At which points exactly has Lebesgue's singular function the derivative zero ?

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December 30, 2010

Abstract

Let $L_a(x)$ be Lebesgue's singular function with a real parameter a $(0 < a < 1, a \neq 1/2)$. As is well known, $L_a(x)$ is strictly increasing and has a derivative equal to zero almost everywhere. However, what sets of $x \in [0,1]$ actually have $L_a'(x) = 0$ or $+\infty$? We give a partial characterization of these sets in terms of the binary expansion of x. As an application, we consider the differentiability of the composition of Takagi's nowhere differentiable function and the inverse of Lebesgue's singular function.

AMS 2000 subject classification: 26A27 (primary); 26A15, 26A30, 60G50 (secondary)

Key words and phrases: Takagi's function, Lebesgue's singular function, Nowhere-differentiable function, Dini derivatives.

1 Introduction

Imagine flipping an unfair coin with probability $a \in (0,1)$ of heads and probability $1-a$ of tails. Note that $a \neq 1/2$. Let the binary expansion of $t \in [0,1]$: $t = \sum_{n=1}^{\infty} \omega_n / 2^n$ be determined by flipping the coin infinitely many

*Supported in part by Japanese GCOE Program G08: "Fostering Top Leaders in Mathematics - Broadening the Core and Exploring New Ground".

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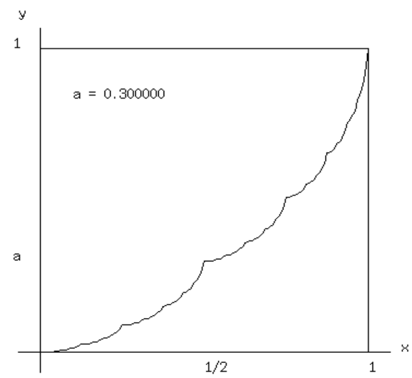

Figure 1: Lebesgue's singular function ($a = 0.3$)

times. More precisely, $\omega_n = 0$ if the n-th toss is heads and $\omega_n = 1$ if it is tails. We define *Lebesgue's singular function* $L_a(x)$ as the distribution function of t:

$$L_a(x) := prob\{t \le x\}, \qquad 0 \le x \le 1.$$

It is well-known that $L_a(x)$ is strictly increasing, but the derivative is 0 almost everywhere. This distribution function $L_a(x)$ was also defined in different ways and studied by a number of authors: Cesaro (1906), Faber (1910), Lomnicki and Ulam(1934), Salem (1943), De Rham (1957) and others. For instance, De Rham [3] studied $L_a(x)$ as a unique continuous solution of the functional equation

$$L_a(x) = \begin{cases} aL_a(2x), & 0 \le x \le \frac{1}{2}, \\ (1-a)L_a(2x-1) + a, & \frac{1}{2} \le x \le 1, \end{cases} \tag{1}$$

where $0 < a < 1$, and $a \neq 1/2$.

From (1), it is clear that the graph of $L_a(x)$ is self-affine. Because of its connection with fractals, several applications have been found in recent years: for instance, in physics [12, 13], real analysis [5, 6], digital sum problems [7, 9] and complex dynamical systems [10]. There is even a connection with the Collatz conjecture [2].

Reconsider the differentiability of $L_a(x)$. It is known that for any $x \in [0, 1]$, $L_a'(x)$ is either zero, or $+\infty$, or it does not exist. Then, it is natural to ask at which points $x \in [0, 1]$ exactly we have $L_a'(x) = 0$ or $+\infty$.

In fact, De Rham [3] gave the following partial answer to this question. Let the binary expansion of $x \in [0, 1]$ be $x = \sum_{k=1}^{\infty} 2^{-k}\varepsilon_k$, where $\varepsilon_k \in \{0, 1\}$.

For those $x \in [0,1]$ having two binary expansions, we choose the expansion which is eventually all zeros. As an exception, fix $\varepsilon_k = 1$ for every k if $x = 1$.

Define

$$I_n := \sum_{k=1}^{n} \varepsilon_k. \tag{2}$$

Note that I_n is the number of 1's occurring in the first n binary digits of x.

