Existence of minimal nodal solutions for the Nonlinear Schroedinger equations with $V(\infty) = 0$

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Abstract

We consider the problem $\Delta u + V(x)u = f'(u)$ in $\mathbb{R}^N$. Here the nonlinearity has a double power behavior and V is invariant under an orthogonal involution, with $V(\infty) = 0$. An existence theorem of one pair of solutions which change sign exactly once is given.

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1 Introduction

It is well known that stationary states of Nonlinear Schroedinger equations lead to problems of the type

$$\begin{cases} -\Delta u + V(x)u = f'(u), & x \in \mathbb{R}^N; \\ E_V(u) < \infty. \end{cases} \tag{$\mathscr{P}$}$$

where the energy functional is defined by

$$E_V(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2(x)dx - \int_{\mathbb{R}^N} f(u)dx. \tag{1}$$

We consider a function $f \in C^2(\mathbb{R}, \mathbb{R})$ even with $f(0) = f'(0) = 0$ that satisfies the following requirements:

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(f_1) there exists $\mu > 2$ such that

$$0 < \mu f(s) \leq f'(s)s < f''(s)s^2 \text{ for all } s \neq 0 \tag{2}$$

(f_2) there exist positive numbers c_0, c_2, p, q with $2 < p < 2^* < q$ such that

$$\begin{cases} c_0|s|^p \leq f(s) & \text{for } |s| \geq 1; \\ c_0|s|^q < f(s) & \text{for } |s| \leq 1; \end{cases} \tag{3}$$

$$\begin{cases} |f''(s)| \leq c_2|s|^{p-2} & \text{for } |s| \geq 1 \\ |f''(s)| \leq c_2|s|^{q-2} & \text{for } |s| \leq 1 \end{cases} \tag{4}$$

where $2^* = \frac{2N}{N-2}$.

We assume $V \in L^{N/2}(\mathbb{R}^N)$ and

$$||V||_{L^{N/2}} < S := \inf_{u \in \mathcal{D}^{1,2}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2}{\left(\int_{\mathbb{R}^N} |u|^{2^*}\right)^{2/2^*}} \tag{5}$$

In the case of a single-power nonlinearity some paper has been devoted to the existence of positive solutions when potential V vanishes at infinity. Among others we recall [2, 6] and we quote the references therein.

In pioneering work Berestycki and Lions [11, 12] showed the existence of a positive solution in the case $V \equiv 0$ when $f''(0) = 0$, f has a supercritical growth near the origin and subcritical at infinity.

More recently in the papers [3, 8, 9, 10, 18] the double-power growth condition (f_2) has been used to obtain the existence of positive solutions for different problems of the tipe $(\mathscr{P})$. In particular, in [8] the authors proved that if $V \geq 0$ and $V > 0$ on a set of positive measure the problem $(\mathscr{P})$ has no ground state solution, i.e. there is no solution u of $(\mathscr{P})$ which minimizes the functional E_V on the Nehari manifold $\mathcal{N}_V$, defined by

$$\mathcal{N}_V = \left\{u \in \mathcal{D}^{1,2} \ : \ \langle \nabla E_V(u), u \rangle = 0, u \neq 0 \right\}. \tag{6}$$

On the contrary there exists a ground state solution either if $V \leq 0$ and $V < 0$ on a set of positive measure, or $V \equiv 0$.

In this paper we are interested in the existence of sign changing solutions. Besides the difficulty posed by the lack of compactness we have another problem: there is no natural regular constraint for sign changing solution of problem $(\mathscr{P})$. To overcome this difficulty we consider the problem

$$\begin{cases} -\Delta u + V(x)u = f'(u), \quad x \in \mathbb{R}^N; \\ E_V(u) < \infty; \\ u(\tau x) = -u(x), \end{cases} \tag{$\mathscr{P}_\tau$}$$

where τ is a non trivial orthogonal involution that is a linear orthogonal transformation on $\mathbb{R}^N$ such that $\tau \neq \mathrm{Id}$ and $\tau^2 = \mathrm{Id}$ (Id being the identity on $\mathbb{R}^N$).

We assume $V(\tau(x)) = V(x)$.

By the nontrivial orthogonal involution τ on $\mathbb{R}^N$ we can define a self adjoint linear isometry on $\mathcal{D}^{1,2}$ which we also denote τ. We define

$$\begin{aligned} &\tau : \mathcal{D}^{1,2} \to \mathcal{D}^{1,2}; \\ &(\tau u)(x) := -u(\tau(x)). \end{aligned} \tag{7}$$

If $u(\tau x) = -u(x)$, it will be called τ-antisymmetric. Note that non trivial antisymmetric solutions are changing sign or nodal solutions. Nodal solutions which change sign exactly once will be called minimal nodal solutions.

We define

$$\mathcal{N}_V^\tau = \mathcal{N}_V \cap \mathcal{D}_\tau^{1,2} \quad \text{where} \quad \mathcal{D}_\tau^{1,2} = \{u \in \mathcal{D}^{1,2} \ : \ \tau u = u\}.$$

The non trivial antisymmetric solutions of $(\mathscr{P}_\tau)$ are the critical points of E_V on $\mathcal{N}_V^\tau$

We set now

$$\mu_V = \mu_V(\mathbb{R}^N) := \inf_{\mathcal{N}_V} E_V; \qquad \mu_0 = \mu_0(\mathbb{R}^N) := \inf_{\mathcal{N}_0} E_0; \tag{8}$$

$$\mu_V^\tau = \mu_V^\tau(\mathbb{R}^N) := \inf_{\mathcal{N}_V^\tau} E_V; \qquad \mu_0^\tau = \mu_0^\tau(\mathbb{R}^N) := \inf_{\mathcal{N}_0^\tau} E_0. \tag{9}$$

We shall prove the following results.

Theorem 1. *If $V(x) > 0$ for almost every x, then*

$$\mu_V^\tau = \mu_0^\tau = 2\mu_0,$$

and μ_V^τ is not achieved. Then the problem $(\mathscr{P}_\tau)$ has no solution of minimal energy.

We consider the following class of potentials.

$$V_y(x) = \begin{cases} a|x-y| - 1 & |x-y| < 1; \\ a|x-\tau y| - 1 & |x-\tau y| < 1; \\ 0 & \text{elsewhere} \end{cases} \tag{10}$$

where $a \in \mathbb{R}$ is chosen such that $||V||_{L^{N/2}} < S$, S as in (5). We can prove the following existence result.

Theorem 2. *For the potential V_y such that $|y - \tau y|$ is sufficiently large we have that $\mu^\tau_V < \mu^\tau_0$ and it is achieved. Then the problem ($\mathscr{P}_\tau$) has at least one pair of antisymmetric solutions which change sign exactly once, and the energy of these solutions is minimal.*

We want to mention some recent work about sign changing solutions. The existence of a sequence of nodal solutions and some properties for the number of their nodal domains has been obtained in [5] considering the problem in a bounded smooth domain Ω with $V \equiv 0$, and in [4] in $\mathbb{R}^N$ with $\operatorname{essinf} V > 0$.

In [15] there is a theorem of multiplicity of solutions for the problem $-\Delta u + Vu = q(x)|u|^{p-2}u$ where $V(x)$ and $q(x)$ tend to some positive number V_∞ and q_∞ respectively as $|x| \to \infty$. However no precise information is given whether there are sign changing solutions or not. If $V \equiv 1$ and $q(x)$ suitable chosen, with $||q - 1||_\infty$ small, Hirano [16] prove the existence of at least two pairs of sign changing solutions.

In [14] the equation $-\Delta u + \lambda u = |u|^{p^*-2}u$, $\lambda > -\lambda_1$ on a symmetric domain is considered and the effect of the domain topology on the number of minimal nodal solutions in studied.

The plan of the paper is the following.

In section 2 we recall some technical result concerning the appropriate function spaces required by the growth properties of f; the proof of these results are contained in [7, 8, 10]. In section 3 we prove a *splitting lemma* which is a variant of a well known result of [19]; this lemma is the ingredient to handle the problem with lack of compactness. In section 4 we prove our results.

