any exact twisted Courant algebroid $(\mathcal{T}, \langle\cdot,\cdot\rangle, \{\cdot,\cdot\}_h, \rho, dh)$ gives rise to a Leibniz 2-algebra, whose degree-1 part is $\Omega^1(M)$, degree-0 part is $\Gamma(\mathcal{T})$, differential is the inclusion $\mathbb{i} : \Omega^1(M) \longrightarrow \Gamma(\mathcal{T})$, the bilinear bracket operation l_2^h is given by*

$$\begin{cases} l_2^h(X+\xi, Y+\eta) &= \{X+\xi, Y+\eta\}_h, \\ l_2^h(X+\xi, \eta) &= \{X+\xi, \eta\}_h = L_X\eta, \\ l_2^h(\eta, X+\xi) &= \{\eta, X+\xi\}_h = -i_X d\eta, \end{cases} \tag{36}$$

and the trilinear map l_3^{dh} is given by

$$l_3^{dh}(X+\xi, Y+\eta, Z+\gamma) = i_{X\wedge Y\wedge Z} dh. \tag{37}$$

If we choose a different splitting for the exact twisted Courant algebroid by a closed 4-form H $(E, \langle\cdot,\cdot\rangle, \{\cdot,\cdot\}, \rho, H)$, we obtain an exact twisted Courant algebroid $(\mathcal{T}, \langle\cdot,\cdot\rangle, \{\cdot,\cdot\}_{h+dB}, \rho, dh)$ for some $B \in \Omega^2(M)$. By Corollary 5.7, we obtain the Leibniz 2-algebra $(\Omega^1(M) \xrightarrow{\mathbb{i}} \Gamma(\mathcal{T}), l_2^{h+dB}, l_3^{dh})$. By (35), we have

Corollary 5.8. *$(f_0 = e^B,\ f_1 = \mathrm{Id},\ f_2 = 0)$ is an isomorphism from the Leibniz 2-algebra $(\Omega^1(M) \xrightarrow{\mathbb{i}} \Gamma(\mathcal{T}), l_2^{h+dB}, l_3^{dh})$ to the Leibniz 2-algebra $(\Omega^1(M) \xrightarrow{\mathbb{i}} \Gamma(\mathcal{T}), l_2^h, l_3^{dh})$.*

In particular, if $dB = 0$, $(f_0 = e^B,\ f_1 = \mathrm{Id},\ f_2 = 0)$ is an automorphism of the Leibniz 2-algebra $(\Omega^1(M) \xrightarrow{\mathbb{i}} \Gamma(\mathcal{T}), l_2^h, l_3^{dh})$. There is a more interesting phenomenon that the Leibniz 2-algebra $(\Omega^1(M) \xrightarrow{\mathbb{i}} \Gamma(\mathcal{T}), l_2^h, l_3^{dh})$ has more automorphisms.

Theorem 5.9. *For any $B \in \Omega^2(M)$, $(f_0 = e^B,\ f_1 = \mathrm{Id},\ f_2)$ is an automorphism of the Leibniz 2-algebra $(\Omega^1(M) \xrightarrow{\mathbb{i}} \Gamma(\mathcal{T}), l_2^h, l_3^{dh})$, where f_2 is given by*

$$f_2(X+\xi, Y+\eta) = i_{X\wedge Y} dB.$$

Proof. First it is obvious that

$$f_0 \circ \mathbb{i} = \mathbb{i} \circ f_1.$$

By straightforward computations, we have

$$\begin{aligned} e^B\{X+\xi, Y+\eta\}_h &= e^B\big([X,Y]+L_X\eta - i_Y d\xi + h(X,Y)\big) \\ &= [X,Y]+L_X\eta - i_Y d\xi + h(X,Y) + i_{[X,Y]}B, \end{aligned}$$

and

$$\begin{aligned} \{e^B(X+\xi), e^B(Y+\eta)\}_h &= \{X+\xi+i_XB, Y+\eta+i_YB\}_h \\ &= [X,Y]+L_X\eta + L_X i_Y B - i_Y d\xi - i_Y d i_X B + h(X,Y). \end{aligned}$$

