IDEALS IN INTRA-REGULAR LEFT ALMOST SEMIGROUPS

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Abstract. In this paper, we have introduced the notion of $(1,2)$-ideal in an LA-semigroup and shown that $(1,2)$-ideal and two-sided ideal coincide in an intra-regular LA-semigroup. We have characterized an intra-regular LA-semigroup by using the properties of left and right ideals. Some natural examples of LA-semigroups have been given. Further we have investigated some useful conditions for an LA-semigroup to become an intra-regular LA-semigroup and given the counter examples to illustrate the converse inclusions. All the ideals (left, right, two-sided, interior, quasi, bi- generalized bi- and $(1,2)$) of an intra-regular LA-semigroup have been characterized. Finally we have given an equivalent statement for a two-sided ideal of an intra-regular LA-semigroup in terms of the intersection of two minimal two-sided ideals of an intra-regular LA-semigroup.

Keywords. LA-semigroups, intra-regular LA-semigroups and $(1,2)$-ideals.

Introduction

The idea of generalization of a commutative semigroup was first introduced by Kazim and Naseeruddin in 1972 (see [3]). They named it as a left almost semigroup (LA-semigroup). It is also called an Abel-Grassmann's groupoid (AG-groupoid) [10].

An LA-semigroup is a non-associative and non-commutative algebraic structure mid way between a groupoid and a commutative semigroup. This structure is closely related with a commutative semigroup, because if an LA-semigroup contains a right identity, then it becomes a commutative semigroup [6]. The connection of a commutative inverse semigroup with an LA-semigroup has been given in [7] as, a commutative inverse semigroup $(S,\circ)$ becomes an LA-semigroup $(S,\cdot)$ under $a\cdot b=b\circ a^{-1}$, for all $a,b\in S$. An LA-semigroup S with left identity becomes a semigroup $(S,\circ)$ defined as, for all $x,\ y\in S$, there exists $a\in S$ such that $x\circ y=(xa)y$ [11]. An LA-semigroup is the generalization of a semigroup theory [6] and has vast applications in collaboration with semigroup like other branches of mathematics.

An LA-semigroup is a groupoid S whose elements satisfy the left invertive law $(ab)c=(cb)a$, for all $a,b,c\in S$. In an LA-semigroup, the medial law [3] $(ab)(cd)=(ac)(bd)$ holds for all $a,b,c,d\in S$. An LA-semigroup may or may not contains a left identity. The left identity of an LA-semigroup allow us to introduce the inverses of

2000 *Mathematics Subject Classification.* 20M10 and 20N99.

elements in an LA-semigroup. If an LA-semigroup contains a left identity, then it is unique [6]. In an LA-semigroup S with left identity, the paramedial law $(ab)(cd) = (dc)(ba)$ holds for all $a, b, c, d \in S$. If an LA-semigroup contains a left identity, then by using medial law, we get $a(bc) = b(ac)$, for all $a, b, c \in S$. Several examples and interesting properties of LA-semigroups can be found in [6] and [11].

In this paper, we have extended the concept of an intra-regular LA-semigroup first considered by M. Khan and N. Ahmad in [5].

Let S be an LA-semigroup. By an LA-subsemigroup of S, we means a non-empty subset A of S such that $A^2 \subseteq A$.

A non-empty subset A of an LA-semigroup S is called a left (right) ideal of S if $SA \subseteq A$ ($AS \subseteq A$).

By two-sided ideal or simply ideal, we mean a non-empty subset of an LA-semigroup S which is both a left and a right ideal of S.

A non empty subset A of an LA-semigroup S is called a generalized bi-ideal of S if $(AS)A \subseteq A$ and an LA-subsemigroup A of S is called a bi-ideal of S if $(AS)A \subseteq A$.

A non-empty subset A of an LA-semigroup S is called an interior ideal of S if $(SA)S \subseteq A$.

A non empty subset A of an LA-semigroup S is called a quasi ideal of S if $SA \cap AS \subseteq A$.

An LA-subsemigroup A of S is called a $(1, 2)$-ideal of S if $(AS)A^2 \subseteq A$.