We will use the following notations

- $u^+ = \max(0, u)$; $u^- = \min(0, u)$;
- $\mathcal{D}^{1,2} = \mathcal{D}^{1,2}(\mathbb{R}^N)$ = completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm
$$||u||_{\mathcal{D}^{1,2}} = \left(\int_{\mathbb{R}^n} |\nabla u|^2\right)^{1/2};$$
- $v_y(x) = v(x + y)$;
- $G_V(u) = \langle \nabla E_V(u), u\rangle = \int |\nabla u|^2 + Vu^2 - f'(u)u$
- $g_u(t) = g_u^V(t) := E_V(tu) = \int_{\mathbb{R}^N} \frac{t^2}{2}(|\nabla u|^2 + Vu^2) - f(tu)dx$
- $B_R = \{x \in \mathbb{R}^N \ : \ |x| < R\}$;

- $B_R(z) = B(z,R) = \{x \in \mathbb{R}^N \ : \ |x-z| < R\}$;
- $A^C = \mathbb{R}^N \smallsetminus A$, where $A \subset \mathbb{R}^N$.

2 Variational Setting

In order to study the functional E_V, by the growth assumption on f, is useful to consider the functional space $L^p + L^q$, where $2 < p < 2^* < q$. Hereafter we recall some result contained in [7, 8, 10].

Given $p \neq q$, we consider the space $L^p + L^q$ made up of the functions $v : \mathbb{R}^N \to \mathbb{R}$ such that

$$v = v_1 + v_2 \quad \text{with} \quad v_1 \in L^p, v_2 \in L^q. \tag{11}$$

The space $L^p + L^q$ is a Banach space equipped with the norm:

$$||v||_{L^p+L^q} = \inf\{ \ ||v_1||_{L^p} + ||v_2||_{L^q} \ : \ v_1 \in L^p, v_2 \in L^q, \ v_1 + v_2 = v\}. \tag{12}$$

It is well known (see, for example [13]) that $L^p + L^q$ coincides with the dual of $L^{p'} \cap L^{q'}$. Then:

$$L^p + L^q = \left(L^{p'} \cap L^{q'}\right)' \quad \text{with} \quad p' = \frac{p}{p-1}, \ q' = \frac{q}{q-1}. \tag{13}$$

We recall some results useful for this paper.

Remark 3. Set $\Gamma_u = \{u \in \mathbb{R}^N \ : \ |u(x)| \geq 1\}$. We have

1. if $v \in L^p + L^q$, the following inequalities hold:

$$\begin{aligned} & \max\left[||v||_{L^q(\mathbb{R}^N \smallsetminus \Gamma_v)} - 1, \frac{1}{1 + |\Gamma_v|^{\frac{1}{\tau}}}||v||_{L^p(\Gamma_v)}\right] \leq \\ \leq \ & ||v||_{L^p+L^q} \leq \\ \leq \ & \max[||v||_{L^q(\mathbb{R}^N \smallsetminus \Gamma_v)}, ||v||_{L^p(\Gamma_v)}] \end{aligned}$$

   when $\tau = \frac{pq}{q-p}$;

2. let $\{v_n\} \subset L^p + L^q$. Then $\{v_n\}$ is bounded in $L^p + L^q$ if and only if the sequences $\{|\Gamma_{v_n}|\}$ and $\{||v||_{L^q(\mathbb{R}^N \smallsetminus \Gamma_{v_n})} + ||v||_{L^p(\Gamma_{v_n})}\}$ are bounded.

3. we have $L^{2^*} \subset L^p + L^q$ when $2 < p < 2^* < q$. Then, by Sobolev inequality, we get the continuous embedding

$$\mathcal{D}^{1,2}(\mathbb{R}^N) \subset L^p + L^q.$$

Remark 4. If f satisfies the hypothesis that we have made in the previous section, we have that

1. f' is a bounded map from $L^p + L^q$ into $L^{p/p-1} \cap L^{q/q-1}$;
2. f'' is a bounded map from $L^p + L^q$ into $L^{p/p-2} \cap L^{q/q-2}$;
3. f'' is a continuous map from $L^p + L^q$ into $L^{p/p-2} \cap L^{q/q-2}$.

At last we recall some result on Nehari manifolds. For the proofs we refer to [8, 10].

Remark 5. The functional E_V is of class C^2 and it holds

$$\langle \nabla E_V(u), v \rangle = \int \nabla u \nabla v + Vuv - f'(u)vdx. \tag{14}$$

Moreover the Nehari manifold defined as

$$\mathcal{N}_V = \left\{ u \in \mathcal{D}^{1,2} \smallsetminus 0 \ : \ \int |\nabla u|^2 + Vu^2 - f'(u)udx = 0 \right\} \tag{15}$$

is of class C^1 and its tangent space at the point u is

$$T_u\mathcal{N}_V = \left\{ u \in \mathcal{D}^{1,2} : \int 2\nabla u \nabla v + 2Vuv - f'(u)vdx - f''(u)uv = 0 \right\}. \tag{16}$$

Remark 6. We have

$$\inf_{u \in \mathcal{N}_V} ||u||_{\mathcal{D}^{1,2}} > 0. \tag{17}$$

Proof. At first notice that, by 5

$$\exists c > 0 \text{ s.t } \int |\nabla u|^2 + Vu^2 \geq c||u||^2_{\mathcal{D}^{1,2}}, \ \forall u \in \mathcal{D}^{1,2}. \tag{18}$$

Now, let $\{u_n\}$ a minimizing sequence in $\mathcal{N}_V$. By contradiction, we suppose that u_n converges to 0. We set $t_n = ||u_n||_{\mathcal{D}^{1,2}}$, hence we can write $u_n = t_n v_n$ where $||v_n||_{\mathcal{D}^{1,2}} = 1$. By claim 3 of Remark 3, the sequence is bounded in

$L^p + L^q$. Since $u_n \in \mathcal{N}_V$ and t_n converges to 0, we have

$$ct_n = \frac{c}{t_n}||u_n||^2_{\mathcal{D}^{1,2}} \leq \frac{1}{t_n} \int\limits_{\mathbb{R}^N} |\nabla u_n|^2 + V u_n^2 = \int\limits_{\mathbb{R}^N} f'(t_n v_n) v_n \leq$$

$$\leq c_1 t_n^{q-1} \int\limits_{\mathbb{R}^N \smallsetminus \Gamma_{t_n v_n}} |v_n|^q + c_1 t_n^{p-1} \int\limits_{\Gamma_{t_n v_n}} |v_n|^p \leq$$

$$\leq c_1 t_n^{q-1} \int\limits_{\mathbb{R}^N \smallsetminus \Gamma_{t_n v_n}} |v_n|^q + c_1 t_n^{p-1} \int\limits_{\Gamma_{v_n}} |v_n|^p \leq$$

$$\leq c_1 t_n^{q-1} \int\limits_{\mathbb{R}^N \smallsetminus \Gamma_{v_n}} |v_n|^q + c_1 t_n^{q-1} \int\limits_{(\mathbb{R}^N \smallsetminus \Gamma_{t_n v_n}) \cap \Gamma_{v_n}} \frac{|v_n|^p}{t_n^{q-p}} + c_1 t_n^{p-1} \int\limits_{\Gamma_{v_n}} |v_n|^p \leq$$

$$\leq c_1 t_n^{q-1} \int\limits_{\mathbb{R}^N \smallsetminus \Gamma_{v_n}} |v_n|^q + 2c_1 t_n^{p-1} \int\limits_{\Gamma_{v_n}} |v_n|^p.$$

Hence we get

$$c \leq c_1 t_n^{q-2} \int\limits_{\mathbb{R}^N \smallsetminus \Gamma_{v_n}} |v_n|^q + 2c_1 t_n^{p-2} \int\limits_{\Gamma_{v_n}} |v_n|^p$$

and by claim 2 of Remark 3 we get the contradiction. □

Remark 7. We have that for any given $u \in \mathcal{D}^{1,2} \smallsetminus \{0\}$, there exists a unique real number $t_u^V > 0$ such that $t_u^V u \in \mathcal{N}_V$ and $E_V(t_u^V u)$ is the maximum for the function

$$t \to E_V(tu), \quad t > 0.$$

Proof. Given $u \neq 0$ we set, for $t \geq 0$:

$$g_u(t) = g_u^V(t) := E_V(tu) = \int\limits_{\mathbb{R}^N} \frac{t^2}{2}(|\nabla u|^2 + Vu^2) - f(tu)dx. \tag{19}$$

