SUPERCUSPIDAL CHARACTERS OF SL_2 OVER A p-ADIC FIELD

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Dedicated to the memory of Joseph Shalika

Abstract. The character formulas of [42] are an early triumph in p-adic harmonic analysis, but, to date, the calculations underlying the formulas have not been available. In this paper, which should be viewed as a precursor of the forthcoming volume [7], we leverage modern technology (for example, the Moy–Prasad theory) to compute explicit character tables. An interesting highlight is the computation of the 'exceptional' supercuspidal characters, i.e., those depth-zero representations not arising by inflation–induction from a Deligne–Lusztig representation of finite SL_2; this provides a concrete application for the recent work of DeBacker and Kazhdan [17].

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1. Introduction

1.1. **History.** Supercuspidal representations of reductive p-adic groups were discovered by F. Mautner in the late 1950s. In fact, one of us (Sally) heard him lecture

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on his discovery at Brandeis in 1959. His construction is contained in the following theorem, which appeared in the *American Journal of Mathematics* in 1964. The notation is explicated in the body of the paper.

Theorem 1.1 ([31, Theorem 9.1])**.** *Let $G = \mathrm{PGL}_2(k)$ and $K = \mathrm{PGL}_2(R)$, where k is a p-adic field and R its ring of integers, and write*

$$N = \left\{ \left(\begin{smallmatrix} 1 & 0 \\ x & 1 \end{smallmatrix}\right) \bigm| x \in R \right\}.$$

Let u be an irreducible, unitary representation of K whose restriction to N does not contain the trivial representation. Let $U = \mathrm{Ind}_K^G u$ be the induced representation (compact induction). Then U is the direct sum of a finite number of irreducible, unitary representations $U^{(j)}$ of G. In a suitable orthonormal basis, the matrix coefficients of each $U^{(j)}$ are functions of compact support on G.

In 1963, I. M. Gel'fand and M. I. Graev published a paper in *Uspekhi* [20] in which they studied the representation theory of p-adic SL_2. Their methods and presentation hewed closely to those used in the study of the discrete series of real SL_2, and did not make use of induction from compact, open subgroups. Their realization of the discrete series can be directly compared to the representation constructed by A. Weil [54], as shown by S. Tanaka in [52]. Gel'fand and Graev presented formulas for the discrete-series characters [20, §§5.3, 5.4] and recovered the Plancherel formula (§6.2 *loc. cit.*), but they made some errors concerning irreducibility of the discrete series and unitary equivalence.

In his thesis [46, §§1.5–1.9], J. A. Shalika constructed the supercuspidal representations of p-adic SL_2 by using the Weil representation. He then proved their irreducibility and identified the equivalences among them in complete detail. In addition, he showed (§§3, 4 *loc. cit.*) that each supercuspidal representation could be induced from a maximal compact subgroup by restricting, picking out an irreducible component, and inducing back up.

The existence of a Frobenius-type inducing formula for supercuspidal characters was shown by T. Shintani [49, Theorem 3], who worked with a group of square matrices over a p-adic field whose determinant is a unit in the ring of integers. In 1968, Sally and Shalika used such a formula (see [45, Theorem 1.9]) to compute the irreducible characters of the supercuspidal representations of SL_2 [42]. Their formulas have a sign error in a few cases, but their later papers [43, 44] are not affected. See Remark 14.15 for details.

There were several additional expositions related to the discrete-series characters of rank-one groups over a p-adic field.

- A. Silberger [50] computed characters for PGL_2 by a type of limit formula;
- H. Jacquet and R. P. Langlands [24] used the information from Sally–Shalika to analyze supercuspidal characters for GL_2;
- H. Shimizu [48] published some character computations for GL_2; and
- Sally gave an example of the inducing construction for SL_2 in [41].

1.2. **Outline.** The aim of this paper is to provide a complete guide for computing the supercuspidal characters of SL_2 over a p-adic field. As discussed in §1.1, these characters have been available in some form at least since the 1960s [20, 42]. Over the intervening half-century, significant advances have been made in our understanding of both reductive p-adic groups and their representation theory. A goal of this paper is to bring these modern, general tools to bear on the problem of explicit character computation.

The formulas in [42] were found by using [46, Theorem 3.1.2] to recognize the supercuspidal representations of $G = \mathrm{SL}_2(k)$ as induced (in the sense of §3.1 *loc. cit.*) from (finite-dimensional) representations of maximal compact subgroups of G, and then employing a p-adic analogue of the Frobenius formula [45, Theorem 1.9]. Broadly speaking, this paper follows the same path. After establishing some basic notation (§§2–7), we discuss how to construct all supercuspidal representations of G (§§8–12), and then finish by computing the characters of these representations (§§13–15).

We begin by establishing some basic facts and notation about fields (§2) and tori (§3). After examining a certain principal-value integral that will appear in the character formulas (§4), we discuss in §5 the pioneering work of Bruhat–Tits [8, 9] and Moy–Prasad [33, 34]. Bruhat–Tits theory underlies nearly everything that we do; indeed, without it, we would not have the language in which to state many of our results. In order not to require of the reader familiarity with the general notions of Bruhat–Tits theory, we specialize them to SL_2, where the group filtrations can be described very concretely (see §5.1), and related to filtrations of tori (see §5.2) and, via the Cayley map, of the Lie algebra (see §5.3). The Cayley map has many of the properties of the exponential map (see Lemma 5.4), but can converge on a larger domain.

After summarizing our choices for normalization of measures (§6)—we will usually use Waldspurger's normalization, adapted to the structure theory of Bruhat–Tits and Moy–Prasad—and discussing the Fourier transform (§7), we turn our attention to the problem of classifying all supercuspidal representations of G. We do this via the theory of types, reviewed in §8. An unrefined minimal K-type is a certain pair (K, ξ) consisting of a compact, open subgroup K and a representation ξ of it. Every irreducible representation of a p-adic group contains an unrefined minimal K-type, unique up to a natural relation (see Definition 8.1).

For SL_2, the unrefined minimal K-type contained in a representation is sufficient to determine whether that representation is supercuspidal; we list all those that can occur in a supercuspidal representation. The final task is to determine from a given unrefined minimal K-type *all* possible supercuspidal representations that contain it. For depth-zero, unrefined minimal K-types, the above plan is carried out in §9; and for positive depth representations it is carried out in §10.

The calculations of [42] were long, involved, and, since the state of the art in structure theory of p-adic groups then (1968) was not nearly so advanced as it is now, somewhat *ad hoc*; and, perhaps most importantly, they have never appeared in print. In the final sections of this paper, we justify the calculations of [42], making use of modern technology whenever it simplifies matters. Two particularly powerful references that are available to us are [16], which handles all but four of the depth-zero, supercuspidal representations; and [6], which handles all positive-depth supercuspidal representations. The remaining four supercuspidals, which we call 'exceptional' (see §15), require a bit more care; but, even in this case, most of the necessary work has already been done, by Waldspurger [53] and DeBacker–Kazhdan [17], and there remains only the (non-trivial!) task of specializing this work to the case of SL_2. See §1.4 for a summary of the results.

In future work [7], the present authors, together with Alan Roche, will continue this program to present a complete picture of harmonic analysis on p-adic SL_2. One of our goals will be to make more accessible some of the general tools that have been developed over the last fifteen years by specializing them to the case of SL_2. Thus, rather than citing major theorems (such as the Bernstein decomposition theorem, or the main theorems of Moy–Prasad), we will prove them in this case wherever doing so has illustrative value. We will construct all irreducible representations and compute their characters. We will treat the principal series in an old-fashioned way, via intertwining operators, and also via the theory of types. We will construct the unitary, smooth, and tempered duals, and describe the discrete series. We will compute the Fourier transforms of nilpotent orbital integrals on the Lie algebra, descibe the local character expansions of all irreducible representations, and compute the Plancherel measure.

1.3. **General notation.** If S is a ring, then we denote by $S^\times$ the group of units in S.

Suppose that k is a non-discrete, non-Archimedean local field with normalized valuation ord (i.e., $\operatorname{ord}(k) = \mathbb{Z} \cup \{+\infty\}$). Let R denote the ring of integers in k and $\wp$ the prime ideal of R. Fix an element $\epsilon \in R^\times \setminus (R^\times)^2$ and a uniformizer $\varpi \in R$.

Let $\mathfrak{f}$ denote the residue field $R/\wp$ of k. Then the image in $\mathfrak{f}^\times$ of ϵ is a non-square in $\mathfrak{f}^\times$. We write $p = \operatorname{char}(\mathfrak{f})$ and $q = \left|\mathfrak{f}\right|$, and assume throughout that $p \neq 2$.

Definition 1.2. If Λ is an (additive) character of k and $b \in k$, then write Λ_b for the additive character given by $t \longmapsto \Lambda(bt)$. If Λ is non-trivial, then the *depth* $\mathrm{d}(\Lambda)$ of Λ is the smallest index $r \in \mathbb{R}$ such that Λ is trivial on $\wp^{\lfloor r \rfloor + 1}$. (If Λ *is* trivial, then we may define $\mathrm{d}(\Lambda) = -\infty$.)

Note that, for Λ an (additive) character of k and $b \in k$, we have

$$\mathrm{d}(\Lambda_b) = \mathrm{d}(\Lambda) - \operatorname{ord}(b). \tag{1.3}$$

We fix, for the remainder of the paper, an additive character Λ of depth 0. Explicitly, Λ is trivial on $\wp$, but not on R. We will use boldface letters to denote algebraic groups, boldface Fraktur letters to denote their Lie algebras, and the corresponding regular letters to denote their groups of rational points. For example, $T = \mathbf{T}(k)$ and $\mathfrak{t} = \operatorname{Lie}(T)$.

Put $\mathbf{G} = \mathrm{SL}_2$. Thus, by our convention, $G = \mathbf{G}(k) = \mathrm{SL}_2(k)$ is the subgroup of all determinant-one matrices in the group $\mathrm{GL}_2(k)$ of invertible 2×2 matrices, and $\mathfrak{g} = \mathfrak{sl}_2(k)$ is the subalgebra of trace-zero matrices in the the Lie algebra $\mathfrak{gl}_2(k)$ of 2×2 matrices over k.

When we are dealing with complicated exponents, we will sometimes write $\exp_t(s)$ instead of t^s, for $t \in \mathbb{R}_{>0}$ and $s \in \mathbb{C}$.

As mentioned, our calculations use rather general results in p-adic harmonic analysis, which, in most cases, have been proven only subject to some restrictions. We discuss those restrictions now.

Since

- $\mathbf{G}$ is split, hence tame;
- $\mathbf{G}$, which is its own derived group, is simply connected; and
- the only bad prime for SL_2 (in the sense of [5, Definition A.5]) is 2,

we have by [5, Remark 1.2] that all the hypotheses of §1 *loc. cit.* hold. We shall demonstrate explicitly that [6, Hypothesis 2.3] holds; see Notations 9.7 and 10.17.

The next hypothesis is only needed when we cite [16, Lemma 12.4.3] (our Lemma 13.10), i.e., in the depth-zero cases of Proposition 13.13 and Theorems 14.20 and 15.2. Although it is possible to remove this restriction in our setting, we have not done so here.

Hypothesis 1.4 ([16, Restriction 12.4.1(2)]). The characteristic of k is 0, and the residual characteristic satisfies $p \geq 2e+3$, where e is the absolute ramification degree of k (i.e., its ramification degree over $\mathbb{Q}_p$).

1.4. **Character formulas.** In this section, we summarize the character values computed in this paper. The formulas which occur use a large amount of notation that has not been defined yet; it is described in detail in §§3–7.

We adopt the parametrization of supercuspidal representations presented in Theorem 11.1. By Remarks 9.8 and 10.19, we may, and do, restrict our attention to tori of the form $T^\theta = T^{\theta,1}$ for some $\theta \in \{\epsilon, \varpi, \epsilon\varpi\}$.

As described in [13], computations of characters of p-adic groups have indicated that, broadly speaking, they have a 'geometric part' near the identity, where they are described in terms of functions associated to (co)adjoint orbits [4, 11, 12, 25, 26, 35–38], and an 'arithmetic part' far from the identity, where they are described by some analogue of Weyl's classical character formula. (Actually, [6, Theorem 7.1] shows that one should usually expect *mixed* arithmetic–geometric behavior, even for supercuspidal characters; but there is a clean separation in the case of SL_2.) To make precise the notion of being near or far from the identity, we use Moy–Prasad's notion of depth; see Definitions 3.5 and 8.3. Specifically, the geometric part of the character of a representation π applies to those elements γ such that $\mathrm{d}(\gamma) > \mathrm{d}(\pi)$, whereas the arithmetic part applies to those elements γ such that $\mathrm{d}(\gamma) < \mathrm{d}(\pi)$. In the intermediate range, where $\mathrm{d}(\gamma) = \mathrm{d}(\pi)$, the character exhibits qualitatively different behavior, related to special functions on p-adic and finite fields (see [51, §7]). We call this range the 'bad shell'; the terminology 'shell' comes from the fact that the depth is the analogue of the valuation on a p-adic field, so that the set of elements of fixed depth may be thought of as an analogue of the set difference of two p-adic balls.

All of the available quantitative information about supercuspidal characters comes from the evaluation of Harish-Chandra's integral formula (13.7). The integral here is taken over the full group G, which is far too large to handle directly; so the main focus in evaluating it is on finding many sub-integrals that equal 0. The remaining terms can then often be related to calculations on a finite field, or a finite group of Lie type. Few details of this part of the character computation are included in the present paper; we refer instead to [6, 16], whose general results we take for granted. The challenge is to interpret these general results as explicitly as possible for the special case of SL_2.

[6, Proposition 5.3.3] defines a crucial ingredient in the Weyl-sum-type formula that gives the arithmetic part of a supercuspidal character; it is a fourth root of unity called a *Gauss sum* (see §5.2 *loc. cit.*). The explicit formula of Proposition 5.2.13 *loc. cit.* describes this fourth root of unity in terms of the Galois action on (absolute) roots. The most technically demanding part of our positive-depth character computations is probably the specialization of this explicit description to our setting; see §14.3. With this in place, we compute the order of a certain coset space, which turns out to be a Weyl discriminant (see Lemma 14.3), to complete our explicit description of the arithmetic part of an ‘ordinary’ (positive-depth) supercuspidal character, in the sense of Definition 10.13.

From the point of view of this paper, the geometric part of the character is a combination of Fourier transforms of semisimple orbital integrals; this is Murnaghan’s version of Kirillov theory (see §13). In the ‘ordinary’ case (see Definitions 9.6 and 10.13), there is only one orbital integral involved; but, in the ‘exceptional’ case, the situation is more complicated. See (13.12). Thus, once we have identified the coefficients occurring in the combination (in particular, the formal degree; see Lemma 14.4), we recall the results of [51, Theorem 11.3] on semisimple orbital integrals to complete the explicit description of the geometric part of the character formula.

We summarize all this below; but, for the sake of brevity, we take some shortcuts. In this section, the letter γ always stands for a regular, semisimple element of G. That is, γ is always a noncentral semisimple element of G. Second, when we write, for example,

$$\Theta_{\pi^\pm}(\gamma) = \frac{1}{2}\Big\{\frac{1}{|D_G(\gamma)|^{1/2}} - 1\Big\}, \quad \gamma \in A_{0+},$$

we are really giving the character value on any G-conjugate of an element of A_{0+}. In this way, we describe the characters ‘shell by shell’. Further, since the central character of $\pi^\pm(T,\psi)$ or $\pi(T,\psi)$ is $\psi|_{Z(G)}$, we have that $\Theta_{\pi^\pm}(z\gamma) = \psi(z)\Theta_\pi(\gamma)$ for all γ and all $z \in Z(G)$, and similarly for $\Theta_{\pi(T,\psi)}(z\gamma)$. That is, the formula above really gives the character value on any G-conjugate of an element of $Z(G)A_{0+}$. Thus, the term ‘otherwise’ in the character formulas below should be understood to mean, not just (for example) that $\gamma \notin A_{0+}$, but in fact that $\gamma \notin Z(G)\cdot\mathrm{Int}(G)A_{0+}$.

From Theorem 14.14, Theorem 14.18, and Theorem 14.20, in the unramified case (see Definition 14.1), a supercuspidal representation $\pi = \pi(T^\epsilon,\psi)$ of depth r has character

$$\Theta_\pi(\gamma) = \begin{cases} \frac{1}{2}\operatorname{sgn}_\epsilon\bigl(\operatorname{Im}_\epsilon(\gamma)\bigr)\dfrac{\psi(\gamma)+\psi(\gamma^{-1})}{|D_G(\gamma)|^{1/2}}\bigl[(-1)^{r+1} + H(\Lambda',k_\epsilon)\bigr] & \gamma \in T^\epsilon \setminus Z(G)T^\epsilon_{r+} \\ c_0(\pi) + H(\Lambda',k_\epsilon)\dfrac{\operatorname{sgn}_\epsilon\bigl(\eta^{-1}\operatorname{Im}_\epsilon(\gamma)\bigr)}{|D_G(\gamma)|^{1/2}} & \gamma \in T^{\epsilon,\eta}_{r+} \\ c_0(\pi) + \dfrac{1}{|D_G(\gamma)|^{1/2}} & \gamma \in A_{r+} \\ c_0(\pi) & \text{otherwise, if } \gamma \in G_{r+} \\ 0 & \text{otherwise, if } \gamma \notin G_{r+}. \end{cases}$$

Here $\eta \in \{1,\varpi\}$, and $c_0(\pi) = -q^r$.

From Theorem 14.14 (along with Lemma 14.11), Theorem 14.19, and Theorem 14.20, for a ramified supercuspidal representation $\pi = \pi(T^\varpi,\psi)$ of depth r we have

$$\Theta_\pi(\gamma)=\begin{cases} \dfrac{\mathrm{sgn}_\varpi\big(\mathrm{Im}_\varpi(\gamma)\big)H(\Lambda',k_\varpi)}{|D_G(\gamma)|^{1/2}}\Big\{\psi(\gamma) \ + \ \psi(\gamma^{-1})\big[\dfrac{\mathrm{sgn}_\varpi(-1)+1}{2}\big]\Big\} & \\ & \gamma\in T^\theta\setminus Z(G)T^\theta_r \\ \dfrac{q^{-1/2}}{2\,|D_G(\gamma)|^{1/2}}\displaystyle\sum_{\substack{\gamma'\in(C_\varpi)_{r:r+}\\ \gamma'\neq\gamma^{\pm1}}}\mathrm{sgn}_\varpi\big(\mathrm{tr}_\varpi(\gamma-\gamma')\big)\psi(\gamma') & \\ \qquad + \quad \dfrac{1}{2}H(\Lambda',k_\varpi)\,\mathrm{sgn}_\varpi\big(\eta^{-1}\,\mathrm{Im}_\varpi(\gamma)\big)\dfrac{\psi(\gamma)+\psi(\gamma^{-1})}{|D_G(\gamma)|^{1/2}} & \\ & \gamma\in T^{\varpi,\eta}_r\setminus T^{\varpi,\eta}_{r+} \\ \dfrac{q^{-1/2}}{2\,|D_G(\gamma)|^{1/2}}\displaystyle\sum_{\gamma'\in(C_\varpi)_{r:r+}}\mathrm{sgn}_\varpi\big(\mathrm{tr}_{\epsilon\varpi}(\gamma)-\mathrm{tr}_\varpi(\gamma')\big)\psi(\gamma') & \gamma\in T^{\epsilon\varpi,\eta}_r\setminus T^{\epsilon\varpi,\eta}_{r+} \\ c_0(\pi)+H(\Lambda',k_\varpi)\dfrac{\mathrm{sgn}_\varpi\big(\eta^{-1}\,\mathrm{Im}_\varpi(\gamma)\big)}{|D_G(\gamma)|^{1/2}} & \gamma\in T^{\varpi,\eta}_{r+} \\ c_0(\pi)+\dfrac{1}{|D_G(\gamma)|^{1/2}} & \gamma\in A_{r+} \\ c_0(\pi) & \text{otherwise, if } \gamma\in G_{r+} \\ 0 & \text{otherwise, if } \gamma\notin G_{r+}. \end{cases}$$

Here $\eta\in\{1,\epsilon\}$ and $c_0(\pi)=-\frac{1}{2}(q+1)q^{r-1/2}$. To obtain character values for a ramified representation $\pi=\pi(T^{\epsilon\varpi},\psi)$, interchange the roles of ϖ and $\epsilon\varpi$ in the formulas above.

From Theorem 15.1 and Theorem 15.2, for the representations $\pi^\pm=\pi^\pm(T^\epsilon,\psi^1_0)$ (see Definition 9.6 for explanation of notation), we have

$$\Theta_{\pi^\pm}(\gamma)=\begin{cases} \dfrac{\mathrm{sgn}_\varpi(\gamma+\gamma^{-1}+2)}{2}\Big\{H(\Lambda',k_\epsilon)\dfrac{\mathrm{sgn}_\epsilon\big(\mathrm{Im}_\epsilon(\gamma)\big)}{|D_G(\gamma)|^{1/2}}-1\Big\} & \gamma\in T^\epsilon\setminus Z(G)T^\epsilon_{0+} \\ \dfrac{1}{2}\Big\{\dfrac{1}{|D_G(\gamma)|^{1/2}}-1\Big\} & \gamma\in A_{0+} \\ \dfrac{1}{2}\Big\{\pm H(\Lambda',k_{\theta'})\dfrac{\mathrm{sgn}_{\theta'}\big(\eta^{-1}\,\mathrm{Im}_{\theta'}(\gamma)\big)}{|D_G(\gamma)|^{1/2}}-1\Big\} & \gamma\in T^{\theta',\eta}_{0+}, \\ 0 & \text{otherwise.} \end{cases}$$

Here $\eta\in\{1,\epsilon\}$ if $\theta'\in\{\varpi,\epsilon\varpi\}$ and $\eta\in\{1,\varpi\}$ if $\theta'=\epsilon$.

Our character formulas for $\pi^\pm=\pi^\pm(T,\psi)$ agree with those of [42] for $\Pi^\pm(\Lambda'_\pi,\psi,k_\epsilon)$. Similarly, our character formulas for $\pi=\pi(T^\theta,\psi)$ agree with those of [42] for $\Pi(\Lambda'_\pi,\psi,k_\theta)$, *except* that, far from the identity (see Theorem 14.14), they differ by a sign in the ramified case when $\mathrm{sgn}_\varpi(-1)=-1$ and the conductor $h=r+1/2$ is odd. See Remark 14.15.

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2. Field extensions

Since $p \neq 2$, every quadratic extension of k is of the form $k_\theta := k(\sqrt{\theta})$ for some non-square $\theta \in k^\times$. Let $\mathrm{Norm}_\theta \colon k_\theta^\times \longrightarrow k^\times$ and $\mathrm{tr}_\theta \colon k_\theta \longrightarrow k$ denote the relevant norm and trace homomorphisms, respectively. We write $C_\theta = \ker \mathrm{Norm}_\theta$ and $V_\theta = \ker \mathrm{tr}_\theta$, and denote by sgn_θ the unique, non-trivial character of $k^\times$ that is trivial on $\mathrm{Norm}_\theta(k_\theta^\times)$. In particular,

$$\mathrm{sgn}_\theta(x) = (-1)^{\mathrm{ord}(x)} \quad \text{for } x \in k^\times \tag{2.1}$$

if k_θ/k is unramified, and

$$\begin{aligned} &\mathrm{sgn}_\theta(\theta) = \mathrm{sgn}_\theta(-1) \\ &\mathrm{sgn}_\theta(x) = \mathrm{sgn}_{R^\times}(x) \quad \text{for } x \in R^\times, \end{aligned} \tag{2.2}$$

where $\mathrm{sgn}_{R^\times}$ is the quadratic character of $R^\times$, if k_θ/k is ramified. Let R_θ, $\wp_\theta$, and $\mathfrak{f}_\theta$ denote the analogues for k_θ of R, $\wp$, and $\mathfrak{f}$, respectively.

For $\alpha = a + b\sqrt{\theta} \in k_\theta$, we write $\mathrm{Re}_\theta(\alpha) = a$ and $\mathrm{Im}_\theta(\alpha) = b$.

Write ord_θ for the valuation on k_θ that extends the one on k. In particular, $\mathrm{ord}_\theta(\sqrt{\theta}) = \frac{1}{2}\mathrm{ord}(\theta)$.

Lemma 2.3. *The map* $\mathrm{c} \colon k_\theta \setminus \{2\} \longrightarrow k_\theta \setminus \{-1\}$ *given by* $\mathrm{c}(x) = (1 + \frac{x}{2})/(1 - \frac{x}{2})$ *is a* $\mathrm{Gal}(k_\theta/k)$*-equivariant bijection, and restricts to a bijection* $V_\theta \cap \wp_\theta \longrightarrow C_\theta \cap (1 + \wp_\theta)$*. For* $Y \in \wp_\theta^n$*, with* $n > 0$*, we have that* $\mathrm{c}(Y) \equiv 1 + Y \pmod{\wp_\theta^{2n}}$.

Lemma 2.4. *For* $\alpha \in k_\theta$*, we have*

$$\mathrm{ord}(\mathrm{Im}_\theta(\alpha)) \geq \mathrm{ord}_\theta(\alpha) - \tfrac{1}{2}\mathrm{ord}(\theta). \tag{$*$}$$

If $\mathrm{ord}_\theta(\alpha) \geq \mathrm{ord}_\theta(\alpha - t)$ *for all* $t \in k$*, then we have equality in* $(*)$*; and if, further,* $\alpha \in \wp_\theta$*, then*

$$\mathrm{Im}_\theta\bigl(\mathrm{c}(\alpha)\bigr) \equiv \mathrm{Im}_\theta(\alpha) \pmod{1 + \wp}.$$

Proof. Write $d = \mathrm{ord}_\theta(\alpha)$, and $\alpha = a + b\sqrt{\theta}$. The inequality $(*)$ follows from the fact that $\mathrm{ord}_\theta(\alpha) = \min\bigl\{\mathrm{ord}(a), \mathrm{ord}(b) + \frac{1}{2}\mathrm{ord}(\theta)\bigr\}$.

If $\mathrm{ord}(b) > d - \frac{1}{2}\mathrm{ord}(\theta)$, then $\mathrm{ord}_\theta(\alpha - a) = \mathrm{ord}_\theta(b\sqrt{\theta}) > d$.

