INVARIANCE PROPERTY OF MORSE HOMOLOGY
ON NONCOMPACT MANIFOLDS

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ABSTRACT. In this article, we focus on the invariance property of Morse homology on noncompact manifolds. We expect to apply outcomes of this article to several types of Floer homology, thus we define Morse homology purely axiomatically and algebraically. The Morse homology on noncompact manifolds generally depends on the choice of Morse functions; it is easy to see that critical points may escape along homotopies of Morse functions on noncompact manifolds. Even worse, homology classes also can escape along homotopies even though critical points are alive. The aim of the article is two fold. First, we give an example which breaks the invariance property by the escape of homology classes and find appropriate growth conditions on homotopies which prevent such an escape. This takes advantage of the bifurcation method. Another goal is to apply the first results to the invariance problem of Rabinowitz Floer homology. The bifurcation method for Rabinowitz Floer homology, however, is not worked out yet. Thus believing that the bifurcation method is applicable to Rabinowitz Floer homology, we study the invariance problems of Rabinowitz Floer homology.

CONTENTS

1. INTRODUCTION

In recent times, several types of Morse and Floer homology have been developed and widely studied. The power of Morse and Floer homologies is the invariance property; that is, these homologies are independent of the choice of the Morse or Hamiltonian functions (or symplectic forms). Unfortunately this is rarely true on noncompact manifolds. One can easily find two Morse functions on $\mathbb{R}$ such that the respective Morse homologies are not

isomorphic. There are two methods to show the invariance property of Morse and Floer theory. The first one is the continuation method; we count gradient flow lines of a homotopy between two Morse functions and this gives a continuation homomorphism between two respective Morse homologies. The other tool is the bifurcation method; we again consider a homotopy between two Morse functions and analyze how the Morse chain varies along the homotopy. It was introduced by Floer [Fl1] to show the invariance of Lagrangian Floer homology though it was not completely justified, but this method was replaced by the continuation argument by himself in [Fl2]. Recently, Hutchings and Lee [Hu, Lee1, Lee2] completed the analysis required in the bifurcation method and it was used in [Co, Us]; to be more specific, Hutchings worked on generalized Morse theory and Lee worked on Floer theories for the torsion invariant of Morse and Floer theories [HL1, HL2]. In particular, Lee proved in the Floer theoretic setting that there exists a "regular homotopy of Floer systems (RHFS)" such that only two types of degeneracies can happen along this homotopy, namely "birth-death" and "handle-slide".

Both methods are painful but useful in the following sense. For the continuation method, we need compactness for gradient flow lines of a time-dependent action functional; but it gives a concrete isomorphism. On the other hand, we need to study gluing and decaying for the bifurcation method. However once this required analysis is worked out, it enables the detection of more general things; for example, invariance of the Reidemeister torsion in Morse and Floer theories has been studied in depth by Hutchings and Lee [Hu, HL1, HL2, Lee1, Lee2] using the bifurcation method.

The purpose of this article is two fold. First, we investigate the invariance problem of Morse homology on noncompact manifolds by using the bifurcation method. As we have already mentioned, Morse homology can change along homotopies on noncompact manifolds. This incident can obviously be caused by the escape of critical points, see Remark 3.2 and Figure 3.4; even worse, homology classes also can escape as described in Theorem A. This is a very surprising phenomenon because homology classes escape to infinity whereas critical points keep alive. How does this happen? Let us change this problem to an interesting story. Suppose that there is no bus to go to heaven, how can we reach heaven? The answer is to transfer infinitely many buses of higher and higher speed and then we eventually arrive in heaven in finite time although no buses arrives at heaven. Using this idea, we will illustrate that homology classes can escape to infinity by infinitely many handle-slides or birth-deaths. It shows that if there are infinitely many generators of chain groups, a homology class may disappear even though generators may not. In the classical Floer theory, chain groups of Floer homology are of finite dimension over a suitable Novikov ring. However it is not true anymore for Rabinowitz Floer homology. So the escape of homology classes is a new phenomenon arising in Rabinowitz Floer theory. In order to prove the invariance property of Rabinowitz Floer homology, Cieliebak-Frauenfelder-Paternain [CFP] and Bae-Frauenfelder [BF] took advantage of the continuation method. On the other hand, one may expect that the invariance of Rabinowitz Floer homology can be proved by means of the bifurcation method.

Question A. Is the bifurcation method applicable to Rabinowitz Floer theory?

We expect that the above question will be answered in the near future. The second aim of the article is to apply the first results to the invariance problem of Rabinowitz Floer homology. Since the bifurcation method for Rabinowitz Floer homology, however, is not worked out yet. Thus believing the bifurcation method is applicable to Rabinowitz Floer

theory, we study the invariance problems of Rabinowitz Floer homology. The following Question B is our starting point.

Question B. Believing the answer to Question A is positive, can we prove the invariance property of Rabinowitz Floer homology using the bifurcation method?

In Theorem B and C, we give sufficient conditions preventing the escape of homology classes; more precisely, in Theorem B we impose an appropriate growth restriction on homotopies so that Morse homology is invariant; moreover, we show that a given homology class never escapes under a mild growth restriction in Theorem C. We apply these results to Rabinowitz Floer homology developed by Cieliebak-Frauenfelder [CF]. The invariance problem of Rabinowitz Floer homology on stable or contact manifolds is not completely known yet. Interestingly, if the answer to Question A is positive then by examining the bifurcation process, we can prove the invariance property of Rabinowitz Floer homology along stable tame homotopies, see section 4; furthermore we are able to slightly relax the tameness condition.

Acknowledgments. I am indebted to Urs Frauenfelder for various suggestions and discussions.

2. Cerf diagram and Morse homology

In this section, we define Morse homology purely axiomatically and algebraically because we hope our results can be applied to all various type of Floer theories which satisfy the basic ingredients of Morse homology theory. Though our story begins with algebraic axioms, the classical Morse and Floer homologies satisfy these axioms.

2.1. **Cerf tuple and Cerf diagram.** We set projection maps

$$\pi_1, \pi_3 : \mathbb{R}^2 \times [0,1] = \mathbb{R} \times [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}, \quad \pi_2 : \mathbb{R} \times [0,1] \times \mathbb{R} \longrightarrow [0,1].$$

Definition 2.1. We call a tuple $\mathscr{C} = (C, F)$ a *Cerf tuple* if the following conditions hold.

- ($\mathscr{C}$1) C is a one dimensional manifold with boundary such that each connected component of C is compact.
- ($\mathscr{C}$2) $F : C \longrightarrow \mathbb{R}^2 \times [0,1]$ is a smooth map with the property: $\pi_3 \circ F$ is proper and, for a connected component $c \subset C$, $F|_c : c \longrightarrow \mathbb{R}^2 \times [0,1]$ is either a Legendrian knot or a Legendrian chord which begins and ends on the pre-Lagrandian submanifold $\mathbb{R}^2 \times \{0\}$ or $\mathbb{R}^2 \times \{1\}$.

We refer to the appendix for the notions of the Legendrian knot and chord and the pre-Lagangian. We denote by $F_i := \pi_i \circ F$, $i = 1, 2, 3$. For a given Cerf tuple, the front projection of parameterized Legendrian curves $F(C)$ is called the *Cerf diagram*:

$$\big\{(F_2(c), F_3(c)) \,|\, c \subset C\big\} \subset [0,1] \times \mathbb{R}.$$

Remark 2.2. Let us take a look at the Cerf triple and the Cerf diagram in the Morse theoretic viewpoint. We have a one-parameter Morse functions $\{f_r\}_{r \in [0,1]}$ on a manifold M. Then a one dimensional manifold C corresponds to $\mathrm{Crit} f_r \subset M \times [0,1]$ and the smooth function F_2 on C indicates the parameter r and F_3 is nothing but the Morse function f_r at critical points.

Remark 2.3. As can be seen in Figure 2.1, cusps appear in the Cerf diagram. About the reason, we refer to Remark 5.7.
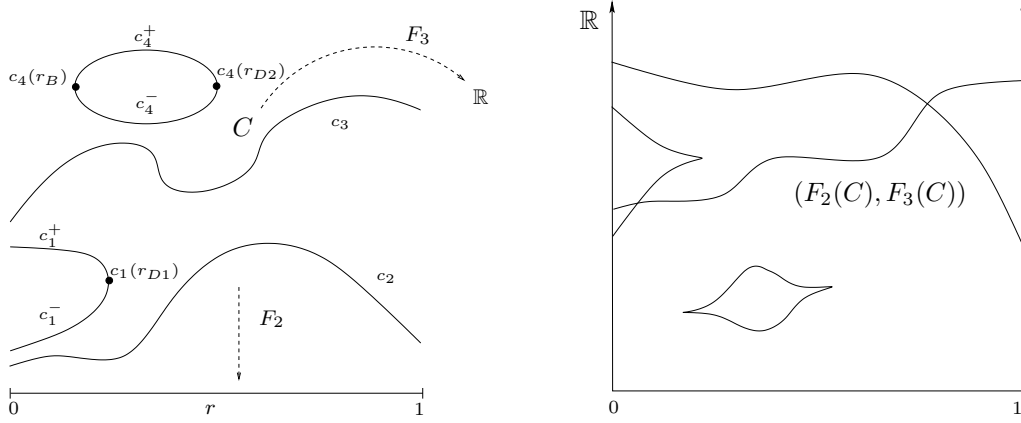
FIGURE 2.1. Cerf tuple and Cerf diagram

Degeneracies. For clarity, we indicate dependence of the parameter $r \in [0,1]$ by $c_1(r) \in c_1$ for $F_2(c_1(r)) = r$. If c_1 has two points with same F_2-value, we denote by $c_1^+(r)$ and $c_1^-(r)$. We often write the subscripts D, B, and H to allude the degenerate types, namely birth-deaths or handle-slides. We also define the *set of deaths* and the *set of birthes* as follows:

- $\mathcal{D}_0 := \big\{ c(r_D) \in c \subset C \,\big|\, c(r_D) \text{ is a local maximum point of } F_2 \big\}$,

- $\mathcal{B}_0 := \big\{ c(r_B) \in c \subset C \,\big|\, c(r_B) \text{ is a local minimum point of } F_2 \big\}$.

We note that the above sets are discrete in C. For $c(r_B) \in \mathcal{B}$, we note that for a small $\epsilon > 0$ then $F_2^{-1}(r_B + \epsilon)|_c$ consists of two distinct points. As mentioned, we denote each of them by $c_B^+(r_B + \epsilon)$ and $c_B^-(r_B + \epsilon)$. We analogously define $c_D^+(r_D - \epsilon)$ and $C_D^-(r_D - \epsilon)$ for deaths. These degeneracies, birth-deaths, is caused by the Cerf tuple itself.

2.2. **Graph structure.**

Definition 2.4. A *graph structure* on a topological space G is a discrete subset V of G such that $G \setminus V$ is a 1-dimensional manifold. A pair (G, V) is called a *graph.* An element in V is called a *vertex* and each connected component of $G \setminus V$ is called an *edge.*

Definition 2.5. A graph structure V_0 on G is called a *supergraph structure* of the graph (G, V) if $V_0 \supseteq V$.

Let (G, V) be a graph. For index sets I and J, we set

$$V = \{v_i \,|\, v_i \in G, \, i \in I\}, \qquad \pi_0(G \setminus V) = \{e_j \,|\, e_j \subset G, \, j \in J\}.$$

Let $\mathbb{F}$ be any principal ideal domain (e.g. $\mathbb{Z}_2$, $\mathbb{Z}$, or $\mathbb{Q}$) with the discrete topology.