We have:

$$g_u'(t) = \int\limits_{\mathbb{R}^N} t|\nabla u|^2 + Vtu^2 - uf'(tu)dx; \tag{20}$$

$$g_u''(t) = \int\limits_{\mathbb{R}^N} |\nabla u|^2 + Vu^2 - u^2 f''(tu)dx. \tag{21}$$

By hypothesis on f, if $g_u'(\bar t) = 0$ we have

$$\bar t^2 g_u''(\bar t) = \int_{\mathbb{R}^N} \bar t u f'(\bar t u) - \bar t^2 u^2 f''(\bar t u) dx < 0, \tag{22}$$

then $\bar t$ is a maximum point for g_u. Furthermore $0 = g_u(0) = g_u'(0)$ and $g_u''(0) > 0$ by the hypothesis on V, then 0 is a local minimum point for g_u. By (3), for $t \geq 1$, we have

$$\begin{aligned} g_u(t) &\leq \int_{\mathbb{R}^N} \frac{t^2}{2}(|\nabla u|^2 + Vu^2)dx - c_0 \int_{\{|tu \leq 1|\}} |tu|^q dx - c_0 \int_{\{|tu>1|\}} |tu|^p dx \leq \\ &\leq \int_{\mathbb{R}^N} \frac{t^2}{2}(|\nabla u|^2 + Vu^2)dx - c_0 \int_{\{|tu>1|\}} |tu|^p dx \leq \\ &\leq \frac{t^2}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + Vu^2)dx - c_0 t^p \int_{\{|u>1|\}} |u|^p dx; \end{aligned} \tag{23}$$

the last quantity diverges negatively as $t \to \infty$, since $p > 2$, and the claim follows. □

We search antisymmetric solutions of $(\mathscr{P}_\tau)$. To do that, we look for critical points of the restriction of E_V to $\mathcal{N}_V^\tau$. In fact, if $\bar u \in \mathcal{N}_V^\tau$ is a critical point of the restriction of E_V to $\mathcal{N}_V^\tau$, then

$$E_V'(\bar u)\varphi = \langle \nabla E_V(\bar u), \varphi \rangle = 0 \qquad \forall \varphi \in T_{\bar u}\mathcal{N}_V \cap \mathcal{D}_\tau^{1,2}(\mathbb{R}^N).$$

But $\nabla E_V(\bar u) = \tau \nabla E_V(\tau \bar u) = \tau \nabla E_V(\bar u)$, so we can see that $\nabla E_V(\bar u) = 0$.

3 A splitting lemma

We recall that a sequence $\{u_n\}_n \in \mathcal{D}^{1,2}$ such that $E_V(u_n) \to c$, and there exists a sequence $\varepsilon_n \to 0$ s.t. $|E_V'(u_n)\varphi| \leq \varepsilon_n ||\varphi||$, for all $\varphi \in \mathcal{D}^{1,2}$ is a Palais-Smale sequence at level c for E_V.

In the same way we say that $\{u_n\}_n \in \mathcal{N}_V^\tau$ such that $E_V(u_n) \to c$, and there exists a sequence $\varepsilon_n \to 0$ s.t. $|E_V'(u_n)\varphi| \leq \varepsilon_n ||\varphi||$, for all $\varphi \in T_{u_n}\mathcal{N}_V \cap \mathcal{D}_\tau^{1,2}$ is a Palais-Smale sequence at level c for E_V restricted to $\mathcal{N}_V^\tau$.

A functional f satisfies the $(PS)_c$ condition if all the Palais-Smale sequences at level c converge.

Unfortunately the functional E_V on $\mathcal{N}_V^\tau$ does not satisfy the PS condition in all the energy range. The following lemma provides a description of the PS sequences in $\mathcal{D}_\tau^{1,2}$.

This splitting lemma is a fundamental tool to obtain the claimed results. The main idea of this result spread over a result of M. Struwe [19] that described all the PS sequences for E_V on $H^1(\Omega)$ when $f(u) = u|u|^{2^*-2}$ and $V(x) = \lambda u$

Lemma 8. *Let $\{u_n\}_n$ a PS sequence at level c for the functional E_V restricted to the manifold $\mathcal{N}_V^\tau$. Then, up to a subsequence, there exist two integers $k_1, k_2 \geq 0$, $k_1 + k_2$ sequences $\{y_n^j\}_n$, an antisymmetric solution u^0 of the problem $-\Delta u + Vu = f'(u)$, and k_1 solutions u^j, $j = 1, \ldots k_1$, and k_2 antisymmetric solutions u^j, $j = k_1+1, \ldots, k_1+k_2$, of the problem $-\Delta u = f'(u)$ such that*

1. *if $j = 1, \ldots, k_1$, then $\tau y_n^j \neq y_n^j$, and $|y_n^j| \to \infty$ as $n \to \infty$*
2. *if $j = k_1 + 1, \ldots, k_2$, then $\tau y_n^j = y_n^j$, and $|y_n^j| \to \infty$ as $n \to \infty$*
3. $u_n(x) = u^0(x) + \sum_{j=1}^{k_1} [u^j(x - y_n^j) + \tau u^j(x - y_n^j)] + \sum_{j=k_1+1}^{k_2} u^j(x - y_n^j) + o(1)$
4. $E_V(u_n) \to E_V(u^0) + 2 \sum_{j=1}^{k_1} E_0(u^j) + \sum_{j=k_1+1}^{k_2} E_0(u^j)$

Proof. Since u_n is a PS sequence for the functional E_V restricted to the manifold $\mathcal{N}_V$, then u_n is a PS sequence for the functional E_V. For Step 1 of [8, Lemma 3.3] we get that u_n converges to u^0 weakly in $\mathcal{D}^{1,2}$ (up to subsequence) and u^0 solves $-\Delta u + Vu = f'(u)$.

Since $\tau u_n = u_n$, we have $\tau u^0 = u^0$. In fact, if $u_n \overset{\mathcal{D}^{1,2}}{\rightharpoonup} u^0$, then, for every $R > 0$, we have that $u_n \overset{L^2(B_R)}{\longrightarrow} u^0$, so $u_n(x) \to u^0(x)$ for almost all x.

We set
$$\psi_n(x) = u_n(x) - u^0(x).$$
Then $\tau \psi_n = \psi_n$, and $\psi_n \rightharpoonup 0$ weakly in $\mathcal{D}^{1,2}$. By Steps 2 and 4 of [8, Lemma 3.3], we get that ψ_n is a PS sequence for E_0. If $\psi_n \nrightarrow 0$ strongly in $\mathcal{D}^{1,2}$, then by Steps 3 and 4 of [8, Lemma 3.3] we get that there exists a sequence $\{\xi_n\} \subset \mathbb{R}^N$ with $|\xi_n| \to \infty$ as $n \to \infty$, and $\psi_n(x + \xi_n) \rightharpoonup u^1(x)$ where u^1 is a weak solution of $-\Delta u = f'(u)$.

We consider in $\mathbb{R}^N = \Gamma \oplus \Gamma^\perp$, where $\Gamma := \{x \in \mathbb{R}^N \ : \ \tau(x) = x\}$. We consider P_Γ the projection on the subspace Γ. At this point we must distinguish two cases

Case I: if $|\xi_n - \tau(\xi_n)|$ is bounded we define $y_n = P_\Gamma \xi_n$.

Case II: if $|\xi_n - \tau(\xi_n)|$ is unbounded we define $y_n = \xi_n$.

Case I In this case there exist a solution $\widetilde{u} \in \mathcal{D}^{1,2}_\tau \setminus \{0\}$ of $-\Delta u = f'(u)$ and a *PS* sequence $\{\widetilde{\psi}_n\}_n$ for E_0 such that

$$\psi_n = \widetilde{\psi}_n + \widetilde{u}(x + y_n), \tag{24}$$

$\widetilde{\psi}_n \rightharpoonup 0$ weakly in $\mathcal{D}^{1,2}$, and

$$E_0(\widetilde{\psi}_n) = E_0(\psi_n) - E_0(\widetilde{u}) + o(1). \tag{25}$$

We can assume, without loss of generality, that $\xi_n = P_\Gamma \xi_n + w$, where $w \in \Gamma^\perp$. We now consider the sequence $\{\psi_n(x + y_n)\}_n$ which is bounded: hence, up to subsequence $\{\psi_n(x + y_n)\}_n$ converges to $\widetilde{u}(x)$ weakly in $\mathcal{D}^{1,2}$, $\tilde{u}(x) = u^1(x - w) \neq 0$, then $-\Delta\widetilde{u} = f'(\widetilde{u})$. Furthermore, because $\tau y_n = y_n$ we have that $\tau\widetilde{u} = \widetilde{u}$. We define

$$\widetilde{\psi}_n(x) := \psi_n(x) - \widetilde{u}(x - y_n). \tag{26}$$

We will verify that $\widetilde{\psi}_n$ is a *PS* sequence for E_0. Indeed by Lemma 2.11 of [10] we get

$$\begin{aligned} E_0(\widetilde{\psi}_n) &= E_0(\psi_n(x) - \widetilde{u}(x - y_n)) = E_0(\psi_n(x + y_n) - \widetilde{u}(x)) = \\ &= E_0(\psi_n) - E_0(\widetilde{u}) + o(1), \end{aligned}$$

and, because $\{\psi_n\}_n$ is a *PS* sequence for E_0, we have that $E_0(\psi_n)$ converges, so $E_0(\widetilde{\psi}_n)$ converges, also.