If $\mathrm{ord}_\theta(\alpha) \geq \mathrm{ord}_\theta(\alpha - t)$ for all $t \in k$, and $\alpha \in \wp_\theta$ (i.e., $d > 0$), then applying $(*)$ to $\mathrm{c}(\alpha) - (1 + \alpha)$ gives (by Lemma 2.3)

$$\mathrm{ord}\bigl(\mathrm{Im}_\theta(\mathrm{c}(\alpha)) - b\bigr) \geq 2d - \tfrac{1}{2}\mathrm{ord}(\theta) = \mathrm{ord}(b) + d.$$

Since $d > 0$, the result follows. □

Remark 2.5. As an illustration of the result, and reminder of our normalization of valuation, recall that $\mathrm{ord}_\varpi(\sqrt{\varpi}) = \frac{1}{2}$. Now fix an odd integer n and put $\alpha = \sqrt{\varpi}^n$. For all $t \in k$, $\mathrm{ord}_\varpi(\alpha - t) = \mathrm{ord}(t) < \mathrm{ord}_\varpi(\alpha)$ if $t \notin \wp^{(n+1)/2}$ and $\mathrm{ord}_\varpi(\alpha - t) = \mathrm{ord}_\varpi(\alpha)$ if $t \in \wp^{(n+1)/2}$. Then $\mathrm{Im}_\varpi(\alpha) = \varpi^{(n-1)/2}$, so that $\mathrm{ord}(\mathrm{Im}_\varpi(\alpha)) = \frac{1}{2}(n-1)$, and $\mathrm{ord}_\varpi(\alpha) = \frac{1}{2}n$; in particular, we have the equality $\mathrm{ord}(\mathrm{Im}_\varpi(\alpha)) = \mathrm{ord}_\varpi(\alpha) - \frac{1}{2}\mathrm{ord}(\varpi)$. Finally,

$$\mathrm{c}(\alpha) = 1 + \sqrt{\varpi}^n + 2(\sqrt{\varpi}^n)^2 + 4(\sqrt{\varpi}^n)^3 + \cdots,$$

so that

$$\mathrm{Im}_\varpi\bigl(\mathrm{c}(\alpha)\bigr) = \varpi^{(n-1)/2} + 4\varpi^{(3n-1)/2} + \cdots \equiv \varpi^{(n-1)/2} = \mathrm{Im}_\varpi(\alpha) \pmod{1 + \wp}.$$

The next two technical lemmas will come in handy in working out the arithmetic of the character formulas.

Notation 2.6. Write ψ_0 for the quadratic character of C_ϵ (so that $\psi_0^2 = 1$, but $\psi_0 \neq 1$).

We shall also use ψ_0 later for quadratic characters on related groups.

Lemma 2.7 ([44, Lemma A.3])**.** *For $\lambda \in C_\epsilon$,*

$$\psi_0(\lambda) = \begin{cases} -\operatorname{sgn}_\varpi(-1), & \lambda = -1 \\ \operatorname{sgn}_\varpi(\lambda + \lambda^{-1} + 2), & \textit{otherwise.} \end{cases}$$

Proof. Clearly,

$$\psi_0(1) = 1 = \operatorname{sgn}_\varpi(1 + 1^{-1} + 2).$$

Now note that ψ_0 takes the value $+1$ on squares in C_ϵ, and -1 on non-squares. In particular, since C_ϵ is the direct product of a cyclic group of order $q+1$ and a pro-p group,

$$\psi_0(-1) = (-1)^{(q+1)/2} = -(-1)^{(q-1)/2} = -\operatorname{sgn}_\varpi(-1).$$

Now fix $\lambda \in C_\epsilon \setminus k^\times = C_\epsilon \setminus \{\pm 1\}$. Since $C_\epsilon \subseteq (k_\epsilon^\times)^2$, we may write $\lambda = (c + d\sqrt{\epsilon})^2$, with $c, d \in k^\times$. In particular, $c^2 - d^2\epsilon \in \{\pm 1\}$; and λ is a square in C_ϵ if and only if $c^2 - d^2\epsilon = +1$. Now $\lambda + \lambda^{-1} + 2 = 2\bigl((c^2 + d^2\epsilon) + 1\bigr)$. If $c^2 - d^2\epsilon = +1$, then

$$\lambda + \lambda^{-1} + 2 = 2\bigl((c^2 + d^2\epsilon) + (c^2 - d^2\epsilon)\bigr) = (2c)^2 \in (k^\times)^2.$$

If $c^2 - d^2\epsilon = -1$, then

$$\lambda + \lambda^{-1} + 2 = 2\bigl((c^2 + d^2\epsilon) - (c^2 - d^2\epsilon)\bigr) = (2d)^2\epsilon \notin (k^\times)^2. \qquad \square$$

Our next lemma discusses traces of norm-one elements, for use in Theorem 14.19.

Lemma 2.8. *If θ is a non-square and $X \in V_\theta \cap \wp_\theta$, then*

$$\operatorname{ord}\bigl(\operatorname{tr}_\theta(\mathrm{c}(X)) - (2 + X^2)\bigr) \geq 4 \operatorname{ord}_\theta(X).$$

Proof. Note that $X^2 \in k$ and $\bar{X} = -X$. By direct computation and Lemma 2.3,

$$\begin{aligned} \operatorname{tr}_\theta\bigl(\mathrm{c}(X)\bigr) &= \mathrm{c}(X) + \overline{\mathrm{c}(X)} = \mathrm{c}(X) + \mathrm{c}(\bar{X}) = \mathrm{c}(X) + \mathrm{c}(-X) \\ &= 2\mathrm{c}\bigl(X^2/2\bigr) \equiv 2 + X^2 \pmod{\wp^{2\operatorname{ord}(X^2)} = \wp^{4\operatorname{ord}_\theta(X)}}, \end{aligned}$$

where $Z \longmapsto \bar{Z}$ is the non-trivial element of $\operatorname{Gal}(k_\theta/k)$. The result follows. $\square$

3. Tori

3.1. **Standard tori and normalizers.** Every maximal k-torus in $\mathbf{G}$ is G-conjugate either to the k-split torus $\mathbf{A} := \bigl\{ \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) \bigm| ad = 1 \bigr\}$, or to an elliptic k-torus (discussed below). The quotient $N_{\mathbf{G}}(\mathbf{A})/\mathbf{A}$ has order 2, with the non-trivial coset represented by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in G$.

We start by defining a few model elliptic tori.

Notation 3.1. For $\beta, \theta, \eta \in k^\times$, define

$$X_\beta^{\theta,\eta} = \begin{pmatrix} 0 & \beta\eta^{-1} \\ \beta\theta\eta & 0 \end{pmatrix}$$

and

$$\mathbf{T}^{\theta,\eta} = \left\{ \begin{pmatrix} a & b\eta^{-1} \\ b\theta\eta & a \end{pmatrix} \;\middle|\; a^2 - b^2\theta = 1 \right\}.$$

Then the Lie algebra $\mathbf{t}^{\theta,\eta}$ of $\mathbf{T}^{\theta,\eta}$ is the 1-dimensional vector space spanned by $X_1^{\theta,\eta}$.

We will only use this notation for θ a non-square, since otherwise $\mathbf{T}^{\theta,\eta}$ is k-split, and thus G-conjugate to $\mathbf{A}$. We write $\mathbf{T}^\theta$ for $\mathbf{T}^{\theta,1}$. There is a natural way to view the torus $\mathbf{T}^\epsilon$ as an $\mathfrak{f}$-group. To emphasise when we are doing so, we shall denote it by T^ϵ.

We have that $T^{\theta,\eta}$ is isomorphic to C_θ, and its Lie algebra $\mathfrak{t}^{\theta,\eta} = \left\{ X_b^{\theta,\eta} \;\middle|\; b \in k \right\}$ is isomorphic to V_θ, in each case via the map $\left(\begin{smallmatrix} a & b\eta^{-1} \\ b\theta\eta & a \end{smallmatrix}\right) \longmapsto a + b\sqrt{\theta}$. We shall therefore freely write $\operatorname{Im}_\theta(\gamma)$ (respectively, $\operatorname{Im}_\theta(Y)$) for $\gamma \in T^{\theta,\eta}$ (respectively, $Y \in \mathfrak{t}^{\theta,\eta}$).

To determine all G-conjugacy classes of elliptic maximal tori, we use the fact that such a torus is determined up to stable conjugacy, or, equivalently, $\mathrm{GL}_2(k)$-conjugacy, by the k-isomorphism class of its splitting field; and that the splitting field of $\mathbf{T}^{\theta,\eta}$ is k_θ. To represent all $\mathbf{G}$-conjugacy classes of elliptic maximal tori, we thus only need to consider values of θ in $\{\epsilon, \varpi, \epsilon\varpi\}$. We shall call a torus *standard* if it is the split torus $\mathbf{A}$, or else of the form $\mathbf{T}^{\theta,\eta}$, with θ as above.

If -1 is not a norm from k_θ, then $N_{\mathbf{G}}(\mathbf{T}^\theta) = \mathbf{T}^\theta$. If $-1 = \operatorname{Norm}_\theta(a + b\sqrt{\theta})$, then $N_{\mathbf{G}}(\mathbf{T}^\theta)/\mathbf{T}^\theta$ has order 2, with the non-trivial coset represented by $\left(\begin{smallmatrix} a & b \\ -b\theta & -a \end{smallmatrix}\right) \in G$. The first case applies if k_θ/k is ramified and -1 is not a square in $\mathfrak{f}^\times$; and the second otherwise (in particular, if k_θ/k is unramified).

Thus, there are two distinct stable conjugacy classes of ramified tori, represented by the tori $\mathbf{T}^\theta$, with $\theta \in \{\varpi, \epsilon\varpi\}$. If -1 is a square in $\mathfrak{f}^\times$, then the stable conjugacy class of $\mathbf{T}^\theta$ splits into two distinct G-conjugacy classes, represented by $\mathbf{T}^{\theta,1}$ and $\mathbf{T}^{\theta,\epsilon}$; otherwise, it is a single G-conjugacy class. In particular, $\mathbf{T}^{\theta,1}$ and $\mathbf{T}^{\theta,\epsilon}$ are G-conjugate in the latter case.

There is a single stable conjugacy class of unramified elliptic maximal k-tori, represented by $\mathbf{T}^\epsilon$. It splits into two distinct G-conjugacy classes, represented by $\mathbf{T}^{\epsilon,1}$ and $\mathbf{T}^{\epsilon,\varpi}$.

Notation 3.2. We write ψ_0^η for the quadratic character of $T^{\epsilon,\eta}$, with $\eta \in \{1, \varpi\}$, and ψ_0 for the quadratic character of $\mathsf{T}^\epsilon(\mathfrak{f})$.

Recall that the notation ψ_0 used below has already been used for a character of a subgroup of C_ϵ (see Notation 2.6). Since an isomorphism of $T^{\epsilon,\eta}$ with C_ϵ intertwines ψ_0^η and ψ_0, this is not a serious ambiguity.

3.2. **Torus filtrations.** Since our tori are isomorphic to subgroups of multiplicative groups of fields, they carry natural filtrations. We specify them explicitly below, since there are some normalization issues, as well as a subtlety to be handled in the depth-zero case.

Definition 3.3. We equip $k^\times$ with the filtration defined by $(k^\times)_0 = R^\times$ and $(k^\times)_n = 1 + \wp^n$ for $n \in \mathbb{Z}_{>0}$, and extend the indexing to $r \in \mathbb{R}_{\geq 0}$ by putting $(k^\times)_r = (k^\times)_{\lceil r \rceil}$. This may be transported, *via* either isomorphism $A \cong k^\times$, to a filtration $\left\{ A_r \;\middle|\; r \in \mathbb{R}_{\geq 0} \right\}$ of A. In particular, $A_0 = \mathrm{SL}_2(R) \cap A$ is the maximal compact subgroup of A.

Any elliptic torus T is conjugate to one of the form $T^{\theta,\eta}$, hence isomorphic to $C_\theta \subseteq k_\theta^\times$. We impose a filtration on $k_\theta^\times$ similar to the one on $k^\times$, except that we omit the maximal term $(k_\theta^\times)_0$, and adjust the indexing to account for ramification. Specifically, if k_θ/k is ramified, then we put $(k_\theta^\times)_{n/2} = 1 + \wp_\theta^n$ for $n \in \mathbb{Z}_{>0}$; whereas, if k_θ/k is unramified, then we put $(k_\theta^\times)_n = 1 + \wp_\theta^n$ for $n \in \mathbb{Z}_{>0}$. As in the split case, we extend the indexing of the filtration to $r \in \mathbb{R}_{>0}$. We obtain a filtration of C_θ by intersection, and then transport it to T to obtain a filtration $\left\{ T_r \;\middle|\; r \in \mathbb{R}_{>0} \right\}$ of T.

Note, however, that we have not yet defined the notation T_0. If T is unramified (i.e., we may take $\theta = \epsilon$), then we put $T_0 = T$, which is the maximal compact subgroup of T. If T is ramified (i.e., we may take $\theta \in \{\varpi, \epsilon\varpi\}$), then we put $T_0 = \bigcup_{r>0} T_r$. (In the notation of the next paragraph, this is T_{0+}.) This is no longer the maximal, compact subgroup of T, but rather an index-2 subgroup. Specifically, $T = Z(G)T_0$.

For any torus H and real number $r \geq 0$, we write H_{r+} for $\bigcup_{s>r} H_s$.

We define filtrations in a similar way (including the adjustment for ramification) on the Lie algebras of maximal tori in G. These filtrations are defined for all $r \in \mathbb{R}$ (not just $r \geq 0$), and the case $r = 0$ no longer needs to be treated separately.

Definition 3.4. If T is a torus, and ψ a character of T, then the *depth* $\mathrm{d}(\psi)$ of ψ is the smallest index $r \in \mathbb{R}_{\geq 0}$ such that the restriction of ψ to T_{r+} is trivial.

Definition 3.5. Let T be a torus, with Lie algebra $\mathfrak{t}$. If Y is a regular, semisimple element of $\mathfrak{t}$ (respectively, γ is a regular, semisimple element of T), then we define the *depth* $\mathrm{d}^{\mathfrak{g}}(Y)$ of Y (respectively, the *depth* $\mathrm{d}^G(\gamma)$ of γ) to be the smallest index $r \in \mathbb{R}$ such that $Y \notin \mathfrak{t}_{r+}$ (respectively, $\gamma \notin T_{r+}$). If it is clear from the context, then we will drop the superscript G or $\mathfrak{g}$. It is also convenient to define the *maximal depth* $\mathrm{d}_+(\gamma)$ of γ to be $\max\left\{ \mathrm{d}(\gamma z) \;\middle|\; z \in Z(G) \right\}$.

Remark 3.6. For example, $\mathrm{d}^{\mathfrak{g}}(X_\beta^{\theta,\eta}) = \mathrm{ord}(\beta) + \frac{1}{2}\mathrm{ord}(\theta)$. Thus, for all $r \in \mathbb{R}$,

$$\mathfrak{t}_r^{\epsilon,\eta} = \varpi^{\lceil r \rceil} R \cdot X_1^{\epsilon,\eta} \quad \text{for } \eta \in \{1, \varpi\}$$

and

$$\mathfrak{t}_{r+1/2}^{\theta,\eta} = \varpi^{\lceil r \rceil} R \cdot X_1^{\theta,\eta} \quad \text{for } \theta \in \{\varpi, \epsilon\varpi\} \text{ and } \eta \in \{1, \epsilon\}.$$

We will see below that the formulas for the supercuspidal characters that we consider depend on the relative depths of a (linear) character of a torus, and an element of that torus. Another basic function is the *Weyl discriminant.*

Definition 3.7. The functions $D_G \colon G \longrightarrow k$ and $D_{\mathfrak{g}} \colon \mathfrak{g} \longrightarrow k$ are defined by letting $D_G(\gamma)$ and $D_{\mathfrak{g}}(Y)$ be the coefficients of the degree-one terms in the characteristic polynomials of $\mathrm{Ad}(\gamma) - 1$ and $\mathrm{ad}(Y)$, respectively, for $\gamma \in G$ and $Y \in \mathfrak{g}$. Concretely,

$$D_G \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a+d)^2 - 4 \quad \text{and} \quad D_{\mathfrak{g}} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = 4(a^2 + bc).$$

An element $Y \in \mathfrak{g}$ (respectively, $\gamma \in G$) is *regular semisimple* if and only if $D_{\mathfrak{g}}(Y) \neq 0$ (respectively, $D_G(\gamma) \neq 0$). We write $\mathfrak{g}^{\mathrm{rss}}$ and G^{rss} for the appropriate sets of regular, semisimple elements.

Lemma 3.8. *The discriminant of a regular, semisimple element of $\mathfrak{g}$ or G with eigenvalues λ and λ' is $(\lambda - \lambda')^2$. If $Y \in \mathfrak{g}^{\mathrm{rss}}$ and $\gamma \in G^{\mathrm{rss}} \cap G_0$, then*

$$|D_{\mathfrak{g}}(Y)| = q^{-2\mathrm{d}(Y)} \quad \textit{and} \quad |D_G(\gamma)| = q^{-2\mathrm{d}_+(\gamma)}.$$

Proof. Note that

- neither $D_{\mathfrak{g}}$ nor D_G is affected by passage to a field extension;
- $D_{\mathfrak{g}}$ (respectively, D_G) is invariant under the adjoint (respectively, conjugation) action of G; and
- neither $\mathrm{d}^{\mathfrak{g}}$, nor the restriction of d^G to G_0, is affected by passage to a quadratic extension [5, Lemma 2.9].

Thus, it suffices to prove the result for $Y \in \mathfrak{a}$ and $\gamma \in A_0$. We shall only consider the group case; the Lie-algebra case is similar.

By Definition 3.7, if $\gamma = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$, then

$$D_G(\gamma) = (\lambda + \lambda^{-1})^2 - 4 = (\lambda - \lambda^{-1})^2,$$

as desired. If $\gamma \in A_0$, then $\lambda \in R^\times$, so $D_G(\gamma) \in R$; and $D_G(\gamma) \in \wp$ if and only if $\lambda \equiv \pm 1 \pmod{1 + \wp}$, in which case $\gamma \in Z(G)A_{0+}$. If $\gamma \in A_{0+}$, then we may write $\lambda = \mathrm{c}(X)$ for some $X \in \wp$. By Lemma 2.3 and direct calculation,

$$\lambda - \lambda^{-1} = \mathrm{c}(X) - \mathrm{c}(X)^{-1} = \mathrm{c}(X) - \mathrm{c}(-X) = \frac{2X}{1 - (X/2)^2},$$

so that

$$\mathrm{ord}\big(D_G(\gamma)\big) = 2\,\mathrm{ord}(\lambda - \lambda^{-1}) = 2\,\mathrm{ord}(X) = 2\mathrm{d}_+(\gamma). \qquad \square$$

4. A principal-value integral

The character formulas of [42] involve a quantity

$$H(\Lambda', k_\theta) := \lim_{m \to -\infty} \int_{\varpi^m R_\theta} \Lambda'(\mathrm{Norm}_\theta(z))\mathrm{d}_{\Lambda'} z$$

(p. 1235 *loc. cit.* and [46, p. 11]), where Λ' is an additive character and $\mathrm{d}_{\Lambda'} z$ is the self-dual Haar measure on k_θ (with respect to the additive character Λ' and the trace pairing on k_θ), in the sense of Definition 7.5 below. We compute the normalization of measure, and the resulting integral, below.

Write r for the depth of Λ'.

The evaluation of the integral will require, as is usual in p-adic harmonic analysis, a fourth root of unity called a *Gauss sum.* We follow [46, p. 5] (see also [51, Definition 6.1]) in making the following definition.

Definition 4.1.

$$\mathcal{G}(\Lambda') := q^{-1/2} \sum_{X \in R/\wp} \Lambda'_{(-\varpi)^r}(X^2).$$

In [53, §I.4], Waldspurger works with an additive character ψ of depth 0. If $r = \mathrm{d}(\Lambda') = 0$, then [51, Lemma 6.2] gives that $\mathcal{G}(\Lambda')$ is the fourth root of unity denoted by $\varepsilon(\Lambda')$ in [53, §V.4].

Considerable information about the transformation laws for this root of unity are available in [51, Lemma 6.2], where our $\mathcal{G}(\Lambda')$ is denoted by $G_\varpi(\Phi)$.

Lemma 4.2.

$$\operatorname{meas}_{\mathrm{d}_{\Lambda'}z}(R_\theta) = \begin{cases} q^{r+1}, & \theta = \epsilon \\ q^{r+1/2}, & \theta = \varpi; \end{cases}$$

and

$$H(\Lambda', k_\theta) = \begin{cases} (-1)^{r+1}, & \theta = \epsilon \\ \mathcal{G}(\Lambda'), & \theta = \varpi. \end{cases}$$

By [51, Lemma 6.3], this formula agrees with the one given in [42, p. 1235].

Proof. Note that $H(\Lambda', k_\theta)$ is just what is called $\mathcal{H}(\Lambda', Q)$ in [46, p. 11], where $Q = \mathrm{Norm}_\theta$ is the norm form on k_θ. In particular, its definition involves a lattice in k_θ (although Lemma 1.5.2 *loc. cit.* shows that the choice does not matter). For definiteness, we take the lattice to be R_θ. By Lemma 1.5.1 *loc. cit.* and the following exposition, the computation of $\mathcal{H}(\Lambda', Q)$ begins with the identification of a Q-orthogonal R-basis for R_θ. In our setting,

$$\left\{\mathbf{x}_1 = 1, \mathbf{x}_2 = \sqrt{\theta}\right\}$$

will do. Note that we have

$$l_1 := Q(\mathbf{x}_1) = 1 \quad \text{and} \quad l_2 := Q(\mathbf{x}_2) = -\theta.$$

Our R-basis for R_θ is also a k-basis for k_θ, hence furnishes a k-isomorphism $k_\theta \cong k \oplus k$. By [46, pp. 5, 11], $\mathrm{d}_{\Lambda'}z$ is the pull-back along the above isomorphism of $\mathrm{d}x_1 \oplus \mathrm{d}x_2$, where

$$\operatorname{meas}_{\mathrm{d}x_1}(R) = q^{(\mathrm{d}(\Lambda'_{l_1})+1)/2} = q^{(r+1)/2}$$

and

$$\operatorname{meas}_{\mathrm{d}x_2}(R) = q^{(\mathrm{d}(\Lambda'_{l_2})+1)/2} = \begin{cases} q^{(r+1)/2}, & \theta = \epsilon \\ q^{r/2}, & \theta = \varpi \end{cases}$$

(by (1.3) and Definition 14.1). Since R_θ is the pull-back of $R \oplus R$, we have the indicated normalization.

By [46, p. 11], $\mathcal{H}(\Lambda', Q) = \mathcal{H}(\Lambda'_{l_1})\mathcal{H}(\Lambda'_{l_2})$, where the notation is as in Lemma 1.3.2 *loc. cit.*

In the unramified case, since $\operatorname{ord}(l_1) = \operatorname{ord}(1) = 0$ and $\operatorname{ord}(l_2) = \operatorname{ord}(-\epsilon) = 0$, we have by (1.3) that

$$\mathrm{d}(\Lambda'_{l_1}) = \mathrm{d}(\Lambda'_{l_2}) = r.$$

On the other hand,

$$\operatorname{sgn}_\varpi(l_1) = \operatorname{sgn}_\varpi(1) = 1 \quad \text{and} \quad \operatorname{sgn}_\varpi(l_2) = \operatorname{sgn}_\varpi(-\epsilon) = -\operatorname{sgn}_\varpi(-1).$$

If $\theta = \epsilon$ and $r = \mathrm{d}(\Lambda')$ is even, then, by [46, Lemma 1.3.2] and [51, Lemma 6.2],

$$\begin{aligned} \mathcal{H}(\Lambda', Q) &= \operatorname{sgn}_\varpi(l_1)\mathcal{G}(\Lambda') \cdot \operatorname{sgn}_\varpi(l_2)\mathcal{G}(\Lambda') \\ &= \operatorname{sgn}_\varpi(-\epsilon)\operatorname{sgn}_\varpi(-1) \\ &= -1 = (-1)^{r+1}. \end{aligned}$$

If $\theta = \epsilon$ and r is odd, then [46, Lemma 1.3.2] gives

$$\mathcal{H}(\Lambda', Q) = 1 \cdot 1 = 1 = (-1)^{r+1}.$$

In the ramified case, $\operatorname{ord}(l_1) = \operatorname{ord}(1) = 0$ and $\operatorname{ord}(l_2) = \operatorname{ord}(-\varpi) = 1$, so

$$\mathrm{d}(\Lambda'_{l_1}) = r \quad \text{and} \quad \mathrm{d}(\Lambda'_{l_2}) = r - 1$$

(meaning that exactly one is even); whereas

$$\operatorname{sgn}_\varpi(l_1) = \operatorname{sgn}_\varpi(1) = 1 \quad \text{and} \quad \operatorname{sgn}_\varpi(l_2) = \operatorname{sgn}_\varpi(-\varpi) = 1.$$

Thus, using [46, Lemma 1.3.2] and [51, Lemma 6.2] again, we obtain the desired formula for $\mathcal{H}(\Lambda', Q)$ (regardless of the parity of r). $\square$

5. The building and filtrations

In the 1990s, Moy and Prasad [33] used Bruhat–Tits theory to initiate a major advance in the study of the harmonic analysis of reductive p-adic groups. In particular, Adler [1], Yu [55], and Kim [27] built upon this foundation to provide a classification of all supercuspidal representations of a reductive p-adic group (under some tameness restrictions). In this paper, we will not assume any familiarity with Bruhat–Tits theory, but we shall use the notation of Moy and Prasad for certain objects that we will describe explicitly in §5.1.

5.1. **Lattices and filtrations.** Recall that a *lattice* in a finite-dimensional k-vector space is a compact, open R-submodule. For example, for each $r \in \mathbb{R}$,

$$\mathfrak{gl}_2(k)_{x_\mathrm{L},r} := \begin{pmatrix} \wp^{\lceil r \rceil} & \wp^{\lceil r \rceil} \\ \wp^{\lceil r \rceil} & \wp^{\lceil r \rceil} \end{pmatrix} \quad \text{and} \quad \mathfrak{gl}_2(k)_{x_\mathrm{C},r} := \begin{pmatrix} \wp^{\lceil r \rceil} & \wp^{\lceil r-1/2 \rceil} \\ \wp^{\lceil r+1/2 \rceil} & \wp^{\lceil r \rceil} \end{pmatrix}$$

are examples of lattices in $\mathfrak{gl}_2(k)$. In each case, if we allow r to vary, we obtain a filtration of $\mathfrak{gl}_2(k)$. Because these two filtrations have played an important role in the representation theory of both $\mathrm{SL}_2(k)$ and $\mathrm{GL}_2(k)$, they were often given their own special notations in the literature. For example, when $r \in \mathbb{Z}$, the first lattice above was often called $\mathfrak{k}_r$, and, when $r \in \frac{1}{2}\mathbb{Z}$, the second was often called $\mathfrak{b}_{2r}$.