Example 1. *Let us consider an LA-semigroup $S = \{a, b, c, d, e, f\}$ with left identity e in the following Clayey's table.*

.	a	b	c	d	e	f
a	a	a	a	a	a	a
b	a	b	b	b	b	b
c	a	b	f	f	d	f
d	a	b	f	f	c	f
e	a	b	c	d	e	f
f	a	b	f	f	f	f

Example 2. *Let us consider the set $(\mathbb{R}, +)$ of all real numbers under the binary operation of addition. If we define $a * b = b - a - r$, where $a, b, r \in R$, then $(\mathbb{R}, *)$ becomes an LA-semigroup as,*

$$(a * b) * c = c - (a * b) - r = c - (b - a - r) - r = c - b + a + r - r = c - b + a$$

and

$$(c * b) * a = a - (c * b) - r = a - (b - c - r) - r = a - b + c + r - r = a - b + c.$$

Since $(\mathbb{R}, +)$ is commutative so $(a * b) * c = (c * b) * a$ and therefore $(\mathbb{R}, *)$ satisfies a left invertive law. It is easy to observe that $(R, *)$ is non-commutative and non-associative. The same is hold for set of integers and rationals. Thus $(\mathbb{R}, *)$ is an LA-semigroup which is the generalization of an LA-semigroup given in 1988 (*see* [7]). *Similarly if we define $a * b = ba^{-1}r^{-1}$, then $(\mathbb{R}\backslash\{0\}, *)$ becomes an LA-semigroup and the same holds for the set of integers and rationals. This LA-semigroup is also the generalization of an LA-semigroup given in* 1988 (*see* [7]).

An element a of an LA-semigroup S is called an intra-regular if there exist $x, y \in S$ such that $a = (xa^2)y$ and S is called intra-regular, if every element of S is intra-regular.

Example 3. *Let $S = \{a, b, c, d, e\}$ be an LA-semigroup with left identity b in the following multiplication table.*

.	a	b	c	d	e
a	a	a	a	a	a
b	a	b	c	d	e
c	a	e	b	c	d
d	a	d	e	b	c
e	a	c	d	e	b

Clearly S is intra-regular because, $a = (aa^2)a$, $b = (cb^2)e$, $c = (dc^2)e$, $d = (cd^2)c$, $e = (be^2)e$.

An element a of an LA-semigroup S with left identity e is called a left (right) invertible if there exits $x \in S$ such that $xa = e$ $(ax = e)$ and a is called invertible if it is both a left and a right invertible. An LA-semigroup S is called a left (right) invertible if every element of S is a left (right) invertible and S is called invertible if it is both a left and a right invertible.

Note that in an LA-semigroup S with left identity, $S = S^2$.

Theorem 1. *Every LA-semigroup S with left identity is an intra-regular if S is left (right) invertible.*

Proof. Let S be a left invertible LA-semigroup with left identity, then for $a \in S$ there exists $a^{'} \in S$ such that $a^{'}a = e$. Now by using left invertive law, medial law with left identity and medial law, we have

$$\begin{aligned} a &= ea = e(ea) = (a^{'}a)(ea) \in (Sa)(Sa) = (Sa)((SS)a) \\ &= (Sa)((aS)S) = (aS)((Sa)S) = (a(Sa))(SS) \\ &= (a(Sa))S = (S(aa))S = (Sa^2)S. \end{aligned}$$

Which shows that S is intra-regular. Similarly in the case of right invertible. □

Theorem 2. *An LA-semigroup S is intra-regular if $Sa = S$ or $aS = S$ holds for all $a \in S$.*

Proof. Let S be an LA-semigroup such that $Sa = S$ holds for all $a \in S$, then $S = S^2$. Let $a \in S$, therefore by using medial law, we have

$$a \in S = (SS)S = ((Sa)(Sa))S = ((SS)(aa))S \subseteq (Sa^2)S.$$

Which shows that S is intra-regular.

Let $a \in S$ and assume that $aS = S$ holds for all $a \in S$, then by using left invertive law, we have

$$a \in S = SS = (aS)S = (SS)a = Sa.$$

Thus $Sa = S$ holds for all $a \in S$, therefore it follows from above that S is intra-regular. □

The converse is not true in general from Example 3.