Definition 2.6. A function $\phi : G \longrightarrow \mathbb{F}$ is called a *step function* on (G, V) if it can be written as

$$\phi(g) = \sum_{i \in I} f_i \chi_{v_i}(g) + \sum_{j \in J} f_j \chi_{e_j}(g), \quad g \in G, \quad f_i, \, f_j \in \mathbb{F}$$

where χ_{v_i} and χ_{e_j} are the indicator functions defined by

$$\chi_{v_i}(g) = \begin{cases} 1 \text{ if } g = v_i \\ 0 \text{ if } g \neq v_i \end{cases} \qquad \chi_{e_j}(g) = \begin{cases} 1 \text{ if } g \in e_j \\ 0 \text{ if } g \notin e_j \end{cases}$$

We define the *fiber product* of $F_2 : C \longrightarrow [0,1]$ as follows:
$$C \times_{F_2} C := \big\{ (c_1, c_2)(r) \in C \times C \mid F_2(c_1(r)) = r = F_2(c_2(r)) \big\}.$$
This fiber product has a natural graph structure given by the Cerf tuple. The sets $\mathcal{D}_0$ and $\mathcal{B}_0$ defined in the previous subsection give the following subsets of the fiber product of F_2.

- $\mathcal{D} := \big\{ (c_1, c_2)(r_D) \in C \times_{F_2} C \mid c_1(r_D) \text{ or } c_2(r_D) \in \mathcal{D}_0 \big\}$,

- $\mathcal{B} := \big\{ (c_1, c_2)(r_B) \in C \times_{F_2} C \mid c_1(r_B) \text{ or } c_2(r_B) \in \mathcal{B}_0 \big\}$.

In particular, we define the diagonals of $\mathcal{D}$ and $\mathcal{B}$ as follows:

- $\triangle \mathcal{D} := \big\{ (c, c)(r_D) \in C \times_{F_2} C \mid c(r_D) \in \mathcal{D}_0 \big\}$,

- $\triangle \mathcal{B} := \big\{ (c, c)(r_B) \in C \times_{F_2} C \mid c(r_B) \in \mathcal{B}_0 \big\}$.

It is easy to see that $C \times_{F_2} C$ is topologically nothing but an union of closed intervals and wedge sums of closed intervals where points in $\triangle \mathcal{D} \cup \triangle \mathcal{B}$ are identified. Furthermore the following discrete set $\mathcal{V}_{\mathscr{C}}$ induced by the cerf tuple $\mathscr{C}$ endows a natural graph structure on $C \times_{F_2} C$.
$$\mathcal{V}_{\mathscr{C}} := \mathcal{D} \cup \mathcal{B} \subset C \times_{F_2} C.$$
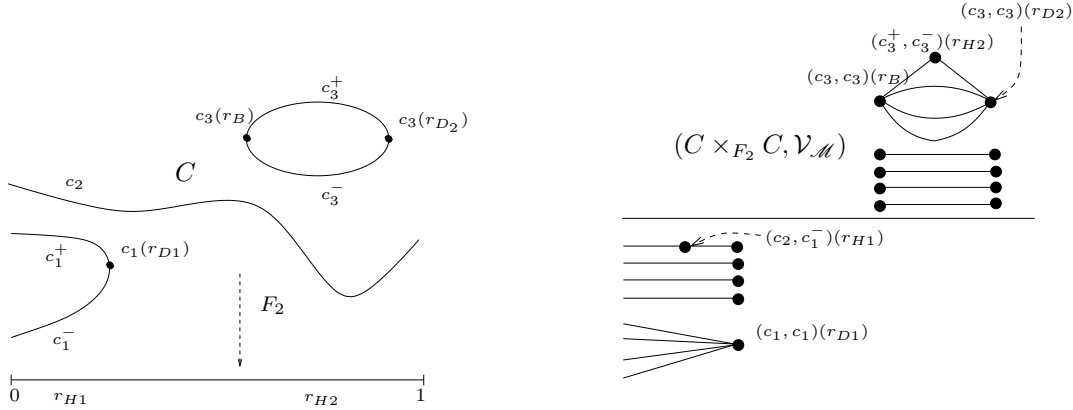

FIGURE 2.2. Graph structure

2.3. Axioms on γ.

Definition 2.7. A function
$$\gamma : C \times_{F_2} C \longrightarrow \mathbb{F}$$
is called the *flow line counter* if it is a step function on a supergraph $(C \times_{F_2} C, \mathcal{V}_{\mathscr{M}})$ of $(C \times_{F_2} C, \mathcal{V}_{\mathscr{C}})$ such that the following holds. There exists function $\delta : \mathcal{H} := \mathcal{V}_{\mathscr{M}} \setminus \mathcal{V}_{\mathscr{C}} \longrightarrow \mathbb{F}$ such that γ together with δ satisfy the following five axioms $(\gamma 1) - (\gamma 5)$. The set $\mathcal{H}$ is called the *set of handle-slides* and the function δ is called the *jump function*. We set $\gamma = 0$ for convention when γ has the infinite value in $\mathbb{F}$.

We write $\gamma_r(c_1, c_2) = \gamma(c_1, c_2)(r)$ and $\delta_{r_H}(c_H^+, c_H^-) = \delta(c_H^+, c_H^-)(r_H)$ for brevity.

Remark 2.8. As can be seen in $(\gamma 3)$, the jump function δ measures the discontinuity of γ at $\mathcal{H}$.

The important data is the value of γ on edges of $C \times_{F_2} C$. γ is constant on each edges, but the value may jump at $\mathcal{V}_{\mathscr{M}} = \mathcal{D} \cup \mathcal{B} \cup \mathcal{H}$. Thus we need to examine how the value of γ changes at $\mathcal{V}_{\mathscr{M}}$. For such a reason, we define the approximated value of γ to compare the value of γ before and after Λ defined by

$$\Lambda := \{r \in [0,1] \,|\, (c_1, c_2)(r) \in \mathcal{V}_{\mathscr{M}}\}.$$

In particular we indicates the type of degeneracies of *degenerate points* in Λ as below:

- $F_2(\mathcal{D}) := \{r_D \in [0,1] \,|\, (c_1, c_2)(r_D) \in \mathcal{D}\}$,

- $F_2(\mathcal{B}) := \{r_B \in [0,1] \,|\, (c_1, c_2)(r_B) \in \mathcal{B}\}$,

- $F_2(\mathcal{H}) := \{r_H \in [0,1] \,|\, (c_1, c_2)(r_H) \in \mathcal{H}\}$.

We assume that those points are **disjoint** in $[0,1]$. By definition,

$$\Lambda = F_2(\mathcal{D}) \cup F_2(\mathcal{B}) \cup F_2(\mathcal{H}) \subset [0,1].$$

Definition 2.9. $(c_1, c_2)(r - \epsilon) \in C \times_\pi C$ is the *left approximation* of $(c_1, c_2)(r) \in C \times_\pi C$ if

- $\epsilon > 0$,
- $\lim_{\epsilon \to 0}(c_1, c_2)(r - \epsilon) = (c_1, c_2)(r)$.

Then we define for the left approximation of $(c_1, c_2)(r)$,

$$\gamma_r^-(c_1, c_2) := \lim_{\epsilon \to 0} \gamma_{r-\epsilon}(c_1, c_2).$$

Since the non-degenerate function γ is constant on each edges, $\gamma_r^-(c_1, c_2)$ is well-defined. Analogously, we also define $(c_1, c_2)(r + \epsilon)$ the *right approximation* of $(c_1, c_2)(r) \in C \times_\pi C$ together with $\gamma_r^+(c_1, c_2) \in \mathbb{F}$.

The non-degenerate function γ satisfies the following five axioms.

(γ1) For $r \in [0,1] \setminus \Lambda$, $\gamma_r(c_1, c_2) = 0$ if $F_3(c_2) \geq F_3(c_1)$;

for $(c_1, c_2)(r_H) \in \mathcal{H}$, $\delta_{r_H}(c_1, c_2) = 0$ if $F_3(c_2) \geq F_3(c_1)$.

(γ2) For $(c_1, c_3)(r) \in C \times_{F_2} C$, $r \in [0,1] \setminus \Lambda$,

$$\sum_{c_2 \subset C} \gamma_r(c_1, c_2)\gamma_r(c_2, c_3) = 0.$$

(γ3) At $r_H \in F_2(\mathcal{H})$, the following holds.

$$\gamma_{r_H}^+(c_1, c_3) = \gamma_{r_H}^-(c_1, c_3) + \sum_{\substack{c_2 \subset C; \\ (c_1, c_2)(r_H) \in \mathcal{H}}} \delta_{r_H}(c_1, c_2)\gamma_{r_H}^-(c_2, c_3) - \sum_{\substack{c_2 \subset C; \\ (c_2, c_3)(r_H) \in \mathcal{H}}} \delta_{r_H}(c_2, c_3)\gamma_{r_H}^+(c_1, c_2).$$

(γ4) At $r_B \in F_2(\mathcal{B})$, $\gamma_{r_B}^+(c_B^+, c_B^-)$ is an invertible element in $\mathbb{F}$ and the following holds.

$$\gamma_{r_B}^+(c_1, c_2) = \gamma_{r_B}^-(c_1, c_2) + \gamma_{r_B}^+(c_1, c_B^-)\gamma_{r_B}^+(c_B^+, c_B^-)^{-1}\gamma_{r_B}^+(c_B^+, c_2);$$
$$\gamma_{r_B}^+(c_1, c_B^+) = \gamma_{r_B}^+(c_B^-, c_1) = \gamma_{r_B}^+(c_B^+, c_2) = 0, \quad c_2 \neq c_B^-.$$

(γ5) At $r_D \in F_2(\mathcal{D})$, $\gamma_{r_D}^-(c_D^+, c_D^-)$ is an invertible element in $\mathbb{F}$ and the following holds.

$$\gamma_{r_D}^+(c_1, c_2) = \gamma_{r_D}^-(c_1, c_2) - \gamma_{r_D}^-(c_1, c_D^-)\gamma_{r_D}^-(c_D^+, c_D^-)^{-1}\gamma_{r_D}^-(c_D^+, c_2);$$
$$\gamma_{r_D}^-(c_1, c_D^+) = \gamma_{r_D}^-(c_D^-, c_1) = \gamma_{r_D}^-(c_D^+, c_2) = 0, \quad c_2 \neq c_D^-.$$

Remark 2.10. (γ1) guarantees that $\gamma_r(c, c) = 0$, $\gamma_{r_B}^+(c_B^-, c_B^+) = 0$, and $\gamma_{r_D}^-(c_D^-, c_D^+) = 0$. By the properness of F_3, the formula in (γ2) is a finite sum, i.e. for fixed $(c_1, c_3)(r) \in C \times_{F_2} C$, there are only finitely many $c_2 \in F_2^{-1}(r)$ such that $\gamma_r(c_1, c_2)\gamma_r(c_2, c_3)$ is nonzero.

Remark 2.11. In the Morse theoretic framework, the axioms and functions can be interpreted as the following.

- δ counts the number of degenerate gradient flow lines between critical points with same indices (non-generic phenomenon).
- γ counts the number of gradient flow lines.
- (γ1) implies that the action value decreases along gradient flow lines.
- (γ2) means that the function γ gives the boundary operator of the Morse chain complex by counting gradient flow lines.
- (γ3) $-$ (γ5) signify how the value of γ (or the boundary operator) changes at degenerate points in Λ, see Figure 3.2 and Figure 3.3.

2.4. Morse homology.

Definition 2.12. We call a pair $\mathcal{M} = (\mathcal{C}, \gamma)$ the *Morse tuple* which consists of a Cerf tuple $\mathcal{C} = (C, F)$ and a flow line counter γ on $\mathcal{C}$.