We may view these two filtrations as part of a large family, indexed by elements x of the (reduced) Bruhat–Tits building $\mathcal{B}(\mathrm{GL}_2, k)$ [8, Définition 7.4.2]. We shall not concern ourselves with a description of the building; for the case $\mathbf{G} = \mathrm{GL}_2$, a point in the building essentially *is* a filtration as above [8, Proposition 10.2.10]. Thus, we may regard $\{x_\mathrm{L}, x_\mathrm{C}\}$ as a subset of $\mathcal{B}(\mathrm{GL}_2, k)$; it contains (up to $\mathrm{GL}_2(k)$-conjugacy) all the *optimal points* [33, §6.1], and so is, in a sense, "all that we need". When we are dealing with SL_2, we shall also require the point x_R with associated filtration

$$\mathfrak{gl}_2(k)_{x_\mathrm{R},r} := \operatorname{Int}\begin{pmatrix} 1 & 0 \\ 0 & \varpi \end{pmatrix} \mathfrak{gl}_2(k)_{x_\mathrm{L},r} = \begin{pmatrix} \wp^{\lceil r \rceil} & \wp^{\lceil r \rceil - 1} \\ \wp^{\lceil r \rceil + 1} & \wp^{\lceil r \rceil} \end{pmatrix}, \quad r \in \mathbb{R};$$

it is $\mathrm{GL}_2(k)$-, but not $\mathrm{SL}_2(k)$-, conjugate to x_L. We put $\mathcal{B}^\mathrm{opt} = \{x_\mathrm{L}, x_\mathrm{C}, x_\mathrm{R}\}$, and, for $(x, r) \in \mathcal{B}^\mathrm{opt} \times \mathbb{R}$, define

$$\mathfrak{g}_{x,r} = \mathfrak{sl}_2(k)_{x,r} = \mathfrak{sl}_2(k) \cap \mathfrak{gl}_2(k)_{x,r}.$$

Remark 5.1. For $x \in \mathcal{B}^\mathrm{opt}$ and $r, s \in \mathbb{R}$, we have that

$$\mathfrak{gl}_2(k)_{x,r} \cdot \mathfrak{gl}_2(k)_{x,s} \subseteq \mathfrak{gl}_2(k)_{x,r+s},$$

where $\cdot$ is the usual matrix multiplication.

In particular, the set of invertible elements of $\mathfrak{gl}_2(k)_{x,0}$ forms a group. This motivates the following definitions, for pairs $(x,r) \in \mathcal{B}^{\mathrm{opt}} \times \mathbb{R}$ with $r \geq 0$:

$$\begin{aligned} \mathrm{GL}_2(k)_{x,r} &:= \mathrm{GL}_2(k) \cap \mathfrak{gl}_2(k)_{x,r}, && r = 0 \\ \mathrm{GL}_2(k)_{x,r} &:= \mathrm{GL}_2(k) \cap (1 + \mathfrak{gl}_2(k)_{x,r}), && r > 0 \\ G_{x,r} = \mathrm{SL}_2(k)_{x,r} &:= \mathrm{SL}_2(k) \cap \mathrm{GL}_2(k)_{x,r}, && r \geq 0. \end{aligned}$$

Remark 5.2. Our definitions here seem rather *ad hoc*, but Moy and Prasad [33, 34] have shown how to fit them into a uniform framework that applies to all reductive, p-adic groups.

Next, we define a family of G-domains in G (i.e., open and closed subsets, invariant under the conjugation action of G) as follows. For $r \in \mathbb{R}_{\geq 0}$, put

$$G_r := \bigcup_{x \in \mathcal{B}^{\mathrm{opt}}} \mathrm{Int}(G) G_{x,r}.$$

We define $\mathfrak{g}_r$ similarly, for $r \in \mathbb{R}$. In all cases, replacing an index r by $r+$ indicates taking the union over all indices $s > r$. For example,

$$G_{r+} := \bigcup_{s>r} G_s \quad \text{and} \quad \mathfrak{g}_{x,r+} := \bigcup_{s>r} \mathfrak{g}_{x,s}.$$

Note that $\mathfrak{g}_{x,r+} \subseteq \mathfrak{g}_{x,r}$. The quotient $\mathfrak{g}_{x,r}/\mathfrak{g}_{x,r+}$ is always a finite-dimensional $\mathfrak{f}$-vector space, and we will soon see (Lemma 5.4(c)) that the quotient $G_{x,r}/G_{x,r+}$ is isomorphic to $\mathfrak{g}_{x,r}/\mathfrak{g}_{x,r+}$ if $r > 0$. The quotient $G_{x,0}/G_{x,0+}$ is the group of $\mathfrak{f}$-rational points of a reductive $\mathfrak{f}$-group G_x. If $x \in \{x_{\mathrm{L}}, x_{\mathrm{R}}\}$, then $\mathsf{G}_x \cong \mathrm{SL}_{2/\mathfrak{f}}$. If $x = x_{\mathrm{C}}$, then $\mathsf{G}_x \cong \mathbb{G}_{m/\mathfrak{f}}$, the 1-dimensional, multiplicative group scheme over $\mathfrak{f}$.

For $r, s \in \mathbb{R}$ with $s \geq r$, it is convenient to define $\mathfrak{g}_{x,r:s}$ to be the quotient $\mathfrak{g}_{x,r}/\mathfrak{g}_{x,s}$. If r and s are non-negative, then set $G_{x,r:s} := G_{x,r}/G_{x,s}$. Below we will use similar notation to denote quotients of other filtration groups.

5.2. **Group filtrations and torus filtrations.** Recall that we have already defined filtrations on tori. It is natural to wonder how they fit into the framework that we have just described. It turns out that the filtration of an elliptic torus is 'associated to' a unique point in the building of $\mathbf{G}$ over k. For standard tori (see §3.1), that point lies in our preferred set $\mathcal{B}^{\mathrm{opt}}$.

Definition 5.3. For $\mathbf{T}$ a standard, elliptic torus, let $x = x_{\mathbf{T}}$ be the unique element of $\mathcal{B}^{\mathrm{opt}}$ such that

$$T_r = T \cap G_{x,r} \quad \text{for all } r \in \mathbb{R}_{>0} \quad \text{and} \quad \mathfrak{t}_r = \mathfrak{t} \cap \mathfrak{g}_{x,r} \qquad \text{for all } r \in \mathbb{R}.$$

Explicitly,

$$x_{\mathbf{T}^{\epsilon,1}} = x_{\mathrm{L}}, \quad x_{\mathbf{T}^{\epsilon,\varpi}} = x_{\mathrm{R}},$$
$$\text{and} \quad x_{\mathbf{T}^{\varpi,\eta}} = x_{\mathrm{C}} \quad \text{for } \eta \in \{1, \epsilon\}.$$

On the other hand, for the split torus, we have that

$$A_r = A \cap G_{x,r} \quad \text{for all } (x,r) \in \mathcal{B}^{\mathrm{opt}} \times \mathbb{R}_{\geq 0}.$$

Further, we have defined the depth of a regular, semisimple element (see Definition 3.5). By [5, Lemma 2.9] and [3, Lemmas 3.5.3 and 3.7.25], if $\mathrm{d}(Y) = r$, then $Y \in \mathfrak{g}_r \setminus \mathfrak{g}_{r+}$; and, if $\mathrm{d}(\gamma) = r$, then $\gamma \in G_r \setminus G_{r+}$. (Alternatively, in our situation, one could use [3, §3.6].) That is, our definition of depth is a special case of the usual one [3, §§3.3, 3.7.3]. Actually, one must take a little care if γ lies in a ramified, elliptic torus T, but not in $T_0 = T_{0+}$; but, even in this situation, the definitions agree, since $T \subseteq G_{x_{\mathrm{C}},0}$.

5.3. **Group filtrations and Lie-algebra filtrations.** We will use the *Cayley transform* defined on an open subset of $\mathfrak{g}$ by $\mathrm{c}\colon X \longmapsto \frac{1+X/2}{1-X/2}$, to relate the filtrations on $\mathfrak{g}$ to those on G. The normalizing factor $\frac{1}{2}$ makes sure that c satisfies [6, Hypothesis A.7].

The biggest possible domain for c is the set of matrices X which do not have 2 as an eigenvalue. Note that a trace-zero matrix with this property also does not have -2 as an eigenvalue. However, we need this enlarged domain only once (in the proof of Lemma 12.4). Everywhere else, we shall be concerned only with the restriction of c to $\mathfrak{g}_{0+}$.

Lemma 5.4. *The Cayley transform* c *has the following properties for any* $x \in \mathcal{B}^{\mathrm{opt}}$ *and* $r, s \in \mathbb{R}$ *with* $r > 0$*:*

(a) It is equivariant under the adjoint and conjugation actions of G *on the domain and codomain.*
(b) It maps $\mathfrak{g}_{x,r}$ *bijectively onto* $G_{x,r}$*.*
(c) If $0 < s \le r \le 2s$*, then it induces an isomorphism* $\mathfrak{g}_{x,s:r} \longrightarrow G_{x,s:r}$*.*
(d) If $\mathbf{T}$ *is a maximal* k*-torus in* $\mathbf{G}$*, then* c *restricts to a bijection* $\mathfrak{t}_{0+} \longrightarrow T_{0+}$*. If* $T \cong C_\theta$*, with* θ *a non-square, then* c *agrees with the map of Lemma 2.3.*
(e) If Y *is in the domain of* c*, then* $\mathrm{c}(-Y) = \mathrm{c}(Y)^{-1}$*.*
(f) If $X \in \mathfrak{g}_{x,r}$ *and* $Y \in \mathfrak{g}_{x,s}$*, then*
$$\mathrm{Ad}(\mathrm{c}(X))Y \equiv Y + [X,Y] \pmod{\mathfrak{g}_{x,r+2s}}.$$
(g) If $X \in \mathfrak{g}_{x,r}$*,* $Y \in \mathfrak{g}_{x,s}$*, and* $s > 0$ *(as well as* $r > 0$*), then*
$$\bigl[\mathrm{c}(X), \mathrm{c}(Y)\bigr] \equiv \mathrm{c}\bigl([X,Y]\bigr) \pmod{G_{r+s+\min\{r,s\}}}.$$
(h) If $Y \in \mathfrak{g}_{0+}$*, then* $\mathrm{d}(Y) = \mathrm{d}(\mathrm{c}(Y))$ *and* $|D_{\mathfrak{g}}(Y)| = |D_G(\mathrm{c}(Y))|$*.*

Proof. By Lemma 3.8 and Remark 5.1, the result follows from straightforward calculations, using that
$$\mathrm{c}(X) = 1 + 2\sum_{i=1}^{\infty}\Bigl(\frac{1}{2}X\Bigr)^i, \quad X \in \mathfrak{g}_{0+}. \qquad \square$$

The isomorphisms of Lemma 5.4(c) are often called *Moy–Prasad maps.* These are the same isomorphisms that appear in Yu's construction of tame, supercuspidal representations [55] (see Lemma 1.3 *loc. cit.*).

6. Haar measure

Especially with calculations involving several orbital integrals (see Proposition 13.14), it is necessary to be very careful about Haar measures. We describe the ones that we use here.

Waldspurger [53, §I.4] defines a canonical way to normalize the measure on a reductive, p-adic group. In our setting, this gives measures $\mathrm{d}g$, $\mathrm{d}t^\theta$, and $\mathrm{d}z$ on G, T^θ (for θ a non-square), and $Z(G)$, respectively, such that
$$\mathrm{meas}_{\mathrm{d}g}(\mathrm{SL}_2(R)) = \frac{q^2-1}{q^{1/2}}, \qquad \mathrm{meas}_{\mathrm{d}t^\epsilon}(T_0^\epsilon) = \frac{q+1}{q^{1/2}},$$
$$\mathrm{meas}_{\mathrm{d}t^\varpi}(T_0^\varpi) = 1, \qquad \mathrm{meas}_{\mathrm{d}z}(\{1\}) = 1.$$

Similarly, the Haar measures $\mathrm{d}t^{\varpi,\epsilon}$, $\mathrm{d}t^{\epsilon\varpi,1}$, and $\mathrm{d}t^{\epsilon\varpi,\epsilon}$ on the obvious tori T all have $\mathrm{meas}(T_0) = 1$.

Note that these normalizations have pleasant properties with respect to Moy–Prasad filtrations. For example, we have defined the measure on G so that $\mathrm{meas}_{\mathrm{d}g}(G_{x_{\mathrm{L}},0}) = \left|\mathsf{G}_{x_{\mathrm{L}}}(\mathfrak{f})\right| \cdot \left|\mathrm{Lie}(\mathsf{G}_{x_{\mathrm{L}}})(\mathfrak{f})\right|^{-1/2}$; but it is in fact true that $\mathrm{meas}_{\mathrm{d}g}(G_{x,0}) = \left|\mathsf{G}_x(\mathfrak{f})\right| \cdot \left|\mathrm{Lie}(\mathsf{G}_x)(\mathfrak{f})\right|^{-1/2}$ for all $x \in \mathcal{B}^{\mathrm{opt}}$, and, indeed, this is the definition that Waldspurger offers.

The computations of [51] use a different normalization of quotient measure. Namely, they involve the measures $\mathrm{d}_\theta\dot{g}$ on G/T^θ defined by

$$\mathrm{meas}_{\mathrm{d}_\epsilon\dot{g}}(\mathrm{SL}_2(R)/T^\epsilon) = \frac{q-1}{q} \quad \text{and} \quad \mathrm{meas}_{\mathrm{d}_\theta\dot{g}}(\mathrm{SL}_2(R)/T^\theta) = \frac{q^2-1}{2q^2} \quad \text{for } \theta \in \{\varpi, \epsilon\varpi\}.$$

Thus,

$$\frac{\mathrm{d}g}{\mathrm{d}t^\epsilon} = q \cdot \mathrm{d}_\epsilon\dot{g} \quad \text{and} \quad \frac{\mathrm{d}g}{\mathrm{d}t^\varpi} = q^{3/2} \cdot \mathrm{d}_\varpi\dot{g}.$$

For $\theta \in \{\epsilon, \varpi\}$, we write $\mathrm{d}'_\theta\dot{g}$ for the measure on $G/Z(G)$ such that

$$\mathrm{meas}_{\mathrm{d}'_\theta\dot{g}}(\mathrm{SL}_2(R)/Z(G)) = \mathrm{meas}_{\mathrm{d}_\theta\dot{g}}(\mathrm{SL}_2(R)/T^\theta).$$

Thus,

$$\frac{\mathrm{d}g}{\mathrm{d}z} = \frac{1}{2}q^{1/2}(q+1) \cdot \mathrm{d}'_\epsilon\dot{g} \quad \text{and} \quad \frac{\mathrm{d}g}{\mathrm{d}z} = q^{3/2} \cdot \mathrm{d}'_\varpi\dot{g}.$$

To reduce to a minimum the symbol-juggling asked of the reader, we will often abuse notation by writing $\mathrm{d}_\theta\dot{g}$ instead of $\mathrm{d}'_\theta\dot{g}$ when the context makes it clear which measure is needed.

7. Duality, Fourier transforms, and orbital integrals

Let V be a finite-dimensional k-vector space equipped with a non-degenerate, symmetric, bilinear pairing $\langle\ ,\ \rangle$. We shall use this pairing to identify the dual vector space $V^* := \mathrm{Hom}_k(V, k)$ with V.

7.1. **Duality.**

Notation 7.1. If $\mathcal{L}$ is a lattice in V, then we write

$$\mathcal{L}^\bullet := \left\{ X \in V \,\middle|\, \langle X, \mathcal{L}\rangle \subseteq \wp \right\}.$$

(The requirement that $\langle X, \mathcal{L}\rangle \subseteq \wp$, rather than, say, that $\langle X, \mathcal{L}\rangle \subseteq R$, is a result of our choice of a depth-zero additive character Λ.)

For example, the reader can check directly that $(\mathfrak{g}_{x,r})^\bullet = \mathfrak{g}_{x,(-r)+}$, hence that $(\mathfrak{g}_{x,r+})^\bullet = \mathfrak{g}_{x,-r}$, for all $(x, r) \in \mathcal{B}^{\mathrm{opt}} \times \mathbb{R}$.

For any lattices $\mathcal{L}$ and $\mathcal{M}$ in $\mathfrak{g}$, we have an isomorphism from $\mathcal{M}^\bullet/\mathcal{L}^\bullet$ to the Pontryagin dual $(\mathcal{L}/\mathcal{M})\hat{}$ of $\mathcal{L}/\mathcal{M}$, given by $X \longmapsto \chi_X$, where $\chi_X \colon Y \longmapsto \Lambda(\langle X, Y\rangle)$. Suppose that for some $x \in \mathcal{B}^{\mathrm{opt}}$ and positive $r \in \mathbb{R}$, we have that $\mathfrak{g}_{x,r+} \subseteq \mathcal{L} \subseteq \mathfrak{g}_{x,(r/2)+}$. Lemma 5.4(b) implies that the Cayley transform c induces an isomorphism $\mathcal{L}/\mathfrak{g}_{x,r+} \xrightarrow{\sim} \mathrm{c}(\mathcal{L})/G_{x,r+}$ (which we will also denote by c). Thus, we have an isomorphism

$$\mathfrak{g}_{x,-r}/\mathcal{L}^\bullet \xrightarrow{\sim} (\mathrm{c}(\mathcal{L})/G_{x,r+})\hat{} \tag{7.2}$$

given by $X \longmapsto \chi_X$, where

$$\chi_X \colon \mathrm{c}(Y) \longmapsto \Lambda\bigl(\mathrm{tr}(X \cdot Y)\bigr).$$

In particular, we have isomorphisms

$$\mathfrak{g}_{x,-r}/\mathfrak{g}_{x,(-r)+} \xrightarrow{\sim} (G_{x,r}/G_{x,r+})^\frown \qquad \text{and} \qquad \mathfrak{g}_{x,(-r)+}/\mathfrak{g}_{x,(-r/2)+} \xrightarrow{\sim} (G_{x,(r/2)+}/G_{x,r+})^\frown; \tag{7.3}$$

and, similarly,

$$\mathfrak{t}_{-r}/\mathfrak{t}_{(-r)+} \xrightarrow{\sim} (T_r/T_{r+})^\frown \quad \text{and} \quad \mathfrak{t}_{(-r)+}/\mathfrak{t}_{(-r/2)+} \xrightarrow{\sim} (T_{(r/2)+}/T_{r+})^\frown \tag{7.4}$$

for any maximal torus T with Lie algebra $\mathfrak{t}$.

7.2. **Fourier transforms and orbital integrals.**

Definition 7.5. Fix a Haar measure $\mathrm{d}v$ on V. The *Fourier transform* of a function $f \in C_{\mathrm{c}}^\infty(V)$ is the function $\hat{f} \in C_{\mathrm{c}}^\infty(V)$ defined by

$$\hat{f}(w) = \int_V f(v)\Lambda(\langle v, w\rangle)\mathrm{d}v, \quad w \in V.$$

We say that $\mathrm{d}v$ is *self-dual* (with respect to the additive character Λ and pairing $\langle\ ,\ \rangle$) if

$$\hat{\hat{f}}(v) = f(-v) \quad \text{for all } f \in C_{\mathrm{c}}^\infty(V) \text{ and } v \in V.$$

If T is a distribution on V (i.e., a linear functional on $C_{\mathrm{c}}^\infty(V)$), then the *Fourier transform* of T is the distribution $\widehat{T}$ on V defined by

$$\widehat{T}(f) = T(\hat{f}), \quad f \in C_{\mathrm{c}}^\infty(V).$$

Every vector space supports a unique self-dual Haar measure.

We leave to the reader the verification that the Fourier transform does, indeed, carry $C_{\mathrm{c}}^\infty(V)$ to itself. In fact, $C(\mathcal{L}/\mathcal{M})$ is carried to $C(\mathcal{M}^\bullet/\mathcal{L}^\bullet)$ [3, p. 282].

Now suppose that $V = \mathfrak{h}$ is the Lie algebra of a reductive subgroup H of $G = \mathrm{SL}_2(k)$. Then we may, and do, take the pairing on V to be the trace form, given by

$$\langle X, Y\rangle := \mathrm{tr}(X \cdot Y), \quad X, Y \in \mathfrak{g}. \tag{7.6}$$

Definition 7.7. If $X \in \mathfrak{h}$ is regular and semisimple, say with $T = C_H(X)$, then there is an H-invariant measure $\mathrm{d}\dot{h}$ on H/T (which we may also view as a measure on the H-adjoint orbit of X in $\mathfrak{h}$). The *orbital integral* of X is the distribution μ_X^H on $\mathfrak{h}$ defined by

$$\mu_X^H(f) = \int_{H/T} f(\mathrm{Ad}(h)X)\mathrm{d}\dot{h}, \quad f \in C_{\mathrm{c}}^\infty(\mathfrak{h}).$$

As a special case of Definition 7.5, we define

$$\hat{\mu}_X^H(f) := \mu_X^H(\hat{f}) = \int_{H/T} \hat{f}(\mathrm{Ad}(h)X)\mathrm{d}\dot{h}, \quad f \in C_{\mathrm{c}}^\infty(\mathfrak{h}).$$

The integral defining μ_X^H converges because the H-adjoint orbit of X is closed in $\mathfrak{h}$. (In fact, it can be shown to converge under weaker conditions [40, Theorem 2]; but we do not need this fact.)

Let $\mathrm{d}Y$ be the self-dual Haar measure on $\mathfrak{h}$. By [23, Theorem 1.1], there is a function on $\mathfrak{h}$, which we shall again denote by $\hat{\mu}_X^H$, such that

$$\hat{\mu}_X^H(f) = \int_{\mathfrak{h}} f(Y)\hat{\mu}_X^H(Y)\mathrm{d}Y, \quad f \in C_{\mathrm{c}}^\infty(\mathfrak{h}).$$

Although we have specified a choice of Haar measure on $\mathfrak{h}$, it is unimportant here, since the *function* $\hat{\mu}^H_X$ does not depend on the choice (although the *distribution* does). The function *does* depend on the normalization chosen for the measure $\mathrm{d}\dot{h}$ on H/T (as well as on Λ), so we shall be careful to specify this normalization.

The case $\mathbf{H} = \mathbf{G}$ is covered in [51]. The only other case of interest to us is handled by the lemma below.

Lemma 7.8. *If* $\mathbf{H} = \mathbf{T}$ *is a torus, and* $\operatorname{meas}_{\mathrm{d}\dot{h}}(H/T) = 1$*, then*

$$\hat{\mu}^T_X(Y) = \Lambda(\langle X, Y\rangle), \quad X, Y \in \mathfrak{t}.$$

In particular, $\hat{\mu}^T_X(0) = 1$.

Note that, in this setting, H/T is a singleton.

Proof. Since $\mu^T_X(f) = f(X)$ for all $f \in C^\infty_c(\mathfrak{t})$ and $X \in \mathfrak{t}$, this follows immediately from Definition 7.7. □

8. Unrefined minimal K-types

Definition 8.1 ([33, Definition 5.1])**.** Suppose $x \in \mathcal{B}^{\mathrm{opt}}$, $r \geq 0$, and let χ be an irreducible representation of $G_{x,r}$, trivial on $G_{x,r+}$. We say that the pair $(G_{x,r}, \chi)$ is an *(unrefined) minimal K-type of depth r* if

(a) $r = 0$ and χ is the inflation to $G_{x,0}$ of a cuspidal representation of $\mathsf{G}_x(\mathfrak{f}) = G_{x,0:0+}$; or
(b) $r > 0$ and the coset $\Sigma \in \mathfrak{g}_{x,-r}/\mathfrak{g}_{x,(-r)+}$ corresponding to χ (see §7.1) contains no nilpotent elements.

Two minimal K-types $(G_{x,r}, \chi)$ and $(G_{y,s}, \xi)$ are *associate* if $r = s$ and

(a) $r = 0$, $x = y$, and χ is equivalent to ξ; or
(b) $r > 0$, and the G-orbit of the coset that realizes χ intersects the coset that realizes ξ.

For arbitrary reductive, p-adic groups, one must call depth-zero K-types associate even under some circumstances when $x \neq y$; but working with $\mathbf{G} = \mathrm{SL}_2$, and restricting to $x \in \mathcal{B}^{\mathrm{opt}}$, avoids this complication.

Theorem 8.2 ([33, Theorem 5.2] and [34, Theorem 3.5])**.** *Let (π, V) be an irreducible admissible representation of G. Then there is a non-negative, rational number r with the following properties:*

(1) For some $x \in \mathcal{B}^{\mathrm{opt}}$, the space $V^{G_{x,r+}}$ of $G_{x,r+}$-fixed vectors is non-zero, and r is the smallest non-negative real number with this property.
(2) For any $y \in \mathcal{B}^{\mathrm{opt}}$, if $W := V^{G_{y,r+}} \neq \{0\}$, then
  (a) if $r = 0$, then every irreducible $G_{y,r}$-submodule of W contains an unrefined minimal K-type of depth zero.
  (b) if $r > 0$, then every irreducible $G_{y,r}$-submodule of W is an unrefined minimal K-type.

Moreover, any two unrefined minimal K-types contained in π are associate.

Definition 8.3. The number r in Theorem 8.2 (denoted by $\rho(\pi)$ in [33]) is called the *depth* $\mathrm{d}(\pi)$ of π.

Remark 8.4. If the representation π of G contains an unrefined minimal K-type of the form $(G_{x_{\mathrm{R}},0}, \chi_{\mathrm{R}})$, then $\pi \circ \operatorname{Int}\left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi \end{smallmatrix}\right)$ contains the unrefined minimal K-type $(G_{x_{\mathrm{L}},0}, \chi_{\mathrm{L}})$, where $\chi_{\mathrm{L}} = \chi_{\mathrm{R}} \circ \operatorname{Int}\left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi \end{smallmatrix}\right)$.

In order to classify the representations of G, we start by listing the unrefined minimal K-types for G, and check which items on the list are associate.

9. Representations of depth zero

9.1. **Cuspidal representations of** $\mathrm{SL}_2(\mathfrak{f})$**.** Write $\mathsf{G} = \mathrm{SL}_{2/\mathfrak{f}}$. The torus T^ϵ of §3.1 is, up to $\mathrm{SL}_2(\mathfrak{f})$-conjugacy, the unique maximal elliptic $\mathfrak{f}$-torus in G; and its (relative) Weyl group is $\{1, \sigma_\epsilon\}$, where $\sigma_\epsilon(\psi) = \psi^{-1}$ for any character ψ of $\mathsf{T}^\epsilon(\mathfrak{f})$.

Definition 9.1. For any character ψ of $\mathsf{T}^\epsilon(\mathfrak{f})$ such that $\psi \neq \psi^{-1}$, we have from [18, Theorems 6.8 and 8.3] that the Deligne–Lusztig virtual representation $R^{\mathsf{G}}_{\mathsf{T}^\epsilon,\psi}$ is irreducible and cuspidal. Let $|R^{\mathsf{G}}_{\mathsf{T}^\epsilon,\psi}| = -R^{\mathsf{G}}_{\mathsf{T}^\epsilon,\psi}$ denote the corresponding (non-virtual) representation.

Definition 9.2. By [18, Theorem 6.8], $R^{\mathsf{G}}_{\mathsf{T}^\epsilon,\psi_0}$ is a sum of two inequivalent, irreducible (virtual) representations, which we will denote by $R^{\pm}_{\mathsf{T}^\epsilon,\psi_0}$. By [19, pp. 70–73], where $R^{\mathsf{G}}_{\mathsf{T}^\epsilon,\psi_0}$ is denoted by X_{ψ_0} and its two components by X' and X'', we may give an explicit description of the virtual representations. It is convenient to choose signs in such a way that

$$R^{\pm}_{\mathsf{T}^\epsilon,\psi_0} = \tfrac{1}{2}R^{\mathsf{G}}_{\mathsf{T}^\epsilon,\psi_0} \mp \tfrac{1}{2}q^{1/2}\mathcal{G}(\Lambda)^{-1} \cdot {}^0 f, \tag{9.3}$$

where ${}^0 f$ is as in Definition 13.2 below, and, as usual, we have identified the finite-group representation $R^{\pm}_{\mathsf{T}^\epsilon,\psi_0}$ with its character.

