On the Existence of the Fundamental Eigenvalue of an Elliptic Problem in $\mathbb{R}^N$

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Abstract

We study an eigenvalue problem for functions in $\mathbb{R}^N$ and we find sufficient conditions for the existence of the fundamental eigenvalue. This result can be applied to the study of the orbital stability of the standing waves of the nonlinear Schrödinger equation.

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1 Introduction

In this paper, we study the following eigenvalue problem:

$$\begin{cases} -\Delta u + F'(u) = \lambda u \\ u > 0 \end{cases} \tag{*}$$

where $u \in H^1(R^n)$ and $F : \mathbb{R} \to \mathbb{R}$ be a even C^2 function such that $F(0) = F'(0) = F''(0) = 0$.

In particular, we are interested in the existence of the *fundamental eigenvalue*; namely, the Lagrange multiplier of the following minimization problem:

Minimization problem: *Find the minimum point of the functional*

$$J(u) = \int_{N} \frac{1}{2}|\nabla u|^2 + F(u)dx$$

constrained to the manifold

$$M_\rho = \{v \in H^1(R^N),\ ||v||_{L^2} = \rho\}.$$

One of the motivations in studying the above problem is the application to the following nonlinear Schrödinger equation:

$$i\frac{\partial \psi}{\partial t} + \Delta\psi - W'(|\psi|)\frac{\psi}{|\psi|} = 0 \tag{1}$$

where

$$W(u) = \frac{1}{2}\Omega u^2 + F(u).$$

We get the following result, which is a generalization of Theorem II.2 of Cazenave and Lions [3]:

Theorem 1 *The nonlinear Schrödinger equation (1) admits orbitally stable standing waves if the above minimization problem admits a solution with $\lambda < 0$. These standing waves have the form*

$$\psi(t,x) = u(x)e^{-i\omega t}$$

where $\omega = \lambda - \Omega$, and where u and λ are the first eigenfunction and the first eigenvalue of problem (*) respectively.

We notice that if $\lambda > 0$, it has been proved in [2] that (*) has no radial solution. Clearly the above result has a limited interest if we do not know when the fundamental solution of (*) exists.

Now we will state the main result of this paper concerning problem (*). We state the hypothesis

$$|F'(s)| \leq c_1|s|^{q-1} + c_2|s|^{p-1} \text{ for some } 2 < q \leq p < 2^*. \qquad (F_p)$$

We also assume

$$F(s) \geq -c_1 s^2 - c_2|s|^\gamma \text{ for some } c_1, c_2 \geq 0,\ \gamma < 2 + \frac{4}{N} \qquad (F_0)$$

and

$$\text{there exists } s_0 \in \text{ such that } F(s_0) < 0. \qquad (F_1)$$

Theorem 2 *Let F satisfy (F_p), (F_0) and (F_1). Then, $\exists\ \bar{\rho}$ such that $\forall\ \rho > \bar{\rho}$ there exists $\bar{u} \in H^1$ satisfying*

$$J(\bar{u}) = \inf_{\{v \in H^1,\ ||v||_{L^2} = \rho\}} J(v),$$

with $||\bar{u}||_{L^2} = \rho$. Then, there exist λ and $\bar{u}$ that solve (*), with $\lambda < 0$ and $\bar{u}$ positive radially symmetric.

In order to have stronger results, we can replace (F_1) with the following hypothesis

$$F(s) < -s^{2+\epsilon},\ 0 < \epsilon < \frac{4}{N} \text{ for small } s. \qquad (F_2)$$

In this case we find the following results concerning the existence of the minimizer of $J(u)$ for any ρ.