First of all, we define the following set which is finite by the properness of F_3. For $a \le b \in \mathbb{R}$, $r \in [0, 1] \setminus \Lambda$,

$$C^{(a,b)}(\mathcal{M}, r) := \big\{ c(r) \in C \,\big|\, F_3(c(r)) \in (a, b) \big\}.$$

Then we have the following $\mathbb{F}$-module by tensoring $\mathbb{F}$.

$$\mathrm{CM}^{(a,b)}(\mathcal{M}, r) := C^{(a,b)}(\mathcal{M}, r) \otimes \mathbb{F}.$$

Next, we define a boundary operator ∂ using γ.

$$\partial_r^{(a,b)} : \mathrm{CM}^{(a,b)}(\mathcal{M}, r) \longrightarrow \mathrm{CM}^{(a,b)}(\mathcal{M}, r)$$
$$c_1(r) \longmapsto \sum_{c_2 \subset C} \gamma_r(c_1, c_2) \cdot c_2(r).$$

Recall that we set $\gamma_r(c_1, c_2) = 0$ if it equals to infinity. We note that $(\mathrm{CM}^{(a,b)}(\mathcal{M}, r), \partial_r^{(a,b)})$ is indeed a chain complex due to Axiom (γ2); therefore, we get *filtered Morse homology*:

$$\mathrm{HM}^{(a,b)}(\mathcal{M}, r) := \mathrm{H}\big(\mathrm{CM}^{(a,b)}(\mathcal{M}, r), \partial_r^{(a,b)}\big), \qquad r \in [0, 1] \setminus \Lambda$$

and then taking direct and inverse limits, we obtain *(full) Morse homology*:

$$\mathrm{HM}(\mathcal{M}, r) := \varinjlim_{b \to \infty} \varprojlim_{a \to -\infty} \mathrm{HM}^{(a,b)}(\mathcal{M}, r), \qquad r \in [0, 1] \setminus \Lambda.$$

2.5. Invariance.
Thanks to the fact that γ is constant at each edges, we easily derive the invariance property of Morse homology on a non-degenerate interval.

$$\mathrm{HM}(\mathcal{M}, r_1) \cong \mathrm{HM}(\mathcal{M}, r_2) \quad \text{whenever} \ \ [r_1, r_2] \cap \Lambda = \emptyset.$$

In next three propositions, we shall show that Morse homology is unchanged even after a handle-slide and a birth-death. Due to the axioms (γ_3), (γ_4), and (γ_5) together with ($\mathcal{C}$2), we know that how Morse chain and the boundary operator vary by passing through those degenerate points. In fact, Lee [Lee1, Lee2] completes all the necessary analysis of the bifurcation method in Floer theory argued originally by Floer [Fl1]; accordingly she proved that all axioms and hypotheses of this article hold in Floer theory, but she did not explicitly

prove the invariance property even though it immediately follows. Instead, she concerned with the torsion invariants in Floer theory, (see the introduction). Usher [Us] stated and proved the invariance property described below.

Proposition 2.13. [Lee1, Lee2, Us] *If* $[r_0, r_1] \cap \Lambda = \{r_H \in F_2(\mathcal{H})\}$, $\mathrm{HM}(\mathscr{M}, r_0) \cong \mathrm{HM}(\mathscr{M}, r_1)$.

PROOF. We choose continuous functions $a(r)$, $b(r) : [0, 1] \longrightarrow \mathbb{R}$ such that the images of a and b do not intersect with the Cerf diagram. We set the map

$$A : \mathrm{CM}^{(a(r_1), b(r_1))}(\mathscr{M}, r_1) \longrightarrow \mathrm{CM}^{(a(r_0), b(r_0))}(\mathscr{M}, r_0)$$

$$c(r_1) \longmapsto c(r_0) + \sum_{\substack{c_H^- \subset C; \\ (c, c_H^-)(r_H) \in \mathcal{H}}} \delta_{r_H}(c, c_H^-) \cdot c_H^-(r_0). \qquad (2.1)$$

Since A is invertible, it suffices to show that A is a chain map then it gives an isomorphism on the homology level. We abbreviate $\partial_{r_0}^{(a(r_0), b(r_0))}$ resp. $\partial_{r_1}^{(a(r_1), b(r_1))}$ by ∂_- resp. ∂_+.

Claim: A is a chain map, i.e. $A \circ \partial_+ = \partial_- \circ A$.

Proof of the claim. We compute for $c(r_1) \in C^{(a(r_1), b(r_1))}(\mathscr{M}, r_1)$,

$$A \circ \partial_+ - \partial_- \circ A(c(r_1))$$

$$= A\Big(\sum_{c' \subset C} \gamma_{r_1}(c, c') \cdot c'(r_1) \Big) - \partial_-\Big(c(r_0) + \sum_{c_H^- \subset C} \delta_{r_H}(c, c_H^-) \cdot c_H^-(r_0) \Big)$$

$$= \sum_{c' \subset C} \gamma_{r_1}(c, c') \cdot \Big(c'(r_0) + \sum_{c_H^- \subset C} \delta_{r_H}(c', c_H^-) \cdot c_H^-(r_0) \Big)$$

$$\quad - \sum_{c' \subset C} \Big(\gamma_{r_0}(c, c') - \sum_{c_H^- \subset C} \delta_{r_H}(c, c_H^-) \gamma_{r_0}(c_H^-, c') \Big) \cdot c'(r_0)$$

$$= 0$$

The last equality follows from the axiom $(\gamma 3)$ and this computation finishes the proof of the claim, hence the proposition. $\square$

Proposition 2.14. [Lee1, Lee2, Us] *If* $[r_0, r_1] \cap \Lambda = \{r_B \in F_2(\mathcal{B})\}$, $\mathrm{HM}(\mathscr{M}, r_0) \cong \mathrm{HM}(\mathscr{M}, r_1)$.

PROOF. We choose again continuous functions $a(r)$, $b(r) : [0, 1] \longrightarrow \mathbb{R}$ such that the images of a and b do not intersect with the Cerf diagram. We abbreviate $C^{(a(r_0), b(r_0))}(\mathscr{M}, r_0)$ resp. $C^{(a(r_1), b(r_1))}(\mathscr{M}, r_1)$ by C_- resp. C_+. We note that there exists the natural bijection

$$\Psi : C_- \cup \{c_B^+(r_1), c_B^-(r_1)\} \longrightarrow C_+$$

$$c(r_0) \in C_- \longmapsto c(r_1)$$

$$c_B^{\pm}(r_1) \longmapsto c_B^{\pm}(r_1)$$

and it gives the isomorphism

$$\mathrm{CM}^{(a(r_1), b(r_1))}(\mathscr{M}, r_1) \cong \mathrm{CM}^{(a(r_0), b(r_0))}(\mathscr{M}, r_0) \oplus \mathbb{F}\langle c_B^+(r_1), c_B^-(r_1) \rangle.$$

For convenience we identify C_- with $\Psi(C_-) \subset C_+$; but one can easily distinguish elements in C_- or $\Psi(C_-)$ by the parameters r_0 or r_1. We set the chain maps:

$$i : \mathrm{CM}^{(a(r_0),b(r_0))}(\mathscr{M}, r_0) \longrightarrow \mathrm{CM}^{(a(r_1),b(r_1))}(\mathscr{M}, r_1)$$
$$c(r_0) \longmapsto c(r_1) - \gamma_{r_1}(c, c_B^-)\gamma_{r_B}^+(c_B^+, c_B^-)^{-1} \cdot c_B^+(r_1)$$

$$p : \mathrm{CM}^{(a(r_1),b(r_1))}(\mathscr{M}, r_1) \longrightarrow \mathrm{CM}^{(a(r_0),b(r_0))}(\mathscr{M}, r_0)$$
$$c(r_1) \in C_- \longmapsto c(r_0)$$
$$c_B^+(r_1) \longmapsto 0$$
$$c_B^-(r_1) \longmapsto - \sum_{c \subset C_-} \gamma_{r_B}^+(c_B^+, c_B^-)^{-1}\gamma_{r_1}(c_B^+, c) \cdot c(r_0)$$

where $\gamma_{r_B}^+(c_B^+, c_B^-)^{-1}$ is the inverse of $\gamma_{r_B}^+(c_B^+, c_B^-)$ in $\mathbb{F}$.

<u>Claim 1</u>: i and p are indeed chain maps, namely $\partial_+ \circ i = i \circ \partial_-$ and $\partial_- \circ p = p \circ \partial_+$.

Proof of Claim 1. Using the axioms (γ_2) and (γ_4), we compute that for any $c(r_0) \in C_-$,

$$\sum_{c' \subset C_-} \gamma_{r_0}(c, c')\gamma_{r_1}(c', c_B^-) = \sum_{c' \subset C_-} \gamma_{r_1}(c, c')\gamma_{r_1}(c', c_B^-)$$
$$- \sum_{c' \subset C_-} \gamma_{r_1}(c, c_B^-)\gamma_{r_B}^+(c_B, c_B^-)\gamma_{r_1}(c_B^+, c')\gamma_{r_1}(c', c_B^-) \tag{2.2}$$
$$= 0.$$

Similarly, we also can show that for $c(r_1) \in C_-$,

$$\sum_{c' \subset C_-} \gamma_{r_1}(c_B^+, c')\gamma_{r_0}(c', c) = 0. \tag{2.3}$$

With the axioms (γ_2) and $(\gamma 4)$ again, we calculate for $c(r_0) \in C_-$,

- $i \circ \partial_- - \partial_+ \circ i \, (c(r_0))$

$$= i\Big(\sum_{c' \subset C_-} \gamma_{r_0}(c, c') \cdot c'(r_0) \Big) - \partial_+ \Big(c(r_1) + \gamma_{r_1}(c, c_B^-) \gamma_{r_B}^+(c_B^+, c_B^-)^{-1} \cdot c_B^+(r_1) \Big)$$

$$= \sum_{c' \subset C_-} \gamma_{r_0}(c, c') \cdot c'(r_1) + \underbrace{\sum_{c' \subset C_-} \gamma_{r_0}(c, c') \gamma_{r_1}(c', c_B^-) \, \gamma_{r_B}^+(c_B^+, c_B^-)^{-1} \cdot c_B^+(r_1)}_{=0 \text{ by } (2.2)}$$

$$- \sum_{c' \subset C_+} \gamma_{r_1}(c, c') \cdot c'(r_1) + \sum_{c' \subset C_+} \gamma_{r_1}(c, c_B^-) \gamma_{r_B}^+(c_B^+, c_B^-)^{-1} \gamma_{r_1}(c_B^+, c') \cdot c'(r_1)$$

$$= \sum_{c' \subset C_-} \gamma_{r_0}(c, c') \cdot c'(r_1) - \sum_{c' \subset C_-} \gamma_{r_1}(c, c') \cdot c'(r_1) + \gamma_{r_1}(c, c_B^-) \cdot c_B^-(r_1)$$

$$+ \sum_{c' \subset C_+} \gamma_{r_1}(c, c_B^-) \gamma_{r_B}^+(c_B^+, c_B^-)^{-1} \gamma_{r_1}(c_B^+, c') \cdot c'(r_1)$$

$$= \sum_{c' \subset C_-} \gamma_{r_0}(c, c') \cdot c'(r_1) - \sum_{c' \subset C_-} \Big\{ \gamma_{r_0}(c, c') - \gamma_{r_1}(c, c_B^-) \gamma_{r_B}^+(c_B^+, c_B^-)^{-1} \gamma_{r_1}(c_B^+, c') \Big\} \cdot c'(r_1)$$

$$- \gamma_{r_1}(c, c_B^-) \cdot c_B^-(r_1) + \sum_{c' \subset C_-} \gamma_{r_1}(c, c_B^-) \gamma_{r_B}^+(c_B^+, c_B^-)^{-1} \gamma_{r_1}(c_B^+, c') \cdot c'(r_1)$$

$$+ \gamma_{r_1}(c, c_B^-) \gamma_{r_B}^+(c_B^+, c_B^-)^{-1} \gamma_{r_B}^+(c_B^+, c_B^-) \cdot c_B^-(r_1)$$

$$= 0.$$

The fourth equality follows from $(\gamma 4)$. Similarly, we show $p \circ \partial_+ = \partial_- \circ p$ for $c(r_1) \in C_-$, $c_B^-(r_1)$, and $c_B^+(r_1)$.