Thus we have

$$\begin{aligned} \{e^B(X+\xi), e^B(Y+\eta)\}_h - e^B\{X+\xi, Y+\eta\}_h &= L_X i_Y B - i_Y d i_X B - i_{[X,Y]}B \\ &= L_X i_Y B - i_Y d i_X B - L_X i_Y B + i_Y L_X B \\ &= i_{X\wedge Y} dB. \end{aligned}$$

This shows that (9) in Definition 2.9 holds. At last, we show that (10) in Definition 2.9 also holds. In fact, for any $X+\xi, Y+\eta, Z+\gamma \in \Gamma(\mathcal{T})$, first we have

$$l_3^{dh}(X+\xi, Y+\eta, Z+\gamma) = l_3^{dh}(e^B(X+\xi), e^B(Y+\eta), e^B(Z+\gamma)).$$

Thus the left hand side of (10) is equal to

$$\begin{aligned} &\{e^B(X+\xi), f_2(Y+\eta, Z+\gamma)\}_h - \{e^B(Y+\eta), f_2(X+\xi, Z+\gamma)\}_h \\ &-\{f_2(X+\xi, Y+\eta), e^B(Z+\gamma)\}_h - f_2(\{X+\xi, Y+\eta\}_h, Z+\gamma) \\ &+f_2(X+\xi, \{Y+\eta, Z+\gamma\}_h) - f_2(Y+\eta, \{X+\xi, Z+\gamma\}_h), \end{aligned}$$

which is equal to

$$L_X i_{Y\wedge Z} dB - L_Y i_{X\wedge} Z dB + i_Z d i_{X\wedge Y} dB - i_{[X,Y]\wedge Z} dB + i_{X\wedge [Y,Z]} dB - i_{Y\wedge [X,Z]} dB.$$

Acting on arbitrary $W \in \mathfrak{X}(M)$, we get

$$d(dB)(X,Y,Z,W),$$

which is zero since $d^2 = 0$. Thus (10) in Definition 2.9 holds. Therefore, ($f_0 = e^B$, $f_1 = \mathrm{Id}$, f_2), where f_2 is given by

$$f_2(X+\xi, Y+\eta) = i_{X\wedge Y} dB,$$

is a morphism of the Leibniz 2-algebra $(\Omega^1(M) \xrightarrow{\mathrm{i}} \Gamma(\mathcal{T}), l_2^h, l_3^{dh})$. It is an automorphism of Leibniz 2-algebras follows from the fact that ($f_0 = e^B$, $f_1 = \mathrm{Id}$) is an automorphism of the underlying complex. ■

6 Dirac structures of twisted Courant algebroids

Dirac structures of a twisted Courant algebroid by a closed 4-form can be defined as usual.

Definition 6.1. *A Dirac structure of the twisted Courant algebroid* $(E, \langle\cdot,\cdot\rangle, \{\cdot,\cdot\}, \rho, H)$ *is a maximal isotropic subbundle* L *such that the section space* $\Gamma(L)$ *is closed under the bracket operation* $\{\cdot,\cdot\}$.

By (16), the restriction of the bracket operation $\{\cdot,\cdot\}$ on $\Gamma(L)$ is skew-symmetric. In general, for Courant algebroids, the restriction of $\{\cdot,\cdot\}$ on a Dirac structure is a Lie bracket. Denote the set $(\rho^*)^{-1}\Gamma(L)$ by $\Omega^1_L(M)$. Now we have

Theorem 6.2. *Let L be a Dirac structure of the twisted Courant algebroid by a closed 4-form H $(E,\langle\cdot,\cdot\rangle,\{\cdot,\cdot\},\rho,H)$. Then*

$$(\Omega^1_L(M)\xrightarrow{\rho^*}\Gamma(L),l_2,l_3^H)$$

is a 2-term L_∞-algebra (Lie 2-algebra), in which the degree-1 part is $\Omega^1_L(M)\triangleq(\rho^*)^{-1}\Gamma(L)$, the degree-0 part is $\Gamma(L)$, l_2 and l_3^H are given by (19) and (20) respectively.