Again, since $\{\psi_n\}$ is a *PS* sequence, we have that exists an $\varepsilon_n \to 0$ such that, for all $\varphi \in C_0^\infty$

$$\begin{aligned} &\left|(E_0)'(\widetilde{\psi}_n)[\varphi]\right| = \left|\int \nabla\psi_n\nabla\varphi - \int \nabla\widetilde{u}_{-y_n}\nabla\varphi - \int f'(\psi_n - \widetilde{u}_{-y_n}\varphi)\right| = \\ &= \left|\int [f'(\psi_n(x)) - f'(\widetilde{u}(x - y_n)) - f'(\psi_n(x) - \widetilde{u}(x - y_n))]\varphi(x)dx + \varepsilon_n||\varphi||\right| = \\ &= \left|\int [f'(\psi_n(z + y_n)) - f'(\widetilde{u}(z)) - f'(\psi_n(z + y_n) - \widetilde{u}(z))]\varphi_{y_n}(z)dz + \varepsilon_n||\varphi||\right|. \end{aligned}$$

Now we can choose an $R > 0$ and split this integral as follows.

$$\begin{aligned} \left|(E_0)'(\widetilde{\psi}_n)[\varphi]\right| &= \Big|\int_{B_R} [f'(\psi_n(x + y_n)) - f'(\widetilde{u})]\varphi_{y_n}(z)dz + \\ &+ \int_{\mathbb{R}^N \setminus B_R} [f'(\psi_n(x + y_n)) - f'(\psi_n(x + y_n) - \widetilde{u})]\varphi_{y_n}dz - \\ &- \int_{\mathbb{R}^N \setminus B_R} f'(\widetilde{u})\varphi_{y_n} - \int_{B_R} f'(\psi_n(x + y_n) - \widetilde{u})\varphi_{y_n}dz + \varepsilon_n||\varphi||\Big| \leq \\ &\leq A_n\ \gamma_{n,R}\ ||\varphi||_{\mathcal{D}^{1,2}} + B_n\ M_R\ ||\varphi||_{\mathcal{D}^{1,2}} + \varepsilon_n||\varphi||_{\mathcal{D}^{1,2}} \end{aligned}$$

where

$$A_n = ||f''(\theta\psi_n(\cdot + y_n) + (1-\theta)\widetilde{u}) - f''(\theta\psi_n(\cdot + y_n) - \theta\widetilde{u})||_{L^{p/p-2}};$$
$$\gamma_{n,R} = ||\psi_n(\cdot - y_n) - \widetilde{u}||_{L^p(B_R)};$$
$$B_n = ||f''(\psi_n(\cdot + y_n) + \theta\widetilde{u}) - f''(\theta\widetilde{u})||_{L^{p/p-2}\cap L^{q/q-2}};$$
$$M_R = ||\widetilde{u}||_{L^p+L^q(\mathbb{R}^N\setminus B_R)},$$

for some $0 < \theta < 1$.

By [10, Lemma 2.11] we have that both A_n and B_n are bounded. Since $M_R \to 0$ as $R \to +\infty$ and, given R, $\gamma_{n,R} \to 0$ as $n \to +\infty$, we get the claim.

Case II In this case there exist a solution $u^1 \neq 0$ of $-\Delta u = f'(u)$ and a *PS* sequence $\{\widetilde{\psi}_n\}_n \subset \mathcal{D}^{1,2}_\tau$ for E_0 such that

$$\psi_n(x) = \widetilde{\psi}_n(x) + u^1(x - y_n) - u^1(\tau x - y_n) + o(1); \tag{27}$$
$$E_0(\widetilde{\psi}_n) = E_0(\psi_n) - 2E_0(u^1) + o(1). \tag{28}$$

We define $\widetilde{\psi}_n = \psi_n - \gamma_n$,

$$\gamma_n(x) = u^1(x - y_n)\chi\left(\frac{|x - y_n|}{\rho_n}\right) - u^1(\tau x - y_n)\chi\left(\frac{|x - \tau y_n|}{\rho_n}\right), \tag{29}$$

where $\rho_n := \frac{|y_n - \tau y_n|}{2} \to \infty$ for $n \to \infty$, and, as usual, $\chi : \mathbb{R}^+_0 \to [0,1]$ is a C^∞ function such that $\chi(s) \equiv 0$ for all $s \geq 2$, $\chi(s) \equiv 1$ for all $s \leq 1$ and $|\chi'(s)| \leq 2$ for all s.

It is trivial that $\tau\gamma_n = \gamma_n$, so $\tau\widetilde{\psi}_n = \widetilde{\psi}_n$. Furthermore, easily we have

$$\psi_n(x) = \widetilde{\psi}_n(x) + u^1(x - y_n) - u^1(\tau x - y_n) + o(1).$$

Now we have to prove (28), and to show that $\widetilde{\psi}_n$ is a *PS* sequence.

At first we prove that

$$||\widetilde{\psi}_n||^2_{\mathcal{D}^{1,2}} = ||\psi_n - \gamma_n||^2 = ||\psi_n||^2 + 2||u^1||^2 + o(1). \tag{30}$$

In fact we have that $||\psi_n - \gamma_n||^2 = ||\psi_n||^2 + ||\gamma_n^2||^2 - 2(\psi_n, \gamma_n)_{\mathcal{D}^{1,2}}$, and it is easy to see that $||\gamma_n^2||^2 \to 2||u^1||^2$. Furthermore

$$(\psi_n, \gamma_n)_{\mathcal{D}^{1,2}} = \int \nabla\psi_n \nabla\left(u^1(x - y_n)\chi\left(\frac{|x - y_n|}{\rho_n}\right)\right) + \tag{31}$$
$$+ \int \nabla\psi_n \nabla\left(u^1(\tau x - y_n)\chi\left(\frac{|\tau x - y_n|}{\rho_n}\right)\right),$$

and the first term converges to $\int|\nabla u^1|$. For the second term we have

$$\begin{aligned}&\int \nabla\psi_n\nabla\left(u^1(\tau x-y_n)\chi\left(\frac{|\tau x-y_n|}{\rho_n}\right)\right)=\\&=\int\left(\nabla\psi_n\nabla u^1(\tau x-y_n)\right)\chi(\cdot)+\int\left(\nabla\psi_n\nabla\chi(\cdot)\right)u^1(\tau x-y_n).\end{aligned}$$

The last term of the equation vanishes when $n\to\infty$, while, remembering that ψ_n is symmetric, and setting $z=\tau x-y_n$, we have

$$\begin{aligned}&\int\left(\nabla\psi_n(x)\nabla u^1(\tau x-y_n)\right)\chi(\cdot)dx=\\&\int\left(\tau\nabla\psi_n(z+y_n)\nabla u^1(z)\right)\chi\left(\frac{z}{\rho_n}\right)dz\to-\int|\nabla u^1|^2,\end{aligned}$$

so we have proved (30).