Suppose that I_n/n tends to a limit l as $n \to \infty$, and let

$$l_0 := \frac{\log 2a}{\log a - \log(1-a)}. \tag{3}$$

Then the derivative $L_a'(x)$ exists and is zero, when $(l - l_0)(a - 1/2) > 0$. An English translation of De Rham's original paper is included in Edgar's book [4].

Unfortunately, De Rham's paper did not contain a proof. The main purpose of this note is to give a proof of De Rham's statement and extend his result. The paper is organized as follows. Section 2 states and proves the main results. The key to the proof is to use Lomnicki and Ulam's expression from 1934 [8]. De Rham might have had a different proof in mind, as he did not mention Lomnicki and Ulam's paper. In Section 3, as an application, we consider a question about the differentiability of the composition of Takagi's nowhere differentiale function and the inverse of Lebesgue's singular function.

2 The main result

For convenience, define the right-hand and left-hand derivatives of $L_a(x)$ as follows.

$$L'_{a+}(x) : = \lim_{h \to 0+} \frac{L_a(x+h) - L_a(x)}{h},$$
$$L'_{a-}(x) : = \lim_{h \to 0-} \frac{L_a(x+h) - L_a(x)}{h},$$

provided the limits exist.

From the self-affinity of the graph, we have

Lemma 2.1. *For any $x \in [0,1]$ for which $L'_{a+}(x)$ exists,*

$$L'_{a+}(x) = L'_{(1-a)-}(1-x).$$

Define

$$D_1(x) := \lim_{n\to\infty} \frac{I_n}{n} = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \varepsilon_k, \tag{4}$$

provided the limit exists, and put $D_0(x) := 1 - D_1(x)$. In other words, $D_i(x)$ is the density of the digit i in the binary expansion of x, for $i = 0, 1$.

Theorem 2.2. *1. If $x \in [0,1]$ is dyadic, then $L'_{a+}(x) \neq L'_{a-}(x)$.*

2. If $x \in [0,1]$ is not dyadic and $0 < D_1(x) < 1$, then

$$L'_a(x) = \begin{cases} 0, & \textit{if} \quad a^{D_0(x)}(1-a)^{D_1(x)} < 1/2, \\ +\infty, & \textit{if} \quad a^{D_0(x)}(1-a)^{D_1(x)} > 1/2. \end{cases}$$

Remark 2.3. *De Rham's statement is equivalent to the following. For a value of x for which $D_1(x)$ exists, $L'_a(x) = 0$ when $a^{D_0(x)}(1-a)^{D_1(x)} < 1/2$.*

Remark 2.4. *If x is a binary normal, that is, $D_0(x) = D_1(x) = 1/2$, then Theorem 2.2 gives $L'_a(x) = 0$, since $\sqrt{a(1-a)} < 1/2$.*

Proof of Theorem 2.2. First, suppose $x \in [0,1]$ is a dyadic point, say $x = j/2^N$. Let $2^{-(k+1)} \leq h \leq 2^{-k}$ where $k > N$. Since L_a is increasing, this implies that

$$\frac{L_a(x+2^{-(k+1)}) - L_a(x)}{2^{-k}} \leq \frac{L_a(x+h) - L_a(x)}{h} \leq \frac{L_a(x+2^{-k}) - L_a(x)}{2^{-(k+1)}}. \tag{5}$$

The key to the proof is to use the following expression for $L_a(x)$, given by Lomnicki and Ulam [8]:

$$L_a(x) = \frac{a}{1-a}\sum_{n=1}^{\infty} \varepsilon_n a^{n-I_n}(1-a)^{I_n}, \tag{6}$$

where I_n is defined by (2). By (6), we have

$$L_a(x+2^{-k}) - L_a(x) = a^{k-I_N}(1-a)^{I_N}.$$

Since $(1-a)/a$ is a positive constant,

$$\begin{aligned} \lim_{k\to\infty} \frac{L_a(x+2^{-k}) - L_a(x)}{2^{-k}} &= \lim_{k\to\infty} (2a)^k \left(\frac{1-a}{a}\right)^{I_N} \\ &= \begin{cases} 0, & \text{if} \quad 0 < a < 1/2, \\ +\infty, & \text{if} \quad 1/2 < a < 1. \end{cases} \end{aligned}$$

By (5), it follows that

$$L'_{a+}(x) = \begin{cases} 0, & \text{if} \quad 0 < a < 1/2, \\ +\infty, & \text{if} \quad 1/2 < a < 1. \end{cases}$$

Since $1-x$ is also a dyadic, the left-hand derivative follows from Lemma 2.1:

$$L'_{a-}(x) = L'_{(1-a)+}(1-x) = \begin{cases} +\infty, & \text{if} \quad 0 < a < 1/2, \\ 0, & \text{if} \quad 1/2 < a < 1. \end{cases}$$

Therefore, $L'_a(x)$ does not exist if x is dyadic.