Remark 9.4. An explicit computation shows that

$$\sum_u {}^0 f(u) = 0,$$

where the sum runs over $\left\{ \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) \,\middle|\, n \in \mathfrak{f} \right\}$. By (9.3), [18, (8.3.2)], and [10, Corollary 9.1.2], $R^{\pm}_{\mathsf{T}^\epsilon,\psi_0}$ is cuspidal.

By [18, Proposition 8.2] and [10, p. 457], all irreducible, cuspidal representations of $\mathrm{SL}_2(\mathfrak{f})$ arise in this way. Moreover, by [18, Theorem 6.8], if ψ and ψ' satisfy $\psi \neq \psi^{-1}$ and $\psi' \neq \psi'^{-1}$, then $|R^{\mathsf{G}}_{\mathsf{T}^\epsilon,\psi}|$ is not isomorphic to $|R^{\pm}_{\mathsf{T}^\epsilon,\psi_0}|$, and is isomorphic to $|R^{\mathsf{G}}_{\mathsf{T}^\epsilon,\psi'}|$ if and only if $\psi' = \psi^{\pm 1}$.

9.2. **Lifting finite-field representations to depth-zero representations.**

Proposition 9.5. *Let π be an irreducible, depth-zero, supercuspidal representation of G. Then π contains an unrefined minimal K-type $(G_{x,0}, \sigma)$, where $x \in \{x_{\mathrm{L}}, x_{\mathrm{R}}\}$, and π is equivalent to $\mathrm{Ind}_{G_{x,0}}^G \sigma$.*

Proof. This follows from [34, Proposition 6.6] upon noting that $G_{x,0}$ is self-normalizing in G for $x \in \{x_{\mathrm{L}}, x_{\mathrm{R}}\}$. □

Suppose $x \in \{x_{\mathrm{L}}, x_{\mathrm{R}}\}$, so that $\mathsf{G}_x(\mathfrak{f}) \cong \mathrm{SL}_2(\mathfrak{f})$. There is an unramified, elliptic torus T in G (unique up to $G_{x,0+}$-conjugacy) such that the image of $T_0 \subseteq G_{x,0}$ in $G_{x,0:0+}$ is $\mathsf{T}^\epsilon(\mathfrak{f})$ [15, Lemma 2.2.2]. Specifically, we take $T = T^{\epsilon,\eta}$, where $\eta = 1$ if $x = x_{\mathrm{L}}$ and $\eta = \varpi$ if $x = x_{\mathrm{R}}$. Thus, a character of $\mathsf{T}^\epsilon(\mathfrak{f}) \cong T_{0:0+} = T/T_{0+}$ may be viewed in a natural way as a character of T trivial on T_{0+}, i.e., a depth-zero character (in the sense of Definition 3.4). Note that the character ψ_0 of T^ϵ inflates to the character ψ_0^η of T (with notation as in Notation 3.2).

Definition 9.6. A *depth-zero, supercuspidal parameter* is

- a pair (T, ψ), where $T = T^{\epsilon,\eta}$, with $\eta \in \{1, \varpi\}$, and ψ is a depth-zero character of T such that $\psi \neq \psi^{-1}$; or
- a triple $(T, \psi, \pm)$, where $T = T^{\epsilon,\eta}$ with $\eta \in \{1, \varpi\}$, and $\psi = \psi_0^\eta$.

Given such a datum, put

$$\pi(T, \psi) = \mathrm{Ind}_{G_{x,0}}^G |R_{T,\psi}^G|; \quad \text{respectively, } \pi^\pm(T, \psi) = \mathrm{Ind}_{G_{x,0}}^G |R_{T,\psi}^\pm|,$$

where $|R_{T,\psi}^G|$ (respectively, $|R_{T,\psi}^\pm|$) is the inflation to $G_{x,0}$ of the appropriate finite-field representation. We call the various $\pi(T, \psi)$ (and all the positive-depth, supercuspidal representations, which we will construct later) *ordinary* (see §14), and the four possible $\pi^\pm(T, \psi)$ *exceptional* (see §15).

The distinction between 'ordinary' and 'exceptional' is just an *ad hoc* one reflecting the different techniques needed in their character computations.

The following *ad hoc* definition allows us to state Proposition 13.13 uniformly. All that is important for us is that

- $X_\pi \in \mathfrak{g}_{x,0}$;
- the image $\overline{X_\pi}$ of X_π in $\mathfrak{g}_{x,0:0+} = \mathrm{Lie}(\mathsf{G}_x)(\mathfrak{f})$ satisfies $C_{\mathsf{G}_x}(\overline{X_\pi}) = \mathsf{T}^\epsilon$; and
- $C_{\mathbf{G}}(X_\pi) = \mathbf{T}^{\epsilon,\eta}$.

(Recall that $\eta = 1$ if $x = x_{\mathrm{L}}$, and $\eta = \varpi$ if $x = x_{\mathrm{R}}$.)

Notation 9.7. If π is a depth-zero, supercuspidal representation, then we write $X_\pi := X_1^{\epsilon,\eta}$. Put $\Lambda'_\pi = \Lambda$.

Remark 9.8. Note that, if (T, ψ) (respectively, $(T, \psi, \pm)$) and (T', ψ') (respectively, $(T', \psi', \pm)$) are depth-zero, supercuspidal parameters, and $g \in \mathrm{GL}_2(k)$ is such that $T' = \mathrm{Int}(g)T$ and $\psi' = \psi \circ \mathrm{Int}(g)$, then $\pi' = \pi \circ \mathrm{Int}(g)$, where $\pi = \pi(T, \psi)$ and $\pi' = \pi(T', \psi')$ (respectively, $\pi = \pi^\pm(T, \psi)$ and $\pi' = \pi^\pm(T', \psi')$). In particular, if $g \in G$, then $\pi(g)$ intertwines π and π'. Further, $\mathrm{Ad}(g)X_\pi = X_{\pi'}$.

Note that this applies in particular when $T = T^{\epsilon,\varpi}$; $g = \left(\begin{smallmatrix} \varpi & 0 \\ 0 & 1 \end{smallmatrix}\right)$; and $T' = T^{\epsilon,1}$. In this setting, $x_{\mathbf{T}'} = x_{\mathrm{L}}$.

10. Representations of positive depth

10.1. **Unrefined minimal K-types of positive depth.** Now that we have classified the representations of depth zero, we turn to those of positive depth (Definition 8.1(b)). Theorem 8.2 suggests that we start by classifying the unrefined minimal K-types that they contain. As before, we may confine our attention to the three filtrations associated to elements of $\mathcal{B}^{\mathrm{opt}}$. We begin by listing the K-types associated to the filtration coming from x_{L}.

Let $r \in \mathbb{Z}$, since otherwise the quotient $\mathfrak{g}_{x_{\mathrm{L}},-r}/\mathfrak{g}_{x_{\mathrm{L}},(-r)+}$ is trivial. The quotient $G_{x_{\mathrm{L}},0:0+} = \mathsf{G}_{x_{\mathrm{L}}}(\mathfrak{f})$ is isomorphic to $\mathrm{SL}_2(\mathfrak{f})$. Every coset in $\mathfrak{g}_{x_{\mathrm{L}},-r}/\mathfrak{g}_{x_{\mathrm{L}},(-r)+}$ can be written in the form

$$\varpi^{-r}X + \mathfrak{g}_{x_{\mathrm{L}},(-r)+},$$

where X has one of the following forms (up to $G_{x_{\mathrm{L}},0}$-conjugacy):

$$\begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}, \qquad X_\beta^{\mathrm{split}} = \begin{pmatrix} \beta & 0 \\ 0 & -\beta \end{pmatrix}, \qquad X_\beta^{\epsilon,1} = \begin{pmatrix} 0 & \beta \\ \epsilon\beta & 0 \end{pmatrix},$$

where $\beta \in R^\times$. Since β is determined only modulo $\wp$, we will think of it as lying in $\mathfrak{f}^\times$. In the first example above, X is nilpotent, so the corresponding coset does not correspond to an unrefined minimal K-type.

Now consider the K-types arising from the Iwahori filtration (i.e., the filtration associated to the point x_{C}). If $r \in \mathbb{Z}$, then every coset in the quotient space $\mathfrak{g}_{x_{\mathrm{C}},-r}/\mathfrak{g}_{x_{\mathrm{C}},(-r)+}$ has the form $\varpi^{-r}X_\beta^{\mathrm{split}}+\mathfrak{g}_{x_{\mathrm{C}},(-r)+}$. As before, we may take β to lie in $\mathfrak{f}^\times$. Similarly, if $r \in \mathbb{Z}+\frac{1}{2}$, then every coset in the quotient space $\mathfrak{g}_{x_{\mathrm{C}},-r}/\mathfrak{g}_{x_{\mathrm{C}},(-r)+}$ either contains a nilpotent element or has the form (up to conjugation by $G_{x_{\mathrm{C}}}$) $\varpi^{-\lceil r\rceil}X_\beta^{\theta,\eta} + \mathfrak{g}_{x_{\mathrm{C}},(-r)+}$, where $\theta \in \{\varpi, \epsilon\varpi\}$ and $\eta \in \{1, \epsilon\}$.

Suppose $\chi_{\beta,r}^{\mathrm{unr,split}}$ is the character of $G_{x_{\mathrm{L}},r}/G_{x_{\mathrm{L}},r+}$ corresponding to $\varpi^{-r}X_\beta^{\mathrm{split}} + \mathfrak{g}_{x_{\mathrm{L}},(-r)+}$, and $\chi_{\beta,r}^{\mathrm{split}}$ is the character of $G_{x_{\mathrm{C}},r}/G_{x_{\mathrm{C}},r+}$ corresponding to $\varpi^{-r}X_\beta^{\mathrm{split}} + \mathfrak{g}_{x_{\mathrm{C}},(-r)+}$. (See §7.1.) Then any representation π of G that contains $\chi_{\beta,r}^{\mathrm{unr,split}}$ must contain $\chi_{\beta,r}^{\mathrm{split}}$, as the latter is the restriction of the former to $G_{x_{\mathrm{C}},r}$. From [55, Corollary 17.3], a representation of G that contains such an unrefined minimal K-type cannot be supercuspidal. Therefore, we may ignore this family of K-types.

The situation for x_{R} is the same as that for x_{L}, after conjugation by $\left(\begin{smallmatrix}1 & 0\\ 0 & \varpi\end{smallmatrix}\right)$.

Thus, every irreducible supercuspidal representation of G of positive depth must contain a K-type whose corresponding coset has the form

$$\varpi^{-\lceil r\rceil}X + \mathfrak{g}_{x,(-r)+},$$

where the possibilities for X, x, and r are as follows:

(10.1)

X	x	r
$X_\beta^{\epsilon,1}$	x_{L}	$\mathbb{Z}_{>0}$
$X_\beta^{\epsilon,\varpi}$	x_{R}	$\mathbb{Z}_{>0}$
$X_\beta^{\theta,\eta}$	x_{C}	$\mathbb{Z}_{\geq 0}+\frac{1}{2}$

Here, β ranges over $\mathfrak{f}^\times$, $\theta \in \{\varpi, \epsilon\varpi\}$, and $\eta \in \{1, \epsilon\}$. Thus, we have six families of K-types, each parametrized by $\mathfrak{f}^\times$ and $\mathbb{Z}_{>0}$ (or $\mathbb{Z}_{\geq 0}+\frac{1}{2}$).

Next we will determine which of the K-types in the six families above are associate.

Given two cosets Σ_1 and Σ_2 of the form above, when does there exist $g \in G$ such that $\mathrm{Int}(g)\Sigma_1 \cap \Sigma_2 \neq \emptyset$? In each case, one can answer this through direct computation. A less cumbersome method requires appealing to a few general theorems, each of which is easy to prove in our special situation.

Proposition 10.2. *For $i = 1, 2$, let $\Sigma_i = \varpi^{-\lceil r_i\rceil}X_i + \mathfrak{g}_{x_i,(-r_i)+}$ be a coset listed in* (10.1). *Then Σ_1 and Σ_2 are associate if and only if $r_1 = r_2$, and X_1 and X_2 are G-conjugate.*

Proof. If $r_1 = r_2$ and X_1 and X_2 are G-conjugate, then $\varpi^{-\lceil r_1\rceil}X_1$ is conjugate to $\varpi^{-\lceil r_2\rceil}X_2$, so Σ_1 and Σ_2 are associate.

Conversely, suppose that Σ_1 and Σ_2 are associate. Then $r_1 = r_2$ by the definition of 'associate'. By [2, Proposition 9.3], $\varpi^{-\lceil r_1\rceil}X_1$ is conjugate to $\varpi^{-\lceil r_2\rceil}X_2$. Therefore, X_1 and X_2 are conjugate. □

Thus, we only need to check which of our chosen coset representatives in (10.1) are conjugate. From easy calculations, we have the following conjugacy relations, and no other non-trivial ones:

(10.3)

$$\begin{array}{lll} X_\beta^{\epsilon,\eta} \sim X_{-\beta}^{\epsilon,\eta} & (\eta \in \{1, \varpi\}) & \text{via an element of } G_x \\ X_\beta^{\theta,\eta} \sim X_{-\beta}^{\theta,\eta} & (\theta \in \{\varpi, \epsilon\varpi\}, \quad \eta \in \{1, \epsilon\}) & \text{if and only if } q \equiv 1 \pmod 4 \\ X_\beta^{\theta,1} \sim X_{-\beta}^{\theta,\epsilon} & (\theta \in \{\varpi, \epsilon\varpi\}) & \text{if and only if } q \equiv 3 \pmod 4. \end{array}$$

10.2. **From positive-depth K-types to inducing data.** Fix a triple (X, x, r) from the list in (10.1), let Σ be the corresponding coset, and let $\chi_\Sigma = \chi_X \in (G_{x,r}/G_{x,r+})^\frown$ be the character corresponding to Σ (see §7.1). We will describe all irreducible representations of G that contain $(G_{x,r}, \chi_\Sigma)$.

Let $T = C_G(X)$, and $\mathfrak{t} = C_{\mathfrak{g}}(X) = \mathrm{Lie}(T)$. These are independent of β.

Let

$$\mathfrak{t}^\perp = \{ Y \in \mathfrak{g} \mid \mathrm{tr}(Y \cdot Z) = 0 \text{ for all } Z \in \mathfrak{t} \},$$

and $\mathfrak{t}_s^\perp = \mathfrak{t}^\perp \cap \mathfrak{g}_{x,s}$. Then it is easy to verify that for all $s \in \mathbb{R}$,

$$\mathfrak{g}_{x,s} = \mathfrak{t}_s \oplus \mathfrak{t}_s^\perp \quad \text{and} \quad \mathfrak{g}_{x,s+} = \mathfrak{t}_{s+} \oplus \mathfrak{t}_{s+}^\perp.$$

Define lattices $\mathcal{J}$ and $\mathcal{J}_+$ in $\mathfrak{g}$ by

$$\mathcal{J} = \mathfrak{t}_r + \mathfrak{t}_{r/2}^\perp \quad \text{and} \quad \mathcal{J}_+ = \mathfrak{t}_r + \mathfrak{t}_{(r/2)+}^\perp.$$

By Lemma 5.4(b, c), the images $J = \mathrm{c}(\mathcal{J})$ and $J_+ = \mathrm{c}(\mathcal{J}_+)$ of these lattices under the Cayley transform are groups, and we have isomorphisms $J/G_{x,r} \xrightarrow{\sim} \mathcal{J}/\mathfrak{g}_{x,r}$ and $J_+/G_{x,r+} \xrightarrow{\sim} \mathcal{J}_+/\mathfrak{g}_{x,r+}$, which we will again denote by c. In particular, by (7.2),

$$(J_+/G_{x,r+})^\frown \cong (\mathcal{J}_+/\mathfrak{g}_{x,r+})^\frown \cong \mathfrak{g}_{x,-r}/(\mathfrak{t}_{-r} + \mathfrak{t}_{-r/2}^\perp). \tag{10.4}$$

Since χ_Σ is trivial on $\mathfrak{t}_r^\perp$, we may extend it to a character $\bar\chi$ of $\mathcal{J}_+/\mathfrak{g}_{x,r+}$ (or $J_+/G_{x,r+}$) by setting $\bar\chi$ to be trivial on $\mathfrak{t}_{(r/2)+}^\perp$. By inflation, we may regard χ_Σ as a character of $G_{x,r}$, and $\bar\chi$ as a character of J_+. Explicitly,

$$\bar\chi\big(\mathrm{c}(Y)\big) = \Lambda\big(\mathrm{tr}(X \cdot Y)\big), \quad Y \in \mathcal{J}_+. \tag{10.5}$$

In terms of (10.4), this extension–inflation process corresponds to following the projection

$$\mathfrak{g}_{x,-r}/(\mathfrak{t}_{-r} + \mathfrak{t}_{-r/2}^\perp) \longrightarrow \mathfrak{g}_{x,-r:-r/2} \cong (G_{x,(r/2)+:r+})^\frown.$$

Proposition 10.6. *Any representation of G that contains χ must contain $\bar\chi$.*

Proof. This is a special case of Corollary 6.5 of [2], though earlier versions exist. □

We have shown that in order to classify the irreducible representations of G that contain χ, it is enough to classify the irreducible representations containing each character $\bar\chi$.

Proposition 10.7. *There exists a unique irreducible representation ρ_Σ of J that contains $\bar\chi$. Moreover, $\rho_\Sigma|_{J_+}$ is a sum of $[J\colon J_+]^{1/2}$ copies of $\bar\chi$. The character of ρ_Σ is given by*

$$\Theta_{\rho_\Sigma}(g) = \begin{cases} [J\colon J_+]^{1/2}\,\bar\chi(g) & g \in J_+ \\ 0 & g \in J \setminus J_+. \end{cases}$$

The proof will require two lemmas along the way. If $J = J_+$, then there is nothing to prove, so assume that $J \neq J_+$. Define an alternating form $\langle\ ,\ \rangle$ on J/J_+ by $\langle a, b\rangle = \bar{\chi}([a, b])$ for all $a, b \in J$.

Lemma 10.8 (special case of [1, Lemma 2.6.1])**.** *The form $\langle\ ,\ \rangle$ is non-degenerate.*

Let $N = \ker \bar{\chi}_\Sigma$. It follows from Lemma 10.8 that J/N is a two-step nilpotent group, and that its center and derived group are both J_+/N. The representation theory of such groups is well known. (The corresponding result over $\mathbb{R}$ is called the Stone–von Neumann theorem.)

Lemma 10.9 ([21, Lemma 1.2])**.** *Let H be a finite group, let A be the center of H, and suppose that A is also the derived group of H. Let ξ be a non-trivial character of A. Then there exists a unique (up to equivalence) representation ρ_ξ of H with central character ξ. Moreover, $\dim \rho_\xi = [H \colon A]^{1/2}$, and the character of ρ_ξ is supported on A.*

Proof of Proposition 10.7. This follows from Lemma 10.9, setting $H = J/N$, $A = J_+/N$, and $\xi = \bar{\chi}$. □

Corollary 10.10. *Every representation of G that contains χ_Σ also contains ρ_Σ.*

Thus, in order to determine the postive-depth, supercuspidal representations, it is enough to classify the irreducible representations of G that contain ρ_Σ. We start by noting that T normalizes J, and classify the irreducible representations of TJ that contain ρ_Σ.

Proposition 10.11. *The representation ρ_Σ extends to TJ. There is an explicit bijection between the set of characters of T that contain $\chi_\Sigma|_{T_r}$ and the set of such extensions.*

If $J = J_+$, then $TJ/N \cong T/T_r$, so there is nothing to prove. The only case for which $J \neq J_+$ is where T is unramified and r is even. Since we are interested in computing characters explicitly, it will be convenient to imitate a method of Moy [32] instead. We defer this to §12.

Remark 10.12. For future reference, we note that the group TJ is equal to $TG_{x,s}$; and that ρ_Σ has dimension q if $r \in 2\mathbb{Z}$, and dimension 1 otherwise.

10.3. **Constructing positive-depth, supercuspidal representations.** Remember that we have constructed representations $\pi(T^{\epsilon,\eta}, \psi)$, where ψ is a depth-zero character of $T^{\epsilon,\eta}$ such that $\psi \neq \psi^{-1}$, and $\pi^\pm(T^{\epsilon,\eta}, \psi_0^\eta)$ (see Definition 9.6). We now complete our construction of the supercuspidal representations of G by defining representations $\pi(T, \psi)$ when T is any maximal, standard, elliptic torus, and ψ is a positive-depth character of T.

Definition 10.13. A *positive-depth, supercuspidal parameter* is a pair (T, ψ), where $T = T^{\theta,\eta}$ is a standard torus and ψ is a positive-depth character of T. Given such a parameter, put $r = \mathrm{d}(\psi)$. Using Lemma 5.4(c) and (7.2), we may deduce from the restriction to ψ of $T_{(r/2)+}$ an element of

$$(T_{(r/2)+:r+})^\wedge \cong (\mathfrak{t}_{(r/2)+:r+})^\wedge \cong \mathfrak{t}_{-r:-r/2}. \tag{10.14}$$

Thus, there exists $\beta \in R^\times$ (uniquely determined only modulo some power of the prime ideal) such that

$$\psi\bigl(\mathrm{c}(Y)\bigr) = \Lambda\bigl(\mathrm{tr}(\varpi^{-\lceil r\rceil} X_\beta^{\theta,\eta} \cdot Y)\bigr) \quad \text{for all } Y \in \mathfrak{t}_{(r/2)+}. \tag{10.15}$$

We then put $\Sigma = \varpi^{-\lceil r \rceil} X_\beta^{\theta,\eta} + \mathfrak{g}_{x,(-r)+}$, which depends only on the image of β in $\mathfrak{f}^\times$, and construct χ_Σ, J, J_+, $\bar{\chi}$, and ρ_Σ as in §10.2. By Proposition 10.11, the extensions of ρ_Σ to TJ are parametrized by characters of T containing $\chi_\sigma|_{T_r}$. By definition, ψ is such a character, hence affords an extension $\sigma(T,\psi)$ of ρ_Σ to TJ. Put $\pi(T,\psi) := \mathrm{Ind}_{TJ}^G \sigma(T,\psi)$. The positive-depth, supercuspidal representations constructed this way (as well as some of the depth-zero, supercuspidal representations constructed in Definition 9.6) are called *ordinary.*

Remark 10.16. By a direct computation (or, given Proposition 12.8, by [55, Remark 3.6]), $r = \mathrm{d}(\psi)$ is also the depth of π, in the sense of Definition 8.3.

Notation 10.17. In the notation of Definition 10.13 (specifically, (10.14)), if $\pi = \pi(T,\psi)$, then put $X_\pi = \varpi^{-\lceil r \rceil} X_\beta^{\theta,\eta}$ and $\Lambda'_\pi = \Lambda_{\varpi^{-\lceil r \rceil}\beta\theta}$.

Proposition 10.18. *Let (T,ψ) be a positive-depth, supercuspidal parameter. Then $\pi(T,\psi)$ is supercuspidal; and an irreducible, smooth representation π of G contains $\sigma(T,\psi)$ if and only if it is equivalent to $\pi(T,\psi)$.*

Proof. From [1, §2.5] or [55, Theorem 15.1], $\pi(T,\psi)$ is irreducible. Mautner observed [31, Theorem 9.1] that it is therefore supercuspidal if it has a non-zero, compactly supported matrix coefficient. The function

$$g \longmapsto \begin{cases} \langle \sigma(\psi)(g)v, w\rangle & g \in TJ \\ 0 & g \in G \setminus TJ, \end{cases}$$

where v and w are any non-zero vectors in the space of ψ, and $\langle\ \ ,\ \ \rangle$ is a non-trivial TJ-invariant pairing, is one such matrix coefficient.

By Frobenius reciprocity, any irreducible smooth representation of G that contains $\sigma(T,\psi)$ is equivalent to $\pi(T,\psi)$. $\square$

Remark 10.19. As in Remark 9.8, we observe that, if (T,ψ) and (T',ψ') are positive-depth, supercuspidal parameters, and $g \in \mathrm{GL}_2(k)$ is such that $T' = \mathrm{Int}(g)T$ and $\psi' = \psi \circ \mathrm{Int}(g)$, then $\pi' = \pi \circ \mathrm{Int}(g)$, where $\pi = \pi(T,\psi)$ and $\pi' = \pi(T',\psi')$. In particular, if $g \in G$, then $\pi(g)$ intertwines π and π'. Further, all relevant data (χ, J, etc.) behave well with respect to the conjugation,

Note that this applies in particular when $T = T^{\theta,\eta}$; with $\theta = \epsilon$ and $\eta = \varpi$ or $\theta \in \{\varpi, \epsilon\varpi\}$ and $\eta = \epsilon$; $g = \left(\begin{smallmatrix}\eta & 0\\ 0 & 1\end{smallmatrix}\right)$; and $T' = T^{\theta,1}$. In this setting, $x_{\mathbf{T}'} \in \{x_{\mathrm{L}}, x_{\mathrm{C}}\}$.

11. Parametrization of supercuspidal representations

Theorem 11.1. *Every supercuspidal representation of G is of the form $\pi(T,\psi)$ or $\pi^\pm(T,\psi)$ for some (depth-zero or positive-depth) supercuspidal parameter (T,ψ) or $(T,\psi,\pm)$.*

The only non-trivial isomorphisms among supercuspidal representations are as follows. We have $\pi(T,\psi) \cong \pi(T,\psi^{-1})$ if

- $T = T^{\epsilon,\eta}$ *with $\eta \in \{1,\varpi\}$, or*
- $T = T^{\theta,\eta}$ *with $\theta \in \{\varpi,\epsilon\varpi\}$ and $\eta \in \{1,\epsilon\}$, and $q \equiv 1 \pmod 4$.*

If $q \equiv 3 \pmod 4$, then we have $\pi(T^{\theta,1},\psi^1) \cong \pi(T^{\theta,\epsilon},\psi^\epsilon)$, where $\theta \in \{\varpi,\epsilon\varpi\}$ and $\psi^\epsilon = \psi^1 \circ \mathrm{Int}\begin{pmatrix}\sqrt{-\epsilon} & 0\\ 0 & \sqrt{-\epsilon}^{-1}\end{pmatrix}$.

Proof. By Proposition 9.5 and §9.1 (in the depth-zero case), and Proposition 10.18 and §10.1 (in the positive-depth case), we have the desired exhaustion.

We now identify equivalences. That all the stated equivalences hold follows from (10.3) and Remarks 9.8 and 10.19. To show that there are no others, we use Theorem 8.2.

No depth-zero, supercuspidal representation is equivalent to any positive-depth, supercuspidal representation.

That the stated isomorphisms among depth-zero representations are the only ones follows from §9.1.