We want now to estimate

$$\int f(\widetilde{\psi_n})=\int f(\psi_n-\gamma_n).$$

Set

$$\begin{aligned}I_1&=\int_{|x-y_n|<2\rho_n}f\left(\psi_n(x)-u^1(x-y_n)\chi\left(\frac{|x-y_n|}{\rho_n}\right)\right);\\I_2&=\int_{|\tau x-y_n|<2\rho_n}f\left(\psi_n(x)+u^1(\tau x-y_n)\chi\left(\frac{|\tau x-y_n|}{\rho_n}\right)\right);\\I_3&=\int_{\{B(y_n)_{2\rho_n}\cup B(\tau y_n)_{2\rho_n}\}^C}f(\psi_n(x)),\end{aligned}$$

we have

$$\int f(\widetilde{\psi_n})=I_1+I_2+I_3. \tag{32}$$

We have that

$$\left[\psi_n(z+y_n)-u^1(z)\chi\left(\frac{|z|}{\rho_n}\right)\right]\chi\left(\frac{|z|}{\rho_n}\right)\rightharpoonup 0\text{ in }\mathcal{D}^{1,2}. \tag{33}$$

By [10, Lemma 2.11], then we have that

$$\begin{aligned}I_1&=\int_{|z|<2\rho_n}f\left(\psi_n(z+y_n)-u^1(z)\chi\left(\frac{|z|}{\rho_n}\right)\right)=\\&=\int_{|z|<2\rho_n}f(\psi_n(z+y_n))-\int_{|z|<2\rho_n}f\left(u^1(z)\chi\left(\frac{|z|}{\rho_n}\right)\right)+o(1)=\\&=\int_{|z|<2\rho_n}f(\psi_n(z+y_n))-\int_{\mathbb{R}^n}f\left(u^1(z)\right)+o(1).\end{aligned}$$

In the same way, because ψ_n is symmetric,

$$\left[\psi_n(\tau z + \tau y_n) - u^1(z)\chi\left(\frac{|z|}{\rho_n}\right)\right]\chi\left(\frac{|z|}{\rho_n}\right) \rightharpoonup 0 \text{ in } \mathcal{D}^{1,2}, \tag{34}$$

and

$$I_2 = \int_{|\tau x - y_n|<2\rho_n} f(\psi_n(x)) - \int_{\mathbb{R}^n} f\left(u^1(x)\right) + o(1).$$

At last we have

$$\begin{aligned} \int f(\widetilde{\psi}_n) &= \int\limits_{|x-y_n|<2\rho_n} f(\psi_n(x)) + \int\limits_{|\tau x-y_n|<2\rho_n} f(\psi_n(x)) + \\ &+ \int\limits_{\{B(y_n)_{2\rho_n}\cup B(\tau y_n)_{2\rho_n}\}^C} f(\psi_n(x)) - 2\int\limits_{\mathbb{R}^n} f(u^1(x)) + o(1) = \\ &= \int_{\mathbb{R}^N} f(\psi_n(x)) - 2\int_{\mathbb{R}^n} f(u^1(x)) + o(1). \end{aligned} \tag{35}$$

From (30) and (35) we obtain, as claimed

$$E_0(\widetilde{\psi}_n) = E_0(\psi_n - \gamma_n) = E_0(\psi_n) - 2E_0(u^1) + o(1); \tag{36}$$

furthermore, because $\{\psi_n\}_n$ is a PS sequence for E_0, we have that $E_0(\widetilde{\psi}_n) \to c$ for some $c \in \mathbb{R}$.

To complete the proof we must show that

$$|(E_0)'(\widetilde{\psi}_n)\varphi| \leq \varepsilon_n ||\varphi||_{\mathcal{D}^{1,2}}, \tag{37}$$

where $\varepsilon_n \to 0$. Set

$$\begin{aligned} I_n^1 &= \int\limits_{|x-y_n|<2\rho} \left[f'(\psi_n(x)) - f'\left(\psi_n(x) - u^1(x-y_n)\chi\left(\frac{|x-y_n|}{\rho_n}\right)\right)\right]\varphi(x)dx - \\ &\quad - \int \nabla\left(u^1(x-y_n)\chi\left(\frac{|x-y_n|}{\rho_n}\right)\right)\nabla\varphi(x); \\ I_n^2 &= \int\limits_{|\tau x-y_n|<2\rho} \left[f'(\psi_n(x)) - f'\left(\psi_n(x) - u^1(\tau x-y_n)\chi\left(\frac{|\tau x-y_n|}{\rho_n}\right)\right)\right]\varphi(x)dx - \\ &\quad - \int \nabla\left(u^1(\tau x-y_n)\chi\left(\frac{|\tau x-y_n|}{\rho_n}\right)\right)\nabla\varphi(x); \\ I_n^3 &= \int\limits_{\{B(y_n)_{2\rho_n}\cup B(\tau y_n)_{2\rho_n}\}^C} [\nabla\psi_n - f'(\psi_n)]\varphi, \end{aligned}$$

we have that

$$|(E_0)'(\widetilde{\psi}_n)\varphi| = I_n^1 + I_n^2 + I_n^3. \tag{38}$$

Immediately we have that $I_n^3 \leq \varepsilon_n||\varphi||$; furthermore, we can estimate I_n^1 as before, obtaining

$$\begin{aligned} I_n^1 &= \int\limits_{|z<2\rho_n|} \left[f'(\psi_n(z+y_n)) - f'\left(\psi_n(z+y_n) - u^1(z)\chi\left(\frac{|z|}{\rho_n}\right)\right)\right]\varphi(z+y_n)dz - \\ &\quad - \int\limits_{|z<2\rho_n|} f'(u^1\chi)\varphi(z+y_n)\; dz + \varepsilon_n||\varphi||. \end{aligned}$$

Setting

$$\alpha_n(z) := \psi_n(z+y_n) - u^1(z)\;\chi\left(\frac{|z|}{\rho_n}\right), \tag{39}$$

we have

$$|I_n^1| \;=\; \Big| \int\limits_{B_{\rho_n}} \left[f'(u^1\chi+\alpha_n) - f'(\alpha_n) - f'(u^1\chi)\right]\varphi_{y_n}dz + \varepsilon_n||\varphi||\;\Big|,$$

and, chosen an $R > 0$,

$$\begin{aligned} |I_n^1| \;=\; \Big| &\int\limits_{B_{\rho_n}} \left[f'(u^1\chi+\alpha_n) - f'(u^1\chi)\right]\varphi_{y_n}dz + \\ &+ \int\limits_{(\mathbb{R}^N\setminus B_R)\cap B_{2\rho_n}} \left[f'(u^1\chi+\alpha_n) - f'(\alpha_n)\right]\varphi_{y_n}dz - \\ &- \int\limits_{(\mathbb{R}^N\setminus B_R)\cap B_{2\rho_n}} f'(u^1\chi)\varphi_{y_n}dz - \int\limits_{B_{\rho_n}} f'(\alpha_n)\varphi_{y_n}dz + \varepsilon_n||\varphi||\;\Big|. \end{aligned}$$

Using that $f'(0) = 0$ at last we have

$$\begin{aligned} &|I_n^1| \leq ||f''(u^1\chi+\theta_1\alpha_n) - f''(\theta_1\alpha_n)||_{L^{p/p-2}(\mathbb{R}^N)}||\varphi||_{L^p(\mathbb{R}^N)}||\alpha_n||_{L^p(B_R)} + \\ &+||f''(\alpha_n+\theta u^1\chi) - f''(\theta_1 u^1\chi)||_{L^{\frac{p}{p-2}}\cap L^{\frac{q}{q-2}}(\mathbb{R}^N)}||\varphi||_{L^p+L^q}||u^1\chi||_{L^p+L^q(B_R^C)} + \\ &+\varepsilon_n||\varphi||, \end{aligned}$$

where $0 < \theta, \theta_1 < 1$. By Remark 4 we get

$$|I_n^1| \leq \varepsilon_n||\varphi||_{\mathcal{D}^{1,2}}. \tag{40}$$

In the same way we can estimate $|I_n^2|$, and this concludes the proof. □

Remark 9. If $u_n \in \mathcal{N}_V^\tau$ is a Palais-Smale sequence of the restriction of E_V to $\mathcal{N}_V^\tau$, that is $E_V(u_n)$ converges and

$$|E_V'(u_n)w| \leq \varepsilon_n ||w||_{\mathcal{D}^{1,2}} \quad \forall w \in T_{u_n}\mathcal{N}_V \cap \mathcal{D}_\tau^{1,2},$$

where $\varepsilon_n \to 0$, then u_n is a Palais-Smale sequence for the functional E_V.

This remark, combined with the splitting lemma, provides a complete description of the *PS* sequences in our case.

4 The main result

At this point we prove some technical lemmas.