Next, suppose $x \in [0,1]$ is not dyadic and $0 < D_1(x) < 1$. Let p_k be the address of the k-th "0" in the binary expansion of x, and $2^{-p_{k+1}} \leq h \leq 2^{-p_k}$. Since L_a is increasing, this implies that

$$\frac{L_a(x+2^{-p_{k+1}}) - L_a(x)}{2^{-p_k}} \leq \frac{L_a(x+h) - L_a(x)}{h} \leq \frac{L_a(x+2^{-p_k}) - L_a(x)}{2^{-p_{k+1}}}. \quad (7)$$

Using (6), we have

$$L_a(x+2^{-p_k}) - L_a(x) = a^k(1-a)^{p_k-k} + \left(1 - \frac{a}{1-a}\right) \sum_{n=p_k+1}^{\infty} \varepsilon_n a^{n-I_n}(1-a)^{I_n}. \quad (8)$$

Let $n(l)$ be the address of the l-th "1" appearing after position p_k in the binary expansion of x. Then we have

$$\sum_{n=p_k+1}^{\infty} \varepsilon_n a^{n-I_n}(1-a)^{I_n} = a^k(1-a)^{p_k-k} \sum_{l=1}^{\infty} a^{n(l)-p_k-l}(1-a)^l.$$

Since $n(l) - p_k - l \geq 0$ and $0 < a < 1$, the series in the right hand side above converges, say to $C(x,k)$.

For convenience, define

$$C_1(x,k) := 1 + \left(1 - \frac{a}{1-a}\right) C(x,k).$$

Then we can write (8) as

$$L_a(x+2^{-p_k}) - L_a(x) = a^k(1-a)^{p_k-k} C_1(x,k). \quad (9)$$

Since $C(x,k) \leq \sum_{l=1}^{\infty}(1-a)^l$, it follows that

$$\min\left\{1, \frac{1-a}{a}\right\} \leq C_1(x,k) \leq \max\left\{1, \frac{1-a}{a}\right\}. \tag{10}$$

By (9), we have

$$\begin{aligned}
\frac{L_a(x+2^{-p_k}) - L_a(x)}{2^{-p_{k+1}}} &= \left\{2^{\frac{p_{k+1}}{p_k}} a^{\frac{k}{p_k}} (1-a)^{1-\frac{k}{p_k}}\right\}^{p_k} C_1(x,k),\\
\frac{L_a(x+2^{-p_{k+1}}) - L_a(x)}{2^{-p_k}} &= \left\{2^{\frac{p_k}{p_{k+1}}} a^{\frac{k+1}{p_{k+1}}} (1-a)^{1-\frac{k+1}{p_{k+1}}}\right\}^{p_{k+1}} C_1(x,k).
\end{aligned}$$

Since k/p_k tends to a nonzero limit $D_0(x)$ as $k \to \infty$, we have $p_{k+1}/p_k \to 1$ as $k \to \infty$. Therefore, it follows from (7) and (10) that

$$L'_{a+}(x) = \begin{cases} 0, & \text{if} \quad a^{D_0(x)}(1-a)^{D_1(x)} < 1/2,\\ +\infty, & \text{if} \quad a^{D_0(x)}(1-a)^{D_1(x)} > 1/2. \end{cases}$$

Finally, for the left-hand derivative, it follows from Lemma 2.1 that

$$L'_{a-}(x) = L'_{(1-a)+}(1-x) = \begin{cases} 0, & \text{if} \quad a^{D_0(x)}(1-a)^{D_1(x)} < 1/2,\\ +\infty, & \text{if} \quad a^{D_0(x)}(1-a)^{D_1(x)} > 1/2, \end{cases}$$

since $D_i(x) = D_j(1-x)$ when $i \neq j$. This concludes the proof. □

Remark 2.5. *A careful study of the above proof shows that the existence of the full limit* $D_1(x) = \lim_{n\to\infty}(I_n/n)$ *is not necesary. The following generalization is straightforward:*