Proof. First it is not hard to see that $(\Omega^1_L(M)\xrightarrow{\rho^*}\Gamma(L),l_2,l_3^H)$ is a sub-Leibniz 2-algebra of $(\Omega^1(M)\xrightarrow{\rho^*}\Gamma(\mathcal{T}),l_2,l_3^H)$. In fact, for any $e_1,e_2\in\Gamma(L)$, by the definition of Dirac structures, we have $l_2(e_1,e_2)\in\Gamma(L)$. It is also obvious that for any $e\in\Gamma(L)$ and $\xi\in\Omega^1_L(M)$, we have

$$\rho^*l_2(e,\xi)=l_2(e,\rho^*(\xi))\in\Gamma(L),\quad \rho^*l_2(\xi,e)=l_2(\rho^*(\xi),e)\in\Gamma(L),$$

which implies that

$$l_2(e,\xi)\in\Omega^1_L(M),\quad l_2(\xi,e)\in\Omega^1_L(M).$$

For any $e_1,e_2,e_3\in\Gamma(L)$, on one hand, we have

$$l_3^H(e_1,e_2,e_3)=i_{\rho(e_1)\wedge\rho(e_2)\wedge\rho(e_3)}H.$$

On the other hand, we have

$$\rho^*i_{\rho(e_1)\wedge\rho(e_2)\wedge\rho(e_3)}H=\{e_1,\{e_2,e_3\}\}-\{\{e_1,e_2\},e_3\}-\{e_2,\{e_1,e_3\}\}\in\Gamma(L).$$

Thus $l_3^H(e_1,e_2,e_3)\in\Omega^1_L(M)$. Therefore, $(\Omega^1_L(M)\xrightarrow{\rho^*}\Gamma(L),l_2,l_3^H)$ is a sub-Leibniz 2-algebra of $(\Omega^1(M)\xrightarrow{\rho^*}\Gamma(\mathcal{T}),l_2,l_3^H)$.

Since the Dirac structure L is maximal isotropic, l_2 is skew-symmetric. l_3^H is also skew-symmetric. Thus $(\Omega^1_L(M)\xrightarrow{\rho^*}\Gamma(L),l_2,l_3^H)$ is a 2-term L_∞-algebra. ■

For a bi-vector field $\pi\in\mathfrak{X}^2(M)$, let $\pi^\sharp:T^*M\longrightarrow TM$ be the induced bundle map given by

$$\pi^\sharp(\xi)=i_\xi\pi,\quad\forall\ \xi\in\Omega^1(M).$$

Similar to the discussion in [21], the graph of a bundle map $\pi^\sharp$ is a Dirac structure of the exact twisted Courant algebroid $(\mathcal{T},\langle\cdot,\cdot\rangle,\{\cdot,\cdot\}_h,\rho,dh)$ if and only if

$$[\pi,\pi]=\frac{1}{2}\wedge^3\pi^\sharp h. \tag{38}$$

Definition 6.3. *A bi-vector field $\pi\in\mathfrak{X}^2(M)$ is called an h-twisted Poisson structure if (38) holds. (M,π) is called an h-twisted Poisson manifold if π is an h-twisted Poisson structure.*

For the case that the 3-form h is closed, i.e. $dh=0$, it is discussed by Ševera and Weinstein in [21]. See [6, 15] for more details.