Let $u \in \mathcal{N}_V$, then u^+ and u^- belong to $\mathcal{N}_V$. Furthermore, if u is antisymmetric, we have $E_V(u^+) = E_V(u^-)$. So, if $u \in \mathcal{N}_V^\tau$, we get

$$E_V(u) = E_V(u^+) + E_V(u^-) = 2E_V(u^+) \geq 2 \inf_{\mathcal{N}_V} E_V = 2\mu_V.$$

This implies that

$$\mu_V^\tau \geq 2\mu_V; \tag{41}$$
$$\mu_0^\tau \geq 2\mu_0. \tag{42}$$

Remark 10. We have $\mu_0^\tau = 2\mu_0$

Proof. We have to proof that $\mu_0^\tau \leq 2\mu_0$. It is possible to find a sequence $\{u_k\}_k \subset \mathcal{N}_0^\tau$ such that $E_0(u_k) \to 2\mu_0$. So

$$\mu_0^\tau \leq \inf_k E_0(u_k) \leq 2\mu_0.$$

The construction of $\{u_k\}_k$ is quite similar to the construction of $\{z_k\}_k$ in the next theorem. So also the proof that $E_0(u_k) \to 2\mu_0$. Therefore, for the sake of simplicity, we omit the detailed proof of this result. □

We are ready now to prove the main lemma of this section

Lemma 11. *We have that $\mu_V^\tau \leq \mu_0^\tau$*

Proof. We prove it by steps
Step I We know that w exists such that $\mu_0 = E_0(w)$. Let $\chi(x)$ a smooth, real function such that

$$\chi = \begin{cases} 1 & B(0,1); \\ 0 & \mathbb{R}^N \setminus B(0,3). \end{cases}$$

We also ask that $\chi(x) = \chi(|x|)$ and that $|\nabla \chi| \leq 1$.

Let $\{y_k\} \subset \mathbb{R}^N$ s.t. $|y_k| \to \infty$ and $|\tau(y_k) - y_k| \to \infty$. Let ρ_k be defined as

$$\rho_k := \frac{|\tau(y_k) - y_k|}{6}.$$

At last we define a function in $\mathcal{D}^{1,2}$

$$z_k = z_k^1 + z_k^2, \tag{43}$$

where

$$z_k^1 = w(x - y_k)\chi\left(\frac{x - y_k}{\rho_k}\right); \tag{44}$$

$$z_k^2 = -w(\tau(x) - y_k)\chi\left(\frac{x - \tau(y_k)}{\rho_k}\right). \tag{45}$$

Obviously we have that $\tau z_k^1 = z_k^2$ and $\tau z_k^2 = z_k^1$, so $z_k \in \mathcal{D}_\tau^{1,2}\ \forall k$. Furthermore z_k^1 and z_k^2 have disjoint supports, so

$$E_V(t \cdot z_k) = E_V(t \cdot z_k^1) + E_V(t \cdot z_k^2)\ \forall t > 0. \tag{46}$$

We know, from Remark 7, that it exists a $t_k > 0$ s.t. $t_k \cdot z_k^1 \in \mathcal{N}_V$. It's easy to see that, for such t_k we have that $t_k \cdot z_k^2 \in \mathcal{N}_V$ and $t_k \cdot z_k \in \mathcal{N}_V^\tau$.

In the next we will prove that $E_V(t_k z_k) \to 2\mu_0$, when $k \to \infty$.

Step II We prove that $||z_k^1(x) - w(x - y_k)||_{\mathcal{D}^{1,2}} \to 0$ for $k \to \infty$.

Set $w_k := w(x - y_k)$, and $\gamma_k := \left(1 - \chi\left(\frac{x - y_k}{\rho_k}\right)\right)$, we have

$$\begin{aligned} ||z_k^1(x) - w_k||_{\mathcal{D}^{1,2}}^2 &= \int |\nabla [\gamma_k w_k]|^2 \leq 2\int \gamma_k^2 |\nabla w_k|^2 + 2\int |\nabla \gamma_k|^2 w_k^2 \leq \\ &\leq \int\limits_{|x - y_k| > 3\rho_k} |\nabla w_k|^2 + \frac{2}{\rho_k^2} \int\limits_{\rho_k < |x - y_k| < 3\rho_k} w_k^2 \to 0, \end{aligned}$$

as $k \to \infty$.

Step III We prove that it exists $c, C > 0$ such that $c < t_k < C$ for all k.

By Remark 6, we know that, if $t_k z_k^1 \in \mathcal{N}_V$, then it exists $M > 0$ such that, for all k, $M \leq ||t_k z_k^1||$. Furthermore, for the above step

$$||z_k^1|| \to ||w_k|| = ||w||. \tag{47}$$

This implies that c exists such that $t_k > c > 0$ for all k.

For the other inequality we must prove that

$$\int V(x)w_k^2(x)dx \to 0 \text{ when } k \to \infty, \tag{48}$$

in fact, fixed an $R > 0$ we have

$$\begin{aligned}\left|\int V w_k^2\right| &\leq \int_{B_R} |V|w_k^2 + \int_{\mathbb{R}^N \setminus B_R} |V|w_k^2 \leq \\ &\leq ||V||_{L^{\frac{N}{2}}(B_R)} ||w_k||^2_{L^{2^*}(B_R)} + ||V||_{L^{\frac{N}{2}}(\mathbb{R}^N \setminus B_R)} ||w_k||^2_{L^{2^*}(\mathbb{R}^N)}.\end{aligned}$$

We have that $||V||_{L^{\frac{N}{2}}(\mathbb{R}^N \setminus B_R)} \to 0$ as $R \to \infty$; furthermore

$$\int_{B_R} w^{2^*}(x - y_k)dx = \int_{B_R(-y_k)} w^{2^*}(s)ds \leq \int_{\mathbb{R}^N \setminus B_{(|y_k|-R)}} w^{2^*}(s)ds, \tag{49}$$

thus $||w_k||^2_{L^{2^*}} \to 0$ as $k \to \infty$, that proves (48).

Now set a function

$$g^V_{z^1_k}(t) := E_V(tz^1_k). \tag{50}$$

Obviously $g^0_w(t) = E_0(tw) = \frac{1}{2}t^2 \int |\nabla w|^2 - \int f(tw)$. For Remark 7 we know that there exists a $\bar{t}$ such that $g^0_{w_k}(\bar{t}) = g^0_w(\bar{t}) < 0$ for all k. We want to prove that, for k sufficiently big, we have also $g^V_{z^1_k}(\bar{t}) < 0$.

$$\begin{aligned} g^V_{z^1_k}(\bar{t}) - g^0_w(\bar{t}) &= g^V_{z^1_k}(\bar{t}) - g^0_{w_k}(\bar{t}) = \frac{1}{2}\int |\nabla \bar{t} z^1_k|^2 + \frac{1}{2}\int V(x)\bar{t}^2(z^1_k)^2 - \\ &- \int f(\bar{t}z^1_k) - \frac{1}{2}\int |\nabla \bar{t} w_k|^2 + \int f(\bar{t}w_k) = \\ &= \frac{\bar{t}^2}{2}\left[\int |\nabla z^1_k|^2 - |\nabla w_k|^2 + V(x)(z^1_k)^2\right] - \int f(\bar{t}z^1_k) - f(\bar{t}w_k). \end{aligned}$$

By (47) and by (48) we have that the first integral of the right hand side of the equation vanishes when $k \to \infty$. We estimate the last term.

$$\begin{aligned} \int f(\bar{t}z^1_k) - f(\bar{t}w_k) &= \int_{|x-y_k|>\rho_k} f\left(\bar{t}w_k\chi\left(\frac{x-y_k}{\rho_k}\right)\right) - f(\bar{t}w_k) = \\ &= \int_{|s|>\rho_k} f\left(\bar{t}w\chi\left(\frac{s}{\rho_k}\right)\right) - f(\bar{t}w) = \\ &= \int_{|s|>\rho_k} f'\left(\left[\theta\chi\left(\frac{s}{\rho_k}\right) + (1-\theta)\right]\bar{t}w\right)\left(\chi\left(\frac{s}{\rho_k}\right) - 1\right)\bar{t}w = \\ &\leq \bar{t} \int_{|s|>\rho_k} wf'\left(\left[\theta\chi\left(\frac{s}{\rho_k}\right) + (1-\theta)\right]\bar{t}w\right)\left(\chi\left(\frac{s}{\rho_k}\right) - 1\right), \end{aligned}$$

so

$$\left|\int f(\bar{t}z_k^1) - f(\bar{t}w_k)\right| \leq \bar{t} \int\limits_{|s|>\rho_k} \left|f'\left(\left[\theta\chi\left(\frac{s}{\rho_k}\right) + (1-\theta)\right]\bar{t}w\right)\right| \left|\chi\left(\frac{s}{\rho_k}\right) - 1\right| w$$

and the last term vanishes when $k \to \infty$. In fact, for Remark 4, f' is bounded in $L^{\frac{p}{p-1}} \cap L^{\frac{q}{q-1}}$ and $\left|\chi\left(\frac{s}{\rho_k}\right) - 1\right| w \to 0$ in L^{2^*}. So, for k_0 big enough, we have that

$$\exists c, C \in \mathbb{R}^+ \text{ s.t. } 0 < c < t_k < C \ , \ \forall k > k_0$$