Corollary 1. *If S is an LA-semigroup such that $aS = S$ holds for all $a \in S$, then $Sa = S$ holds for all $a \in S$.*

Theorem 3. *If S is intra-regular LA-semigroup with left identity, then $(BS)B = B \cap S$, where B is a bi-(generalized bi-) ideal of S.*

Proof. Let S be an intra-regular LA-semigroup with left identity, then clearly $(BS)B \subseteq B \cap S$. Now let $b \in B \cap S$, which implies that $b \in B$ and $b \in S$. Since S is intra-regular so there exist $x, y \in S$ such that $b = (xb^2)y$. Now by using medial law with left identity, left invertive law, paramedial law and medial law, we have

$$\begin{aligned} b &= (x(bb))y = (b(xb))y = (y(xb))b = (y(x((xb^2)y)))b \\ &= (y((xb^2)(xy)))b = ((xb^2)(y(xy)))b = (((xy)y)(b^2x))b \\ &= ((bb)(((xy)y)x))b = ((bb)((xy)(xy)))b = ((bb)(x^2y^2))b \\ &= ((y^2x^2)(bb))b = (b((y^2x^2)b))b \in (BS)B. \end{aligned}$$

This shows that $(BS)B = B \cap S$. □

The converse is not true in general. For this, let us consider an LA-semigroup S with left identity e in Example 1. It is easy to see that $\{a, b, f\}$ is a bi-(generalized bi-) ideal of S such that $(BS)B = B \cap S$ but S is not an intra-regular because $d \in S$ is not an intra-regular.

Corollary 2. *If S is intra-regular LA-semigroup with left identity, then $(BS)B = B$, where B is a bi-(generalized bi-) ideal of S.*

Theorem 4. *If S is intra-regular LA-semigroup with left identity, then $(SB)S = S \cap B$, where B is an interior ideal of S.*

Proof. Let S be an intra-regular LA-semigroup with left identity, then clearly $(SB)S \subseteq S \cap B$. Now let $b \in S \cap B$, which implies that $b \in S$ and $b \in B$. Since S is an intra-regular so there exist $x, y \in S$ such that $b = (xb^2)y$. Now by using paramedial law and left invertive law, we have

$$b = ((ex)(bb))y = ((bb)(xe))y = (((xe)b)b)y \in (SB)S.$$

Which shows that $(SB)S = S \cap B$. □

The converse is not true in general. It is easy to see that form Example 1 that $\{a, b, f\}$ is an interior ideal of an LA-semigroup S with left identity e such that $(SB)S = B \cap S$ but S is not an intra-regular because $d \in S$ is not an intra-regular.

Corollary 3. *If S is intra-regular LA-semigroup with left identity, then $(SB)S = B$, where B is an interior ideal of S.*

Let S be an LA-semigroup, then $\emptyset \neq A \subseteq S$ is called semiprime if $a^2 \in A$ implies $a \in A$.

Theorem 5. *An LA-semigroup S with left identity is intra-regular if $L \cup R = LR$, where L and R are the left and right ideals of S respectively such that R is semiprime.*

Proof. Let S be an LA-semigroup with left identity, then clearly Sa and a^2S are the left and right ideals of S such that $a \in Sa$ and $a^2 \in a^2S$, because by using paramedial law, we have

$$a^2S = (aa)(SS) = (SS)(aa) = Sa^2.$$

Therefore by given assumption, $a \in a^2S$. Now by using left invertive law, medial law, paramedial law and medial law with left identity, we have

$$\begin{aligned} a &\in Sa \cup a^2S = (Sa)(a^2S) = (Sa)((aa)S) = (Sa)((Sa)(ea)) \\ &\subseteq (Sa)((Sa)(Sa)) = (Sa)((SS)(aa)) \subseteq (Sa)((SS)(Sa)) \\ &= (Sa)((aS)(SS)) = (Sa)((aS)S) = (aS)((Sa)S) \\ &= (a(Sa))(SS) = (a(Sa))S = (S(aa))S = (Sa^2)S. \end{aligned}$$

Which shows that S is intra-regular. □

The converse is not true in general. In Example 1, the only left and right ideal of S is $\{a, b\}$, where $\{a, b\}$ is semiprime such that $\{a, b\} \cup \{a, b\} = \{a, b\}\{a, b\}$ but S is not an intra-regular because $d \in S$ is not an intra-regular.