1. *Suppose* $0 < a < 1/2$. *Then*

$$L'_a(x) = \begin{cases} 0, & \textit{if} \quad \lim_{n\to\infty} \sup I_n/n < l_0,\\ +\infty, & \textit{if} \quad \lim_{n\to\infty} \inf I_n/n > l_0. \end{cases}$$

2. *Suppose* $1/2 < a < 1$. *Then*

$$L'_a(x) = \begin{cases} 0, & \textit{if} \quad \lim_{n\to\infty} \inf I_n/n > l_0,\\ +\infty, & \textit{if} \quad \lim_{n\to\infty} \sup I_n/n < l_0, \end{cases}$$

where l_0 *is defined by* (3).

Note that Theorem 2.2 left out the boundary case; that is, those numbers x for which $a^{D_0(x)}(1-a)^{D_1(x)} = 1/2$; in other words, numbers x which have the following densities:

$$D_1(x) = \frac{\log 2a}{\log a - \log(1-a)}, \qquad D_0(x) = \frac{\log 2(1-a)}{\log(1-a) - \log a}.$$

Let us define some additional notation. As a complement of I_n, define O_n to be the number of 0's occurring in the first n binary digits of x:

$$O_n := \sum_{k=1}^{n}(1-\varepsilon_k).$$

Let q_k be the address of the k-th "1" in the binary expansion of x as a complement of p_k. Observe that

$$\begin{array}{lll} q_k \le n & \text{if and only if} & I_n \ge k, \\ p_k \le n & \text{if and only if} & O_n \ge k. \end{array}$$

Then, it is easy to prove the following lemma by contradiction.

Lemma 2.6. *Let $f(k) = p_k - \frac{k}{D_0(x)}$ and $g(k) = \frac{k}{D_1(x)} - q_k$. If $f(k) \to \infty$ as $k \to \infty$, then $g(k) \to \infty$.*

Theorem 2.7. *Suppose $x \in [0,1]$ satisfies $a^{D_0(x)}(1-a)^{D_1(x)} = 1/2$. Let $f(k) = p_k - \frac{k}{D_0(x)}$ and suppose $f(k+1)/f(k) \to 1$.*

1. *If $f(k) \to \infty$ as $k \to \infty$, then*

$$L_a'(x) = \begin{cases} +\infty, & \text{if} \quad 0 < a < 1/2, \\ 0, & \text{if} \quad 1/2 < a < 1. \end{cases}$$

2. *If $f(k) \to -\infty$ as $k \to \infty$, then*

$$L_a'(x) = \begin{cases} 0, & \text{if} \quad 0 < a < 1/2, \\ +\infty, & \text{if} \quad 1/2 < a < 1. \end{cases}$$

Proof of Theorem 2.7.

We follow the same argument for non-dyadic points $x \in [0,1]$ as in the proof of Theorem 2.2. Let $f(k) = p_k - \frac{k}{D_0(x)}$. Since k/p_k tends to a nonzero limit $D_0(x)$ as $k \to \infty$, $f(k)$ is of smaller order than k. Then, it follows from (9) that

$$\begin{aligned}\frac{L_a(x+2^{-p_k})-L_a(x)}{2^{-p_k}} &= \left[\{2(1-a)\}^{\frac{1}{D_0(x)}}\, a(1-a)^{-1}\right]^k \{2(1-a)\}^{f(k)}\, C_1(x,k)\\ &= \{2(1-a)\}^{f(k)}\, C_1(x,k),\end{aligned}$$

because

$$\{2(1-a)\}^{\frac{1}{D_0(x)}} a(1-a)^{-1} = 1, \qquad \text{when } a^{D_0(x)}(1-a)^{D_1(x)} = 1/2.$$