- $p \circ \partial_+(c(r_1)) = p\Big(\sum_{c' \subset C_+} \gamma_{r_1}(c, c') \cdot c'(r_1) \Big)$

$$= p\Big(\sum_{c' \subset C_-} \gamma_{r_1}(c, c') \cdot c'(r_1) + \gamma_{r_1}(c, c_B^-) \cdot c_B^-(r_1) \Big)$$

$$= \sum_{c' \subset C_-} \gamma_{r_1}(c, c') \cdot c'(r_0) - \sum_{c' \subset C_-} \gamma_{r_B}^+(c_B^+, c_B^-)^{-1} \gamma_{r_1}(c_B^+, c') \gamma_{r_1}(c, c_B^-) \cdot c'(r_0)$$

$$= \sum_{c' \subset C_-} \gamma_{r_0}(c, c') \cdot c'(r_0)$$

$$= \partial_-(c(r_0))$$

$$= \partial_- \circ p(c(r_1)).$$

- $\partial_- \circ p(c_B^-(r_1)) = \partial_-\big(-\sum\limits_{c' \subset C_-} \gamma_{r_B}^+(c_B^+, c_B^-)^{-1}\gamma_{r_1}(c_B^+, c') \cdot c'(r_0)\big)$

$$= -\gamma_{r_B}^+(c_B^+, c_B^-)^{-1} \sum\limits_{c'' \subset C_-} \underbrace{\sum\limits_{c' \subset C_-} \gamma_{r_1}(c_B^+, c')\gamma_{r_0}(c', c'')}_{=0 \text{ by } (2.3)} \cdot c''(r_0)$$

$$= 0$$

$$= p \circ \partial_+(c_B^-(r_1)).$$

- $p \circ \partial_+(c_B^+(r_1)) = p\big(\sum\limits_{c' \subset C_+} \gamma_{r_1}(c_B^+, c') \cdot c'(r_1)\big)$

$$= p\big(\sum\limits_{c'' \subset C_-} \gamma_{r_1}(c_B^+, c'') \cdot c''(r_1) + \gamma_{r_B}^+(c_B^+, c_B^-) \cdot c_B^-(r_1)\big)$$

$$= \sum\limits_{c'' \subset C_-} \gamma_{r_1}(c_B^+, c'') \cdot c''(r_0) - \sum\limits_{c'' \subset C_-} \gamma_{r_B}^+(c_B^+, c_B^-)^{-1}\gamma_{r_1}(c_B^+, c'')\gamma_{r_B}^+(c_B^+, c_B^-) \cdot c''(r_0)$$

$$= 0$$

$$= \partial_- \circ p(c_B^+(r_1)).$$

Claim 2: i and p are chain homotopic.

Proof of Claim 2. It obviously holds that $p \circ i = \mathrm{id}$. Thus it remains to show that $i \circ p \simeq \mathrm{Id}$, so we set the chain homotopy D below.

$$D : \mathrm{CM}^{(a(r_1),b(r_1))}(\mathscr{M}, r_1) \longrightarrow \mathrm{CM}^{(a(r_1),b(r_1))}(\mathscr{M}, r_1)$$
$$c(r_1), c_B^+(r_1) \longmapsto 0 \tag{2.4}$$
$$c_B^-(r_1) \longmapsto \gamma_{r_B}^+(c_B^+, c_B^-)^{-1} \cdot c_B^+(r_1)$$

Then the following three simple calculations complete the proof of Claim 2 and hence the proposition. For $c(r_1) \in C_-$, and $c_B^\pm(r_1)$, we compute

- $\partial^+ \circ D + D \circ \partial^+(c(r_1)) = 0 + D\big(\sum\limits_{c' \subset C^-} \gamma_{r_1}(c, c') \cdot c'(r_1) + \gamma_{r_1}(c, c_B^-) \cdot c_B^-(r_1)\big)$

$$= \gamma_{r_1}(c, c_B^-)\gamma_{r_B}^+(c_B^+, c_B^-)^{-1} \cdot c_B^+(r_1)$$
$$= c(r_1) - \big(c(r_1) - \gamma_{r_1}(c, c_B^-)\gamma_{r_B}^+(c_B^+, c_B^-)^{-1} \cdot c_B^+(r_1)\big)$$
$$= \mathrm{Id} - i \circ p(c(r_1)).$$

- $\partial^+ \circ D + D \circ \partial^+(c_B^+(r_1)) = 0 + D\big(\sum\limits_{c' \subset C_-} \gamma_{r_1}(c_B^+, c') \cdot c'(r_1) + \gamma_{r_B}^+(c_B^+, c_B^-) \cdot c_B^-(r_1)\big)$

$$= \gamma_{r_B}^+(c_B^+, c_B^-)\gamma_{r_B}^+(c_B^+, c_B^-)^{-1} \cdot c_B^+(r_1)$$
$$= c_B^+(r_1)$$
$$= \mathrm{Id} - i \circ p(c_B^+(r_1)).$$

- $\partial^+ \circ D + D \circ \partial^+ (c_B^-(r_1)) = \gamma_{r_B}^+ (c_B^+ c_B^-)^{-1} \cdot c_B^+(r_1) + 0$

$$= \sum_{c' \subset C_+} \gamma_{r_B}^+ (c_B^+, c_B^-)^{-1} \gamma_{r_1}(c_B^+, c') \cdot c'(r_r)$$

$$+ \gamma_{r_B}^+ (c_B^+, c_B^-)^{-1} \gamma_+ (c_B^+, c_B^-) \cdot c_B^-(r_1)$$

$$= \sum_{c' \subset C_-} \gamma_{r_B}^+ (c_B^+, c_B^-)^{-1} \gamma_{r_1}(c_B^+, c') \cdot c'(r_1) + c_B^-(r_1)$$

$$= c_B^-(r_1) - \Big(- \sum_{c' \subset C_-} \gamma_{r_B}^+ (c_B^+, c_B^-)^{-1} \gamma_{r_1}(c_B^+, c') \cdot c'(r_1) \Big)$$

$$= \mathrm{Id} - i \circ p(c_B^-(r_1)).$$

$\square$

Proposition 2.15. [Lee1, Lee2, Us] *If* $[r_0, r_1] \cap \Lambda = \{r_D \in F_2(\mathcal{D})\}$, $\mathrm{HM}(\mathcal{M}, r_0) \cong \mathrm{HM}(\mathcal{M}, r_1)$.

PROOF. It follows immediately from the proof of the previous proposition by reversing arrows and signs. $\square$

3. STATEMENT OF THE MAIN RESULTS

In the previous section, we showed that Morse homology is unchanged even if a handle-slide or a birth-death takes place. Unfortunately, however, this invariance property may not hold as passing through infinitely many degenerate points in Λ. The following shocking example describes that infinitely many transfer of the spectral value can make a homology class escape to infinity. As mentioned in the introduction, the following phenomenon only happen in Morse homology on noncompact manifolds and Rabinowitz Floer homology because Morse homology on compact manifolds or the classical Floer theory has finite dimensional chain groups over a suitable ring.

Theorem A. (Escape of a homology class.) *A homology class* $h \in \mathrm{HM}(\mathcal{M}, 0)$ *may be able to escape and thus there is no homology class in* $\mathrm{HM}(\mathcal{M}, 1)$ *corresponding to* h.

PROOF. The only possible accident breaking the invariance property is the escape of a homology class since we have assumed that C is compact; otherwise a critical point also can escape to infinity, see Remark 3.2. It is caused by infinitely many "transfer of the spectral value" which occur by handle-slides and birth-deaths. For simplicity, we argue only with handle-slides.

For each homology class $h \in \mathrm{HM}(\mathcal{M}, r)$, the spectral value is defined by

$$\rho(h, r) := \inf_{\substack{\alpha \in \mathrm{CM}(\mathcal{M}, r) \\ [\alpha] = h}} \sigma(\alpha, r) \tag{3.1}$$

where

$$\sigma(\alpha, r) := \sup_i \big\{ F_3(c_i(r)) \,\big|\, \alpha = \sum_i f_i \cdot c_i(r), \ 0 \neq f_i \in \mathbb{F}, \ c_i(r) \in C \big\}.$$

Moreover we set $\rho(0, r) = -\infty$ for convention.

We recall that even though $[r_1, r_2] \cap \Lambda = \{r_H \in F_2(\mathcal{H})\}$ we have the chain map (2.1)

$$A : \mathrm{CM}^{(a(r_2), b(r_2))}(\mathscr{M}, r_2) \longrightarrow \mathrm{CM}^{(a(r_1), b(r_1))}(\mathscr{M}, r_1)$$

$$c_1(r_2) \longmapsto c_1(r_1) + \sum_{\substack{c_H^- \subset C; \\ (c, c_H^-)(r_H) \in \mathcal{H}}} \delta_{r_H}(c_1, c_H^-) \cdot c_H^-(r_1)$$

which gives an isomorphism A_* between $\mathrm{HM}(\mathscr{M}, r_2)$ and $\mathrm{HM}(\mathscr{M}, r_1)$.

For clarity, we assume that $[c_1]$ is a homology class in $\mathrm{HM}(\mathscr{M}, r_1)$ such that $(c_1, c_2)(r_H) \in \mathcal{H}$ with $\delta_{r_H}(c_1, c_2) = 1 \in \mathbb{F}$. Thus after passing through a degenerate point r_H, the homology class $[c_1]$ changes to $A_*[c_1] = [c_1 - c_2] \in \mathrm{HM}(\mathscr{M}, r_2)$ so called "bifurcation of a homology class", see Figure 3.2. Moreover we note that the spectral value changes along r as below:

$$\rho([c_1], r_1) = F_3(c_1(r_1)) \longrightarrow \rho(A_*[c_1]), r_2) = \max\{F_3(c_1(r_2)), F_3(c_2(r_2))\}.$$

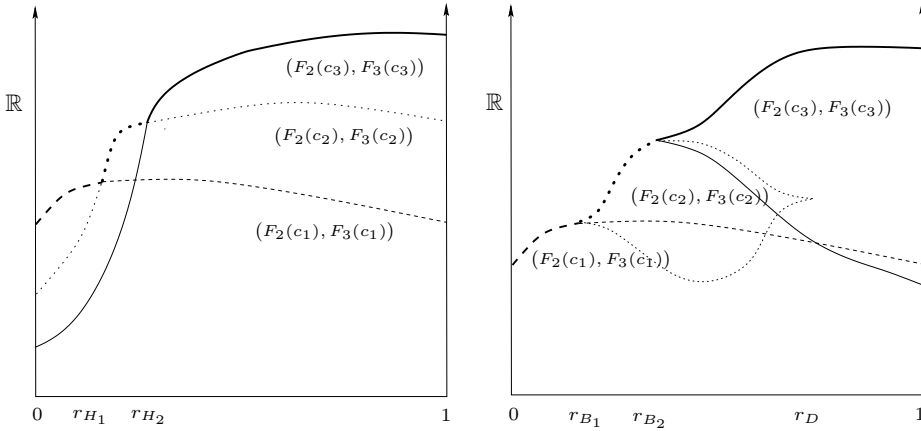We refer to this change "transfer of the spectral value".

FIGURE 3.1. transfer of the spectral value

For example, in Figure 3.1 there are two degenerate points $r_{H_1}, r_{H_2} \in F_2(\mathcal{H})$ and homology class $[c_1]$ changes to $[c_1 - c_2]$ and to $[c_1 - c_2 - c_3]$. Here dashed line, dotted line, and solid line are the front projections of $F(c_1), F(c_2)$, and $F(c_3)$ respectively; moreover bold line indicates critical points which give the spectral value of the homology class $[c_1]$ at each time. In the extremal case that there are infinitely many degeneration points $F_2(\mathcal{H})$ and the spectral value $\rho([c_1], \cdot)$ transfers infinitely many times so that it finally diverges to infinity, the homology class $[c_1]$ escapes to infinity and it gives an example described in Theorem A. It is conceivable that infinitely many birth-deaths are also able to cause the escape of a homology class by the analogous argument, see Figure 3.3. $\square$

Remark 3.1. We note that the spectral value never transfers at a point in $F_2(\mathcal{H})$. To $\delta_{r_H}(c_1, c_H^-)$ be nonzero, $F_3(c_H^-)$ has to be less than $F_3(c_1)$; thus the transfer of the spectral value takes place after a handle-slide. By the same reason, this is true for birth-deaths.