Now suppose that (T, ψ) and (T', ψ') are two positive-depth, supercuspidal parameters, and put $\pi = \pi(T, \psi)$ and $\pi' = \pi(T', \psi')$. Let the data X, X' and ψ, ψ' be as in Definition 10.13.

We have that $\pi \cong \pi'$ only if X is G-conjugate to X'. (Remember that X and X' are well defined only modulo some Moy–Prasad filtration lattice; so we are really claiming that there exist G-conjugate elements in the appropriate cosets.) By Remark 10.19, it therefore suffices to assume that $\pi \cong \pi'$ and $X = X'$, and show that $\psi = \psi'$. Note (under our assumption) that $T = T'$; that the J-groups for π and π' are the same, as are the J_+-groups; and, by (10.5), that $\bar{\chi} = \bar{\chi}'$.

Now

$$\operatorname{Hom}_G(\pi, \pi') = \bigoplus_{g \in TJ\backslash G/TJ} \operatorname{Hom}_{{}^g\cap(TJ)}\bigl(\sigma(T, \psi), \sigma(T, \psi') \circ \operatorname{Int}(g)^{-1}\bigr),$$

where ${}^g\cap(TJ) := TJ \cap \operatorname{Int}(g)(TJ)$. Since the left-hand side is non-zero, there exists $g \in G$ such that the corresponding summand on the right-hand side is non-zero; i.e., g intertwines $\sigma(T, \psi)$ and $\sigma(T', \psi')$. Since $\sigma(T, \psi)|_{J_+}$ is $\bar{\chi}$-isotypic and $\sigma(T, \psi')|_{J_+}$ is $\bar{\chi}'$-isotypic, g intertwines $\bar{\chi}$ and $\bar{\chi}'$; that is, $\bar{\chi} = \bar{\chi}' \circ \operatorname{Int}(g)^{-1}$ on ${}^g\cap(TJ)$. Since $\bar{\chi} = \bar{\chi}'$, a calculation shows that we must have $g \in JTJ = TJ$. Then ${}^g\cap(TJ) = TJ$, so the irreducible representations $\sigma(T, \psi)$ and $\sigma(T, \psi')$ intertwine, and hence are equivalent. It follows from Proposition 10.11 that $\psi = \psi'$. $\square$

12. Inducing representations

Recall from Proposition 10.7 that a character ψ' of T of depth $r > 0$ uniquely determines a representation ρ_χ of J (there denoted by ρ_Σ), where $\chi = \psi'|_{T_r}$ is the restriction to T_r of ψ'. In this section, imitating a method of Moy [32], we give an explicit bijection between the set of characters of T that contain χ and the set of irreducible representations of TJ that contain ρ_χ, thus proving Proposition 10.11. We also show that the resulting parametrization of representations of TJ agrees with that in Yu's construction ([55]).

Suppose first that $J = J_+$. As remarked before, there is nothing to prove in this case: given a character ψ of T that extends χ, we clearly have a corresponding character of TJ, since $TJ/N \cong T/T_r$ (where $N = \ker \bar{\chi}$, as before).

The character of TJ is easy to describe. As in Definition 10.13, given ψ, there exists $X \in \mathfrak{t}_{-r}$ (well defined modulo $\mathfrak{t}_{-r/2}$) such that

$$\psi\bigl(\mathrm{c}(Y)\bigr) = \Lambda\bigl(\operatorname{tr}(X \cdot Y)\bigr) \qquad \text{for all } Y \in \mathfrak{t}_{(r/2)+}.$$

The character $\sigma(T, \psi)$ of TJ that corresponds to ψ is then given by the formula

$$t \cdot \mathrm{c}(Y) \longmapsto \psi(t) \cdot \Lambda\bigl(\operatorname{tr}(X \cdot Y)\bigr), \qquad t \in T,\ Y \in \mathcal{J}. \tag{12.1}$$

For the rest of this section, assume that $J \neq J_+$. That is, T is unramified and r is even. Note that then $[J\colon J_+] = q^2$.

Put $Z = Z(G)$. Consider the following diagram of four groups:

$$\begin{tikzcd} ZT_{0+}J \arrow[r, dash, "\text{normal, cyclic}", "\frac{q+1}{2}"'] \arrow[d, dash, "q^2"'] & TJ \arrow[d, dash, "q^2"] \\ ZT_{0+}J_+ \arrow[ur, dash, "\text{normal}"] \arrow[r, dash, "\frac{q+1}{2}"'] & TJ_+ \end{tikzcd}$$

Given a character ψ of T that extends χ, we want to construct a corresponding representation $\sigma(T, \psi)$ of TJ and compute its character. Using formula (12.1), one can define an extension of ψ to TJ_+; we will denote it again by ψ. It is clear that such extensions are parametrized by the characters of T that extend χ.

Meanwhile, applying Lemma 10.9 (with H and A the quotients of $ZT_{0+}J$ and $ZT_{0+}J_+$, respectively, by their common normal subgroup $\ker(\psi|_{ZT_{0+}J_+})$), we see that there is a unique irreducible representation κ_ψ of $ZT_{0+}J$ containing $\psi|_{ZT_{0+}J_+}$. Moreover, $\kappa_\psi|_{ZT_{0+}J_+}$ is a sum of q copies of $\psi|_{ZT_{0+}J_+}$, and the character of κ_ψ is supported on $ZT_{0+}J_+$.

Remark 12.2. Note that κ_ψ contains $\psi|_{J_+} = \bar{\chi}$, in the notation of §10.1; hence, by Proposition 10.7, contains the representation ρ_χ. Since κ_ψ and ρ_χ both have dimension $[ZT_{0+}J\colon ZT_{0+}J_+]^{1/2} = q = [J\colon J_+]^{1/2}$, we actually have that κ_ψ extends ρ_χ.

Lemma 12.3. *The character $\psi|_{ZT_{0+}J_+}$ is fixed under conjugation by TJ.*

Proof. Since $\psi|_Z$ is clearly fixed, it is enough to show that $\psi|_{T_{0+}J_+}$ is, too.

We show first that $G_{x,r/2}$, which contains J, stabilizes $\psi|_{T_{0+}J_+}$. Since

$$[G_{x,r/2}, G_{x,(r/2)+}] \subseteq G_{x,r+} \subseteq \ker(\psi),$$

the group $G_{x,r/2}$ fixes $\psi|_{G_{x,(r/2)+}}$. Suppose $g \in G_{x,r/2}$ and $t \in T_{0+}$. Then, putting $Y = \mathrm{c}^{-1}(t)$ and $W = \mathrm{c}^{-1}(g)$, we have by Lemma 5.4(g) that

$$\begin{aligned}\psi\bigl(\mathrm{Int}(g)t\bigr) &= \psi(t)\psi([t^{-1}, g]) \\ &= \psi(t)\Lambda\bigl(\mathrm{tr}(X \cdot \mathrm{c}^{-1}([t^{-1}, g]))\bigr) = \psi(t)\Lambda\bigl(\mathrm{tr}([X, Y] \cdot W)\bigr) = \psi(t).\end{aligned}$$

Finally, we show that T stabilizes $\psi|_{T_{0+}J_+}$. Let $e \in T$, $t \in T_{0+}$, and $k \in J_+ \subseteq G_{x,(r/2)+}$. Write $W = \mathrm{c}^{-1}(k)$. Then, using Lemma 5.4(a, g),

$$\begin{aligned}\psi({}^e(tk)) = \psi(t)\psi({}^e k) = \psi(t)\Lambda\bigl(\mathrm{tr}(X \cdot \mathrm{c}^{-1}(\mathrm{Int}(e)k))\bigr) &= \psi(t)\Lambda\bigl(\mathrm{tr}(X \cdot \mathrm{Ad}(e)W)\bigr) \\ &= \psi(t)\Lambda\bigl(\mathrm{tr}(\mathrm{Ad}(e)X \cdot W)\bigr) = \psi(t)\psi(k) = \psi(tk). \quad \square\end{aligned}$$

Since the character $\psi|_{ZT_{0+}J_+}$ is fixed under conjugation by TJ, the representation κ_ψ is also fixed by TJ. Therefore, κ_ψ may be extended to a representation of TJ. The number of such extensions is $[TJ\colon ZT_1J] = \frac{q+1}{2}$.

We will show how our choice of ψ picks out one of these extensions.

Lemma 12.4. *If $g \in TJ \setminus TJ_+$, then ${}^{g\cap}(TJ_+) = ZT_{0+}J_+$.*

As in the proof of Theorem 11.1, we have written ${}^{g\cap}(TJ_+)$ for $TJ_+ \cap \mathrm{Int}(g)(TJ_+)$.

Proof. Since $ZT_{0+}J_+$ is normal in TJ, the right-hand side is contained in the left-hand side.

Let $g = kt'$, with $k \in J \setminus J_+$ and $t' \in T$. Then

$$\mathrm{Int}(g)(TJ_+) = \mathrm{Int}(kt')(TJ_+) = \mathrm{Int}(k)(TJ_+) = \mathrm{Int}(k)(T) \cdot J_+.$$

Now let $t \in T \setminus ZT_{0+}$. It will be enough to show that $\mathrm{Int}(k)t \notin TJ_+$. Since $t \notin -1 \cdot T_{0+}$, one can check directly that there is some $a \in \mathfrak{t} \setminus \mathfrak{t}_{0+}$ such that $\mathrm{c}(a) = t$. Put $b = \mathrm{c}^{-1}(k) \in \mathcal{J} \setminus \mathcal{J}_+$. From Lemma 5.4(f), $\mathrm{Int}(k)a \equiv a + [b, a] \pmod{\mathfrak{g}_{x,(r/2)+}}$. A calculation shows that $[b, a] \in \mathcal{J} \setminus \mathcal{J}_+$. Therefore, $\mathrm{Int}(k)a \in (\mathfrak{t} + \mathcal{J}) \setminus (\mathfrak{t} + \mathcal{J}_+)$, so, by Lemma 5.4(a), $\mathrm{Int}(k)t \notin TJ_+$. $\square$

Corollary 12.5. *TJ_+ has $2q - 1$ double cosets in TJ.*

Proof. If $g \in TJ$, then

$$\left|TJ_+ g TJ_+/TJ_+\right| = \left|TJ_+/({}^g(TJ_+) \cap TJ_+)\right| = \begin{cases} \frac{1}{2}(q+1) & \text{if } g \notin TJ_+, \\ 1 & \text{if } g \in TJ_+. \end{cases}$$

Therefore, the number of double cosets is $1+m$, where $1+\frac{1}{2}(q+1)m = [TJ \colon TJ_+] = q^2$. The result follows. $\square$

Consider the representation $I_\psi = \mathrm{Ind}_{TJ_+}^{TJ} \psi$. Every irreducible component of I_ψ must contain $\psi|_{T_{0+}J_+}$, and therefore κ_ψ. That is, as a TJ-module, I_ψ is a sum of extensions of κ_ψ, with various multiplicities. Let $a_1, a_2, \ldots, a_{(q+1)/2}$ denote these multiplicities. Then

$$\sum a_i = (\dim I_\psi) \cdot (\dim \kappa_\psi)^{-1} = q,$$

and so

$$\begin{aligned} \sum a_i^2 &= \dim \mathrm{Hom}_{TJ}(I_\psi, I_\psi) \\ &= \sum_{g \in TJ_+ \setminus TJ/TJ_+} \dim \mathrm{Hom}_{{}^{g\cap}(TJ_+)}(\psi, \psi \circ \mathrm{Int}(g)^{-1}). \end{aligned}$$

From Lemma 12.3, each term in this last sum is 1. From Corollary 12.5, the sum is $2q - 1$.

Lemma 12.6. *The multiplicities a_i are all equal to 2, except for one of them, which is equal to 1.*

Proof. Apply [32, Lemma 3.5.4] with $\Delta = \frac{1}{2}(q+1)$, and $r = q$. $\square$

Let $\sigma(T, \psi)$ denote the unique extension of κ_ψ to TJ having multiplicity one in I_ψ.

Proposition 12.7. *We have*

$$\Theta_{\sigma(T,\psi)}(g) = \begin{cases} -\psi(g) & g \in TJ_+ \setminus ZT_1J_+ \\ q\cdot\psi(g) & g \in ZT_1J_+ \\ 0 & \textit{if } g \textit{ is not conjugate to an element of } TJ_+. \end{cases}$$

Proof. Since $ZT_{0+}J$ is normal in TJ and $TJ/ZT_{0+}J$ is cyclic, we have that the $\frac{1}{2}(q+1)$ extensions of κ_ψ are of the form $\sigma(T,\psi)\otimes\nu$ for $\nu \in (TJ/ZT_{0+}J)\widehat{\ } \cong (T/ZT_{0+})\widehat{\ }$. Thus, in the Grothendieck ring, we have

$$(*)\qquad \begin{aligned} I_\psi = \operatorname{Ind}_{TJ_+}^{TJ}\psi &= 2\Big(\sum_{\nu\neq 1}\sigma(T,\psi)\otimes\nu\Big) + \sigma(T,\psi) \\ &= 2\Big(\sum_{\nu}\sigma(T,\psi)\otimes\nu\Big) - \sigma(T,\psi). \end{aligned}$$

Let $\dot\psi$ denote the function on TJ that is equal to ψ on TJ_+ and is zero on $TJ\setminus TJ_+$. Then for all $g \in TJ$,

$$(**)\qquad \begin{aligned} \Theta_{I_\psi}(g) &= \sum_{s\in TJ/TJ_+}\dot\psi\big(\operatorname{Int}(s)g\big) \\ &= \begin{cases} q^2\psi(g) & \text{if } g \in ZT_{0+}J_+, \\ \psi(g) & \text{if } g \in TJ_+ \setminus ZT_{0+}J_+, \\ 0 & \text{if } g \text{ is not conjugate to an element of } TJ_+. \end{cases} \end{aligned}$$

Combining $(*)$ and $(**)$ gives the desired result. $\square$

Proof of Proposition 10.11. Proposition 12.7 shows that $\psi \longmapsto \sigma(T,\psi)$ is an injective map from the set of characters of T that extend $\chi|_{T_r}$ to the set of representations of TJ; and, together with Proposition 10.7, that the image of the map is contained in the set of representations of TJ that extend ρ_χ. (This latter fact can also be observed directly; see Remark 12.2.) We show that it is a surjection by a counting argument. *A priori*, we do not know how many extensions there are of ρ_χ from J to TJ, but certainly there are no more than

$$[TJ\colon J] = [T\colon T\cap J] = [T\colon T_r]\,.$$

Since there are *exactly* $[T\colon T_r]$ extensions of $\chi|_{T_r}$ to T, the (injective) map must be surjective. $\square$

It remains to show that our parametrization agrees with that of Yu [55].

Proposition 12.8. *The representation $\sigma(T,\psi)$ of TJ is equivalent to the inducing representation constructed in* [55, §4] *from the datum $\big((\mathbf{T},\mathbf{G}), x_{\mathbf{T}}, (\psi,1)\big)$ (see §3 loc. cit.).*

Our argument uses results from §§4, 14.3 that will only be proven later; but the reader can check that there is no circularity involved.

Proof. We write $\sigma'(T,\psi)$ for Yu's inducing datum; it is denoted in [55] by ρ_1.

All extensions of (TJ_+,ψ) to TJ agree on $ZT_{0+}J_+$, and all have characters supported on the conjugacy classes that meet TJ_+ (see [55, §11]). Thus, $\sigma'(T,\psi) =$

$\sigma(T,\psi')$, where ψ' agrees with ψ on $ZT_{0+}J_+$. To determine ψ', it is enough to compute the character of $\sigma'(T,\psi)$ at an element of $TJ_+ \setminus ZT_{0+}J_+$. On that domain, the explicit construction of [55, §4] shows that $\sigma'(T,\psi) = \psi \otimes \tilde{\psi}$ (i.e., that $\sigma'(T,\psi)(tj_+) = \psi(t)\tilde{\psi}(t \ltimes j_+)$ for $t \in T$ and $j_+ \in J_+$), where $\tilde{\psi}$ is a representation of $T \ltimes J_+$ that is trivial on $T_{0+} \ltimes \{1\}$ and $\bar{\chi}$-isotypic on $1 \ltimes J_+$ (see Theorem 11.5 *loc. cit.*). From [6, Proposition 3.8], the character of $\tilde{\psi}$ at g is $\varepsilon(\psi, g)$. From Proposition 14.9 and Lemma 4.2, we see that $\varepsilon(\psi,\gamma) = (-1)^{r+1} = -1$. Thus, $\sigma(\psi)$ and $\sigma'(\psi)$ have the same character. □

13. Murnaghan–Kirillov theory

Our calculations of character values near the identity (see Theorems 14.20 and 15.2) rely on Murnaghan–Kirillov theory, i.e., on asymptotic descriptions of the character in terms of Fourier transforms of orbital integrals (see §7.2). In the ordinary case, we have a 'single-orbit' theory, i.e., only one orbital integral is involved (see Proposition 13.13); but, in the exceptional case, the situation is more complicated (see Proposition 13.14).

In the depth-zero case, we use results of [16], which require Hypothesis 1.4.

13.1. **Lusztig's generalized Green functions.** Put $\mathsf{G} = \mathrm{SL}_{2/\mathfrak{f}}$, and write U for the set of unipotent elements in G. Lusztig [29, Lemma 25.4] has described the space of class functions on $\mathsf{U}(\mathfrak{f})$ (or, rather, on the set of unipotent elements in any finite group of Lie type) in terms of *generalized Green functions* [28, (8.3.1)]. In our setting, there are only two such functions that we need to consider.

Definition 13.1 ([18, Definition 4.1])**.** The *(elliptic) Green function* $Q^{\mathsf{G}}_{\mathsf{T}^\epsilon}$ is the restriction to $\mathsf{U}(\mathfrak{f})$ of $R^{\mathsf{G}}_{\mathsf{T}^\epsilon,\psi}$ for any character ψ of $\mathsf{T}^\epsilon(\mathfrak{f})$ (for example, $\psi = 1$).

Definition 13.2. The *Lusztig function* ${}^0f$ is defined by

$$ {}^0f := \left[\mathrm{Int}(\mathrm{SL}_2(\mathfrak{f}))\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right] - \left[\mathrm{Int}(\mathrm{SL}_2(\mathfrak{f}))\begin{pmatrix}0 & \epsilon\\ 0 & 0\end{pmatrix}\right] $$

(i.e., the difference of the characteristic functions of the two non-trivial nilpotent orbits in $\mathfrak{sl}_2(\mathfrak{f})$). We will view this as a function only on the set of nilpotent elements, or on all of $\mathfrak{sl}_2(\mathfrak{f})$, as convenient. See [53, p. 7]. By abuse of notation, we will also write ${}^0f$ for the function $u \longmapsto {}^0f(\mathrm{c}^{-1}(u))$ on $\mathsf{U}(\mathfrak{f})$.

When convenient, we think of a generalized Green function on $\mathsf{U}(\mathfrak{f})$ as a function on $\mathsf{G}(\mathfrak{f})$ by setting it equal to zero off the set of unipotent elements.

The definition of the Fourier transform (Definition 7.5) also makes sense for vector spaces V over finite fields; in that setting, the self-dual Haar measure assigns to V measure $\left|V\right|^{1/2}$. We also need a choice of non-trivial additive character on $\mathfrak{f} = R/\wp$. Since Λ is trivial on $\wp$, but non-trivial on R, it induces such a character in a natural way. The Fourier transform in the next lemma is taken with respect to the specified measure and character; and the constant $\mathcal{G}(\Lambda)$ is as in Definition 4.1.

Lemma 13.3. $\widehat{{}^0f} = \mathrm{sgn}_\varpi(-1)\mathcal{G}(\Lambda)\cdot{}^0f$.

Proof. This is [53, Proposition V.8] in the symplectic case, with $k = 1$. (See also [30, Corollary 10].) □

13.2. **Lifting from finite to local fields.** Suppose that σ is a cuspidal representation of $\mathsf{G}(\mathfrak{f}) = \mathrm{SL}_2(\mathfrak{f})$ [10, §9.1]. Then we may write the restriction to $\mathsf{U}(\mathfrak{f})$ of the character χ_σ of σ as a $\mathbb{C}$-linear combination

$$\chi_\sigma|_{\mathsf{U}(\mathfrak{f})} = \sum_{\mathcal{G}} c_\sigma(\mathcal{G})\mathcal{G}, \tag{13.4}$$

where the sum runs over $\left\{Q^{\mathsf{G}}_{\mathsf{T}^\epsilon}, {}^0 f\right\}$. (As with all the results of this section, an appropriate analogue of this statement holds in a very general setting; we shall exhibit an explicit linear combination in all cases of interest.)

We will show how to lift this finite-field formula (over $\mathfrak{f}$) to the local-field setting (over k). We warn the reader that our discussion will involve both generalized Green functions, denoted as above by $\mathcal{G}$, and a fourth root of unity, denoted by $\mathcal{G}(\Lambda)$. Context should make clear which is meant.

For the remainder of this section, put $x = x_{\mathrm{L}}$, so that $G_{x,0:0+} \cong \mathrm{SL}_2(\mathfrak{f})$.

Notation 13.5. If f is any function on $\mathrm{SL}_2(\mathfrak{f}) = G_{x,0:0+}$, then we denote by $\dot{f}$ the function on G defined by

$$\dot{f}(g) = \begin{cases} f(\bar{g}), & \text{if } g \in G_{x,0} \text{ has image } \bar{g} \in \mathsf{G}_x(\mathfrak{f}) \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, if f is any function on $\mathfrak{sl}_2(\mathfrak{f})$, then we inflate and extend it to a function $\dot{f}$ on $\mathfrak{g}$. (This is the function denoted by $f_{\{x\}}$ in [17, p. 3].)

Notation 13.6. Let $\mathrm{d}\ell$ be the Haar measure on $G_{x,0}$ that assigns it total mass 1. (We shall preserve the notation $\mathrm{d}g$ for the Haar measure on G specified in §6.)

Now we may inflate σ to a representation $\dot{\sigma}$ of $\mathrm{SL}_2(R) = G_{x,0}$. From, for example, Proposition 9.5, the representation $\pi = \mathrm{Ind}_{G_{x,0}}^G \dot{\sigma}$ is an irreducible, hence supercuspidal, representation of G, so its character may be computed by Harish-Chandra's integral formula [22, p. 94]:

$$\Theta_\pi(\gamma) = \frac{\deg_{\mathrm{d}g/\mathrm{d}z}(\pi)}{\chi_\sigma(1)} \int_{G/Z(G)} \int_{G_{x,0}} \dot{\chi}_\sigma\bigl(\mathrm{Int}(g\ell)\gamma\bigr)\mathrm{d}\ell\,\frac{\mathrm{d}g}{\mathrm{d}z}, \quad \gamma \in G^{\mathrm{rss}}. \tag{13.7}$$

We will use the decomposition (13.4) to evaluate this integral formula on the topologically unipotent set G_{0+} (see [17, (5)]), but first we need a convergence result. Fix $\gamma \in G^{\mathrm{rss}} \cap G_{0+}$, and write $\gamma = \mathrm{c}(Y)$ with $Y \in \mathfrak{g}^{\mathrm{rss}} \cap \mathfrak{g}_{0+}$.

The intersection with $G_{x,0}$ of the G-orbit of γ projects to $\mathsf{U}(\mathfrak{f})$ in $G_{x,0:0+}$, so that the expression $\dot{\mathcal{G}}\bigl(\mathrm{Int}(g\ell)\gamma\bigr)$ makes sense (see Notation 13.5).

Lemma 13.8. *If $\mathcal{G} \in \left\{Q^{\mathsf{G}}_{\mathsf{T}^\epsilon}, {}^0 f\right\}$, then*

$$g \longmapsto \int_{G_{x,0}} \dot{\mathcal{G}}\bigl(\mathrm{Int}(g\ell)\gamma\bigr)\mathrm{d}\ell$$

is a compactly supported function on G.

Proof. From [22, Lemma 23], we know that both

$$g \longmapsto \int_{G_{x,0}} \dot{\chi}_\sigma\bigl(\mathrm{Int}(g\ell)\gamma\bigr)\mathrm{d}\ell \quad \text{and} \quad g \longmapsto \int_{G_{x,0}} \dot{R}^{\mathsf{G}}_{\mathsf{T}^\epsilon,\psi}\bigl(\mathrm{Int}(g\ell)\gamma\bigr)\mathrm{d}\ell$$

are compactly supported functions on G (when $\psi \neq \psi^{-1}$). The result for $\mathcal{G} = Q^{\mathsf{G}}_{\mathsf{T}^\epsilon}$ follows (see also [16, Lemma 10.0.6]).

For $\mathcal{G} = {}^0 f$, choose a cuspidal representation σ for which $c_\sigma({}^0 f) \neq 0$ (namely, $\sigma = R^{\pm}_{\mathsf{T}^\epsilon,\psi_0}$). Since

$$ {}^0 f = \frac{1}{c_\sigma({}^0 f)}\bigl(\chi_\sigma - c_\sigma(Q^{\mathsf{G}}_{\mathsf{T}^\epsilon})Q^{\mathsf{G}}_{\mathsf{T}^\epsilon}\bigr), $$

the result follows. □

By (13.4) and Lemma 13.8, since

$$ \frac{\deg_{\mathrm{d}g/\mathrm{d}z}(\pi)}{\chi_\sigma(1)} = \mathrm{meas}_{\mathrm{d}g/\mathrm{d}z}(\mathrm{SL}_2(R)/Z(G))^{-1} = \frac{2q^{1/2}}{q^2-1}, $$

(13.7) becomes

$$ \Theta_\pi(\gamma) = \frac{2q^{1/2}}{q^2-1}\sum_{\mathcal{G}} c_\sigma(\mathcal{G})\int_{G/Z(G)}\int_{G_{x,0}}\dot{\mathcal{G}}\bigl(\mathrm{Int}(g\ell)\gamma\bigr)\mathrm{d}\ell\,\frac{\mathrm{d}g}{\mathrm{d}z}, \tag{13.9} $$

so we wish to understand these integrals of Green functions.