Corollary 3 *If (F_p), (F_0) and (F_2) hold, then for all ρ, there exists $\bar{u} \in H^1$, with $||\bar{u}||_{L^2} = \rho$, such that*

$$J(\bar{u}) = \inf_{\{v \in H^1, ||v||_{L^2} = \rho\}} J(v).$$

In particular, for $N = 3$ we have

Corollary 4 *Let $N = 3$. If (F_p) and (F_0) hold and $F \in C^3$, with $F'''(0) < 0$, then for all ρ, there exists $\bar{u} \in H^1$ with $||\bar{u}||_{L^2} = \rho$ such that*

$$J(\bar{u}) = \inf_{\{v \in H^1, ||v||_{L^2} = \rho\}} J(v).$$

There is a lot of literature for nonlinear eigenvalue problems. We refer to Rabinowitz [10] and the references therein. However, as far as we know, there are no results in the case when the nonlinearity is not a compact perturbation of the Laplace operator.

In [2], in order to prove the existence of a solution for the problem (*) with λ negative and fixed, the authors used a slightly weaker version of (F_p) and a slightly different version of (F_1). In fact, (F_0) is used in [3] in order to obtain that the Cauchy problem associated to equation (1) has a solution for all time, and also to prove the orbital stability of the standing wave relative to the ground state solution.

The paper is organized as follows. In section 2 we prove theorem 2. In section 3 we prove theorem 1. This theorem is a generalization of theorem II.2 of [3]. As a matter of fact, theorem 1 can be obtained following the same type of argument as in Theorem II.2, as claimed in [3]. Here we give a complete and different proof which is based on the "splitting lemma". In the appendix we prove the splitting lemma in the form used here.

2 Proofs of the main results

Lemma 5 *If F satisfies (F_1), then $\exists\ \bar{\rho}$ such that $\forall\ \rho > \bar{\rho}$*

$$\inf_{||u||_{L^2} = \rho} J(u) < 0.$$

Otherwise, if F satisfies (F_2) then $\inf_{||u||_{L^2} = \rho} J(u) < 0$ *for all ρ.*

Proof. We build a sequence of radial functions u_n in H^1 such that $J(u_n) < 0$ for large n. The sequence is defined as follows:

$$u_n(r) = \begin{cases} s_0 & r < R_n; \\ s_0 - s_0(r - R_n) & R_n \le r \le R_n + 1; \\ 0 & r > R_n + 1. \end{cases} \tag{2}$$

We show that $J(u_n) < 0$ when $R_n \to +\infty$. Notice that $|\nabla u_n|^2 = |\frac{\partial u_n}{\partial r}|^2 = s_0^2$. We have

$$\int_{\mathbb{R}^N} \frac{1}{2}|\nabla u_n|^2 + F(u_n)dx$$

$$\le C_1 \int_{R_n}^{R_n+1} \left[\left| \frac{\partial u_n}{\partial r} \right|^2 + \sup_{|s| \in [R_n, R_n+1]} F(s) \right] r^{N-1} dr + C_2 \int_0^{R_n} F(s_0) r^{N-1} dr$$

with C_1 and C_2 strictly positive. This proves the first statement.

Now, we want to prove that if (F_2) holds, then

$$\inf_{||u||_{L^2} = \rho} J(u) < 0$$

for all ρ. We use the same approach as before; we build a sequence of radial functions that are constant in a ball with a suitable cut-off. Let u_n be

$$u_n(r) = \begin{cases} s_n & r < R_n; \\ s_n - \frac{s_n}{R_n}(r - R_n) & R_n \le r \le 2R_n; \\ 0 & r > 2R_n. \end{cases} \tag{3}$$