Now we describe the phenomenon illustrated in Figure 3.2. At time r_1, there is only one gradient flow line v between c_2 and c_3; it means that $\gamma_{r_1}(c_1, c_2) = \gamma_{r_1}(c_1, c_3) = 0$ and $\gamma_{r_1}(c_2, c_3) = 1$. At this moment $[c_1] = h \in \mathrm{HM}(\mathscr{M}, r_1)$ is a nonzero homology class and the spectral value of h is $\rho(h, r_1) = F_3(c_1(r_1))$. A handle-slide takes place at r_H; a degenerate
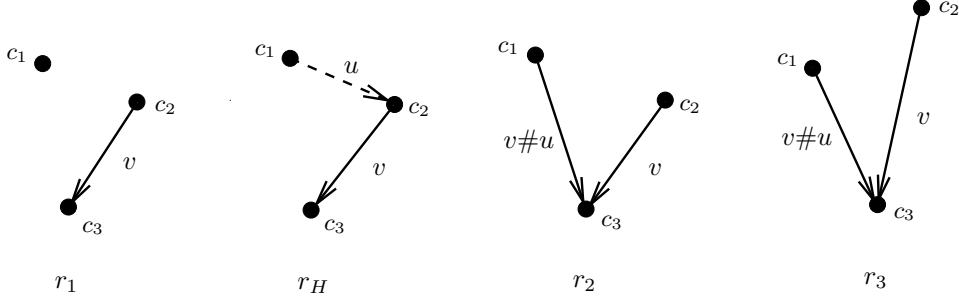
FIGURE 3.2. Handle-slide

gradient flow line u interchanging c_1 and c_2 emerges, it means that $(c_1, c_2)(r_H) \in \mathcal{H}$ and $\delta_{r_H}(c_1, c_2) = 1$. After a handle-slide, by gluing two gradient flow lines v and u, a new gradient flow line $v \# u$ between c_1 and c_3 appears at r_2, see Axiom $(\gamma 3)$; the homology class $[c_1]$ represented by $c_1 - c_2$ at r_2 but the spectral value is still $F_3(c_1(r_2))$. However after some time, the action value of c_2 goes over the action value of c_1, thus the spectral value of h at r_3 is changed to $\rho(h, r_3) = F_3(c_2(r_3))$.
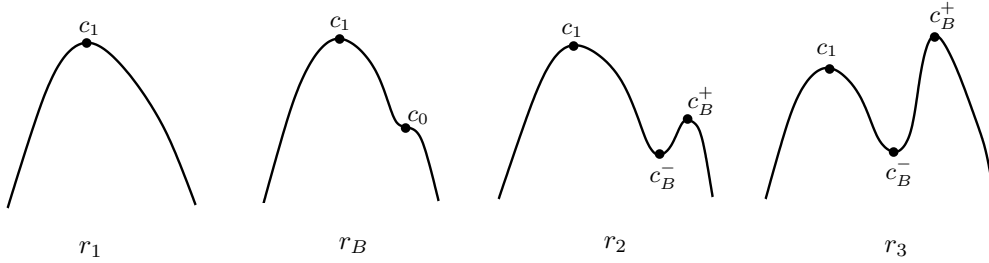

FIGURE 3.3. Birth

Let above Figure 3.3 be graphs of one parameter family of functions $\{f_r\}_{r \in [0,1]}$ on a one dimensional manifold, embedded in a plane; note that these functions are always Morse except at time $r_B \in [0, 1]$. At r_1, $h = [c_1]$ is a nonzero homology class in $\mathrm{HM}(\mathcal{M}, r_1)$ and $\rho(h, r_1) = f_{r_1}(c_1)$. At the moment of birth, $r_B \in [0, 1]$, a new critical point c_0 born which is a degenerate point thus Morse homology cannot be defined at r_B. After that, it bifurcates into two critical points c_B^+ and c_B^-. Now the homology class h is represented by $c_1 - c_B^+$; and the spectral value may transfer after some time, in this example $\rho(h, r_2) = f_{r_2}(c_B^+)$. Figure 3.1 illustrates how the spectral value varies along certain homotopies.

Remark 3.2. (Escape of a critical point.) In the general Morse theory on noncompact manifolds, critical points may escape to infinity during homotopies and this also violates the invariance property of Morse homology. For instance, let our manifold be $\mathbb{R}^2 - l$ where $l := \{(x, 0) \mid \frac{1}{2} \leq x \leq 1\}$ and an one-parameter family of Morse functions be $f_r(x, y) := (x - r)^2 + y^2$. Then $\mathrm{Crit} f_r = \{(r, 0) \mid r \in [0, \frac{1}{2})\}$, accordingly this critical point escape ever after the time $1/2$, see Figure 3.4 below. However in this paper we have assumed that each connected component of C is compact, it means that each critical point of the Morse function stays in a compact region of a manifold during homotopies.

To exclude the escape of homology classes, we need the following hypothesis:
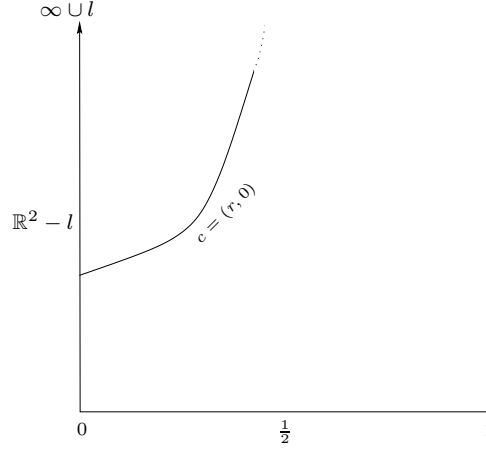
FIGURE 3.4. escape of a critical point

(H1) There exists a continuous function $\Phi(s) : \mathbb{R} \setminus (a, b) \longrightarrow \mathbb{R}_{>0}$ such that for all $c \subset C$,

$$\left| \frac{\partial F_3(c(r))}{\partial r} \right| \le \Phi\big(F_3(c(r))\big) \quad \& \quad \int_b^\infty \frac{1}{\Phi(s)} ds = \infty \quad \& \quad \int_{-\infty}^a \frac{1}{\Phi(s)} ds = \infty.$$

Remark 3.3. In the hypothesis (H1), (a, b) can be empty. For example, $\Phi(s) = 1/|s|$ satisfies the hypothesis.

Theorem B. *Under the hypothesis* (H1), *Morse homology is invariant.*

PROOF. Assume on the contrary that for a give homology class $h \in \mathrm{HM}(\mathscr{M}, 0)$, there exist $\{c_\nu \subset C\}_{\nu \in \mathbb{N}}$ components of C and $\{r_\nu \in [0, 1]\}_{\nu \in \mathbb{N}}$ such that

- $r_1 = 0$, $r_{\nu+1} \ge r_\nu$,
- $\lim_{\nu \to \infty} \rho(h, r_\nu) = \infty$.
- the F_3-value of c_ν transfers to $c_{\nu+1}$ at r_ν and it diverges to infinity; that is, $F_3(c_\nu(r_\nu)) = F_3(c_{\nu+1}(r_\nu))$ and $\lim_{\nu \to \infty} F_3(c_\nu(r_\nu)) = \infty$.
- there exists $k \in \mathbb{N}$ such that $F(c_\nu(r_\nu)) > b$ for all $\nu \ge k$.

Assuming the spectral value goes to the positive infinity, we compute

$$\lim_{\substack{n \to \infty \\ n \in \mathbb{N}}} r_n - r_k = \lim_{n \to \infty} \int_{r_k}^{r_n} 1 dr$$

$$= \lim_{n \to \infty} \sum_{\nu=k}^{n-1} \int_{r_\nu}^{r_{\nu+1}} 1 dr$$

$$\ge \lim_{n \to \infty} \sum_{\nu=k}^{n-1} \int_{r_\nu}^{r_{\nu+1}} \frac{1}{\Phi(F_3)} \frac{\partial F_3(c_{\nu+1}(r))}{\partial r} dr$$

$$\ge \lim_{n \to \infty} \sum_{\nu=k}^{n-1} \int_{F_3(c_{\nu+1}(r_\nu))}^{F_3(c_{\nu+1}(r_{\nu+1}))} \frac{1}{\Phi(s)} ds$$

$$= \lim_{n \to \infty} \int_{F_3(c_{k+1}(r_k))}^{F_3(c_n(r_n))} \frac{1}{\Phi(s)} ds = \infty.$$

It contradicts to the fact $r_k, r_n \in [0,1]$. On the other hand, if $\lim_{\nu \to \infty} \rho(h, r_\nu) = -\infty$, we may assume that

- $\lim_{\nu \to \infty} F_3(c_\nu(r_\nu)) = -\infty$.
- there exists $k \in \mathbb{N}$ such that $F(c_\nu(r_\nu)) < a$ for all $\nu \geq k$.

Then the following similar computation holds.

$$
\begin{aligned}
\lim_{\substack{n \to \infty \\ n \in \mathbb{N}}} r_n - r_1 &= \lim_{n \to \infty} \sum_{\nu=k}^{n-1} \int_{r_\nu}^{r_{\nu+1}} 1 dr \\
&\geq \lim_{n \to \infty} \sum_{\nu=k}^{n-1} \int_{F_3(c_{\nu+1}(r_\nu))}^{F_3(c_{\nu+1}(r_{\nu+1}))} -\frac{1}{\Phi(s)} ds \\
&= \lim_{n \to \infty} \int_{F_3(c_{k+1}(r_k))}^{F_3(c_n(r_n))} -\frac{1}{\Phi(s)} ds = \infty.
\end{aligned}
$$

The above two contradictory computation complete the proof of the theorem. □

Remark 3.4. In a compact manifold M, $\mathrm{Crit} f_r$ for $f_r \in C^\infty(M)$, $r \in [0,1]$ consists of finite components thus (H1) holds with $\Phi \equiv O$ for some constant $O \in \mathbb{R}$ and thus, neither critical points nor homology classes never escape. Therefore, Morse homology on compact manifold is independent of the choice of Morse functions since we can always homotop two Morse functions. We can also find a constant function Φ in the classical Floer theory. Let (M, ω) be a weakly monotone closed symplectic manifold and $H \in C^\infty(S^1 \times M)$ be a time-dependent Hamiltonian function. For each contractible loop $v \in C^\infty(S^1, M)$, we choose a map $w \in C^\infty(D^2, M)$ with $w|_{\partial D^2} = v$. With additional equivalences and conditions, the Floer action functional is defined as below (refer to [HS, Sa] for a rigorous framework):

$$
\mathcal{A}_H(v, w) = -\int_{D^2} w^* \omega - \int_0^1 H_t(v(t)) dt.
$$

Along the homotopy $\{H_r\}_{r \in [0,1]}$ and corresponding critical points $\{(v_r, w_r)\}_{r \in [0,1]}$ we calculate

$$
\begin{aligned}
\left| \frac{\partial}{\partial r} \mathcal{A}_{H_r}(v_r, w_r) \right| &= \left| d\mathcal{A}_{H_r}(v_r, w_r)[\partial_r v_r, \partial_r w_r] + \partial_r \mathcal{A}_{H_r}(v_r, w_r) \right| \\
&= \left| \int_0^1 \partial_r H_r(v_r(t)) dt \right| \\
&\leq \|\partial_r H_r\|_{L^\infty}.
\end{aligned}
$$

Thus the value $|\frac{\partial}{\partial r} \mathcal{A}_{H_r}(v_r, w_r)|$ is uniformly bounded by some constant for all critical points (v_r, w_r). Accordingly, the invariance property of Floer homology can be proved by the bifurcation method. However for the Rabinowitz action functional, this argument does not hold anymore by the effect of the Lagrange multiplier. Nevertheless if we assume tameness then we get Φ, not necessarily constant, satisfying (H1), see section 4.