Lemma 13.10. *Under Hypothesis 1.4,*

$$ \frac{2q^{1/2}}{q^2-1}\int_{G/Z(G)}\int_{G_{x,0}}\dot{Q}^{\mathsf{G}}_{\mathsf{T}^\epsilon}(\mathrm{Int}(g\ell)\gamma)\mathrm{d}\ell\,\frac{\mathrm{d}g}{\mathrm{d}z} = -\hat{\mu}^G_{X_1^{\epsilon,1}}(Y), $$

where the orbital integral is taken with respect to the measure $\mathrm{d}g/\mathrm{d}t^\epsilon$ *on* G/T^ϵ*.*

Proof. This follows from [16, §9.2 and Lemma 12.4.3] upon noting that the measure with respect to $\mathrm{d}z$ of $Z(G)_0 = \{1\}$ (there denoted by Z_J) is 1; that the measure with respect to $\mathrm{d}g$ of $G_{x,0} = \mathrm{SL}_2(R)$ is $q^{-1/2}(q^2-1)$; and that

$$ 2\int_{G/Z(G)} f(g)\frac{\mathrm{d}g}{\mathrm{d}z} = \int_G f(g)\mathrm{d}g, \quad f \in C_{\mathrm{c}}^\infty(G/Z(G)). $$ □

Lemma 13.11.

$$ \begin{aligned} \frac{2q^{1/2}}{q^2-1}&\int_{G/Z(G)}\int_{G_{x,0}}({}^0 f)^{\cdot}(\mathrm{Int}(g\ell)\gamma)\mathrm{d}\ell\,\frac{\mathrm{d}g}{\mathrm{d}z} \\ &= \frac{\mathcal{G}(\Lambda)}{2q}\Bigl[\bigl(\hat{\mu}^G_{X_1^{\varpi,1}} - \hat{\mu}^G_{X_1^{\varpi,\epsilon}}\bigr) + \bigl(\hat{\mu}^G_{X_1^{\epsilon\varpi,1}} - \hat{\mu}^G_{X_1^{\epsilon\varpi,(\epsilon^{-1})}}\bigr)\Bigr](Y), \end{aligned} $$

where the orbital integrals are taken with respect to the measures $\mathrm{d}g/\mathrm{d}t^{\theta,\eta}$ *on* $G/T^{\theta,\eta}$*, with* $\theta \in \{\varpi, \epsilon\varpi\}$ *and* $\eta \in \{1, \epsilon\}$*.*

Proof. Following the proof of [17, Lemma 5.3.1], define the distribution $D_{{}^0 f}$ on $\mathfrak{g}$ by

$$ D_{{}^0 f}(F) = \int_{\mathfrak{g}}\int_G\int_{G_{x,0}}({}^0 f)^{\cdot}\bigl(\mathrm{Ad}(g\ell)Y\bigr)F(Y)\mathrm{d}\ell\,\mathrm{d}g\,\mathrm{d}Y, \quad F \in C_{\mathrm{c}}^\infty(\mathfrak{g}), $$

where $\mathrm{d}Y$ is the self-dual Haar measure on $\mathfrak{g}$ (so that $\mathrm{meas}_{\mathrm{d}Y}(\mathfrak{sl}_2(R)) = q^{3/2}$).

By Lemma 13.8, the innermost integral defines a compactly supported function on $\mathfrak{g}\times G$, so that we may switch the order of the outer two integrals (over $\mathfrak{g}$ and G). For fixed $g \in G$, the integrand is compactly supported as a function on $G_{x,0}\times\mathfrak{g}$, so we may switch the (now) inner two integrals (over $\mathfrak{g}$ and $G_{x,0}$), obtaining

$$ D_{{}^0 f}(F) = \int_G\int_{G_{x,0}}\int_{\mathfrak{g}}({}^0 f)^{\cdot}\bigl(\mathrm{Ad}(g\ell)Y\bigr)F(Y)\mathrm{d}Y\,\mathrm{d}\ell\,\mathrm{d}g, \quad F \in C_{\mathrm{c}}^\infty(\mathfrak{g}). $$

From [53, Lemme III.2], the innermost integral defines a compactly supported function on $G\times G_{x,0}$, so that we may switch the order of the outer integrals. Absorbing the integral over $G_{x,0}$ into that over G, and recalling that $x = x_{\mathrm{L}}$, we find that our distribution $D_{{}^0 f}$ is equal to

$$|\mathsf{G}(\mathfrak{f})|\, q^{-3/2}\phi_{(1,0,\emptyset,\emptyset,\emptyset)} = \frac{q^2-1}{q^{1/2}}\phi_{(1,0,\emptyset,\emptyset,\emptyset)},$$

where $\phi_{(1,0,\emptyset,\emptyset,\emptyset)}$ is as defined by Waldspurger [53, p. 53].

By Lemma 13.3, for all $F \in C_{\mathrm{c}}^\infty(\mathfrak{g})$, we have

$$\begin{aligned}\operatorname{sgn}_\varpi(-1)\mathcal{G}(\Lambda)\, D_{{}^0 f}(F) &= \int_G \int_{\mathfrak{g}} \widehat{({}^0 f)^{\cdot}}\bigl(\mathrm{Ad}(g)Y\bigr)F(Y)\mathrm{d}Y\,\mathrm{d}g \\ &= \int_G \int_{\mathfrak{g}} ({}^0 f)^{\cdot}\bigl(\mathrm{Ad}(g)Y\bigr)\hat{F}(Y)\mathrm{d}Y\,\mathrm{d}g \\ &= D_{{}^0 f}(\hat{F}) \\ &= \frac{q^2-1}{q^{1/2}}\phi_{(1,0,\emptyset,\emptyset,\emptyset)}(\hat{F})\end{aligned}$$

By [51, Lemma 6.2], the above equality becomes

$$\frac{q^{1/2}}{q^2-1}D_{{}^0 f} = \mathcal{G}(\Lambda)\hat{\phi}_{(1,0,\emptyset,\emptyset,\emptyset)}.$$

In the notation of [53, §I.9], we have $\widehat{\mathcal{H}} = C_{\mathrm{c}}^\infty(\mathfrak{g}_{\mathrm{tn}}) = C_{\mathrm{c}}^\infty(\mathfrak{g}_{0+})$. The result now follows from [53, p. 70 and Proposition IV.3], upon adjusting (as in Lemma 13.10) for the difference between integration over G and $G/Z(G)$. (See also [53, p. 7], where the constants are not completely explicated.) □

Lemmas 13.10 and 13.11 allow us to re-write (13.9) as

(13.12)
$$\begin{aligned}\Theta_\pi(\gamma) = &-c_\sigma(Q^{\mathsf{G}}_{\mathsf{T}^\epsilon})\hat{\mu}^G_{X_1^{\epsilon,1}}(Y) \\ &+ c_\sigma({}^0 f)\frac{\mathcal{G}(\Lambda)}{2q}\bigl[(\hat{\mu}^G_{X_1^{\varpi,1}} - \hat{\mu}^G_{X_1^{\varpi,\epsilon}}) + (\hat{\mu}^G_{X_1^{\epsilon\varpi,1}} - \hat{\mu}^G_{X_1^{\epsilon\varpi,(\epsilon^{-1})}})\bigr](Y).\end{aligned}$$

13.3. **Character expansions.**

Proposition 13.13. *Suppose that π is an ordinary supercuspidal representation of G, and put $r = \mathrm{d}(\pi)$. If*

- *$\gamma \in G^{\mathrm{rss}} \cap G_{r+}$ and Hypothesis 1.4 holds, or*
- *$r > 0$ and $\gamma \in G^{\mathrm{rss}} \cap G_r$,*

then

$$\Theta_\pi(\gamma) = \deg(\pi)\cdot\hat{\mu}^G_{X_\pi}\bigl(\mathrm{c}^{-1}(\gamma)\bigr).$$

Here, X_π is the regular, semisimple element associated to π in Notations 9.7 and 10.17. Put $T = C_G(X_\pi)$.

Recall from Definition 7.7 that $\hat{\mu}^G_{X_\pi}$ depends on a choice of measure on G/T, and from §14.2 that $\deg(\pi)$ depends on a choice of measure on $G/Z(G)$. The actual choices of measure in the above proposition are unimportant; they need only be *consistent*, in the sense that the measure of a subset of G/T is the same as the measure of its pull-back (along the natural projection) in $G/Z(G)$; i.e., that the 'quotient measure' on $T/Z(G)$ assigns it total mass 1. (Note that the measures $\mathrm{d}g/\mathrm{d}t^\epsilon$ and $\mathrm{d}g/\mathrm{d}z$ of §6 are *not* consistent in this sense.)

Proof. If $r > 0$, then this follows from [4, Theorem 6.3.1 and Remark 6.3.2] or [6, Corollary]. (There are other, similar results in the literature, but most of them make stronger assumptions on p [25, 26, 35–39].)

If $r = 0$, then recall from Remark 9.8 that it suffices to consider the case where π is induced from the inflation to $G_{x_{\mathrm{L}},0}$ of the representation $\sigma = -R^{\mathsf{G}_{x_{\mathrm{L}}}}_{\mathsf{T}^\epsilon,\psi}$ of $\mathrm{SL}_2(\mathfrak{f}) \cong \mathsf{G}_{x_{\mathrm{L}}}(\mathfrak{f})$. In this case, by Definition 13.1, $c_\sigma(Q^{\mathsf{G}_{x_{\mathrm{L}}}}_{\mathsf{T}^\epsilon}) = -1$ and $c_\sigma({}^0 f) = 0$; so (13.12) and [16, §5.3] give

$$\begin{aligned}\Theta_\pi(\gamma) &= \hat\mu^G_{X_\pi}\bigl(\mathrm{c}^{-1}(\gamma)\bigr)\\ &= \frac{2q^{1/2}}{q+1}\cdot \deg_{\mathrm{d}g/\mathrm{d}z}(\pi)\hat\mu^G_{X_\pi}\bigl(\mathrm{c}^{-1}(\gamma)\bigr),\end{aligned}$$

where the orbital integral is computed with respect to the measure $\mathrm{d}g/\mathrm{d}t^\epsilon$ on G/T^ϵ. (As in the proof of Lemma 13.10, we adapt the results of [16, §5.3] to account for the fact that we are working on $G/Z(G)$, whereas they work with G/Z, where $\mathbf{Z} = \{1\}$ is the maximal k-split torus in $Z(G)$.) As mentioned above, the measures $\mathrm{d}g/\mathrm{d}z$ and $\mathrm{d}g/\mathrm{d}t^\epsilon$ are not consistent; in fact,

$$\mathrm{meas}_{\mathrm{d}t^\epsilon/\mathrm{d}z}(T^\epsilon/Z(G)) = \frac{q+1}{2q^{1/2}}.$$

Adjusting either measure to achieve consistency thus gives the desired result. □

Proposition 13.14. *If $\gamma \in G^{\mathrm{rss}} \cap G_{0+}$ and Hypothesis 1.4 holds, then*

$$\Theta_{\pi^\pm}(\gamma) = \tfrac12\hat\mu^G_{X_1^{\epsilon,1}}\bigl(\mathrm{c}^{-1}(\gamma)\bigr) \pm \tfrac14 q^{-1/2}\Bigl[\bigl(\hat\mu^G_{X_1^{\varpi,1}} - \hat\mu^G_{X_1^{\varpi,\epsilon}}\bigr) + \bigl(\hat\mu^G_{X_1^{\epsilon\varpi,1}} - \hat\mu^G_{X_1^{\epsilon\varpi,\epsilon}}\bigr)\Bigr],$$

where $\pi^\pm = \pi^\pm(T^{\epsilon,1},\psi_0^1)$.

Proof. By Definitions 9.2 and 9.6, $\pi^\pm$ is induced from the inflation to $G_{x_{\mathrm{L}},0}$ of the representation

$$\sigma = -\tfrac12 R^{\mathsf{G}_{x_{\mathrm{L}}}}_{\mathsf{T}^\epsilon,\psi_0} \pm \tfrac12 q^{1/2}\mathcal{G}(\Lambda)^{-1}\cdot {}^0 f$$

of $\mathrm{SL}_2(\mathfrak{f}) \cong \mathsf{G}_{x_{\mathrm{L}}}(\mathfrak{f})$. In this setting, by Definition 13.1, $c_\sigma(Q^{\mathsf{G}_{x_{\mathrm{L}}}}_{\mathsf{T}^\epsilon}) = -\frac12$ and $c_\sigma({}^0 f) = \pm\frac12 q^{1/2}\mathcal{G}(\Lambda)^{-1}$; so (13.12) gives the desired formula. □

14. 'Ordinary' supercuspidal characters

Let π be an ordinary supercuspidal representation. By Definitions 9.6 and 10.13, $\pi = \pi(T,\psi)$ for some (depth-zero or positive-depth) supercuspidal parameter (T,ψ).

Definition 14.1. By Remarks 9.8 and 10.19, we may, and do, assume that $T = T^{\theta,1}$ for some $\theta \in \{\epsilon, \varpi, \epsilon\varpi\}$. Since the character formulas for the case $\theta = \epsilon\varpi$ are, *mutatis mutandis*, the same as those for the case $\theta = \varpi$, we further restrict to the cases $\theta \in \{\epsilon, \varpi\}$. We say that we are in the *unramified case* if $\theta = \epsilon$, and in the *ramified case* if $\theta = \varpi$.

Put

- $r = \mathrm{d}(\psi)$ and $s = r/2$;
- $x = x_{\mathbf{T}} \in \{x_{\mathrm{L}}, x_{\mathrm{C}}\}$ (see Definition 5.3); and
- $X = X_\pi$ and $\Lambda' = \Lambda'_\pi$ (see Notations 9.7 and 10.17).

Finally, write h for the *conductor* of ψ, in the sense of [42, p. 1232]. Then $h = \lceil r \rceil = r + 1 - \frac{1}{2}\operatorname{ord}(\theta)$, so $h = r + 1$ in the unramified case and $h = r + \frac{1}{2}$ in the ramified case. By (1.3), $\mathrm{d}(\Lambda') = r - \frac{1}{2}\operatorname{ord}(\theta) = h - 1$.

Note that Λ' does not bear the same relationship to Λ as does Φ' to Φ in [51, Notation 5.2].

14.1. **Indices.**

Lemma 14.2. *Suppose that $i > 0$. In the unramified case,*

$$[TG_{x,i} \colon TG_{x,i+}] = \begin{cases} q^2, & i \in \mathbb{Z} \\ 1, & i \notin \mathbb{Z}. \end{cases}$$

In the ramified case,

$$[TG_{x,i} \colon TG_{x,i+}] = \begin{cases} q, & 2i \in \mathbb{Z} \\ 1, & 2i \notin \mathbb{Z}. \end{cases}$$

Proof. By Lemma 5.4(b), we have compatible isomorphisms $G_{x,i:i+} \cong \mathfrak{g}_{x,i:i+}$ and $T_{i:i+} \cong \mathfrak{t}_{i:i+}$, so it suffices to compute

$$\left|\mathfrak{g}_{x,i:i+}\right| \cdot \left|\mathfrak{t}_{i:i+}\right|^{-1} = \exp_q\bigl(\dim_{\mathfrak{f}}(\mathfrak{g}_{x,i:i+}) - \dim_{\mathfrak{f}}(\mathfrak{t}_{i:i+})\bigr).$$

We use the explicit descriptions of §5.1.

In the unramified case, both quotients are trivial unless $i \in \mathbb{Z}$, in which case the $\mathfrak{f}$-vector spaces $\mathfrak{g}_{x,i:i+} \cong \mathfrak{g}_{x,0:0+}$ and $\mathfrak{t}_{i:i+} \cong \mathfrak{t}_{0:0+}$ are isomorphic to $\mathfrak{sl}_2(\mathfrak{f})$ and a Cartan subalgebra, hence are 3- and 1-dimensional, respectively. (Explicitly, they are spanned by $\varpi^i \cdot \bigl\{\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)\bigr\}$ and $\bigl\{\varpi^i \cdot X^\epsilon\bigr\}$).

In the ramified case, both quotients are trivial unless $2i \in \mathbb{Z}$. If $i \in \mathbb{Z}$, then $\mathfrak{g}_{x,i:i+} \cong \mathfrak{g}_{x,0:0+}$ and $\mathfrak{t}_{i:i+} \cong \mathfrak{t}_{0:0+}$, are isomorphic to a *split* Cartan subalgebra $\mathfrak{a}(\mathfrak{f})$ of $\mathfrak{sl}_2(\mathfrak{f})$, and an *elliptic* Cartan subalgebra *of* $\mathfrak{a}(\mathfrak{f})$, hence are 1- and 0-dimensional, respectively. (An explicit basis for $\mathfrak{g}_{x,i:i+}$ is $\bigl\{\varpi^i \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\bigr\}$.) If $i \in \mathbb{Z} + \frac{1}{2}$, then $\mathfrak{g}_{x,i:i+}$ has basis $\varpi^i \cdot \{\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \varpi\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)\}$, and $\mathfrak{t}_{i:i+}$ has basis $\bigl\{\varpi^i \cdot X^\varpi\bigr\}$. □

Lemma 14.3. *If $\gamma \in T \setminus Z(G)T_r$ and $d = \mathrm{d}_+(\gamma)$, in the sense of Definition 3.5, then*

$$\bigl[T_{0+}G_{x,(r-d)/2} \colon T_{0+}G_{x,s}\bigr] \cdot \bigl[T_{0+}G_{x,((r-d)/2)+} \colon T_{0+}G_{x,s+}\bigr] = \left|D_G(\gamma)\right|^{-1}.$$

Proof. Note that we could replace the first index by $\bigl[TG_{x,(r-d)/2} \colon TG_{x,s}\bigr]$, and similarly for the second. By Lemma 14.2, in the unramified case, the product is

$$\begin{aligned}
&\Bigl(\prod_{\substack{(r-d)/2 \le i < s \\ i \in \mathbb{Z}}} q^2\Bigr) \cdot \Bigl(\prod_{\substack{(r-d)/2 < i \le s \\ i \in \mathbb{Z}}} q^2\Bigr) \\
&\quad = \exp_{q^2}\Bigl(\Bigl|\mathbb{Z} \cap [\tfrac{r-d}{2}, s)\Bigr| + \Bigl|\mathbb{Z} \cap (\tfrac{r-d}{2}, s]\Bigr|\Bigr) \\
&\quad = \exp_{q^2}\Bigl|\mathbb{Z} \cap \Bigl([\tfrac{r-d}{2}, s) \cup [-s, -\tfrac{r-d}{2})\Bigr)\Bigr| \\
&\quad = \exp_{q^2}\Bigl|\mathbb{Z} \cap [s-d, s)\Bigr|;
\end{aligned}$$

and, in the ramified case, it is

$$\begin{aligned}
&\Bigl(\prod_{\substack{(r-d)/2\le i<s\\ 2i\in\mathbb{Z}}} q\Bigr)\cdot\Bigl(\prod_{\substack{(r-d)/2<i\le s\\ 2i\in\mathbb{Z}}} q\Bigr)\\
&\qquad=\exp_q\Bigl(\Bigl|\tfrac{1}{2}\mathbb{Z}\cap[\tfrac{r-d}{2},s)\Bigr|+\Bigl|\tfrac{1}{2}\mathbb{Z}\cap(\tfrac{r-d}{2},s]\Bigr|\Bigr)\\
&\qquad=\exp_q\Bigl(\Bigl|\mathbb{Z}\cap[r-d,r)\Bigr|+\Bigl|\mathbb{Z}\cap(-r,-(r-d)]\Bigr|\Bigr)\\
&\qquad=\exp_q\Bigl|\mathbb{Z}\cap[r-2d,r)\Bigr|.
\end{aligned}$$

We translated $\mathbb{Z}\cap[-s,-\frac{r-d}{2})$ by $r-d\in\mathbb{Z}$ (in the first case) and $\mathbb{Z}\cap(-r,-(r-d)]$ by $2(r-d)\in\mathbb{Z}$ (in the second case) without changing any cardinalities. In either case, we use the fact that $\bigl|\mathbb{Z}\cap[a-n,a)\bigr|=n$ for any $a\in\mathbb{R}$ and $n\in\mathbb{Z}$ (with $n=d$ in the unramified case, and $n=2d$ in the ramified case) to conclude by Lemma 3.8 that the index is

$$q^{2d}=|D_G(\gamma)|^{-1},$$

as desired. □

14.2. **Formal degrees.** The Schur orthogonality relations for finite groups have an analogue for supercuspidal (or even discrete-series) representations of p-adic groups; see [22, Theorem 1]. The analogue of the dimension of a finite-group representation is the so called *formal degree* of a supercuspidal representation π. This definition depends on a choice of Haar measure $\mathrm{d}\dot{g}$ on $G/Z(G)$, so we denote it by $\deg_{\mathrm{d}\dot{g}}(\pi)$. We have that $\deg_{c\cdot\mathrm{d}\dot{g}}(\pi)=c^{-1}\deg_{\mathrm{d}\dot{g}}(\pi)$ for $c\in\mathbb{R}_{>0}$.

Lemma 14.4. *In the unramified case,*

$$\deg_{\mathrm{d}_\epsilon\dot{g}}(\pi)=q^{r+1}.$$

In the ramified case,

$$\deg_{\mathrm{d}_\varpi\dot{g}}(\pi)=q^{h+1}.$$

Proof. If $r=0$, then $T=T^\epsilon$, and [16, §5.3] gives

$$\deg_{\mathrm{d}g/\mathrm{d}z}(\pi)=\frac{q^{1/2}}{(q+1)/2}=\frac{2q^{1/2}}{q+1}.$$

By §6, $\mathrm{d}_\epsilon\dot{g}=\dfrac{2}{q^{1/2}(q+1)}\cdot\dfrac{\mathrm{d}g}{\mathrm{d}z}$, so that

$$\deg_{\mathrm{d}_\epsilon\dot{g}}(\pi)=\frac{q^{1/2}(q+1)}{2}\cdot\frac{2q^{1/2}}{q+1}=q=q^{r+1}.$$

Now suppose that $r>0$. Write $\mathrm{d}\dot{g}$ for $\mathrm{d}_\epsilon\dot{g}$ or $\mathrm{d}_\varpi\dot{g}$, as appropriate; and recall the notation $K=TJ$ and $\rho=\rho_\chi$ from §10.2. By Remark 10.12, $K=TG_{x,s}$ and $\dim(\rho)=q^{1-(\lceil s\rceil-\lfloor s\rfloor)}$. Since $\pi=\operatorname{Ind}_K^G\rho$, we have that

$$(*)\qquad\begin{aligned}\deg_{\mathrm{d}\dot{g}}(\pi)&=\operatorname{meas}_{\mathrm{d}\dot{g}}(K/Z(G))^{-1}\cdot\dim(\rho)\\
&=[\mathrm{SL}_2(R)\colon K]\cdot\operatorname{meas}_{\mathrm{d}\dot{g}}(\mathrm{SL}_2(R)/Z(G))^{-1}\cdot q^{1-(\lceil s\rceil-\lfloor s\rfloor)}.\end{aligned}$$

In the unramified case, $\mathbf{T}=\mathbf{T}^\epsilon$, $x=x_{\mathrm{L}}$, and $G_{x,s}=G_{x,\lceil s\rceil}$. A direct computation shows that

$$[\mathrm{SL}_2(R)\colon TG_{x,1}]=[\mathrm{SL}_2(\mathfrak{f})\colon\mathsf{T}(\mathfrak{f})]=q(q-1),$$

so Lemma 14.2 gives

$$(**_{\mathrm{un}})\qquad [\mathrm{SL}_2(R)\colon K] = q(q-1)\cdot\bigl[TG_{x,1}\colon TG_{x,\lceil s\rceil}\bigr] = q(q-1)q^{2(\lceil s\rceil-1)}.$$

In the ramified case, $\mathbf{T} = \mathbf{T}^{\varpi}$, $x = x_{\mathrm{C}}$, and $G_{x,s} = G_{x,\lceil r\rceil/2}$. Further, $r \in \mathbb{Z} + \frac{1}{2}$, so $\lceil r\rceil = r + \frac{1}{2}$ and $1 - (\lceil s\rceil - \lfloor s\rfloor) = 0$. This time, a direct computation shows that

$$\bigl[\mathrm{SL}_2(R)\colon TG_{x,1/2}\bigr] = [\mathrm{SL}_2(\mathfrak{f})\colon Z(\mathrm{SL}_2)(\mathfrak{f})\mathsf{U}(\mathfrak{f})] = \tfrac{1}{2}(q^2-1),$$

where $\mathsf{U}(\mathfrak{f}) = \bigl\{\left(\begin{smallmatrix}1 & a\\ 0 & 1\end{smallmatrix}\right) \bigm| a \in \mathfrak{f}\bigr\}$. Now Lemma 14.2 gives

$$(**_{\mathrm{ram}})\qquad \begin{aligned}[\mathrm{SL}_2(R)\colon K] &= \tfrac{1}{2}(q^2-1)\cdot\bigl[TG_{x,1/2}\colon TG_{x,\lceil r\rceil/2}\bigr]\\ &= \tfrac{1}{2}(q^2-1)q^{\lceil r\rceil-1}\\ &= \tfrac{1}{2}(q^2-1)q^{r-1/2}.\end{aligned}$$

Combining $(*)$, $(**_{\mathrm{un}})$ *or* $(**_{\mathrm{ram}})$, and §6 gives the desired result. □

14.3. **Roots of unity.** The character formulas of [6] involve a number of roots of unity, defined in terms of roots (i.e., weights for the adjoint action of a maximal torus).

Notation 14.5. Write α_+ for the element

$$\begin{pmatrix} a & b\\ b\theta & a\end{pmatrix} \longmapsto (a^2 + b^2\theta) + 2ab\sqrt{\theta}$$

of $\mathrm{Hom}_{k_\theta}(\mathbf{T}, \mathrm{GL}_1)$, and $\alpha_- = -\alpha_+$. Then the set $\Phi(\mathbf{G}, \mathbf{T})$ of absolute roots of $\mathbf{T}$ in $\mathbf{G}$ is $\{\alpha_\pm\}$, and the set $\dot\Phi(\mathbf{G}, \mathbf{T})$ of orbits of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ on $\Phi(\mathbf{G}, \mathbf{T})$ is a singleton.

Remark 14.6. We have that $\alpha_\pm(\gamma) = \gamma^{\pm 2}$ for $\gamma \in T \cong C_\theta$ (see §3.1).

Notation 14.7. Throughout this section, we adopt [6, Notation 3.6]. In particular,

$$F_\alpha = k_\theta, \quad F_{\pm\alpha} = k, \quad \text{and} \quad \mathbf{G}_{\alpha_+} = \mathbf{G}_{\alpha_-} = \mathbf{G};$$

and, if $\theta = \epsilon$, then

$$\mathfrak{f}_\alpha^1 = \ker \mathrm{Norm}_{\mathfrak{f}_\alpha/\mathfrak{f}_{\pm\alpha}} = \ker \mathrm{Norm}_{\mathfrak{f}_\theta/\mathfrak{f}} =: \mathfrak{f}_\theta^1.$$

The notations on the left are as in [6], and those on the right are ours.

Two roots of unity enter into the character formulas in [6, Corollary 6.4], namely, $\varepsilon(\psi, \gamma)$ and $\mathfrak{G}(\psi, \gamma)$. (There is an unfortunate near-conflict between the notation ϵ, for an element of $R^\times$, and ε, which we use to stand for various signs. We hope that context will allow the reader to distinguish them.) In this section, we compute these quantities (and, more importantly, their product, in Corollary 14.13) in our special case (i.e., the group $\mathbf{G} = \mathrm{SL}_2$).

We begin by computing the 'depth-zero sign' $\varepsilon(\psi, \gamma)$.