We study $J(u_n)$ when $R_n \to +\infty$; and, due to the constraint $||u_n||_{L^2} = \rho$, we have

$$\lim_{R_n \to \infty} s_n^2 R_n^N = \gamma > 0. \tag{4}$$

We can choose R_n sufficiently large such that $F(u_n) \le 0$. Therefore,

$$\begin{aligned} J(u_n) &= \int_{\mathbb{R}^N} \frac{1}{2}|\nabla u_n|^2 + F(u_n)dx = \\ &= C\int_{R_n}^{2R_n} \left|\frac{\partial u}{\partial r}\right|^2 r^{N-1}dr + C\int_0^{R_n} F(u_n)r^{N-1}dr + \\ &+ C\int_{R_n}^{2R_n} F(u_n)r^{N-1}dr \le \\ &\le C\int_{R_n}^{2R_n} \left|\frac{\partial u}{\partial r}\right|^2 r^{N-1}dr + C\int_0^{R_n} F(u_n)r^{N-1}dr = \\ &= C\int_{R_n}^{2R_n} \frac{s_n^2}{R_n^2} r^{N-1}dr + C\int_0^{R_n} F(s_n)r^{N-1}dr \le \\ &\le Cs_n^2 R_n^{N-2} - C\int_0^{R_n} s^{2+\epsilon} r^{N-1}dr = \\ &= Cs_n^2 R_n^{N-2} - Cs^{2+\epsilon} R_n^N, \end{aligned}$$

where C are positive constants. By (4) we have $R_n = O(s_n^{-2/N})$ and thus

$$J(u_n) \le O(s_n^{4/N}) - O(s_n^\epsilon) \to 0^-, \tag{5}$$

if $0 < \epsilon < 4/N$. ■

Remark 6 *In the proof of previous lemma we have used radially symmetric functions. So, in the same way we can obtain that*

$$\inf_{\{u \in H_r^1, ||u||_{L^2} = \rho\}} J(u) < 0 \tag{6}$$

with the same hypothesis.

Proposition 7 *If there exist $c_1, c_2 \ge 0$ such that*

$$F(u) \ge -c_1 u^2 - c_2 u^\gamma, \; 2 \le \gamma < 2 + \frac{4}{N} \tag{F_0}$$

then

1. $\inf_{||u||_{L^2} = \rho} J(u) > -\infty$, *and*

2. *any minimizing sequence u_n, i.e. $J(u_n) \to c$, $||u_n||_{L^2} = \rho$, is bounded in H^1.*

Proof. We apply the Sobolev inequality (see[13])

$$||u||_{L^q} \leq b_q ||u||_{L^2}^{1-\frac{N}{2}+\frac{N}{q}} ||\nabla u||_{L^2}^{\frac{N}{2}-\frac{N}{q}} \tag{7}$$

that holds for $2 \leq q \leq 2^*$ when $N \geq 3$.

From equation (7) we have that any function u such that $||u||_{L^2} = \rho$ fulfills the following equation:

$$||u||_{L^q}^q \leq b_{q,\rho} ||\nabla u||_{L^2}^{\frac{qN}{2}-N}. \tag{8}$$

Now we notice that, by (8), for all $u \in H^1$ with $||u||_{L^2} = \rho$

$$\begin{aligned} J(u) &\geq \int \frac{1}{2}|\nabla u|^2 - c_1 u^2 - c_2 u^\gamma dx \qquad (9) \\ &\geq \int \frac{1}{2}|\nabla u|^2 dx - c_2 b_{\gamma,\rho} \left(\int |\nabla u|^2 dx \right)^{\frac{\gamma N}{2}-N} - c_1 \rho^2. \end{aligned}$$

If $\frac{\gamma N}{2} - N < 2$, i.e if $\gamma < 2 + \frac{4}{N}$, we have

$$J(u) \geq \frac{1}{2}||\nabla u||_{L^2}^2 + O(||\nabla u||_{L^2}^2). \tag{10}$$