Remark 3.5. Note that (H1) is a sufficient condition to prevent the escape of a homology class, but not a necessary condition. As an easy example, if we know that there is no intersection points in the Cerf diagram then the transfer of the spectral value never occurs at all; thus every homology class remains along homotopies without any hypothesis. Besides, if we also have an information about the grading of critical points, it is also useful.

On the other hand, if we know the data of the spectral value of a given homology calss at the initial point, we can show that the homology class survives under a mild hypothesis rather than (H1).

(H2) For a given homology class $h \in \mathrm{HM}(\mathscr{M}, 0)$, there exists a continuous function $\Phi_h(s) :$ $\mathbb{R} \setminus (a, b) \longrightarrow \mathbb{R}_{>0}$ and $\kappa > 0$ such that

$$\left| \frac{\partial F_3(c(r))}{\partial r} \right| \leq \Phi_h\big(F(c(r))\big) \quad \& \quad \int_M^\infty \frac{1}{\Phi_h(s)} ds \geq 1 + \kappa \quad \& \quad \int_{-\infty}^m \frac{1}{\Phi_h(s)} ds \geq 1 + \kappa$$

where $M := \max\{b, \rho(h, 0)\}$, $m := \min\{a, \rho(h, 0)\}$, and $\rho(h, 0)$ is the spectral value of h at 0 defined in (3.1).

Remark 3.6. In the hypothesis (H2), (a, b) can be empty. For example, h with $|\rho(h, 0)| < 1$ and $\Phi_h(s) = 1/s^2$ satisfy the hypothesis.

Theorem C. *The homology class $h \in \mathrm{HM}(\mathscr{M}, 0)$ satisfying the hypothesis* (H2) *survives along homotopies.*

PROOF. Similar as the proof of Theorem B, we assume by contradiction that there exists sequences $c_\nu \subset C$ and $r_\nu \in [0, 1]$, $\nu \in \mathbb{N}$ with the following properties.

- $r_1 = 0$, $r_{\nu+1} > r_\nu$.
- $\lim_{\nu \to \infty} \rho(h, r_\nu) = \infty$; without loss of generality, we may assume that $\rho(h, r)$ is increasing as r becomes bigger.
- The action value of c_ν transfers to $c_{\nu+1}$ at r_ν and it diverges to infinity; that is, $F_3(c_\nu(r_\nu)) = F_3(c_{\nu+1}(r_\nu))$ and $\lim_{\nu \to \infty} F_3(c_\nu(r_\nu)) = \infty$.
- Moreover, if $\rho(h, 0) \geq b$, it holds that

$$\int_{F_3(c_1(r_1))}^\infty \frac{1}{\Phi_h(s)} ds \geq 1 + \frac{\kappa}{2}.$$

If $\rho(h, 0) < b$ there exists $r' \in [0, 1]$ such that $r_k \leq r' \leq r_{k+1}$ for some $k \in \mathbb{N}$, $F_3(c_k(r')) = b$, and $F_3(c_\nu(r)) > b$ for $r' > r$ and $\nu > k$.

Then we compute along the same lines as the proof of Theorem B. If $\rho(h, 0) \geq b$,

$$\begin{aligned}
\lim_{\substack{n \to \infty \\ n \in \mathbb{N}}} r_n - r_1 &= \lim_{n \to \infty} \int_{r_1}^{r_n} 1 \, dr \\
&= \lim_{n \to \infty} \sum_{\nu=1}^{n-1} \int_{r_\nu}^{r_{\nu+1}} 1 \, dr \\
&\geq \lim_{n \to \infty} \sum_{\nu=1}^{n-1} \int_{r_\nu}^{r_{\nu+1}} \frac{1}{\Phi_h(F_3)} \frac{\partial F_3(c_{\nu+1}(r))}{\partial r} \, dr \\
&\geq \lim_{n \to \infty} \sum_{\nu=1}^{n-1} \int_{F_3(c_{\nu+1}(r_\nu))}^{F_3(c_{\nu+1}(r_{\nu+1}))} \frac{1}{\Phi_h(s)} \, ds \\
&= \lim_{n \to \infty} \int_{F_3(c_2(r_1))}^{F_3(c_n(r_n))} \frac{1}{\Phi_h(s)} \, ds \geq 1 + \frac{\kappa}{2} > 1.
\end{aligned}$$

It contradicts to $r_n, r_1 \in [0,1]$. If $\rho(h,0) < b$, similarly we compute

$$\lim_{\substack{n \to \infty \\ n \in \mathbb{N}}} r_n - r' = \lim_{n \to \infty} \int_{r'}^{r_n} 1 dr$$

$$= \lim_{n \to \infty} \sum_{\nu=k+1}^{n-1} \int_{r_\nu}^{r_{\nu+1}} 1 dr + \int_{r'}^{r_{k+1}} 1 dt$$

$$= \lim_{n \to \infty} \int_{F_3(c_k(r'))}^{F_3(c_n(r_n))} \frac{1}{\Phi_h(s)} ds \geq 1 + \kappa > 1.$$

The computation for the case $\lim_{\nu \to \infty} \rho(h, r_\nu) = -\infty$ analogously follows. These contradictory computations conclude the proof. □

4. Application to Rabinowitz Floer homology

In this section we discuss the invariance problem of Rabinowitz Floer homology. Very roughly, Rabinowitz Floer homology is a semi-infinite dimensional Morse homology of the Rabinowitz action functional. The invariance problem of Rabinowitz Floer homology is highly nontrivial; it turns out that Rabinowitz Floer homology is invariant under a suitable condition, but a counterexample is not yet known in more general case. First, we recall the notion of Rabinowitz Floer homology and formulate the invariance problems. We refer to [AF] for a brief survey on Rabinowitz Floer homology theory. In fact, the Rabinowitz action functional and Rabinowitz Floer homology have nice properties on restricted contact submanifolds, but they still can be defined on stable manifolds and have significant roles in the magnetic field theory, see [CFP]. In this paper, we focus on stable hypersurfaces, yet our story continues to hold on any stable coisotropic submanifolds; we refer to [Ka1] for Rabinowitz Floer theory on coisotropic submanifolds.

4.1. **Stability and tameness.** In this subsection we briefly recall the notions of stability and tameness; for further discussions, we refer to [CFP, CV] and cited therein.

Definition 4.1. A *Hamiltonian structure* on a $(2n-1)$-dimensional manifold Σ is a closed 2-form $\omega \in \Omega^2(\Sigma)$ such that $\omega^{n-1} \neq 0$. This Hamiltonian structure is called *stable* if there exists a *stabilizing 1-form* $\lambda \in \Omega^1(\Sigma)$ such that

- $\ker \omega \subset \ker d\lambda$;
- $\lambda|_{\ker \omega} \neq 0$.

Furthermore two equations $\lambda(R) = 1$ and $i_R \omega \neq 0$ characterize the unique vector field R on Σ, so called the *Reeb vector field*.

There are several equivalent formulations of stability.

Theorem 4.2. [Wa] *A Hamiltonian structure (Σ, ω) is stable if and only if its characteristic foliation is geodesible.*

Theorem 4.3. [Su] *A Hamiltonian structure (Σ, ω) is non-stable if and only if there exists a foliation cycle which can be arbitrary well approximated by boundaries of singular 2-chains tangent to the foliation.*

Definition 4.4. A closed hypersurface Σ in a symplectic manifold (M, ω) is called *stable* if Σ separates M and $\omega_{|\Sigma}$ defines a stable Hamiltonian structure. A *stable homotopy* is a smooth homotopy $\{(\Sigma_r, \lambda_r)\}_{r \in [0,1]}$ of stable hypersurfaces together with associated stabilizing one forms.

Proposition 4.5. *A closed hypersurface Σ in (M, ω) is stable if and only if there exists a tubular neighborhood $\Sigma \times (-\epsilon, \epsilon) \subset M$ of $\Sigma \times \{0\}$ such that $\ker \omega|_{\Sigma \times \{r\}} = \ker \omega|_{\Sigma \times \{0\}}$.*

PROOF. A closed 2-form $\omega' = \omega_{|M} + d(r\lambda)$ endows a sympelctic structure on $\Sigma \times (-\epsilon, \epsilon)$ for enough small $\epsilon > 0$. By the Weinstein neighborhood theorem, $\Sigma \times (-\epsilon, \epsilon)$ is symplecto-morphic to a neighborhood of Σ. Conversely a 1-form $\lambda := i_{\frac{\partial}{\partial r}} \omega_{|M}$ is a stabilizing 1-form on Σ. $\qquad\square$

Definition 4.6. Let (Σ, λ) be a stable hypersurface in a symplectic manifold (M, ω) being symplectically aspherical, i.e. $\omega|_{\pi_2(M)} \equiv 0$. We denote by $X(\Sigma)$ the set of closed Reeb orbits in Σ which is contractible in M. Then we define a function $\Omega : X(\Sigma) \longrightarrow \mathbb{R}$ by

$$\Omega(v) = \int_{D^2} \bar{v}^* \omega.$$

where $\bar{v}$ is any filling disk of v, i.e. $\bar{v}|_{\partial D^2} = v$. The symplectically asphericity condition guarantees that the value of this action functional is independent of the choice of filling disks. (Σ, λ) is called *tame* if for all $v \in X(\Sigma)$ there exists a constant $c > 0$ satisfying

$$\left| \int_0^1 v^* \lambda \right| \le c \, |\Omega(v)|.$$

A stable homotopy $\{(\Sigma_r, \lambda_r)\}_{r \in [0,1]}$ is said to be *tame* if each (Σ_r, λ_r) is tame with a constant $c > 0$ independent of $r \in [0, 1]$.

Remark 4.7. There are numerous examples of stable tame or non-tame hypersurfaces in [CFP, CV].

Remark 4.8. If our stable hypersurface (Σ, λ) is of restricted contact type, that is $\lambda \in \Omega^1(M)$ is a 1-form on M and a primitive of ω on whole M, then it is tame with a constant $c = 1$.

$$\left| \int_0^1 v^* \lambda \right| = \left| \int_{D^2} \bar{v}^* \omega \right| = |\Omega(v)|.$$

4.2. **Rabinowitz Floer homology.** Let (Σ, λ) be a stable hypersurface in a symplectic manifold (M, ω) being symplectically aspherical and $\mathscr{L}$ be a component of contractible loop in $C^\infty(S^1, M)$. We choose a *defining Hamiltonian function $H \in C^\infty(M)$* carefully (see [CFP] for details) so that $H^{-1}(0) = \Sigma$, $X_H|_\Sigma = R$, and X_H is compactly supported. Then the Rabinowitz action functional $\mathcal{A}^H : \mathscr{L} \times \mathbb{R} \longrightarrow \mathbb{R}$ is defined by

$$\mathcal{A}^H(v, \eta) := -\int_{D^2} \bar{v}^* \omega - \eta \int_0^1 H(v(t)) dt$$

where $\bar{v}$ is a filling disk of v. In a restricted contact manifold this Rabinowitz action functional itself gives compactness of gradient flow lines and thus Rabinowitz Floer homology can be defined. But in a stable manifold we need an aid of the auxiliary action functional $\widehat{\mathcal{A}}^H : \mathscr{L} \times \mathbb{R} \longrightarrow \mathbb{R}$

$$\widehat{\mathcal{A}}^H(v, \eta) = -\int_{D^2} \bar{v}^* d\beta - \eta \int_0^1 H(v(t)) dt.$$

where β is a 1-form globally defined on M such that $\beta|_\Sigma = \lambda$, see [CFP] for a rigorous construction of β. A critical point of $\mathcal{A}^H$, $(v, \eta) \in \mathrm{Crit}\mathcal{A}^H$, satisfies

$$\begin{cases} \partial_t v(t) = \eta X_H(v(t)), \\ H(v(t)) = 0. \end{cases} \tag{4.1}$$

It is noteworthy that each critical point $(v, \eta) \in \mathrm{Crit}\mathcal{A}^H$ gives rise to a closed Reeb orbit with period η in the following way: let $v_\eta(t) := v(t/\eta)$ for $t \in \mathbb{R}$, then it is η-periodic. By the second equation in (4.1), $v_\eta(t)$ lies in Σ and it solves $\partial_t v_\eta(t) = X_H(v_\eta(t)) = R(v_\eta(t))$.