One can introduce a bilinear skew-symmetric bracket operation on the cotangent bundle of an h-twisted Poisson manifold (M,π) by

$$[\xi,\eta]_{\pi,h}=L_{\pi^\sharp\xi}\eta-L_{\pi^\sharp\eta}\xi+d\pi(\eta,\xi)+i_{\pi^\sharp\xi\wedge\pi^\sharp\eta}h. \tag{39}$$

Then we have

$$\pi^\sharp[\xi,\eta]_{\pi,h}=[\pi^\sharp\xi,\pi^\sharp\eta], \tag{40}$$

where $[\cdot,\cdot]_{\pi,h}$ is given by (39). It is well known that if $dh=0$, then $[\xi,\eta]_{\pi,h}$ is a Lie bracket, consequently, $(T^*M,[\xi,\eta]_{\pi,h},\pi^\sharp)$ is a Lie algebroid. Instead of a Lie algebroid, for an h-twisted Poisson structure, we obtain

Theorem 6.4. *Associated to any h-twisted Poisson structure π, there is a 2-term L_∞-algebra, of which the degree-0 part is $\Omega^1(M)$, the degree-1 part is $\Gamma(\mathrm{Ker}(\pi^\sharp))$, the differential is the inclusion $\mathbb{i}:\Gamma(\mathrm{Ker}(\pi^\sharp))\longrightarrow\Omega^1(M)$, l_2 and l_3 are given by*

$$\begin{array}{rcl} l_2(\xi,\eta)&=&[\xi,\eta]_{\pi,h},\quad\forall\ \xi,\eta\in\Omega^1(M),\\ l_2(\xi,u)&=&[\xi,u]_{\pi,h},\quad\forall\ \xi\in\Omega^1(M),u\in\Gamma(\mathrm{Ker}(\pi^\sharp)),\\ l_3(\xi,\eta,\gamma)&=&i_{\pi^\sharp\xi\wedge\pi^\sharp\eta\wedge\pi^\sharp\gamma}dh,\quad\forall\ \xi,\eta,\gamma\in\Omega^1(M).\end{array}$$

Proof. It is obvious that l_2 and l_3 are all skew-symmetric. For any $\xi,\eta,\gamma\in\Omega^1(M)$, it is straightforward to deduce that

$$[\xi,[\eta,\gamma]_{\pi,h}]_{\pi,h}-[[\xi,\eta]_{\pi,h},\gamma]_{\pi,h}-[\eta,[\xi,\gamma]_{\pi,h}]_{\pi,h}=i_{\pi^\sharp\xi\wedge\pi^\sharp\eta\wedge\pi^\sharp\gamma}dh. \tag{41}$$

Thus we have

$$l_3(\xi,\eta,\gamma)=[\xi,[\eta,\gamma]_{\pi,h}]_{\pi,h}-[[\xi,\eta]_{\pi,h},\gamma]_{\pi,h}-[\eta,[\xi,\gamma]_{\pi,h}]_{\pi,h}.$$

On the other hand, by (40), we have

$$\begin{array}{rcl}\pi^\sharp l_3(\xi,\eta,\gamma)&=&\pi^\sharp([\xi,[\eta,\gamma]_{\pi,h}]_{\pi,h}-[[\xi,\eta]_{\pi,h},\gamma]_{\pi,h}-[\eta,[\xi,\gamma]_{\pi,h}]_{\pi,h})\\ &=&[\pi^\sharp\xi,[\pi^\sharp\eta,\pi^\sharp\gamma]]-[[\pi^\sharp\xi,\pi^\sharp\eta],\pi^\sharp\gamma]-[\pi^\sharp\eta,[\pi^\sharp\xi,\pi^\sharp\gamma]]\\ &=&0.\end{array}$$