Notation 14.8. Adopt [6, Notation 3.7]. In particular, $\Xi^1(\psi)$ is the set of (absolute) roots of $\mathbf{T}$ in $\mathbf{G}$ that "occur in the filtration (of $\mathfrak{sl}_2(k_\theta)$) associated to x at depth s", and whose value at γ is not a principal unit. By our explicit description of Moy–Prasad filtrations (see §5.1), we have that, if $x = x_{\mathrm{L}}$, then

$$\Xi^1(\psi) = \begin{cases} \Phi(\mathbf{G}, \mathbf{T}), & s \in \mathbb{Z} \text{ and } \gamma \notin Z(G)T_{0+}\\ \emptyset, & \text{otherwise;}\end{cases}$$

and, if $x = x_{\mathrm{C}}$, then

$$\Xi^1(\psi) = \begin{cases} \Phi(\mathbf{G}, \mathbf{T}), & s \in \mathbb{Z} + \frac{1}{2} \\ \emptyset, & \text{otherwise.} \end{cases}$$

Proposition 14.9. *Suppose that $\gamma \in T \setminus Z(G)T_r$. Put $d = \mathrm{d}_+(\gamma)$, in the notation of Definition 3.5. The root of unity $\varepsilon(\psi, \gamma)$ defined in* [6, Proposition 3.8] *is given by*

$$\varepsilon(\psi, \gamma) = \begin{cases} H(\Lambda', k_\theta) \operatorname{sgn}_\theta\bigl(\operatorname{Im}_\theta(\gamma)\bigr), & d = 0 \\ 1, & d > 0. \end{cases}$$

The factor $\operatorname{sgn}_\theta\bigl(\operatorname{Im}_\theta(\gamma)\bigr)$ will be shown in the course of the proof to be 1, so that we could leave it out; but we find it convenient to include it, for consistency with Proposition 14.12.

Proof. As remarked after [6, Proposition 3.8], since all roots are symmetric, we may use [21, Corollary 4.8.1], rather than Theorem 4.9.1 *loc. cit.*, to obtain the alternate formula

$$(*) \qquad \varepsilon(\psi, \gamma) = (-1)^{\left|\dot{\Xi}^1_{\mathrm{symm}}(\psi,\gamma)\right|} \prod_{\alpha \in \dot{\Xi}^1_{\mathrm{symm}}(\psi,\gamma)} \operatorname{sgn}_{\mathfrak{f}^1_\alpha}(\alpha(\gamma)),$$

where $\operatorname{sgn}_{\mathfrak{f}^1_\alpha}$ is the unique (non-trivial) order-2 character of $\mathfrak{f}^1_\alpha$. (The notation $\dot{\Xi}$ for a set of orbits is as in Notation 14.5.)

We have $\Xi^1(\psi, \gamma) = \emptyset$ if $d > 0$, i.e., $\gamma \in Z(G)T_{0+}$; in particular, this holds whenever $\theta \neq \epsilon$. Then $(*)$ becomes $\varepsilon(\psi, \gamma) = 1$, as desired. Therefore, we focus on the case where $d = 0$. In particular, $\Xi^1(\psi, \gamma) = \Xi(\psi)$.

By Lemma 2.4, $\operatorname{ord}\bigl(\operatorname{Im}_\epsilon(\gamma)\bigr) = 0$, so $\operatorname{sgn}_\epsilon\bigl(\operatorname{Im}_\epsilon(\gamma)\bigr) = 1$. Thus, by Lemma 4.2, it suffices to show that $\varepsilon(\psi, \gamma) = (-1)^{r+1}$.

If $s \notin \mathbb{Z}$, i.e., $r \notin 2\mathbb{Z}$, then $\Xi(\psi, \gamma) = \emptyset$, so that $(*)$ becomes $\varepsilon(\psi, \gamma) = 1 = (-1)^{r+1}$.

If $s \in \mathbb{Z}$, i.e., $r \in 2\mathbb{Z}$, then $\Xi^1_{\mathrm{symm}}(\psi, \gamma) = \Xi(\psi, \gamma) = \Phi(\mathbf{G}, \mathbf{T})$, whence $\dot{\Xi}^1_{\mathrm{symm}}(\psi, \gamma)$ is a singleton, and $\Xi^{\mathrm{symm}}(\psi, \gamma) = \emptyset$, so $(*)$ again becomes

$$\varepsilon(\psi, \gamma) = (-1)^1 \cdot 1 \cdot \operatorname{sgn}_{\mathfrak{f}^1_\epsilon}(\alpha_+(\gamma)) = -\operatorname{sgn}_{\mathfrak{f}^1_\epsilon}(\gamma^2) = -1 = (-1)^{r+1}. \qquad \square$$

Now we compute the 'positive-depth sign' $\mathfrak{G}(\psi, \gamma)$.

Notation 14.10. Adopt [6, Notation 5.2.11]. In particular, $\Upsilon(\psi, \gamma)$ is empty if $d = 0$, and otherwise is the set of (absolute) roots of $\mathbf{T}$ in $\mathbf{G}$ that "occur in the filtration (of $\mathfrak{sl}_2(k_\theta)$) associated to x at depth $(r - d)/2$". (Recall that d is the (maximal) depth of γ.) As in Notation 14.8, we have that, if $x \in \{x_{\mathrm{L}}, x_{\mathrm{R}}\}$, then

$$\Upsilon(\psi, \gamma) = \begin{cases} \Phi(\mathbf{G}, \mathbf{T}), & (r - d)/2 \in \mathbb{Z} \\ \emptyset, & \text{otherwise;} \end{cases}$$

and, if $x = x_{\mathrm{C}}$, then

$$\Upsilon(\psi, \gamma) = \begin{cases} \Phi(\mathbf{G}, \mathbf{T}), & r - d \in \mathbb{Z} \\ \emptyset, & \text{otherwise.} \end{cases}$$

We set $\Upsilon_{\mathrm{symm,unram}}(\psi,\gamma)$ (respectively, $\Upsilon_{\mathrm{symm,ram}}(\psi,\gamma)$) equal to $\Upsilon(\psi,\gamma)$ in the unramified (respectively, ramified) case, and to the empty set otherwise.

Our calculation of $\mathfrak{G}(\psi,\gamma)$ in the ramified case (Proposition 14.12) will involve the quantity $S(\psi)$ defined in [42, p. 1234]. The measure used in its definition is not specified there, but the statement (on p. 1235 *loc. cit.*) that $S(\psi)^2 = \operatorname{sgn}_\varpi(-1)$ holds only for the normalization chosen below.

Lemma 14.11. *In the ramified case,*

$$\begin{aligned} S(\psi) &:= q^{1/2}\int_{R^\times} \operatorname{sgn}_\varpi(Y)\psi\Big(\frac{1+\varpi^{h-1}\sqrt{\varpi}Y}{1-\varpi^{h-1}\sqrt{\varpi}Y}\Big)\mathrm{d}Y \\ &= \operatorname{sgn}_\varpi(-1)^{h-1}H(\Lambda', k_\varpi), \end{aligned}$$

where $\mathrm{d}Y$ *is the Haar measure on* k *such that* $\operatorname{meas}_{\mathrm{d}Y}(R) = 1$*.*

Proof. By the definitions of c (see Lemma 2.3) and X, β, and Λ' (see Definitions 10.13 and 14.1),

$$\begin{aligned} \psi\Big(\frac{1+\varpi^{h-1}\sqrt{\varpi}Y}{1-\varpi^{h-1}\sqrt{\varpi}Y}\Big) &= \psi(\mathrm{c}(2\varpi^{h-1}\sqrt{\varpi}Y)) \\ &= \Lambda\big(\operatorname{tr}(X\cdot 2\varpi^{h-1}\sqrt{\varpi}Y\rangle)\big) = \Lambda(2\beta\varpi\cdot 2\varpi^{h-1}Y) = \Lambda'_{4\varpi^{h-1}}(Y). \end{aligned}$$

Since $\operatorname{meas}_{\mathrm{d}Y}(1+\wp) = q^{-1}$, and since the restriction of the additive Haar measure $\mathrm{d}Y$ to $R^\times$ is a multiplicative Haar measure,

$$\begin{aligned} S(\psi) &= q^{-1/2}\sum_{Y\in R^\times/(1+\wp)} \operatorname{sgn}_\varpi(Y)\Lambda'_{4\varpi^{h-1}}(Y) \\ &= \mathcal{G}(\Lambda'_{4\varpi^{h-1}}). \end{aligned}$$

By [51, Lemma 6.2] and Lemma 4.2,

$$\begin{aligned} S(\psi) &= \operatorname{sgn}_\varpi(-1)^{h-1}\mathcal{G}(\Lambda') \\ &= \operatorname{sgn}_\varpi(-1)^{h-1}H(\Lambda', k_\varpi). \end{aligned}$$

□

Proposition 14.12. *Suppose that* $\gamma \in T\setminus Z(G)T_r$*. Put* $d = \mathrm{d}_+(\gamma)$*, in the notation of Definition 3.5. The root of unity* $\mathfrak{G}(\psi,\gamma)$ *defined in* [6, Proposition 5.2.13] *is given by*

$$\mathfrak{G}(\psi,\gamma) = \begin{cases} H(\Lambda', k_\theta)\operatorname{sgn}_\theta\big(\operatorname{Im}_\theta(\gamma)\big), & d > 0 \\ 1, & d = 0. \end{cases}$$

Proof. We use the following formula from the cited proposition (adapted to our situation, per Notation 14.7) to compute $\mathfrak{G}(\psi,\gamma)$:

$$(*)\qquad \begin{aligned} \mathfrak{G}(\psi,\gamma) = (-1)^{\left|\dot\Upsilon_{\mathrm{symm}}(\psi,\gamma)\right|}\big(-\mathcal{G}(\Psi)\big)^{f(\dot\Upsilon_{\mathrm{symm,ram}}(\psi,\gamma))}\times \\ \prod_{\alpha\in\dot\Upsilon_{\mathrm{symm,ram}}(\psi,\gamma)} \operatorname{sgn}_{\mathfrak{f}}\big[\operatorname{Norm}_\varpi(w_\varpi)\mathrm{d}\alpha(X_\Psi)(\alpha(\gamma)-1)\big], \end{aligned}$$

where

- Ψ is a certain (additive) character of k (specified in [6, §1.1] and denoted there by Λ);
- the notation $f(\cdot)$ is defined by $f(\dot\Phi(\mathbf{G},\mathbf{T})) = f(k_{\alpha_+}/k) = 1$ and $f(\dot\emptyset) = 0$;

- X_Ψ is an element such that
$$\psi(\mathrm{c}(Y)) = \Psi\bigl(\mathrm{tr}(X_\Psi \cdot Y)\bigr) \quad \text{for all } Y \in \mathfrak{t}_r;$$
- $w_\varpi = \sqrt{\varpi}^{\,r-d}$; and
- the argument of $\mathrm{sgn}_{\mathfrak{f}}$, which lies in $R_\varpi^\times$, is implicitly regarded as an element of $\mathfrak{f}^\times$.

(The notation $\dot\Upsilon$ for a set of orbits is as in Notation 14.5.) The condition on w_ϖ can be satisfied only if $r - d \in \mathbb{Z}$; but this is always the case when $\Upsilon_{\mathrm{symm,ram}}(\psi, \gamma) \neq \emptyset$. This formula differs from the one in [6] in several ways.

- The original formula had $\mathfrak{G}_\Psi(\mathfrak{f})$ in place of $\mathcal{G}(\Psi)$; but [6, Definition 5.2.1 and Lemma 5.2.2] and [51, Lemma 6.2] show that they are equal (since $\mathrm{d}(\Psi) = 0$).
- The argument of $\mathrm{sgn}_{\mathfrak{f}_\alpha}$ in the original formula had a factor of $\frac{1}{2}e_\alpha$, where e_α is the ramification degree of k_α/k; but this factor collapses to 1 whenever $\alpha \in \Upsilon_{\mathrm{symm,ram}}(\psi, \gamma)$.
- The original formula had $\mathrm{d}\alpha^\vee(X^*)$ in place of $\mathrm{d}\alpha(X_\Psi)$; but these are the same once we taken into account our identification of $\mathfrak{g}$ and $\mathfrak{g}^*$.
- The product included an extra factor $\mathrm{sgn}_k(\mathbf{G}_{\pm\alpha})$, defined to be $+1$ if $\mathbf{G}_{\pm\alpha}$ is k-split and -1 otherwise. Since $\mathbf{G}_{\pm\alpha} = \mathrm{SL}_2$, this factor is 1.

Note that $(*)$ collapses to 1 unless $d > 0$, so we assume that.

In the unramified case, $\Upsilon_{\mathrm{symm,ram}}(\psi, \gamma) = \emptyset$, so $(*)$ becomes

$$(*_{\mathrm{un}}) \qquad \mathfrak{G}(\psi, \gamma) = (-1)^{\left|\dot\Upsilon_{\mathrm{symm}}(\psi,\gamma)\right|}.$$

If $(r - d)/2 \in \mathbb{Z}$, then $\dot\Upsilon_{\mathrm{symm}}(\psi, \gamma) = \dot\Phi(\mathbf{G}, \mathbf{T})$ is a singleton, so

$$(\dagger_{\mathrm{even}}) \qquad \mathfrak{G}(\psi, \gamma) = -1 = (-1)^{r+1}(-1)^d.$$

If $(r - d)/2 \notin \mathbb{Z}$, then $\dot\Upsilon_{\mathrm{symm}}(\psi, \gamma) = \emptyset$, so

$$(\dagger_{\mathrm{odd}}) \qquad \mathfrak{G}(\psi, \gamma) = 1 = (-1)^{r+1}(-1)^d.$$

In either case, Lemma 2.4 shows that $\mathrm{sgn}_\epsilon\bigl(\mathrm{Im}_\epsilon(\gamma)\bigr) = (-1)^d$; so that, by Lemma 4.2, $(\dagger_{\mathrm{even}})$ and $(\dagger_{\mathrm{odd}})$ both simplify to the desired formula.

In the ramified case, recall that $r - d \in \mathbb{Z}$; in particular, $\dot\Upsilon_{\mathrm{symm,ram}}(\psi, \gamma) = \dot\Upsilon_{\mathrm{symm}}(\psi, \gamma) = \dot\Phi(\mathbf{G}, \mathbf{T})$ is a singleton, and $f(\dot\Upsilon_{\mathrm{symm,ram}}(\psi, \gamma)) = f(k_\varpi/k) = 1$. Thus, $(*)$ becomes

$$(*_{\mathrm{ram}}) \qquad \mathfrak{G}(\psi, \gamma) = \mathcal{G}(\Psi)\,\mathrm{sgn}_{\mathfrak{f}}\bigl(\mathrm{Norm}_\varpi(w_\varpi)\mathrm{d}\alpha_+(X_\Psi)(\alpha_+(\gamma) - 1)\bigr).$$

Since Ψ and Λ are both non-trivial, additive characters of k, we have $\Lambda = \Psi_b$ for some $b \in k^\times$. Then we may take

$$(\ddagger) \qquad X_\Psi = bX = b\beta \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix};$$

and we note for future reference that, by [51, Lemma 6.2],

$$(\ddagger\ddagger) \qquad \mathrm{sgn}_\varpi(b\beta\varpi)\mathcal{G}(\Psi) = \mathrm{sgn}_\varpi(\beta\varpi)\mathcal{G}(\Lambda) = \mathcal{G}(\Lambda_{\beta\varpi}) = \mathcal{G}(\Lambda').$$

Since $\alpha_+(\gamma) = \gamma^2$, we have that $\alpha_+(\gamma) - 1 = (\gamma - \gamma^{-1})\gamma$. Similarly, we can calculate explicitly from Notation 14.5 and (‡) that $\mathrm{d}\alpha_+(X_\Psi) = 2b\beta\sqrt{\varpi}$, so that

$$\begin{aligned}\operatorname{sgn}_{\mathfrak f}\bigl[\operatorname{Norm}_\varpi(w_\varpi)\mathrm{d}\alpha(X_\Psi)(\alpha_+(\gamma)-1)\bigr] &= \operatorname{sgn}_{\mathfrak f}\bigl[\operatorname{Norm}_\varpi(w_\varpi)\cdot 4b\beta\varpi\cdot \tfrac{1}{2}\sqrt{\varpi}^{-1}(\gamma-\gamma^{-1})\cdot\gamma\bigr]\\ &= \operatorname{sgn}_{\mathfrak f}\bigl[\operatorname{Norm}_\varpi(w_\varpi)\cdot 4b\beta\varpi\cdot \operatorname{Im}_\varpi(\gamma)\bigr]\\ &= \operatorname{sgn}_\varpi\bigl[\operatorname{Norm}_\varpi(w_\varpi)\cdot 4b\beta\varpi\cdot \operatorname{Im}_\varpi(\gamma)\bigr]\\ &= \operatorname{sgn}_\varpi(b\beta\varpi)\cdot\operatorname{sgn}_\varpi\bigl(\operatorname{Im}_\varpi(\gamma)\bigr).\end{aligned}$$

(The crucial point in the transition from the second to the third line is that the argument lies in $R^\times$, not just $R_\varpi^\times$.) Thus, by (‡‡) and Lemma 4.2, ($*_{\mathrm{ram}}$) becomes

$$\begin{aligned}\mathfrak{G}(\psi,\gamma) &= \operatorname{sgn}_\varpi(b\beta\varpi)\mathcal{G}(\Psi)\cdot\operatorname{sgn}_\varpi\bigl(\operatorname{Im}_\varpi(\gamma)\bigr)\\ &= H(\Lambda',k_\varpi)\operatorname{sgn}_\varpi\bigl(\operatorname{Im}_\varpi(\gamma)\bigr).\end{aligned}$$ □

Corollary 14.13. *With the notation of Propositions 14.9 and 14.12,*

$$\varepsilon(\psi,\gamma)\mathfrak{G}(\psi,\gamma) = H(\Lambda',k_\theta)\operatorname{sgn}_\theta\bigl(\operatorname{Im}_\theta(\gamma)\bigr).$$

14.4. **Character values far from the identity.**

Theorem 14.14. *If $\gamma \notin Z(G)G_{0+}$, or $r > 0$ and $\gamma \notin Z(G)G_r$, then $\Theta_\pi(\gamma) = 0$ unless some G-conjugate of γ lies in T^θ. If $\gamma \in T^\theta$, then*

$$\Theta_\pi(\gamma) = \frac{1}{2}\operatorname{sgn}_\epsilon\bigl(\operatorname{Im}_\epsilon(\gamma)\bigr)\frac{\psi(\gamma)+\psi(\gamma^{-1})}{|D_G(\gamma)|^{1/2}}\bigl[(-1)^{r+1} + H(\Lambda',k_\epsilon)\bigr]$$

in the unramified case, and

$$\begin{aligned}\Theta_\pi(\gamma) = \frac{\operatorname{sgn}_\varpi\bigl(\operatorname{Im}_\varpi(\gamma)\bigr)}{2\,|D_G(\gamma)|^{1/2}}\Bigl\{&\psi(\gamma)\bigl[\operatorname{sgn}_\varpi(-1)^{h-1}S(\psi) + H(\Lambda',k_\varpi)\bigr] +\\ &\psi(\gamma^{-1})\bigl[\operatorname{sgn}_\varpi(-1)^{h}S(\psi) + H(\Lambda',k_\varpi)\bigr]\Bigr\}\end{aligned}$$

in the ramified case.

Recall that Λ' is as in Definition 14.1, so that $H(\Lambda',k_\theta)$ is computed in Lemma 4.2. We shall use this in the proof.

Proof. First suppose that $r = 0$. By [16, Lemma 9.3.1], we have that $\Theta_\pi(\gamma) = 0$ unless $\gamma \in Z(G)G_0$; so we assume that $\gamma \in G_0$. Then it has a *topological Jordan decomposition* $\gamma = \gamma_{\mathrm{ts}}\gamma_{\mathrm{tu}}$, with γ_{ts} topologically semisimple and γ_{tu} topologically unipotent (see §7 *loc. cit.*); and, if $\gamma \notin Z(G)G_{0+}$, then γ_{ts} is regular. By [16, Lemma 10.0.4], $\Theta_\pi(\gamma) = 0$ unless γ_{ts} is G-conjugate to an element of T^ϵ, so we assume that $\gamma_{\mathrm{ts}} \in T^\epsilon$. Thus the subgroup $\mathbf{G}_{\gamma_{\mathrm{ts}}} := C_{\mathbf{G}}(\gamma)^\circ = C_{\mathbf{G}}^{(0+)}(\gamma)$ of [16, p. 802] is just $\mathbf{T} = \mathbf{T}^\epsilon$, and the set

$$\widehat{\mathcal{T}}(\gamma_{\mathrm{ts}}) := \bigl\{\,(T' = \operatorname{Int}(g)T^\epsilon, \psi' = \psi\circ\operatorname{Int}(g)^{-1}) \bigm| \gamma_{\mathrm{ts}} \in T'\bigr\}$$

of §10 *loc. cit.* is just

$$\bigl\{\,(T^\epsilon,\psi') \bigm| \psi' = \psi\circ\operatorname{Int}(n)^{-1} \text{ for some } n \in N_G(T^\epsilon)\bigr\} = \bigl\{(T^\epsilon,\psi),(T^\epsilon,\psi^{-1})\bigr\}.$$

Further, $\gamma_{\mathrm{tu}} \in G_{\gamma_{\mathrm{ts}}} = T^\epsilon$, so that $\gamma = \gamma_{\mathrm{tu}}\gamma_{\mathrm{ts}} \in T^\epsilon$ as well.

Recall that $\psi \neq \psi^{-1}$. Now combining Lemmas 9.3.1 and Lemma 10.0.4 *loc. cit.* gives

$$(*_0)\qquad \begin{aligned}\Theta_\pi(\gamma) &= \varepsilon(\mathsf{G}_{x_{\mathrm{L}}}, \mathsf{T}) \sum_{(T',\psi')} \psi'(\gamma_{\mathrm{ts}}) R(T, T', 1)(\gamma_{\mathrm{tu}}) \\ &= -\bigl[\psi(\gamma_{\mathrm{ts}}) + \psi(\gamma_{\mathrm{ts}}^{-1})\bigr] R(T, T, 1)(\gamma_{\mathrm{tu}}),\end{aligned}$$

where

- the sum is taken over the orbits in $\widehat{\mathcal{T}}(\gamma_{\mathrm{ts}})$ under the natural (trivial) action of T;
- $\varepsilon(\mathsf{G}_{x_{\mathrm{L}}}, \mathsf{T}) = \varepsilon(\mathrm{SL}_{2/\mathfrak{f}}, \mathsf{T}) = (-1)^{\mathrm{rk}_{\mathfrak{f}}(\mathrm{SL}_2) - \mathrm{rk}_{\mathfrak{f}}(\mathsf{T})} = -1$ is the Kottwitz sign defined on p. 802 *loc. cit.*; and
- $R(T, T, 1)$ is the function defined in §9.2 *loc. cit.*

Since $\mathrm{d}(\psi) = 0$, we have that ψ is trivial on $\gamma_{\mathrm{tu}} \in T_{0+}$, so that

$$\psi(\gamma_{\mathrm{ts}}) = \psi(\gamma) \quad \text{and} \quad \psi(\gamma_{\mathrm{ts}}^{-1}) = \psi(\gamma^{-1}).$$

By §5.1 *loc. cit.*, and Lemma 7.8, since $R_{\mathsf{T}^\epsilon}^{\mathsf{T}^\epsilon}(1) = 1$, we have that

$$R(T, T, 1)(\gamma_{\mathrm{tu}}) = \varepsilon(\mathbf{T}, Z(\mathbf{G})) \hat{\mu}^T_{X^*}\bigl(\mathrm{c}^{-1}(\gamma_{\mathrm{tu}})\bigr) = 1,$$

where $\varepsilon(\mathbf{T}, Z(\mathbf{G}))$ is the Kottwitz sign, as above. By Lemmas 3.8 and 2.4, $|D_G(\gamma)| = 1$ and $\mathrm{sgn}_\epsilon\bigl(\mathrm{Im}_\epsilon(\gamma)\bigr) = 1$. Thus, since $r = 0$, we have by Lemma 4.2 that $(*_0)$ simplifies to the desired formula (in this case).

Now suppose that $r > 0$, and put $d = \mathrm{d}_+(\gamma)$, in the notation of Definition 3.5. Since γ does not lie in the G-domain $Z(G)G_r$, neither do any of its G-conjugates; so, with the notation of [5, §5 and Definition 8.3], we have for all $\gamma' \in \mathrm{Int}(G)\gamma$ that

$$\gamma'_{<r} = \gamma' \quad \text{and} \quad \gamma'_{\geq r} = 1,$$

so that $C^{(r)}_{\mathbf{G}}(\gamma')$ is the unique torus containing γ'. In particular, if $\gamma' \in T \cap \mathrm{Int}(G)\gamma$, then

$$\llbracket\gamma'; x, r\rrbracket = T_{0+}G_{x,(r-d)/2}, \quad \llbracket\gamma'; x, r+\rrbracket = T_{0+}G_{x,((r-d)/2)+},$$
$$\llbracket\gamma'; x, r\rrbracket_T = T_{0+}, \quad \text{and} \quad \llbracket\gamma'; x, r+\rrbracket_T = T_{0+}.$$

(There is a minor notational inconvenience if $r = 1$ in the unramified case, or $r = 1/2$ in the ramified case. In those cases, $\gamma'_{<r} = \gamma'_{\mathrm{ts}}$, and we use Lemma 7.8 to notice that $\hat{\mu}^T_{X_\pi}(\mathrm{c}^{-1}(\gamma'_{\geq r})) = \psi(\gamma'_{\geq r})$.) With the notation of [6, Definition 1.4.1], the set

$$\mathcal{T}\bigl((\mathbf{T}, \mathbf{G}), (r, r)\bigr) \cap \mathrm{Int}(G)\gamma := \bigl\{\, \gamma' \in \mathrm{Int}(G)\gamma \bigm| \gamma'_{<r} \in T \,\bigr\}$$

is $T \cap \mathrm{Int}(G)\gamma$. Since γ is regular, the intersection is just $\mathrm{Int}(N_G(T))\gamma$. Further, the equivalence relation $\overset{0}{\sim}$ on that set, defined in [6, Definition 6.5] by $\gamma' \overset{0}{\sim} \gamma''$ if and only if γ'' is conjugate in $T = C^{(r)}_T(\gamma')$ to γ', is the identity relation.

Therefore, by [6, Proposition 5.3.3 and Corollary 6.4],

$$(*_{>0})\qquad \begin{aligned}\Theta_\pi(\gamma) = \sum_{\gamma'} \bigl[T_{0+}G_{x,(r-d)/2} \colon T_{0+}G_{x,s}\bigr]^{1/2} \bigl[T_{0+}G_{x,((r-d)/2)+} \colon T_{0+}G_{x,s+}\bigr]^{1/2} \times \\ \mathfrak{G}(\psi, \gamma')\varepsilon(\psi, \gamma') \cdot \psi(\gamma') \cdot \hat{\mu}^T_{X^*}(\mathrm{c}^{-1}(1)\bigr),\end{aligned}$$

where the sum runs over $\mathrm{Int}(N_G(T))\gamma$, and $\hat{\mu}^T_{X^*}$ is the Fourier transform of the T-orbital integral of X^* (as in §7.2) with respect to the measure on the singleton

$T/C_T(X^*)$ that assigns it total measure 1. We have made use of two facts that simplify the quoted results.

- The representation τ_0 of [6, §2] is trivial.
- The inducing subgroup $K_\sigma = TG_{x,0+}$ of [6, §2] contains $H' = C_G^{(r)}(\gamma') = T$ for all conjugates γ' of γ.