In addition we observe that for $(v, \eta) \in \mathrm{Crit}\mathcal{A}^H$,

- $\left| \mathcal{A}^H(v, \eta) \right| = \left| \int_{D^2} \bar{v}^* \omega \right| = |\Omega(v)|,$

- $\left| \widehat{\mathcal{A}}^H(v, \eta) \right| = \left| \int_0^1 v^* \lambda \right| = \left| \eta \int_0^1 \lambda(R(v(t)))dt \right| = |\eta|.$

Next, we note that $\mathcal{A}^H$ is never Morse because there is a S^1-symmetry coming from time-shift on the critical point set. However it is known that $\mathcal{A}^H$ is generically Morse-Bott (see [CF]), so we are able to compute its Floer homology by choosing an auxiliary Morse function f on a critical manifold $\mathrm{Crit}\mathcal{A}^H$ and counting gradient flow lines with cascades, refer to [Fr, CF]. Let us set the $\mathbb{Z}/2$-module by

$$\mathrm{CF}^{(a,b)}(\mathcal{A}^H, f) := \mathrm{Crit}^{(a,b)}(f) \otimes \mathbb{Z}/2 \tag{4.2}$$

where

$$\mathrm{Crit}^{(a,b)}(f) := \left\{ (v, \eta) \in \mathrm{Crit}f \subset \mathrm{Crit}\mathcal{A}^H \,\middle|\, f(v, \eta) \in (a, b) \right\}. \tag{4.3}$$

Then it becomes a complex with the boundary operator ∂ defined by counting gradient flow lines with cascades. Then *filtered Rabinowitz Floer homology* is defined by

$$\mathrm{RFH}_a^b(\Sigma, M) := \mathrm{HF}^{(a,b)}(\mathcal{A}^H) = \mathrm{H}\big(\mathrm{CF}^{(a,b)}(\mathcal{A}^H), \partial\big),$$

and (full) *Rabinowitz Floer homology* is defined by

$$\mathrm{RFH}(\Sigma, M) := \varinjlim_{b \to \infty} \varprojlim_{a \to -\infty} \mathrm{RFH}_a^b(\Sigma, M).$$

4.3. **Invariance.** We recall the invariance result of Rabinowitz Floer homology proved by Cieliebak-Frauenfelder-Paternain. They used the continuation method to prove the invariance property and it needed clever but complicated computations. As we mentioned in the introduction, let us believe that the bifurcation method of Rabinowitz Floer theory is worked out:

(H3) There exists a "regular homotopy of Floer systems" in the sense of Lee [Lee1, Lee2] between any two Rabinowitz action functionals.

Then we can easily show that there is no escape of homology classes along stable tame homotopies; moreover we can prove the invariance with the relaxed condition rather that tameness.

Theorem 4.9. [CFP] *Assuming* (H3), *let* $\{(\Sigma_r, \lambda_r)\}_{r \in [0,1]}$ *be a stable tame homotopy. Then there exist an isomorphism:*

$$\Psi : \mathrm{RFH}(\Sigma_0, M) \xrightarrow{\cong} \mathrm{RFH}(\Sigma_1, M).$$

We reemphasize that this theorem is proved by [CFP] using the continuation method without (H3).

PROOF. At first we prove that critical points of the Rabinowitz action functional survive during the homotopy. We choose defining Hamiltonian functions H_r for Σ_r and $||\partial_r H_r||_{L^\infty} <$

∞. We note that if $(v_r, \eta_r) \in \mathrm{Crit}\mathcal{A}^{H_r}$, then $(v_r, \eta_r) \in \mathrm{Crit}\widehat{\mathcal{A}}^{H_r}$ by the stability condition. Using this fact, for $(v_r, \eta_r) \in \mathrm{Crit}\mathcal{A}^{H_r}$ we compute

$$
\begin{aligned}
\left|\frac{\partial}{\partial r}\eta_r\right| &= \left|\frac{\partial}{\partial r}\widehat{\mathcal{A}}^{H_r}(v_r, \eta_r)\right| \\
&= \left|d\widehat{\mathcal{A}}^{H_r}(v_r, \eta_r)[\partial_r v_r, \partial_r \eta_r] + \eta_r \int_0^1 \partial_r H_r(t, v_r(t))dt\right| \\
&\leq \|\partial_r H_r\|_{L^\infty}|\eta_r|.
\end{aligned}
\tag{4.4}
$$

Let $\mathfrak{H} := \max_{r \in [0,1]} \|\partial_r H_r\|_{L^\infty}$. It directly follows that

$$|\eta_r| \leq e^{\mathfrak{H}}|\eta_0|,$$

thus η_r is bounded in terms of the initial value η_0. From the equation $\partial_t v_r(t) = \eta_r X_{H_r}(t, v_r)$, we conclude that a critical point (v_r, η_r) does not escape.

In order to show the invariance property for Rabinowitz Floer homology, it remains to show that there is no escape of homology classes. We observe that the tameness implies the hypothesis (H1). We compute

$$
\begin{aligned}
\left|\frac{\partial}{\partial r}\mathcal{A}^{H_r}(v_r, \eta_r)\right| &= \left|\underbrace{d\mathcal{A}^{H_r}(v_r, \eta_r)}_{=0}[\partial_r v_r, \partial_r \eta_r] + \eta_r \int_0^1 \partial_r H_r(t, v_r(t))dt\right| \\
&\leq \mathfrak{H}|\eta_r| \\
&= \mathfrak{H}\left|\widehat{\mathcal{A}}^{H_r}(v_r, \eta_r)\right| \\
&\leq c\mathfrak{H}\left|\mathcal{A}^{H_r}(v_r, \eta_r)\right|.
\end{aligned}
$$

With $\Phi(s) = c\mathfrak{H} \cdot |s|$ our hypothesis (H1)

$$\frac{1}{c\mathfrak{H}}\int_1^\infty \frac{1}{s}ds = \infty \quad \& \quad \frac{1}{c\mathfrak{H}}\int_{-\infty}^{-1} -\frac{1}{s}ds = \infty$$

holds and hence Rabinowitz Floer homology is invariant by Theorem B. $\qquad\square$

Definition 4.10. We refer to a stable hypersurface (Σ, λ) as *logarithmic-tame* if there exists $c > 0$ such that the following holds: For all $v \in X(\Sigma)$,

$$\left|\int_0^1 v^*\lambda\right| \leq c\,|\Omega(v)| \times \log|\Omega(v)| \times \log\log|\Omega(v)| \times \cdots \times \log\cdots\log|\Omega(v)|.$$

A stable homotopy $\{(\Sigma_r, \lambda_r)\}_{r \in [0,1]}$ is called *logarithmic-tame* if each (Σ_r, λ_r) is logarithmic-tame with a uniform constant $c > 0$.

Theorem 4.11. *Rabinowitz Floer homology is invariant along a stable logarithmic-tame homotopy $\{(\Sigma_r, \lambda_r)\}_{r \in [0,1]}$.*

PROOF. As in the proof of Theorem 4.9, critical points of the Rabinowitz action functional do not escape. It it enough to show that homology classes also never escape. In this case we take a function

$$\Phi(s) = c\mathfrak{H}|s|\log|s| \times \log\log|s| \times \cdots \times \log\cdots\log|s|.$$

It satisfies (H1) and thus Theorem B finishes the proof. $\qquad\square$

Definition 4.12. We refer to a stable hypersurface (Σ, λ) as *square-tame* if there exists $c > 0$ such that the following holds: For all $v \in X(\Sigma)$,

$$\left| \int_0^1 v^* \lambda \right| \leq c |\Omega(v)|^2.$$

A stable homotopy $\{(\Sigma_r, \lambda_r)\}_{r \in [0,1]}$ is called *square-tame* if each (Σ_r, λ_r) is square-tame with a uniform constant $c > 0$.

In the square-tame homotopy case, Theorem B works no longer because we have

$$\int_1^\infty \frac{1}{cs^2} ds = \frac{1}{c} < \infty.$$

On the other hand, if we know the spectral value (defined in (3.1)) of a given homology class and this value is small enough, then the homology class cannot escape during square-tame homotopies.

Theorem 4.13. *Suppose that there exists a constant $\kappa > 0$ such that for a square-tame homotopy $\{(\Sigma_r, \lambda_r)\}_{r \in [0,1]}$ with a tame constant $c > 0$, the following holds.*

$$\frac{-1}{c + c\kappa} \leq \rho(h, 0) \leq \frac{1}{c + c\kappa} \quad \text{for some } h \in \mathrm{RFH}(\Sigma_0, M).$$

Then the homology class $h \in \mathrm{RFH}(\Sigma_0, M)$ survives along the homotopy.

PROOF. The hypothesis (H2) holds with $\Phi(s) = 1/cs^2$ since

$$\int_{1/(c+c\kappa)}^\infty \frac{1}{cs^2} ds = 1 + \kappa \quad \& \quad \int_{-\infty}^{-1/(c+c\kappa)} \frac{1}{cs^2} ds = 1 + \kappa.$$

Therefore Theorem C concludes the proof of the theorem. $\square$

We also can ask if Rabinowitz Floer homology depends on the choice of symplectic forms on M. In general, there is no answer yet but as before Rabinowitz Floer homology is invariant with a suitable stability and tameness condition defined below.

Definition 4.14. Let λ_0 resp. λ_1 be stabilizing 1-forms on (M, Σ, ω_0) resp. (M, Σ, ω_1). A smooth homotopy $\{(\omega_r, \lambda_r)\}_{r \in [0,1]}$ is called *stable* if each ω_r gives symplectic structure on M and λ_r is a stabilizing 1-form on (M, Σ, ω_r).

To define the tameness condition and the Rabinowitz action functional, we assume that each (M, ω_r) is symplectically aspherical.

Definition 4.15. A stable homotopy $\{(\omega_r, \lambda_r)\}_{r \in [0,1]}$ is said to be *tame* if there exists a constant $c > 0$ independent of $r \in [0, 1]$ such that

$$\left| \int_0^1 v^* \lambda_r \right| \leq c |\Omega_r(v)|, \quad v \in X(\Sigma)$$

where $\Omega_r(v) = \int_{D^2} \bar{v}^* \omega_r$ for $r \in [0, 1]$. Instead of the above formula, if it holds that

$$\left| \int_0^1 v^* \lambda_r \right| \leq c |\Omega_r(v)| \times \log |\Omega_r(v)| \times \log \log |\Omega_r(v)| \times \cdots \times \log \cdots \log |\Omega_r(v)|,$$

then we say that a stable homotopy $\{(\omega_r, \lambda_r)\}_{r \in [0,1]}$ is *logarithmic-tame*.

Let us indicate the dependency of the symplectic structure on the Rabinowitz action functional and Rabinowitz Floer homology in the following way. We define the Rabinowitz action functional on (M, ω_r) by

$$\mathcal{A}_{\omega_r}^{\mathrm{H}}(v, \eta) = -\int_{D^2} \bar{v}^* \omega_r - \eta \int_0^1 H(v(t)) dt.$$

With this action funtional, we can define Rabinowitz Floer homology $\mathrm{RFH}(\Sigma, M, \omega_r)$ for a stable hypersurface (Σ, λ_r) in (M, ω_r) as before.