Step IV We want to prove that $E_V(t_k z_k^1) \to \mu_0$. We have

$$|E_V(t_k z_k^1) - E_V(w_k)| \ = \ |E_V'(\theta t_k z_k^1 + (1-\theta)w_k)(z_k^1 - w_k)|.$$

We know, for Step 2, that $||z_k^1 - w_k||_{\mathcal{D}^{1,2}} \to 0$.

Furthermore

$$||\theta t_k z_k^1 + (1-\theta)w_k|| \leq ||z_k^1|| t_k + ||w_k|| \tag{51}$$

that is bounded because t_k is bounded and by Step II.

At this point by Remark 4 we get the claim.

We know also that

$$\begin{aligned} E_V(w_k) - \mu_0 &= E_V(w_k) - E_0(w) = E_V(w_k) - E_0(w_k) = \\ &= \int V(x) w_k^2 \stackrel{k\to\infty}{\longrightarrow} 0 \end{aligned}$$

for (48). Then

$$|E_V(t_k z_k^1) - \mu_0| \ \leq \ |E_V(t_k z_k^1) - E_V(w_k)| + |E_V(w_k) - \mu_0| \to 0 \tag{52}$$

as we wanted to prove.

Conclusion We know that $t_k z_k \in \mathcal{N}_V^\tau$. Then

$$E_V(t_k z_k) \geq \mu_V^\tau := \inf_{u \in \mathcal{N}_V^\tau} E_V(u).$$

Hence

$$\mu_V^\tau \leq E_V(t_k z_k) = E_V(t_k z_k^1) + E_V(t_k z_k^2) \to 2\mu_0 = \mu_0^\tau$$

that gives us the proof. □

We are ready, now, to prove the first result claimed in the introduction.

Proof of Theorem 1. First, we prove that

$$\forall u \in \mathcal{N}_0 \ \exists t_u^V u \in \mathcal{N}_V \quad \text{s.t} \quad E_V(t_u^V u) \geq E_0(u). \tag{53}$$

In fact, by Remark 7, we have that for every $u \in \mathcal{N}_0$, there exist $t_u^V > 0$ such that $t_u^V u \in \mathcal{N}_V$. Then we have:

$$0 \;=\; g_u'(t_u^V u) = \langle \nabla E_0(t_u^V u), u\rangle + t_u^V \int\limits_{\mathbb{R}^N} V u^2.$$

Since $V > 0$ we have that $\int V u^2 > 0$ and $\langle \nabla E_0(t_u^V u), u\rangle < 0$. Hence $t_u^V > t_u^0 = 1$. Let us observe that by (2) the function $s \to \int \frac{1}{2} f'(su)su - f(su)dx$ is strictly increasing, then, remembering that $t_u^V u \in \mathcal{N}_V$, we have:

$$\begin{aligned} E_V(t_u^V u) &= \frac{1}{2}\int f'(t_u^V u)t_u^V u - f(t_u^V u)dx \geq \\ &\geq \frac{1}{2}\int f'(u)u - f(u)dx = E_0(u). \end{aligned}$$

If $u \in \mathcal{N}_0^\tau$, we can prove in the same way that $t_u^V u \in \mathcal{N}_V^\tau$ and that

$$E_V(t_u^V u) \geq E_0(u) \geq \mu_0^\tau. \tag{54}$$

So

$$\inf_{w \in \mathcal{N}_V^\tau} E_V(w) = \inf_{u \in \mathcal{N}_0^\tau} E_V(t_u^V u) \geq E_0(u) \geq \mu_0^\tau. \tag{55}$$

Theorem 11 provides us the other inequality.

Suppose now that there exists $v \in \mathcal{N}_V^\tau$ such that $\mu_V^\tau = E_V(v)$. We know that $\int V(x)v(x)^2 > 0$ and

$$0 = \langle \nabla E_0(v), v\rangle + \int V(x)v(x)^2 dx,$$

so, consequently $\langle \nabla E_0(v), v\rangle < 0$. Then, by Remark 7, we get $t_v^0 < t_v^V = 1$. As said before, the function $s \mapsto \int \frac{1}{2} f'(sv)sv - f(sv)dx$ is strictly increasing, so we have

$$E_0(vt_v^0) = \int \frac{1}{2} f'(t_v^0)t_v^0 v - f(t_v^0 v)dx < \int \frac{1}{2} f'(v)v - f(v)dx = E_V(v) = \mu_V^\tau$$

and we get a contradiction.

□

Now we prove the following preliminary result.

Proposition 12. *There exists a class of potential $V(x)$ such that $\mu_V^\tau < \mu_0^\tau$.*

Proof. We consider the class of potentials defined in (10). We want to show that, when $|y| \to \infty$ and $|y - \tau y| \to \infty$, then $\mu_{V_y}^\tau < \mu_0^\tau$. We prove it by steps.

Take $w \in \mathcal{N}_0$ such that $E_0(w) = \mu_0$, w radially symmetric and $w > 0$ (see [10, 11, 12]). By means of w, we define

$$z_y(x) = w(x - y) - w(x - \tau y). \tag{56}$$

Step I We prove that, for $|y - \tau y| \to \infty$,

$$g_{z_y}^{V_y}(t) \to t^2 \int |\nabla w(x)|^2 dx + at^2 \int [|x| - 1] w^2(x) dx - 2 \int f(tw(x)) dx,$$

where

$$g_{z_y}^{V_y}(t) = E_{V_y}(tz_y) = \int_{\mathbb{R}^N} \frac{t^2}{2}(|\nabla z_y|^2 + V_y z_y^2) - f(tz_y) dx. \tag{57}$$

Now

$$\frac{t^2}{2}\int |\nabla z_y|^2 = \frac{t^2}{2}\int |\nabla w(x-y)|^2 + |\nabla w(\tau x - y)|^2 + \nabla w(x-y) \nabla w(x - \tau y). \tag{58}$$

After a change of variables, the first two terms are equals to $\frac{t^2}{2} \int |\nabla w|^2$, and the last term vanishes. So we have that, for all t,

$$\frac{t^2}{2}\int |\nabla z_y|^2 \to t^2 \int |\nabla w|^2 \qquad \text{when } |y - \tau y| \to \infty. \tag{59}$$

In a similar way consider

$$\begin{aligned} \int V_y z_y^2 &= a \int_{|x-y|<1} (|x - y| - 1) z_y^2 + a \int_{|x-\tau y|<1} (|x - \tau y| - 1) z_y^2 = \\ &= a \int_{|x-y|<1} (|x - y| - 1)[w(x - y) - w(x - \tau y)]^2 + \\ &\quad + a \int_{|x-\tau y|<1} (|x - \tau y| - 1)[w(x - y) - w(x - \tau y)]^2. \end{aligned} \tag{60}$$

By means of a change of variables we obtain

$$\int_{|x-y|<1} (|x - y| - 1) z_y^2 = \int_{|s|<1} (|s| - 1)[w(s) + w(s + y - \tau y)]^2.$$

It is not difficult to prove that

$$\int_{|s|<1} (|s|-1)w^2(s+y-\tau y) \to 0; \tag{61}$$

$$\int_{|s|<1} (|s|-1)w(s)w(s+y-\tau y) \to 0. \tag{62}$$

In the same way we proceed for the second term of the (60), obtaining

$$\frac{t^2}{2}\int V_v z_y^2 \to at^2 \int_{|x|<1} (|x|-1)w^2(x)dx \text{ when } |y| \to \infty. \tag{63}$$

We have to estimate now $\int f(tw)$. Fixed an $R>0$, we have

$$\int f(tz_y) = \int_{B_R(y)} f(tz_y) + \int_{B_R(\tau y)} f(tz_y) + \int_{(B_R(y)\cup B_R(\tau y))^C} f(tz_y). \tag{64}$$

For the first term we have

$$\begin{aligned}\int_{B_R(y)} f(tz_y) &= \int_{B_R} f(tw(s)-tw(s+y-\tau y)) = \int_{B_R} f(tw) + \\ &+\int_{B_R} f'(\theta tw(s)+(1-\theta)tw(s+y-\tau y))[tw(s+y-\tau y)],\end{aligned}$$

for some $\theta \in [0,1]$.