Therefore, we have

$$l_3(\xi,\eta,\gamma)\in\Gamma(\mathrm{Ker}(\pi^\sharp)). \tag{42}$$

Now we only need to show that the Jacobiator identity holds. For any $\theta\in\Omega^1(M)$, by (42), we have

$$\begin{array}{rl}&l_2(\theta,l_3(\xi,\eta,\gamma))+c.p.(\theta,\xi,\eta,\gamma)-(l_3([\theta,\xi]_{\pi,h},\eta,\gamma)+c.p.(\theta,\xi,\eta,\gamma))\\ =&L_{\pi^\sharp\theta}i_{\pi^\sharp\xi\wedge\pi^\sharp\eta\wedge\pi^\sharp\gamma}dh+c.p.(\theta,\xi,\eta,\gamma)-(i_{\pi^\sharp[\theta,\xi]_{\pi,h}\wedge\pi^\sharp\eta\wedge\pi^\sharp\gamma}dh+c.p.(\theta,\xi,\eta,\gamma))\\ =&L_{\pi^\sharp\theta}i_{\pi^\sharp\xi\wedge\pi^\sharp\eta\wedge\pi^\sharp\gamma}dh+c.p.(\theta,\xi,\eta,\gamma)-(i_{[\pi^\sharp\theta,\pi^\sharp\xi]\wedge\pi^\sharp\eta\wedge\pi^\sharp\gamma}dh+c.p.(\theta,\xi,\eta,\gamma)).\end{array}$$

Act on an arbitrary vector field $X\in\mathfrak{X}(M)$, we get

$$\begin{array}{rl}&\Big(l_2(\theta,l_3(\xi,\eta,\gamma))+c.p.(\theta,\xi,\eta,\gamma)-(l_3([\theta,\xi]_{\pi,h},\eta,\gamma)+c.p.(\theta,\xi,\eta,\gamma))\Big)(X)\\ =&\pi^\sharp\theta\big(dh(\pi^\sharp\xi,\pi^\sharp\eta,\pi^\sharp\gamma,X)\big)-dh(\pi^\sharp\xi,\pi^\sharp\eta,\pi^\sharp\gamma,[\pi^\sharp\theta,X])+c.p.(\theta,\xi,\eta,\gamma)\\ &-(dh([\pi^\sharp\theta,\pi^\sharp\xi],\pi^\sharp\eta,\pi^\sharp\gamma,X)+c.p.(\theta,\xi,\eta,\gamma))\\ =&\pi^\sharp\theta\big(dh(\pi^\sharp\xi,\pi^\sharp\eta,\pi^\sharp\gamma,X)\big)+c.p.(\theta,\xi,\eta,\gamma)\\ &-(dh([\pi^\sharp\theta,\pi^\sharp\xi],\pi^\sharp\eta,\pi^\sharp\gamma,X)+c.p.(\theta,\xi,\eta,\gamma,X))\\ =&d(dh)(\pi^\sharp\theta,\pi^\sharp\xi,\pi^\sharp\eta,\pi^\sharp\gamma,X)-X(dh(\pi^\sharp\theta,\pi^\sharp\xi,\pi^\sharp\eta,\pi^\sharp\gamma))\\ =&0.\end{array}$$

The last equality follows from the fact $d^2 = 0$ and

$$dh(\pi^\sharp\theta, \pi^\sharp\xi, \pi^\sharp\eta, \pi^\sharp\gamma) = i_{\pi^\sharp\gamma}dh(\pi^\sharp\theta, \pi^\sharp\xi, \pi^\sharp\eta) = -i_{\pi^\sharp dh(\pi^\sharp\theta,\pi^\sharp\xi,\pi^\sharp\eta)}\gamma = 0.$$

Therefore, l_3 satisfies the Jacobiator identity. This finishes the proof of $(\Gamma(\mathrm{Ker}(\pi)) \xrightarrow{\mathbb{i}} \Omega^1(M), l_2, l_3)$ being a 2-term L_∞-algebra. ■