Theorem 4.16. *Let $\{(\omega_r, \lambda_r)\}_{r \in [0,1]}$ be a stable and logarithmic-tame homotopy. If assuming* (H3), *we have*

$$\mathrm{RFH}(\Sigma, M, \omega_0) \cong \mathrm{RFH}(\Sigma, M, \omega_1).$$

PROOF. As before, we define an auxiliary action functional $\widehat{\mathcal{A}}_{\lambda_r}^H : \mathscr{L} \times \mathbb{R} \longrightarrow \mathbb{R}$ by

$$\widehat{\mathcal{A}}_{\lambda_r}^H(v, \eta) = -\int_{D^2} \bar{v}^* d\beta_r - \eta \int_0^1 H(t, v(t)) dt$$

where $\beta_r \in \Omega^1(M)$ is an extension of $\lambda_r \in \Omega^1(\Sigma)$, i.e. $\beta_r|_\Sigma = \lambda_r$, see [CFP] for a rigorous construction of β_r. For $(v_r, \eta_r) \in \mathrm{Crit}\mathcal{A}_{\omega_r}^H$, it holds that

$$\left| \frac{\partial}{\partial r} \eta_r \right| = \left| \frac{\partial}{\partial r} \widehat{\mathcal{A}}_{\lambda_r}^H(v_r, \eta_r) \right| = \left| \int_{S^1} v_r^* \dot{\lambda}_r \right| = \mathfrak{R}|\eta_r|. \tag{4.5}$$

where $\mathfrak{R} := \max_{r \in [0,1]} ||\dot{\lambda}_r(R_r)||_{L^\infty(\Sigma)}$, R_r is the Reeb vector field with respect to λ_r. As in the proof of Theorem 4.9, the above computation yields that critical points do not escape. Next we show the survival of homology classes. We consider the universal cover $(\widetilde{M}, \widetilde{\omega}_r)$ of M where $\widetilde{\omega}_r$ is the lift of ω_r. We choose a compatible almost complex structure J_r on (M, ω_r) so that $g_r(\cdot, \cdot) := \omega_r(\cdot, J_r \cdot)$ is a Riemannian metric on M. Then we lift g_r to $\widetilde{M}$, say $\tilde{g}_r$. Let $\widetilde{\Sigma}_\star(\cong \Sigma)$ be one of the fundamental domains in $\widetilde{\Sigma} \subset \widetilde{M}$ and $\tilde{v}_r : S^1 \longrightarrow \widetilde{M}$ intersecting $\Sigma_\star$ be the lift of v_r. Since we have assumed the symplectical asphericity of (M, ω_r), there exists a primitive 1-form σ of $\dot{\widetilde{\omega}}_r$. We let

$$\mathfrak{S} := \max_{x \in \tilde{v}(S^1)} \{||\sigma(x)||_{\tilde{g}} \,|\, (v, \eta) \in \mathrm{Crit}\mathcal{A}^H\}.$$

We define an equivalence relation such that $(v, \eta) \sim (v_0, \eta_0)$ if $\big(v(t), \eta\big) = \big(v_0(nt), n\eta_0\big)$ for some $2 \le n \in \mathbb{N}$ or $v(t) = v(t + r)$ for some $r \in S^1$. We note that there are only finitely many nonconstant representative classes and we can lift v and v_0 with $(v, \eta) \sim (v_0, \eta_0)$ so that $\tilde{v}(S^1) = \tilde{v}_0(S^1)$. Thus $\mathfrak{S}$ has finite value since $\bigcup_{(v,\eta) \in \mathrm{Crit}\mathcal{A}^H} \tilde{v}(S^1)$ is compact. Now we

compute

$$\left|\frac{\partial}{\partial r}\mathcal{A}^H_{\omega_r}(v_r,\eta_r)\right| = \left|\int_{D^2}\bar{v}_r^*(\dot{\omega}_r)\right| = \left|\int_{D^2}\bar{\bar{v}}_r^*\dot{\tilde{\omega}}_r\right| = \left|\int_{S^1}\tilde{v}_r^*\sigma\right|$$

$$\leq \mathfrak{S}\int_{S^1}||\partial_t\tilde{v}_r||_{\tilde{g}}dt$$

$$= \mathfrak{S}\int_{S^1}||\partial_t v_r||_g dt$$

$$\leq \mathfrak{S}|\eta_r|\int_{S^1}||X_H(v_r)||_g dt$$

$$\leq \Theta|\eta_r|$$

$$= \Theta|\widehat{\mathcal{A}}^H_{\lambda_r}(v_r,\eta_r)|$$

$$\leq c\Theta|\mathcal{A}^H_{\omega_r}(v_r,\eta_r)| \times \log|\mathcal{A}^H_{\omega_r}(v_r,\eta_r)| \times \cdots \times \log\cdots\log|\mathcal{A}^H_{\omega_r}(v_r,\eta_r)|.$$

where $\Theta = \mathfrak{S}||X_H|_\Sigma||_{L^\infty}$ and c is the tame constant. This computation shows that a stable and logarithmic-tame homotopy satisfies the hypothesis (H1) with the function

$$\Phi(s) = c\Theta|s|\log|s| \times \log\log|s| \times \cdots \times \log\cdots\log|s|$$

and then Theorem B concludes the proof. $\square$

Remark 4.17. We expect that the previous theorem also can be proved by the continuation method without assuming (H3); we refer to [BF] for the continuation method in the virtually contact case. Without doubt, our arguments are also valid in the virtually contact case.

5. Appendix: Legendrian and pre-Lagrangian

In this appendix, we briefly recall a part of the contact geometry, Legendrian curves and pre-Lagrangian submanifolds; we refer to [EHS, Ge] for the deeper and wider concepts.

Definition 5.1. Let M be a manifold of dimension $2n+1$. A contact structure on M is a maximally non-integrable hyperplane field $\xi = \ker\alpha \subset TM$, $\alpha \in \Omega^1(M)$, i.e. $\alpha \wedge (d\alpha)^n \neq 0$. Such a 1-form α is called a *contact form* and the pair (M,ξ) is called a *contact manifold*.

A defining 1-form α is unique up to nowhere vanishing functions, that is, $\ker\alpha = \ker f\alpha$ for any nowhere vanishing function f on M. Let $\mathcal{S}(M,\xi)$ be the trivial subbundle of T^*M whose fiber over $q \in M$ consists of all non-zero linear forms annihilating $\xi_q \subset T_q M$ and defining its coorientation. The bundle $\mathcal{S}(M,\xi)$ is a principal $\mathbb{R}$-bundle with the $\mathbb{R}$-action:

$$r \cdot \Theta = e^r\Theta, \quad r \in \mathbb{R}, \ \Theta \in \mathcal{S}(M,\xi).$$

Furthermore the canonical 1-form $\lambda = pdq$ on T^*M gives a symplectic structure $d\lambda|_{\mathcal{S}(M,\xi)}$ on $\mathcal{S}(M,\xi)$. The symplectic manifold

$$(\mathcal{S}(M,\xi), d\lambda|_{\mathcal{S}(M,\xi)})$$

is called a *symplectization* of (M,ξ). We note that a section of the bundle $\pi : \mathcal{S}(M,\xi) \longrightarrow M$ is a contact form.

Definition 5.2. An $(n+1)$-dimensional submanifold L of a contact manifold (M,ξ) satisfying the following two properties, is called *pre-Lagrangian*.

   (i) L is transverse to ξ,
   (ii) The distribution $\xi \cap TL$ is integrable and can be defined by a closed 1-form.

The motivation of the notion of "pre-Lagrangian" is provided the following proposition.

Proposition 5.3. [EHS] *If L is a pre-Lagrangian submanifold in (M, ξ) then there exists a Lagrangian submanifold $\widetilde{L}$ in the symplectic manifold $\mathrm{Sympl}(M, \xi)$ such that $\pi(\widetilde{L}) = L$. The cohomology class $\lambda \in \mathrm{H}^1(L; \mathbb{R})$ such that $\pi^* \lambda = [\alpha|_{\widetilde{L}}]$ is defined uniquely up to multiplication by a non-zero constant. Conversely if $\widetilde{L} \subset \mathcal{S}(M, \xi)$ is a Lagrangian submanifold then $\pi(\widetilde{L}) = L$ is a pre-Lagrangian in M.*

Thus a pre-Lagrangian submanifold carries a canonical projective class of the form λ. By definition there exists a contact form β with $d\beta|_L = 0$; in fact the desired Lagrangian submanifold $\widetilde{L}$ is the graph of $\beta|_L$.

Definition 5.4. A *Legendrian knot* in a contact 3-manifold (M, ξ) is a Legendrian embedding $\gamma : S^1 \longrightarrow M$, i.e. $\gamma'(\theta) \in \xi_{\gamma(\theta)}$ for all $\theta \in S^1$. A *Legendrian chord* is a Legendrian embedding $\gamma : [a, b] \longrightarrow M$ which begins and ends on pre-Lagrangian submanifolds.

In this appendix, we consider $\mathbb{R}^2 \times [0, 1]$ with standard contact structure $\xi_{\mathrm{st}} = \ker \alpha_{\mathrm{st}}$ where $\alpha_{\mathrm{st}} = dz + xdy$ for $(x, y, z) \in \mathbb{R} \times [0, 1] \times \mathbb{R}$. Let γ be either a Legendrian knot or a Legnedrian chord in $\mathbb{R}^2 \times [0, 1]$ and write $\gamma(s) = (x(s), y(s), z(s))$. Then the Legendrian condition yields

$$\alpha_{\mathrm{st}}(\gamma') = z' + xy' \equiv 0.$$

Definition 5.5. The *front projection* of a curve $\gamma(s) = (x(s), y(s), z(s))$ in $\mathbb{R} \times [0, 1] \times \mathbb{R}$ is a curve

$$\gamma_{\mathrm{F}}(s) = (y(s), z(s)) \subset [0, 1] \times \mathbb{R}.$$

If a curve γ is Legendrian then $y'(s) = 0$ implies $z'(s) = 0$, thus the front projection of γ has singular points where $y' = 0$, so called *cusp points*; moreover it does not have vertical tangencies. We call γ or γ_{F} *generic* if cusp points are isolated.

Lemma 5.6. Let $\gamma_{\mathrm{F}} : (a, b) \longrightarrow [0, 1] \times \mathbb{R}$ be a front projection of a certain Legendrian immersion. Then away from the cusp points we can recover the unique Legendrian immersion $\gamma : (a, b) \longrightarrow (\mathbb{R} \times [0, 1] \times \mathbb{R}, \xi_{\mathrm{st}})$ via

$$x(s) = -\frac{z'(s)}{y'(s)}.$$

The curve is embedded if and only if γ_{F} has only transverse self-intersections.

Remark 5.7. Consider one-parameter family of the functions $\{f_r\}_{r \in [0, 1]}$ on a manifold M, let $(r, f_r(x_r)) \in [0, 1] \times \mathbb{R}$ be one-parameter family of the critical values of f_r where $x_r \in \mathrm{Crit} f_r$. We parameterize this one-dimensional space $(r(s), f_{r(s)}(x_r(s)))$ and compute

$$\frac{d}{ds}\big(r(s), f_{r(s)}(x_r(s))\big) = \big(\dot{r}(s), df_{r(s)}(x_r(s))[\dot{x}_r(s)] + \dot{r}(s)\dot{f}_{r(s)}(x_r(s))\big)$$

$$= \big(\dot{r}(s), \dot{r}(s)\dot{f}_{r(s)}(x_r(s))\big).$$

It is known that there exists a homotopy of two Morse function (the Floer action functional case is proved by Lee [Lee1, Lee2]) so that the curve $(r(s), f_{r(s)}(x_r(s)))$ is generic. Moreover we also may assume that this curve has only transverse self-intersections; thus this curve can be lifted a unique Legendrian curve or chord in $\mathbb{R} \times [0, 1] \times \mathbb{R}$.

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