By Lemma 14.3, the leading product of square roots is $|D_G(\gamma')|^{-1/2} = |D_G(\gamma)|^{-1/2}$. By Corollary 14.13, the product $\mathfrak{G}(\psi,\gamma')\varepsilon(\psi,\gamma')$ of roots of unity is

$$(-1)^{r+1}\operatorname{sgn}_\epsilon\bigl(\operatorname{Im}_\epsilon(\gamma)\bigr) = \tfrac{1}{2}\operatorname{sgn}_\epsilon\bigl(\operatorname{Im}_\epsilon(\gamma)\bigr)\bigl[(-1)^{r+1} + H(\Lambda', k_\epsilon)\bigr]$$

in the unramified case, and

$$\begin{aligned}\operatorname{sgn}_\varpi(-1)^{h-1}S(\psi)\operatorname{sgn}_\varpi\bigl(\operatorname{Im}_\varpi(\gamma)\bigr)&\\ = \tfrac{1}{2}\operatorname{sgn}_\varpi\bigl(\operatorname{Im}_\varpi(\gamma)\bigr)&\bigl[\operatorname{sgn}_\varpi(-1)^{h-1}S(\psi) + H(\Lambda', k_\varpi)\bigr]\end{aligned}$$

in the ramified case. By Lemma 7.8, $\hat\mu^T_{X^*}\bigl(\mathrm{c}^{-1}(1)\bigr) = \hat\mu^T_{X^*}(0) = 1$.

By §3.1:

- in the unramified case, $\operatorname{Int}(N_G(T))\gamma = \bigl\{\gamma^{\pm 1}\bigr\}$;
- in the ramified case, when $\operatorname{sgn}_\varpi(-1) = 1$, again $\operatorname{Int}(N_G(T))\gamma = \bigl\{\gamma^{\pm1}\bigr\}$, and $\mathfrak{G}(\psi,\gamma)\varepsilon(\psi,\gamma) = \operatorname{sgn}_\varpi(-1)^h S(\psi) + H(\Lambda', k_\varpi)$; and finally
- in the ramified case, when $\operatorname{sgn}_\varpi(-1) = -1$, now $\operatorname{Int}(N_G(T))\gamma = \{\gamma\}$, and $\operatorname{sgn}_\varpi(-1)^h S(\psi) + H(\Lambda', k_\varpi) = 0$.

In each case, we see that $(*_{>0})$ simplifies to the desired formula. □

Remark 14.15. The calculation in [42] has an error in the ramified case, when $\operatorname{sgn}_\varpi(-1) = 1$ and h is odd, so that their formulas for the character of $\Pi(\Lambda',\psi,k_\varpi)$ and ours for the character of $\pi(T^\varpi,\psi)$, which agree near the identity, differ by a sign far from the identity. Remarkably, correcting their formulas does not affect their computations in [43], which depend not on the individual characters $\Pi(\Phi,\psi,k_\varpi)$ and $\Pi(\Phi',\psi,k_\varpi)$ but rather on the sum $\Pi(\Phi,\psi,k_\varpi) + \Pi(\Phi',\psi,k_\varpi)$, which is unchanged; and also does not affect their computations in [44]. To see this latter is more complicated; but, fortunately, [44] writes out the necessary calculations explicitly in the ramified case. The only affected parts of the formula for $K_d(t_1,t_2)$ on p. 330–332 *loc. cit.* are (d), (e), and (f), and, even after correcting the error, the argument on p. 334 *loc. cit.* shows that they are all 0. In the formula for $K_d(t_1,t_2)$ on p. 337, the terms (d) and (e) no longer appear, and the term (f) is replaced by (f'). Since (f') involves the product of two characters, both affected by the sign error, nothing is changed.

We have used results of Shelstad [47, Theorem, p. 276], together with the formulas of [13, Appendix A.3–A.4], to confirm our calculations.

14.5. **Character values near the identity.** Recall that we have put $X = X_\pi$. By the formulas in [51], the values $\hat\mu^G_X(Y)$ arising in Proposition 13.13 can often be most conveniently expressed in terms of a number $\gamma_\Lambda(X,Y)$. We reproduce its definition, adapted to our current situation using Definition 14.1 (and Lemma 4.2). Note that $\mathfrak{t}^{\theta,\eta} = \mathfrak{t}^{\eta^2\theta,1}$, and $\operatorname{Im}_{\eta^2\theta}(Y) = \eta^{-1}\operatorname{Im}_\theta(Y)$.

Definition 14.16 ([51, Definition 6.5])**.** Recall that $X_\pi \in \mathfrak{t}^\theta$.

$$\gamma_\Lambda\bigl(X_\pi, Y\bigr) := \begin{cases} H(\Lambda', k_\theta)\operatorname{sgn}_\theta\bigl(\eta^{-1}\operatorname{Im}_\theta(Y)\bigr), & Y \in \mathfrak{t}^{\theta,\eta} \\ 1, & Y \in \mathfrak{a} \\ 0, & \text{otherwise.} \end{cases}$$

The values near the identity of any smooth, irreducible character of G can be described in terms of a linear combination of 5 functions (independent of the representation), namely, the Fourier transforms of nilpotent orbital integrals on G (see [23]). We will not write our character formulas in this form; but we do find it convenient to isolate a particular coefficient in this combination.

Definition 14.17. The *constant term* $c_0(\pi)$ of π is defined as follows. In the unramified case,

$$c_0(\pi) = -q^r.$$

In the ramified case,

$$c_0(\pi) = -\frac{1}{2}q^h\frac{q+1}{q}.$$

Proposition 13.13 and Lemma 14.4 below, together with [51, Definition 6.10], show that this is, indeed, the coefficient for the Fourier transform of the trivial orbit. By Theorem 15.2, we also have $c_0(\pi') = -1/2$ for any 'exceptional' supercuspidal representation π'.

14.5.1. *The bad shell.* The most challenging range in which to understand the character is that where the depth of the elements we consider (modulo centre) is the same as the depth of our representation. This range is colloquially known as the 'bad shell', for precisely this reason. Actually, it turns out that the *unramified* character formulas in this range are the same as those far from the identity (see Theorems 14.14 and 14.18); the complication is only apparent in the ramified case.

Theorem 14.18. *Suppose that $r > 0$, and $\gamma \in G_r \setminus Z(G)G_{r+}$. In the unramified case, $\Theta_\pi(\gamma) = 0$ unless some G-conjugate of γ lies in T^ϵ. If $\gamma \in T^\epsilon$, then*

$$\Theta_\pi(\gamma) = \tfrac{1}{2}\operatorname{sgn}_\epsilon\bigl(\operatorname{Im}_\epsilon(\gamma)\bigr)\frac{\psi(\gamma)+\psi(\gamma^{-1})}{|D_G(\gamma)|^{1/2}}\bigl[(-1)^{r+1} + H(\Lambda', k_\epsilon)\bigr].$$

Proof. By Definition 14.1, $\theta = \epsilon$; in particular, $X \in \mathfrak{t}^\epsilon$. Put $Y = \mathrm{c}^{-1}(\gamma)$.

The vanishing result follows from Proposition 13.13, Lemmas 14.4 and 5.4(d), and [51, Theorem 9.5].

If $\gamma \in T^\epsilon$ (and $\gamma \in G_r \setminus Z(G)G_r$), then Proposition 13.13, Lemmas 14.4 and 5.4(h), [51, Theorem 9.6], Lemma 2.4, and Definition 14.16 give that

$$\Theta_\pi(\gamma) = |D_G(\gamma)|^{-1/2}\, H(\Lambda', k_\epsilon)\operatorname{sgn}_\epsilon\bigl(\operatorname{Im}_\epsilon(\gamma)\bigr)\bigl(\Lambda(\operatorname{tr}(X\cdot Y)) + \Lambda(\operatorname{tr}(-X\cdot Y))\bigr).$$

By Definition 10.13 and Lemma 5.4(e),

$$\Lambda\bigl(\operatorname{tr}(X\cdot Y)\bigr) = \psi(\gamma) \quad\text{and}\quad \Lambda\bigl(\operatorname{tr}(-X\cdot Y)\bigr) = \Lambda\bigl(\operatorname{tr}(X\cdot -Y)\bigr) = \psi(\gamma^{-1}). \qquad \square$$

Theorem 14.19. *Suppose that $r > 0$, and $\gamma \in G_r \setminus Z(G)G_{r+}$. In the ramified case, $\Theta_\pi(\gamma) = 0$ unless some G-conjugate of γ lies in $T^{\theta',\eta}$, with $\theta' \in \{\varpi, \epsilon\varpi\}$ and $\eta \in \{1, \epsilon\}$. If $\theta' = \varpi$, then*

$$\Theta_\pi(\gamma) = \frac{q^{-1/2}}{2\left|D_G(\gamma)\right|^{1/2}} \sum_{\substack{\gamma' \in (C_\varpi)_{r:r+} \\ \gamma' \neq \gamma^{\pm 1}}} \operatorname{sgn}_\varpi\bigl(\operatorname{tr}_\varpi(\gamma - \gamma')\bigr)\psi(\gamma') + \frac{1}{2} H(\Lambda', k_\varpi) \operatorname{sgn}_\varpi\bigl(\eta^{-1}\operatorname{Im}_\varpi(\gamma)\bigr) \frac{\psi(\gamma) + \psi(\gamma^{-1})}{\left|D_G(\gamma)\right|^{1/2}}.$$

If $\theta' = \epsilon\varpi$, then

$$\Theta_\pi(\gamma) = \frac{q^{-1/2}}{2\left|D_G(\gamma)\right|^{1/2}} \sum_{\gamma' \in (C_\varpi)_{r:r+}} \operatorname{sgn}_\varpi\bigl(\operatorname{tr}_{\epsilon\varpi}(\gamma) - \operatorname{tr}_\varpi(\gamma')\bigr)\psi(\gamma').$$

In the first formula, we are regarding γ as an element of C_ϖ, not of $T^{\varpi,\eta}$. *Via* the isomorphism $T^\varpi \cong C_\varpi$, we can then make sense of $\psi(\gamma)$ and $\psi(\gamma')$; and it makes sense to consider the inequality $\gamma' \neq \gamma^{\pm 1}$, even though γ and γ' may lie in different tori.

Proof. By Definition 14.1, $\theta = \varpi$; in particular, $X \in \mathfrak{t}^\varpi$. Put $Y = \mathrm{c}^{-1}(\gamma)$.

The vanishing result is trivial: since $r \notin \mathbb{Z}$, no element of an unramified or split torus can have depth r; i.e., all elements of depth r already lie in some G-conjugate of $T^{\theta',\eta}$, with θ' and η as above.

If $\gamma \in T^{\varpi,\eta}$ (and $\gamma \in G_r \setminus Z(G)G_{r+}$), with $\eta \in \{1, \epsilon\}$, then write $Y = X_c^{\varpi,\eta}$, and note that $\widetilde{Y} := X_c^{\varpi,1} = \operatorname{Ad}\left(\begin{smallmatrix}\sqrt{\eta} & 0 \\ 0 & \sqrt{\eta}^{-1}\end{smallmatrix}\right) Y$ is a stable conjugate of Y that lies in $\mathfrak{t}^\varpi$. By Proposition 13.13, Lemmas 14.4 and 5.4(d, h), [51, Theorem 10.9], Lemma 2.4, and Definition 14.16,

$$\Theta_\pi(\gamma) = \frac{1}{2} H(\Lambda', k_\theta) \operatorname{sgn}_\varpi\bigl(\eta^{-1}\operatorname{Im}_\varpi(\gamma)\bigr) \frac{\Lambda(\operatorname{tr}(X \cdot \widetilde{Y})) + \Lambda(\operatorname{tr}(-X \cdot \widetilde{Y}))}{\left|D_G(\gamma)\right|^{1/2}} + \frac{q^{-1/2}}{2\left|D_G(\gamma)\right|^{1/2}} \sum_{Z \in \mathfrak{t}^\varpi_{r:r+}} \operatorname{sgn}_\varpi\bigl(Y^2 - Z^2\bigr)\Lambda(\operatorname{tr}(X \cdot Z)).$$

By Definition 10.13,

$$\Lambda\bigl(\operatorname{tr}(X \cdot Z)\bigr) = \psi\bigl(\mathrm{c}(Z)\bigr) \quad \text{for all } Z \in \mathfrak{t}^\varpi_r.$$

Further, we have

$$\Lambda\bigl(\operatorname{tr}(X \cdot \widetilde{Y})\bigr) = \psi(\tilde{\gamma}),$$

where $\tilde{\gamma} = \operatorname{Int}\left(\begin{smallmatrix}\sqrt{\eta} & 0 \\ 0 & \sqrt{\eta}^{-1}\end{smallmatrix}\right)\gamma = \mathrm{c}(\widetilde{Y})$; but note that $\gamma \in T^{\theta'}$ and $\tilde{\gamma} \in T^\varpi$ correspond to the same element of C_ϖ, so our notational conventions allow us to write $\psi(\gamma)$ instead of $\psi(\tilde{\gamma})$. Similarly, $\Lambda\bigl(\operatorname{tr}(-X \cdot \widetilde{Y})\bigr) = \psi(\gamma^{-1})$.

Finally, note that, by Lemma 2.8, since Y and Z (regarded as elements of $V_{\theta'} = V_\varpi$) lie in $\wp_\varpi^{2h-1}$ (where h is as in Definition 14.1), we have the additive congruence

$$Y^2 - Z^2 \equiv \operatorname{tr}_\varpi\bigl(\gamma - \mathrm{c}(Z)\bigr) \pmod{\wp^{2h}}.$$

Since $Y^2, Z^2 \in \wp^{2h-1}$ and $Z \not\equiv Y \pmod{\wp_\varpi^{2h}}$, we have that $\operatorname{ord}(Y^2 - Z^2) = 2h - 1$. Thus we can deduce the multiplicative congruence

$$Y^2 - Z^2 \equiv \operatorname{tr}_\varpi\bigl(\gamma - \mathrm{c}(Z)\bigr) \pmod{1 + \wp},$$

hence the equality

$$\operatorname{sgn}_\varpi(Y^2 - Z^2) = \operatorname{sgn}_\varpi\bigl(\operatorname{tr}_\varpi(\gamma - \mathrm{c}(Z))\bigr).$$

The formula now follows (in this case) from Lemma 5.4(b, d) upon putting $\gamma' = \mathrm{c}(Z)$.

The argument in case $\theta' = \epsilon\varpi$ is similar but easier. $\square$

14.5.2. *Character values very near the identity.* Finally, we consider character values *very* near the identity, so that we are within the range of the local character expansion. The Hales–Moy–Prasad conjecture, proven in [14, Theorem 3.5.2] under mild hypotheses on p, describes the precise range of validity for the local character expansion for any smooth, irreducible representation of a reductive, p-adic group; but we shall not need the general result here. For our case ($\mathbf{G} = \mathrm{SL}_2$), it can be verified by direct computation from our formulas that the local character expansion holds on G_{r+} (see [13, Appendix A]).

Theorem 14.20. *Suppose $\gamma \in G_{r+} \cap G^{\mathrm{rss}}$ and, if $r = 0$, that Hypothesis 1.4 holds. Then $\Theta_\pi(\gamma) = c_0(\pi)$ unless some G-conjugate of γ lies in A or $T^{\theta,\eta}$ for some η. If $\gamma \in A$, then*

$$\Theta_\pi(\gamma) = c_0(\pi) + \frac{1}{|D_G(\gamma)|^{1/2}}.$$

If $\gamma \in T^{\theta,\eta}$, then

$$\Theta_\pi(\gamma) = c_0(\pi) + H(\Lambda', k_\theta)\frac{\operatorname{sgn}_\theta\bigl(\eta^{-1}\operatorname{Im}_\theta(\gamma)\bigr)}{|D_G(\gamma)|^{1/2}}.$$

Theorem 14.20 remains true without the extra hypothesis. A proof in that generality will appear in [7].

Proof. This is a combination of Proposition 13.13, Lemma 14.4, [51, Theorems 9.7 and 10.10], Lemmas 5.4(d, h) and Definition 14.16 and Lemma 4.2. $\square$

15. 'Exceptional' supercuspidal characters

By Remark 9.8, the only representations we still need to consider after §14 are $\pi^\pm := \pi^\pm(T^{\epsilon,1}, \psi_0^1)$. (Recall that the character ψ_0^1 is defined in Notation 3.2, and the associated representation in Definition 9.6.)

15.1. **Character values far from the identity.**

Theorem 15.1. *If $\gamma \notin Z(G)G_{0+}$, then $\Theta_\pi(\gamma) = 0$ unless some G-conjugate of γ lies in T^ϵ. If $\gamma \in T^\epsilon$, then*

$$\Theta_{\pi^\pm}(\gamma) = \frac{\operatorname{sgn}_\varpi(\gamma + \gamma^{-1} + 2)}{2}\Bigl\{H(\Lambda', k_\epsilon)\frac{\operatorname{sgn}_\epsilon\bigl(\operatorname{Im}_\epsilon(\gamma)\bigr)}{|D_G(\gamma)|^{1/2}} - 1\Bigr\}.$$

Note that the character values of π^+ and π^- in this range are the same.

Proof. The proof is almost exactly as in the depth-zero case of Theorem 14.20 and [16, §§9–10]. In particular, we may assume that $\gamma \in T^\epsilon \subseteq G_{x,0}$.

We need only make some minor adjustments to account for the fact that ψ_0 is not 'regular', in the sense of §9.3 *loc. cit.*; i.e., that $\psi_0 \circ \operatorname{Int}(\sigma_\epsilon) = \psi_0^{-1} = \psi_0$, where σ_ϵ is the non-trivial element of the Weyl group of T^ϵ.

Under our hypotheses on γ, its image $\bar\gamma$ in $G_{x,0:0+} = \mathrm{SL}_2(\mathfrak{f})$ is a regular, semisimple element, so Definitions 9.2 and 13.2 give

$$R^{\pm}_{\mathsf{T}^\epsilon,\psi_0}(\bar\gamma) = \tfrac{1}{2}R^{\mathsf{G}}_{\mathsf{T}^\epsilon,\psi_0}(\bar\gamma).$$

Therefore, as in the proof of [16, Lemma 9.3.1], using the Harish-Chandra integral formula (§9.1 *loc. cit.*, or [22, Theorem 12]) gives

$$(*)\qquad \Theta_\pi(\gamma) = \tfrac{1}{2}\varepsilon(\mathsf{G}_x,\mathsf{T}^\epsilon)R(G,T^\epsilon,\psi_0)(\gamma) = -\tfrac{1}{2}R(G,T^\epsilon,\psi_0)(\gamma).$$

As in Theorem 14.14,

$$\begin{aligned}\widehat{\mathcal{T}}(\gamma_{\mathrm{ts}}) &= \big\{\, (T^\epsilon,\psi') \,\big|\, \psi' = \psi_0\circ\mathrm{Int}(n)^{-1} \text{ for some } n\in N_G(T^\epsilon)\big\}\\ &= \big\{(T^\epsilon,\psi_0),(T^\epsilon,\psi_0^{-1})\big\}\\ &= \big\{(T^\epsilon,\psi_0)\big\}.\end{aligned}$$

In our setting, however, the map $(d,\bar n)\longmapsto (nd)^{-1}\cdot(T^\epsilon,\psi_0)$ of [16, p. 857] is a double cover, not a bijection; so the formula in Lemma 10.0.4 *loc. cit.* becomes

$$(**)\qquad \begin{aligned}R(G,T^\epsilon,\psi_0)(\gamma) &= 2\sum_{(T',\psi')}\psi'(\gamma_{\mathrm{ts}})R(G_{\gamma_{\mathrm{ts}}},T',1)(\gamma_{\mathrm{tu}})\\ &= 2\psi_0(\gamma_{\mathrm{ts}})R(T^\epsilon,T^\epsilon,1)(\gamma_{\mathrm{tu}})\\ &= 2\psi_0(\gamma),\end{aligned}$$

where the sum again runs over the set of orbits in $\widehat{\mathcal{T}}(\gamma_{\mathrm{ts}})$ under the natural (trivial) action of T^ϵ.

Again as in Theorem 14.14,

$$\frac{\mathrm{sgn}_\epsilon\big(\mathrm{Im}_\epsilon(\gamma)\big)}{|D_G(\gamma)|^{1/2}} = 1,$$

so the result now follows by combining $(*)$ with $(**)$, and using Lemmas 2.7 and 4.2. □

15.2. **Character values near the identity.**

Theorem 15.2. *Suppose that $\gamma\in G^{\mathrm{rss}}\cap G_{0+}$. If $\gamma\in A$, then*

$$\Theta_{\pi^\pm}(\gamma) = \frac{1}{2}\Big\{\frac{1}{|D_G(\gamma)|^{1/2}} - 1\Big\}.$$

If $\gamma\in T^{\theta',\eta}$, where

- *$\theta' = \epsilon$ and $\eta\in\{1,\varpi\}$ or*
- *$\theta'\in\{\varpi,\epsilon\varpi\}$ and $\eta\in\{1,\epsilon\}$,*

then

$$\Theta_{\pi^\pm}(\gamma) = \frac{1}{2}\Big\{\pm H(\Lambda',k_{\theta'})\frac{\mathrm{sgn}_{\theta'}\big(\eta^{-1}\,\mathrm{Im}_{\theta'}(\gamma)\big)}{|D_G(\gamma)|^{1/2}} - 1\Big\}.$$

Proof. We use Proposition 13.14, which writes $\Theta_\pi\circ\mathrm{c}$ in the indicated range as a linear combination of Fourier transforms of orbital integrals.

In order to compute this combination of Fourier transforms of orbital integrals, we adopt the notation of [51, Notation 8.12], so that

$$(\ddagger)\qquad \big[A;B_1,B_\epsilon,B_\varpi,B_{\epsilon\varpi}\big]_{\theta,r'}$$

stands for the function whose value at an element Y of an elliptic Cartan subalgebra $\mathfrak{t}^{\theta',\eta} \cong V_{\theta'}$ is

$$|\theta|^{1/2} A + q^{-(r'+1)} |D_{\mathfrak{g}}(Y)|^{-1/2} B_{\theta'}\big(\eta^{-1} \operatorname{Im}_{\theta'}(Y)\big),$$

and whose value at an element Y of the split Cartan subalgebra $\mathfrak{a}$ is

$$|\theta|^{1/2} A + q^{-(r'+1)} |D_{\mathfrak{g}}(Y)|^{-1/2} B_1.$$

Then Theorems 9.7 and 10.10 *loc. cit.*, combined with Definition 14.16, give

$$\begin{aligned}\hat\mu^G_{X_1^{\epsilon,1}} &= \big[-q^{-1}; \quad 1, H(\Lambda', k_\epsilon)\operatorname{sgn}_\epsilon, 0, \quad 0\big]_{\epsilon,0}\\ \hat\mu^G_{X_1^{\varpi,1}} &= \big[-\tfrac12 q^{-3/2}(q+1); 1, 0, \quad H(\Lambda', k_\varpi)\operatorname{sgn}_\varpi, 0\big]_{\varpi,0};\end{aligned}$$

but it is important to realize that there are two obstacles to combining the formulas. First, the subscripts are different ($(\epsilon, 0)$ versus $(\varpi, 0)$); and, second, the measures with respect to which the orbital integrals are computed are not those in Lemmas 13.10 and 13.11. (It may seem that a third obstacle is the fact that Definition 14.16 is stated only in the positive-depth setting; but, since we are working in the Lie algebra, there is no harm now in multiplying by a scalar to see that, in fact, it remains valid in the depth-zero case.) Our approach to the first problem will be to replace all subscripts with the arbitrarily chosen $(\epsilon, -1)$ (which we then drop from the notation). For the second, we recall that the quoted orbital integrals use the various measures $\mathrm{d}_{\theta'}\dot g$ on $G/T^{\theta'}$, and so use §6 to replace them by the measures $\mathrm{d}g/\mathrm{d}t^{\theta'}$. Making both of these adjustments gives

$$(*)\qquad \begin{aligned}\hat\mu^G_{X_1^{\epsilon,1}} &= \quad \big[-1; \quad 1, H(\Lambda', k_\epsilon)\operatorname{sgn}_\epsilon, 0, \quad 0\big]\\ \hat\mu^G_{X_1^{\varpi,1}} &= q^{1/2}\big[-\tfrac12 q^{-1}(q+1); 1, 0, \quad H(\Lambda', k_\varpi)\operatorname{sgn}_\varpi, 0\big].\end{aligned}$$

It is important to note that the difference between this equation and the previous one is *not* just notational; since we have changed normalizations of measures, we are actually describing different *functions*.

We have that

$$\operatorname{Ad}\begin{pmatrix}\epsilon & 0\\ 0 & 1\end{pmatrix} X_1^{\varpi,\epsilon} = X_1^{\varpi,1},$$

so

$$\hat\mu^G_{X_1^{\varpi,\epsilon}} = \hat\mu^G_{X_1^{\varpi,1}} \circ \operatorname{Ad}\begin{pmatrix}\epsilon & 0\\ 0 & 1\end{pmatrix}.$$

A direct computation shows that this reduces to

$$(**)\qquad \hat\mu^G_{X_1^{\varpi,\epsilon}} = q^{1/2}\big[-\tfrac12 q^{-1}(q+1); 1, 0, -H(\Lambda', k_\varpi)\operatorname{sgn}_\varpi, 0\big];$$

i.e., all that has changed is that $H(\Lambda', k_\varpi)$ has become $-H(\Lambda', k_\varpi)$.

Further, as observed in [51, Remark 6.9], we may adapt formulas involving one choice of uniformizer (such as ϖ) to another choice (such as $\epsilon\varpi$) by simple substitution; so (remembering that the order of the arguments is significant) we find

$$(***)\qquad \begin{aligned}\hat\mu^G_{X_1^{\epsilon\varpi,1}} &= q^{1/2}\big[-\tfrac12 q^{-1}(q+1); 1, 0, 0, \quad H(\Lambda', k_{\epsilon\varpi})\operatorname{sgn}_{\epsilon\varpi}\big];\\ \hat\mu^G_{X_1^{\epsilon\varpi,\epsilon}} &= q^{1/2}\big[-\tfrac12 q^{-1}(q+1); 1, 0, 0, -H(\Lambda', k_{\epsilon\varpi})\operatorname{sgn}_{\epsilon\varpi}\big].\end{aligned}$$

By $(*)$, $(**)$, and $(***)$,

$$\hat{\mu}^G_{X_1^{\varpi,1}} - \hat{\mu}^G_{X_1^{\varpi,\epsilon,1}} + \hat{\mu}^G_{X_1^{\epsilon\varpi,1}} - \hat{\mu}^G_{X_1^{\epsilon\varpi,\epsilon}} \\ = 2q^{1/2}\bigl[0; 0, 0, H(\Lambda', k_\varpi)\,\mathrm{sgn}_\varpi, H(\Lambda', k_{\epsilon\varpi})\,\mathrm{sgn}_{\epsilon\varpi}\bigr].$$

By (‡), and Lemmas 2.4 and 5.4(h), Proposition 13.14 now simplifies to the desired formula. □

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