For an h-twisted Poisson structure π, the graph of $\pi^\sharp$, which we denote by $\mathcal{G}_\pi \subset \mathcal{T}$, is a Dirac structure. The 2-term L_∞-algebra constructed in Theorem 6.4 is isomorphic to the 2-term L_∞-algebra constructed in Theorem 6.2 for the Dirac structure $\mathcal{G}_\pi$. More precisely, for the Dirac structure $\mathcal{G}_\pi$, we have

$$(\rho^*)^{-1}\mathcal{G}_\pi = \mathbb{i}^{-1}\mathcal{G}_\pi = \mathcal{G}_\pi \cap T^*M = \mathrm{Ker}(\pi).$$

Define $f_0 : \Gamma(\mathcal{G}_\pi) \longrightarrow \Omega^1(M)$ by

$$f_0(\pi^\sharp\xi + \xi) = \xi,$$

and define $f_1 : (\rho^*)^{-1}\mathcal{G}_\pi \longrightarrow \mathrm{Ker}(\pi)$ to be the identity map. It is obvious that $f_0 \circ \mathbb{i} = \mathbb{i} \circ f_1$. Moreover, we have

$$\begin{aligned} f_0(\{\pi^\sharp\xi + \xi, \pi^\sharp\eta + \eta\}_h) &= L_{\pi^\sharp\xi}\eta - L_{\pi^\sharp\eta}\xi + d\pi(\eta, \xi) + i_{\pi^\sharp\xi\wedge\pi^\sharp\eta}h \\ &= [\xi, \eta]_{\pi,h} \\ &= [f_0(\pi^\sharp\xi + \xi), f_0(\pi^\sharp\eta + \eta)]_{\pi,h}. \end{aligned}$$

Thus (f_0, f_1) is an isomorphism of 2-term L_∞-algebras.

Remark 6.5. The geometric structure underlying this 2-term L_∞-algebra is actually the H-twisted Lie algebroids introduced by Melchior Grützmann in [7]. An H-twisted Lie algebroid is a quadruple $(E, [\cdot,\cdot], \rho, H)$ consists of a vector bundle $E \longrightarrow M$, a bundle map $\rho : E \longrightarrow TM$, a section $H : \wedge^3\Gamma(E) \longrightarrow \Gamma(\mathrm{Ker}(\rho))$, and a skew-symmetric bracket $[\cdot,\cdot] : \Gamma(E) \wedge \Gamma(E) \longrightarrow \Gamma(E)$ subject to the following axioms:

$$\begin{aligned} [e_1, [e_2, e_3]] + c.p.(e_1, e_2, e_3) &= H(e_1, e_2, e_3), \\ [e_1, fe_2] &= f[e_2, e_2] + \rho(e_1)(f)e_2, \\ DH &= 0, \end{aligned}$$

where $e_i \in \Gamma(E)$, $f \in C^\infty(M)$ and $DH : \wedge^4\Gamma(E) \longrightarrow \Gamma(\mathrm{Ker}(\rho))$ is defined by

$$\begin{aligned} DH(e_1, e_2, e_3, e_4) \triangleq& \sum_{i=1}^{4}(-1)^{i+1}[e_i, H(e_1, \cdots, \widehat{e_i}, \cdots, e_4)] \\ &+ \sum_{i<j}(-1)^{i+j}H([e_i, e_j], e_1 \cdots, \widehat{e_i}, \cdots, \widehat{e_j}, \cdots, e_4). \end{aligned}$$

It is straightforward to see that for any h-twisted Poisson structure π, $(T^*M, [\cdot,\cdot]_{\pi,h}, \pi^\sharp, l_3)$ is an l_3-twisted Lie algebroid.

References

[1] M. Ammar and N. Poncin, Coalgebraic Approach to the Loday Infinity Category, Stem Differential for $2n$-ary Graded and Homotopy Algebras, *Ann. Inst. Four.* 60 no. 1 (2010), 355-387.

[2] J. C. Baez, A. S. Crans, Higher-Dimensional Algebra VI: Lie 2-Algebras, *Theory and Appl. Categ.* 12 (2004), 492-528.