Now, for Remark 4, we have that $f'(\theta tw(x-y)+(1-\theta)tw(\tau x-y))$ is bounded in $L^{p'} \cap L^{q'}$, in fact $\theta tw(x-y)+(1-\theta)tw(\tau x-y)$ is bounded in $\mathcal{D}^{1,2}$ and so in L^p+L^q. Furthermore, $w(s+y-\tau y) \to 0$ strongly in $L^P(B_R)$ when $|y-\tau y| \to +\infty$.

Concluding we get

$$\int_{B_R(y)} f(tz_y) = \int_{B_R} f(tw(s))ds + I_1(R,y), \tag{65}$$

where, given $R>0$, $I_1(R,y) \to 0$ when $|y-\tau y| \to \infty$. In the same way we can conclude that

$$\int_{B_R(\tau y)} f(tz_y) = \int_{B_R} f(tw(s))ds + I_2(R,y), \tag{66}$$

where, again, given $R > 0$, $I_2(R, y) \to 0$ when $|y - \tau y| \to \infty$.

For the last term we have that there exist a $\theta \in [0, 1]$ such that

$$\begin{aligned}\int_{(B_R(y)\cup B_R(\tau y))^C} f(tz_y) &= \int_{(B_R(y)\cup B_R(\tau y))^C} f(t(w(x-y))) + \\ &\quad + \int_{(B_R(y)\cup B_R(\tau y))^C} f'(\theta t(w(x-y)) - (1-\theta)w(x-\tau y))tw(x-\tau y) = \\ &= \int_{(B_R\cup B_R(y-\tau y))^C} f(t(w(s))) + \\ &\quad + \int_{(B_R\cup B_R(\tau y - y))^C} f'(\theta t(w(\xi + \tau y - y)) - (1-\theta)w(\xi))tw(\xi).\end{aligned}$$

Now,

$$\left| \int_{(B_R\cup B_R(\tau y - y))^C} f'(\cdot)tw(\xi) \right| \le t||f'(\cdot)||_{L^{p'}\cap L^{q'}(\mathbb{R}^N)} ||w||_{L^p + L^q(\mathbb{R}^N \smallsetminus B_R)}, \tag{67}$$

and we use that $||w||_{\mathcal{D}^{1,2}(\mathbb{R}^N \smallsetminus B_R)}$ goes to zero when $R \to \infty$ and that $||f'(\cdot)||_{L^{p'}\cap L^{q'}(\mathbb{R}^N)}$ is bounded by Remark 4.

At this point we have that

$$\int f(tz_y) \to 2\int f(w) \qquad \text{when } |y - \tau y| \to \infty, \tag{68}$$

and we get the claim.

Step II There exists a $\bar{t} < 1$ such that

$$t_{z_y}^{V_y} \to \bar{t} \qquad \text{when } |y - \tau y| \to \infty, \tag{69}$$

where $t_{z_y}^{V_y}$ is the maximum point of $g_{z_y}^{V_y}(t)$.

We set

$$\varphi(t) = t^2 \int |\nabla w(x)|^2 dx + at^2 \int [|x| - 1]w^2(x)dx - 2\int f(tw(x))dx. \tag{70}$$

By Remark 7 there exists a unique maximizer $\bar{t} > 0$ for the function $\varphi(t)$.

We know that

$$g_w^0(t) = \frac{1}{2}t^2 \int \nabla w^2 - \int f(tw) \tag{71}$$

reach its maximum for $t = 1$. Thus the maximum of the function

$$\varphi(t) = 2g_w^0(t) + at^2 \int [|x| - 1]w^2(x)dx \tag{72}$$

is achieved for $\bar{t}$, with $0 < \bar{t} < 1$.

Given $t_1 < \bar{t} < t_2$, we can choose a $\delta > 0$ such that

$$\max\{\varphi(t_1), \varphi(t_2)\} + \delta < \varphi(\bar{t}) - \delta. \tag{73}$$

By Step I, for $|y - \tau y|$ sufficiently large, we obtain

$$g_{z_y}^{V_y}(t_i) < \varphi(t_i) + \delta < \varphi(\bar{t}) - \delta < g_{z_y}^{V_y}(\bar{t}). \tag{74}$$

By Remark 7, we know that $g_{z_y}^{V_y}(t)$ has an unique maximum point $t_{z_y}^{V_y}$, thus we conclude that

$$t_1 < t_{z_y}^{V_y} < t_2. \tag{75}$$

Since t_1 and t_2 are arbitrarily chosen, we get the claim.

Step III For $|y - \tau y|$ sufficiently large we have

$$\mu_{V_y}^\tau < \mu_0^\tau. \tag{76}$$

We know that

$$E_{V_y}(t_{z_y}^{V_y}\ z_y) = g_{z_y}^{V_y}(t_{z_y}) \to \varphi(\bar{t}) \qquad \text{for } |y - \tau y| \to \infty, \tag{77}$$

in fact, for all $\varepsilon > 0$ we have that, for $|y - \tau y|$ sufficiently large,

$$\begin{aligned} |g_{z_y}^{V_y}(t_{z_y}) - \varphi(\bar{t})| &= |g_{z_y}^{V_y}(t_{z_y}) - g_{z_y}^{V_y}(\bar{t}) + g_{z_y}^{V_y}(\bar{t}) - \varphi(\bar{t})| \leq \\ &\leq |g_{z_y}^{V_y}(t_{z_y}) - g_{z_y}^{V_y}(\bar{t})| + |g_{z_y}^{V_y}(\bar{t}) - \varphi(\bar{t})| \leq \varepsilon. \end{aligned}$$

By Step I the second term goes to zero when $|y - \tau y| \to \infty$. By Step II, $t_{z_y}^{V_y} \to \bar{t}$, so, arguing as in Step I, we get the claim. We observe that

$$\begin{aligned} \varphi(\bar{t}) &= \bar{t} \int |\nabla w|^2 + a\bar{t}^2 \int [|x| - 1]w^2 - 2 \int f(\bar{t}w) < \\ &< 2E_0(\bar{t}w) < 2\mu_0, \end{aligned}$$

because $\bar{t} < 1$ and $E_0(w) = \mu_0$. By (77) we get

$$\mu_{V_y}^\tau \leq E_{V_y}(t_{z_y}^{V_y}\ z_y) < 2\mu_0 = \mu_0^\tau \qquad \text{for } |y - \tau y| \text{ large enough,} \tag{78}$$

that concludes the proof ☐

Now we are ready to prove the second result claimed in the introduction.

Proof of theorem 2. By the Splitting Lemma and the above Proposition, we get the existence of a minimizer for E_{V_y}, for the class of potential V_y defined by (10), when $|y - \tau y|$ large enough.

Let ω be this minimizer. We know that ω changes sign, because it is antisymmetric by construction. We have to prove that ω changes sign exactly once. Suppose that the set $\{x \in \mathbb{R}^N \ : \ \omega(x) > 0\}$ has k connected components $\Omega_1, \ldots, \Omega_k$. Set

$$\omega_i = \begin{cases} \omega(x) & x \in \Omega_i \cup \tau\Omega_i; \\ 0 & \text{elsewhere} \end{cases} \tag{79}$$

For all i, $\omega_i \in \mathcal{N}^{\tau}_{V_y}$. Furthermore we have

$$E_{V_y}(\omega) = \sum_i E_{V_y}(\omega_i), \tag{80}$$

thus

$$\mu^{\tau}_{V_y} = E_{V_y}(\omega) = \sum_{i=1}^{k} E_{V_y}(\omega_i) \geq k\mu^{\tau}_{V_y}, \tag{81}$$

so $k = 1$, that concludes the proof. □

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