Lemma 1. [5] *If S is intra-regular regular LA-semigroup, then $S = S^2$.*

Theorem 6. *For a left invertible LA-semigroup S with left identity, the following conditions are equivalent.*

(i) S is intra-regular.
(ii) $R \cap L = RL$, where R and L are any left and right ideals of S respectively.

Proof. $(i) \Longrightarrow (ii)$: Assume that S is intra-regular LA-semigroup with left identity and let $a \in S$, then there exist $x, y \in S$ such that $a = (xa^2)y$. Let R and L be any left and right ideals of S respectively, then obviously $RL \subseteq R \cap L$. Now let $a \in R \cap L$ implies that $a \in R$ and $a \in L$. Now by using medial law with left identity, medial law and left invertive law, we have

$$\begin{aligned} a &= (xa^2)y \in (Sa^2)S = (S(aa))S = (a(Sa))S = (a(Sa))(SS) \\ &= (aS)((Sa)S) = (Sa)((aS)S) = (Sa)((SS)a) = (Sa)(Sa) \\ &\subseteq (SR)(SL) = ((SS)R)(SL) = ((RS)S)(SL) \subseteq RL. \end{aligned}$$

This shows that $R \cap L = RL$.

$(ii) \Longrightarrow (i)$: Let S be a left invertible LA-semigroup with left identity, then for $a \in S$ there exists $a' \in S$ such that $a'a = e$. Since a^2S is a right ideal and also a left ideal of S such that $a^2 \in a^2S$, therefore by using given assumption, medial law with left identity and left invertive law, we have

$$\begin{aligned} a^2 &\in a^2S \cap a^2S = (a^2S)(a^2S) = a^2((a^2S)S) = a^2((SS)a^2) \\ &= (aa)(Sa^2) = ((Sa^2)a)a. \end{aligned}$$

Thus we get, $a^2 = ((xa^2)a)a$ for some $x \in S$.

Now by using left invertive law, we have

$$\begin{aligned} (aa)a' &= (((xa^2)a)a)a' \\ (a'a)a &= (a'a)(((xa^2)a) \\ a &= (xa^2)a. \end{aligned}$$

This shows that S is intra-regular. □

Lemma 2. [5] *Every two-sided ideal of an intra-regular LA-semigroup S with left identity is idempotent.*

Theorem 7. *In an LA-semigroup S with left identity, the following conditions are equivalent.*

(i) S is intra-regular.
(ii) $A = (SA)^2$, where A is any left ideal of S.

Proof. $(i) \Longrightarrow (ii)$: Let A be a left ideal of an intra-regular LA-semigroup S with left identity, then $SA \subseteq A$ and by Lemma 2, $(SA)^2 = SA \subseteq A$. Now $A = AA \subseteq SA = (SA)^2$, which implies that $A = (SA)^2$.

$(ii) \Longrightarrow (i)$: Let A be a left ideal of S, then $A = (SA)^2 \subseteq A^2$, which implies that A is idempotent and by using Lemma 4, S is intra-regular. □

Theorem 8. *In an intra-regular LA-semigroup S with left identity, the following conditions are equivalent.*

(i) A is a bi-(generalized bi-) ideal of S.
(ii) $(AS)A = A$ and $A^2 = A$.

Proof. $(i) \Longrightarrow (ii)$: Let A be a bi-ideal of an intra-regular LA-semigroup S with left identity, then $(AS)A \subseteq A$. Let $a \in A$, then since S is intra-regular so there exist x, $y \in S$ such that $a = (xa^2)y$. Now by using medial law with left identity, left invertive law, medial law and paramedial law, we have

$$\begin{aligned} a &= (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a \\ &= (y(x((xa^2)y)))a = (y((xa^2)(xy)))a \\ &= ((xa^2)(y(xy)))a = ((x(aa))(y(xy)))a \\ &= ((a(xa))(y(xy)))a = ((ay)((xa)(xy)))a \\ &= ((xa)((ay)(xy)))a = ((xa)((ax)y^2))a \\ &= ((y^2(ax))(ax))a = (a((y^2(ax))x))a \in (AS)A. \end{aligned}$$