Thus,

$$\begin{aligned}\frac{L_a(x+2^{-p_k})-L_a(x)}{2^{-p_{k+1}}} &= \left\{2^{\frac{f(k+1)}{f(k)}}(1-a)\right\}^{f(k)} \cdot 2^{\frac{1}{D_0(x)}} C_1(x,k),\\ \frac{L_a(x+2^{-p_{k+1}})-L_a(x)}{2^{-p_k}} &= \left\{2^{\frac{f(k)}{f(k+1)}}(1-a)\right\}^{f(k+1)} \cdot 2^{\frac{1}{D_0(x)}} C_1(x,k+1).\end{aligned}$$

Since $f(k+1)/f(k) \to 1$ as $k \to \infty$, it follows from (7) and (10) that if $f(k) \to \infty$,

$$L'_{a+}(x) = \begin{cases} +\infty, & \text{if} \quad 0 < a < 1/2,\\ 0, & \text{if} \quad 1/2 < a < 1.\end{cases}$$

Similary, if $f(k) \to -\infty$ as $k \to \infty$, then

$$L'_{a+}(x) = \begin{cases} 0, & \text{if} \quad 0 < a < 1/2,\\ +\infty, & \text{if} \quad 1/2 < a < 1.\end{cases}$$

Next, consider the left-hand derivative. From Lemma 2.1, we have $L'_{a-}(x) = L'_{(1-a)+}(1-x)$. It is clear that $1-x$ also satisfies $a^{D_0(1-x)}(1-a)^{D_1(1-x)} = 1/2$, since $D_i(x) = D_j(1-x)$ for $i \neq j$. Let $g(k) = \frac{k}{D_1(x)} - q_k$. Since $q_k = p_k(1-x)$, we have $g(k+1)/g(k) \to 1$ if $f(k+1)/f(k) \to 1$. It follows from Lemma 2.6 that if $f(k) \to \infty$ or $-\infty$, then $L'_{a-}(x) = L'_{(1-a)+}(1-x) = L'_{a+}(x)$. This concludes the proof. □

3 Application

We apply the main result to the following simple question. In classical calculus, the chain rule is used to compute the derivative of the composition of two differentiable functions. However, what can we say, for example, about the

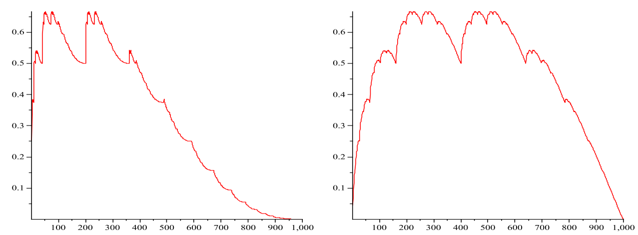

Figure 2: Graphs of $(T \circ L_a^{-1})(x)$ for $a = 0.2$ (left) and $a = 0.4$ (right)

differentiability of the composition of a nowhere differentiable function and a singular function? For instance, let T be Takagi's nowhere differentiable function, which is defined by

$$T(x) = \sum_{k=0}^{\infty} \frac{1}{2^k} |2^k x - \lfloor 2^k x + \tfrac{1}{2} \rfloor|, \qquad 0 \leq x \leq 1.$$

Is $(T \circ L_a^{-1})$ nowhere differentiable? See Figure 2. If $a = 0.4$, the figure of the graph looks somewhat like Takagi's function; on the other hand, if $a = 0.2$, the shape of the graph is more like Lebesgue's singular function. Thus, we can guess that $(T \circ L_a^{-1})$ might not be nowhere differentiable if a is close to 0.

Although T does not have a finite derivative anywhere, it is known to have an improper infinite derivative at many points. In fact, Allaart and Kawamura [1] proved that the set of points where $T'(x) = +\infty$ or $-\infty$ has Hausdorff dimension one. Note that the inverse of Lebesgue's singular function is also singular. Hence, if we try to (naively) use the chain rule to compute the derivative of $(T \circ L_a^{-1})(x)$, we may run into one of the indeterminate products $+\infty \cdot 0$ or $-\infty \cdot 0$.

The following theorem gives an answer to this concrete question: $(T \circ L_a^{-1})(x)$ has a finite but vanishing derivative at uncountably many points.