[3] J. Baez and C. Rogers, Categorified Symplectic Geometry and the String Lie 2-Algebra, *Homology, Homotopy and Applications* 12 (2010), 221-236.

[4] A. Carey, M. Murray and B. L. Wang, Higher Bundle Gerbes and Cohomology Classes in Gauge Theories, arXiv:hep-th/9511169.

[5] J.M. Casas, E. Khmaladze and M. Ladra, Crossed modules for Leibniz n-algebras, *Forum Math.* 20 (2008), 841-858.

[6] A. S. Cattaneo and P. Xu, Integration of twisted Poisson structures, *J. Geom. Phys.* 49 (2004), 187-196.

[7] M. Grutzmann, H-twisted Lie algebroids, *J. Geom. Phys.* 61 (2011), 476-484.

[8] M. Gualtieri, *Generalized Complex Geometry,* PhD thesis, St John's College, University of Oxford, Nov. 2003.

[9] M. Hansena and T. Strobl, First Class Constrained Systems and Twisting of Courant Algebroids by a Closed 4-form, arXiv:0904.0711v1.

[10] C. M. Hull, Generalised Geometry for M-Theory, *JHEP* 0707:079, 2007.

[11] T. Lada and J. Stasheff, Introduction to sh Lie algebras for physicists, *Int. Jour. Theor. Phys.* 32 (7) (1993), 1087-1103.

[12] Z.-J. Liu, A. Weinstein and P. Xu, Manin triples for Lie bialgebroids, *J. Diff. Geom.* 45 (1997), 547-574.

[13] J.-L. Loday, Une version non commutative des algèbres de Lie: les algèbres de Leibniz, *Enseign. Math.* (2), 39 (1993), 269-293.

[14] J.-L. Loday and T. Pirashvili, Universal enveloping algebras of Leibniz algebras and (co)homology, *Math. Ann.* 296 (1993), 139-158.

[15] F. Petalidou, On the geometric quantization of twisted Poisson manifolds, *J. Math. Phys.* 48 (2007), 083502.

[16] C. Rogers, L_∞-algebras from multisymplectic geometry, arXiv:1005.2230.

[17] D. Roytenberg, *Courant algebroids, derived brackets and even symplectic supermanifolds*, PhD thesis, UC Berkeley, 1999, arXiv:math.DG/9910078.

[18] D. Roytenberg, Courant algebroids and strongly homotopy Lie algebras, *Lett. Math. Phys.*, 46 (1) (1998), 81-93.

[19] D. Roytenberg, On weak Lie 2-algebras, In *XXVI Workshop on Geometrical Methods in Physics*, volume 956 of *AIP Conf. Proc.*, pages 180–198. Amer. Inst. Phys., Melville, NY, 2007.

[20] M. Schlessinger and J. Stasheff, The Lie algebra structure of tangent cohomology and deformation theory, *Jour. Pure App. Alg.* 38 (1985), 313-322.

[21] P. Ševera and A. Weinstein, Poisson geometry with a 3-form background. *Prog. Theor. Phys. Suppl.* 144 (2001), 145-154.

[22] Y. Sheng, Z. Liu and C. Zhu, Omni-Lie 2-algebras and their Dirac structures, *J. Geom. Phys.* 61 (2011), pp. 560-575.

[23] Y. Sheng and C. Zhu, Semidirect products of representations up to homotopy, arXiv:0910.2147, to appear in *Pacific J. Math.*.

[24] Y. Sheng and C. Zhu, Integration of semidirect product Lie 2-algebras, arXiv:1003.1348.

[25] K. Uchino, Derived brackets and sh Leibniz algebras, *J. Pure Appl. Algebra*, doi:10.1016/j.jpaa.2010.07.016.

[26] M. Zambon, L_∞-algebras and higher analogues of Dirac structures, arXiv:1003.1004.