Thus $(AS)A = A$ holds. Now by using medial law with left identity, left invertive law, paramedial law and medial law, we have

$$\begin{aligned} a &= (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a = (y(x((xa^2)y)))a \\ &= (y((xa^2)(xy)))a = ((xa^2)(y(xy)))a = ((x(aa))(y(xy)))a \\ &= ((a(xa))(y(xy)))a = (((y(xy))(xa))a)a = (((ax)((xy)y))a)a \\ &= (((ax)(y^2x))a)a = (((ay^2)(xx))a)a = (((ay^2)x^2)a)a \\ &= (((x^2y^2)a)a)a = (((x^2y^2)((x(aa))y))a)a \\ &= (((x^2y^2)((a(xa))y))a)a = (((x^2(a(xa)))(y^2y))a)a \\ &= (((a(x^2(xa)))y^3)a)a = (((a((xx)(xa)))y^3)a)a \\ &= (((a((ax)(xx)))y^3)a)a = ((((ax)(ax^2))y^3)a)a \\ &= ((((aa)(xx^2))y^3)a)a = (((y^3x^3)(aa))a)a \\ &= ((a((y^3x^3)a))a)a \subseteq ((AS)A)A \subseteq AA = A^2. \end{aligned}$$

Hence $A = A^2$ holds.
$(ii) \Longrightarrow (i)$ is obvious. □

Theorem 9. *In an intra-regular LA-semigroup S with left identity, the following conditions are equivalent.*

(i) A is a quasi ideal of S.
(ii) $SQ \cap QS = Q$.

Proof. $(i) \Longrightarrow (ii)$: Let Q be a quasi ideal of an intra-regular LA-semigroup S with left identity, then $SQ \cap QS \subseteq Q$. Let $q \in Q$, then since S is intra-regular so there exist $x, y \in S$ such that $q = (xq^2)y$. Let $pq \in SQ$, then by using medial law with left identity, medial law and paramedial law, we have

$$\begin{aligned} pq &= p((xq^2)y) = (xq^2)(py) = (x(qq))(py) = (q(xq))(py) \\ &= (qp)((xq)y) = (xq)((qp)y) = (y(qp))(qx) \\ &= q((y(qp))x) \in QS. \end{aligned}$$

Now let $qy \in QS$, then by using left invertive law, medial law with left identity and paramedial law, we have

$$\begin{aligned} qp &= ((xq^2)y)p = (py)(xq^2) = (py)(x(qq)) = x((py)(qq)) \\ &= x((qq)(yp)) = (qq)(x(yp)) = ((x(yp))q)q \in SQ. \end{aligned}$$

Hence $QS = SQ$. As by using medial law with left identity and left invertive law, we have

$$q = (xq^2)y = (x(qq))y = (q(xq))y = (y(xq))q \in SQ.$$

Thus $q \in SQ \cap QS$ implies that $SQ \cap QS = Q$.

$(ii) \Longrightarrow (i)$ is obvious. □

Theorem 10. *In an intra-regular LA-semigroup S with left identity, the following conditions are equivalent.*

(i) A is an interior ideal of S.
(ii) $(SA)S = A$.

Proof. $(i) \Longrightarrow (ii)$: Let A be an interior ideal of an intra-regular LA-semigroup S with left identity, then $(SA)S \subseteq A$. Let $a \in A$, then since S is intra-regular so there exist $x, y \in S$ such that $a = (xa^2)y$. Now by using medial law with left identity, left invertive law and paramedial law, we have

$$\begin{aligned} a &= (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a = (y(xa))((xa^2)y) \\ &= (((xa^2)y)(xa))y = ((ax)(y(xa^2)))y = (((y(xa^2))x)a)y \in (SA)S. \end{aligned}$$

Thus $(SA)S = A$.

$(ii) \Longrightarrow (i)$ is obvious. □

Theorem 11. *In an intra-regular LA-semigroup S with left identity, the following conditions are equivalent.*

(i) A is a $(1, 2)$-ideal of S.
(ii) $(AS)A^2 = A$ and $A^2 = A$.