Theorem 3.1. *Let $x \in [0, 1]$, and put $y = L_a^{-1}(x)$. If $0 < D_1(y) < 1$ and $a^{D_0(y)}(1-a)^{D_1(y)} > 1/2$, then*

$$(T \circ L_a^{-1})'(x) = 0. \tag{11}$$

Proof.

Define $\tilde{h} := L_a^{-1}(x+h) - L_a^{-1}(x)$. Then we can write

$$\frac{T(L_a^{-1}(x+h)) - T(L_a^{-1}(x))}{h} = \frac{T(y+\tilde{h}) - T(y)}{\tilde{h}\log_2(1/|\tilde{h}|)} \cdot \frac{\tilde{h}\log_2(1/|\tilde{h}|)}{h}. \tag{12}$$

Allaart and Kawamura [1] proved that if $D_1(x)$ exists and $0 < D_1(x) < 1$, then

$$\lim_{h\to 0} \frac{T(x+h) - T(x)}{h\log_2(1/|h|)} = D_0(x) - D_1(x).$$

Therefore, we have

$$-1 \leq \lim_{h\to 0} \frac{T(y+\tilde{h}) - T(y)}{\tilde{h}\log_2(1/|\tilde{h}|)} \leq 1.$$

A slight modification of the proof of Theorem 2.2 yields

$$\lim_{h\to 0} \frac{\tilde{h}\log_2(1/|\tilde{h}|)}{h} = 0, \qquad \text{if} \quad a^{D_0(y)}(1-a)^{D_1(y)} > 1/2.$$

Substituting these results into (12) gives (11). □

Acknowledgment

This research was done mainly during my visit to RIMS, Kyoto university. I am grateful to Prof. H. Okamoto for his support and warm-hearted hospitality. Also, I would like to thank Prof. P. Allaart for his helpful comments and suggestions in preparing this paper.

Lastly, I wish to dedicate this paper to the memory of Prof. H. Shinya, who taught me a deeper understanding of calculus.

References

[1] P. Allaart and K. Kawamura, The improper infinite derivatives of Takagi's nowhere-differentiable function, *J. Math. Anal. Appl.*, **372**, pp 656-665 (2010).

[2] L. Berg and M. Kruppel, De Rham's singular function and related functions, *Z. Anal. Anwendungen.*, **19**, no. 1, pp 227-237 (2000).

[3] G. de Rham, Sur quelques courbes définies par des équations fonctionnelles, *Rend. Sem. Mat. Torino* **16**, pp. 101-113 (1957).

[4] G. A. Edgar, *Classics on Fractals*, Addison-Wesley, Reading, MA (1993).

[5] M. Hata and M. Yamaguti, Takagi function and its generalization, *Japan J. Appl. Math.*, **1**, pp. 183-199 (1984).

[6] K. Kawamura, On the classification of self-similar sets determined by two contractions on the plane, *J. Math. Kyoto Univ.*, **42**, no. 2, pp. 255-286 (2002).

[7] M. Kruppel, De Rham's singular function, its partial derivatives with respect to the parameter and binary digital sums, *Rostock. Math. Kolloq.*, **64**, pp 57-74 (2009).

[8] Z. Lomnicki and S. Ulam, Sur la théorie de la mesure dans les espaces combinatoires et son application au calcul des probabilités I. Variables indépendantes. *Fund. Math.* **23**, pp. 237-278 (1934).

[9] T. Okada, T. Sekiguchi and Y. Shiota, An explicit formula of the exponential sums of digital sums, *Japan J. Indust. Appl. Math.* **12**, pp. 425-438 (1995).

[10] H. Sumi, Rational semigroups, random complex dynamics and singular functions on the complex plane. *Sugaku* **61**, no. 2, pp. 133-161 (2009).

[11] T. Takagi, A simple example of the continuous function without derivative, *Phys.-Math. Soc. Japan* **1** (1903), 176-177. *The Collected Papers of Teiji Takagi*, S. Kuroda, Ed., Iwanami, pp. 5-6 (1973).

[12] H. Takayasu, Physical models of fractal functions, *Japan J. Appl. Math.*, **1**, pp. 201-205 (1984).

[13] S. Tasaki, I. Antoniou and Z. Suchanecki, Deterministic diffusion, De Rham equation and fractal eigenvectors, *Physics Letter* A 179, pp. 97-102 (1993).