Proof. $(i) \Longrightarrow (ii)$: Let A be a $(1, 2)$-ideal of an intra-regular LA-semigroup S with left identity, then $(AS)A^2 \subseteq A$ and $A^2 \subseteq A$. Let $a \in A$, then since S is intra-regular so there exist $x, y \in S$ such that $a = (xa^2)y$. Now by using medial law with left identity, left invertive law and paramedial law, we have

$$\begin{aligned} a &= (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a \\ &= (y(x((xa^2)y)))a = (y((xa^2)(xy)))a = ((xa^2)(y(xy)))a \\ &= (((xy)y)(a^2x))a = ((y^2x)(a^2x))a = (a^2((y^2x)x))a \\ &= (a^2(x^2y^2))a = (a(x^2y^2))a^2 = (a(x^2y^2))(aa) \in (AS)A^2. \end{aligned}$$

Thus $(AS)A^2 = A$. Now by using medial law with left identity, left invertive law, paramedial law and medial law, we have

$$\begin{aligned}
a &= (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a \\
&= (y(xa))((xa^2)y) = (xa^2)((y(xa))y) = (x(aa))((y(xa))y) \\
&= (a(xa))((y(xa))y) = (((y(xa))y)(xa))a = ((ax)(y(y(xa))))a \\
&= ((((xa^2)y)x)(y(y(xa))))a = (((xy)(xa^2))(y(y(xa))))a \\
&= (((xy)y)((xa^2)(y(xa))))a = ((y^2x)((x(aa))(y(xa))))a \\
&= ((y^2x)((xy)((aa)(xa))))a = ((y^2x)((aa)((xy)(xa))))a \\
&= ((aa)((y^2x)((xy)(xa))))a = ((aa)((y^2x)((xx)(ya))))a \\
&= ((((xx)(ya))(y^2x))(aa))a = ((((ay)(xx))(y^2x))(aa))a \\
&= ((((x^2y)a)(y^2x))(aa))a = (((xy^2)(a(x^2y)))(aa))a \\
&= ((a((xy^2)(x^2y)))(aa))a = ((a(x^3y^3))(aa))a \\
&\in ((AS)A^2)A \subseteq AA = A^2.
\end{aligned}$$

Hence $A^2 = A$.

$(ii) \Longrightarrow (i)$ is obvious. □

Lemma 3. [5]*Every non empty subset A of an intra-regular LA-semigroup S with left identity is a left ideal of S if and only if it is a right ideal of S.*

Theorem 12. *In an intra-regular LA-semigroup S with left identity, the following conditions are equivalent.*

(i) A is a $(1,2)$-ideal of S.
(ii) A is a two-sided ideal of S.

Proof. $(i) \Longrightarrow (ii)$: Assume that S is intra-regular LA-semigroup with left identity and let A be a $(1,2)$-ideal of S then, $(AS)A^2 \subseteq A$. Let $a \in A$, then since S is intra-regular so there exist x, $y \in S$ such that $a = (xa^2)y$. Now by using medial law with left identity, left invertive law and paramedial law, we have

$$\begin{aligned}
sa &= s((xa^2)y) = (xa^2)(sy) = (x(aa))(sy) = (a(xa))(sy) \\
&= ((sy)(xa))a = ((sy)(xa))((xa^2)y) = (xa^2)(((sy)(xa))y) \\
&= (y((sy)(xa)))(a^2x) = a^2((y((sy)(xa)))x) \\
&= (aa)((y((sy)(xa)))x) = (x(y((sy)(xa))))(aa) \\
&= (x(y((ax)(ys))))(aa) = (x((ax)(y(ys))))(aa) \\
&= ((ax)(x(y(ys))))(aa) = ((((xa^2)y)x)(x(y(ys))))(aa) \\
&= (((xy)(xa^2))(x(y(ys))))(aa) = (((a^2x)(yx))(x(y(ys))))(aa) \\
&= ((((yx)x)a^2)(x(y(ys))))(aa) = (((y(ys))x)(a^2((yx)x)))(aa) \\
&= (((y(ys))x)(a^2(x^2y)))(aa) = (a^2(((y(ys))x)(x^2y)))(aa) \\
&= ((aa)(((y(ys))x)(x^2y)))(aa) = (((x^2y)((y(ys))x))(aa))(aa) \\
&= (a((x^2y)(((y(ys))x)a)))(aa) \in (AS)A^2 \subseteq A.
\end{aligned}$$

Hence A is a left ideal of S and by Lemma 3, A is a two-sided ideal of S.

$(ii) \Longrightarrow (i)$: Let A be a two-sided ideal of S. Let $y \in (AS)A^2$, then $y = (as)b^2$ for some $a, b \in A$ and $s \in S$. Now by using medial law with left identity, we have

$$y = (as)b^2 = (as)(bb) = b((as)b) \in AS \subseteq A.$$

Hence $(AS)A^2 \subseteq A$, therefore A is a $(1,2)$-ideal of S. □

Lemma 4. [5] *Let S be an LA-semigroup, then S is intra-regular if and only if every left ideal of S is idempotent.*

Lemma 5. [5]*Every non empty subset A of an intra-regular LA-semigroup S with left identity is a two-sided ideal of S if and only if it is a quasi ideal of S.*

Theorem 13. *A two-sided ideal of an intra-regular LA-semigroup S with left identity is minimal if and only if it is the intersection of two minimal two-sided ideals of S.*

Proof. Let S be intra-regular LA-semigroup and Q be a minimal two-sided ideal of S, let $a \in Q$. As $S(Sa) \subseteq Sa$ and $S(aS) \subseteq a(SS) = aS$, which shows that Sa and aS are left ideals of S, so by Lemma 3, Sa and aS are two-sided ideals of S.

Now

$$\begin{aligned} S(Sa \cap aS) \cap (Sa \cap aS)S &= S(Sa) \cap S(aS) \cap (Sa)S \cap (aS)S \\ &\subseteq (Sa \cap aS) \cap (Sa)S \cap Sa \subseteq Sa \cap aS. \end{aligned}$$

This implies that $Sa \cap aS$ is a quasi ideal of S, so by using 5, $Sa \cap aS$ is a two-sided ideal of S.

Also since $a \in Q$, we have

$$Sa \cap aS \subseteq SQ \cap QS \subseteq Q \cap Q \subseteq Q.$$

Now since Q is minimal, so $Sa \cap aS = Q$, where Sa and aS are minimal two-sided ideals of S, because let I be an two-sided ideal of S such that $I \subseteq Sa$, then $I \cap aS \subseteq Sa \cap aS \subseteq Q$, which implies that $I \cap aS = Q$. Thus $Q \subseteq I$. Therefore, we have

$$Sa \subseteq SQ \subseteq SI \subseteq I, \text{ gives } Sa = I.$$

Thus Sa is a minimal two-sided ideal of S. Similarly aS is a minimal two-sided ideal of S.

Conversely, let $Q = I \cap J$ be a two-sided ideal of S, where I and J are minimal two-sided ideals of S, then by using 5, Q is a quasi ideal of S, that is $SQ \cap QS \subseteq Q$. Let $Q^{'}$ be a two-sided ideal of S such that $Q^{'} \subseteq Q$, then

$$SQ^{'} \cap Q^{'}S \subseteq SQ \cap QS \subseteq Q, \text{ also } SQ^{'} \subseteq SI \subseteq I \text{ and } Q^{'}S \subseteq JS \subseteq J.$$

Now

$$S(SQ^{'}) = (SS)(SQ^{'}) = (Q^{'}S)(SS) = (Q^{'}S)S = (SS)Q^{'} = SQ^{'},$$

which implies that $SQ^{'}$ is a left ideal and hence a two-sided ideal by Lemma 3. Similarly $Q^{'}S$ is a two-sided ideal of S.

But since I and J are minimal two-sided ideals of S, therefore $SQ^{'} = I$ and $Q^{'}S = J$. But $Q = I \cap J$, which implies that, $Q = SQ^{'} \cap Q^{'}S \subseteq Q^{'}$. This give us $Q = Q^{'}$ and hence Q is minimal. □

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