The proof follows easily. ■

Lemma 8 *Let (F_p) hold, and let $\bar{u} \neq 0$, $\bar{u} \in H^1(\mathbb{R}^N)$ and $\bar{\lambda} \in \mathbb{R}$ be such that*

$$-\Delta \bar{u} + F'(\bar{u}) = \bar{\lambda} \bar{u}. \tag{*}$$

If

$$J(\bar{u}) = \int_{\mathbb{R}^N} \frac{1}{2}|\nabla \bar{u}|^2 + F(\bar{u}) dx < 0, \tag{11}$$

then

$$\bar{\lambda} < 0.$$

Proof. By assumption (F_p), if $\bar{u} \in H^1$ solves (*), by a bootstrap argument, $\bar{u} \in H^{2,2}_{loc}(\mathbb{R}^N)$. Furthermore, since $F(u) \in L^1$, we can apply the Derrick-Pohozaev identity (see [4, 9, 2])

$$\int_{\mathbb{R}^N} |\nabla \bar{u}|^2 = \frac{2N}{N-2} \int_{\mathbb{R}^N} \frac{\bar{\lambda}\bar{u}^2}{2} - F(\bar{u}) dx. \tag{12}$$

The function $\bar{u}$ satisfies the equation $-\Delta \bar{u} + F'(\bar{u}) = \bar{\lambda}\bar{u}$. Therefore, by integration, we get

$$\frac{1}{2} \int_{\mathbb{R}^N} |\nabla \bar{u}|^2 dx = \frac{1}{2} \int_{\mathbb{R}^N} \bar{\lambda}\bar{u}^2 - F'(\bar{u})\bar{u} dx. \tag{13}$$

By equation (12) we have

$$\frac{N-2}{2N} \int_{\mathbb{R}^N} |\nabla \bar{u}|^2 dx = \int_{\mathbb{R}^N} \frac{\bar{\lambda}\bar{u}^2}{2} - F(\bar{u}) dx, \tag{14}$$

and subtracting (13) from (14), we get

$$-\frac{1}{N} \int_{\mathbb{R}^N} |\nabla \bar{u}|^2 = \left[\frac{N-2}{2N} - \frac{1}{2} \right] \int_{\mathbb{R}^N} |\nabla \bar{u}|^2 = \int_{\mathbb{R}^N} \frac{1}{2} F'(\bar{u})\bar{u} - F(\bar{u}). \tag{15}$$

This proves that

$$\int_{\mathbb{R}^N} \frac{1}{2}F'(\bar{u})\bar{u} - F(\bar{u})dx < 0. \tag{16}$$

On the other hand, by (11) and (13)

$$2J(\bar{u}) + \int_{\mathbb{R}^N} F'(\bar{u})\bar{u} - 2F(\bar{u})dx = \bar{\lambda}\int_{\mathbb{R}^N} \bar{u}^2 dx, \tag{17}$$

and thus we have $\bar{\lambda} < 0$. ∎

Remark 9 *With the same argument as before, we can prove that*

$$\frac{2}{N-2}\bar{\lambda}\int_{\mathbb{R}^N} \bar{u}^2 dx = \int_{\mathbb{R}^N} 2^* F(\bar{u}) - F'(\bar{u})\bar{u}dx \tag{18}$$

without any assumption on the sign of $J(\bar{u})$. Indeed, we have, as above,

$$\int_{\mathbb{R}^N} |\nabla\bar{u}|^2 dx = \frac{N}{N-2}\bar{\lambda}\int_{\mathbb{R}^N} \bar{u}^2 dx - \frac{2N}{N-2}\int_{\mathbb{R}^N} F(\bar{u})dx \tag{19}$$

$$\int_{\mathbb{R}^N} |\nabla\bar{u}|^2 dx = \int_{\mathbb{R}^N} \bar{\lambda}\bar{u}^2 - F'(\bar{u})\bar{u}dx. \tag{20}$$

By subtraction we obtain (18). Notice that this is an Ambrosetti-Rabinowitz type inequality.

Remark 10 *We notice that the infimum of $J(u)$ for $||u||_{L^2} = \rho, u \in H^1(\mathbb{R}^N)$ or $||u||_{L^2} = \rho, u \in H^1_r(\mathbb{R}^N)$ has the same value. If we call*

$$c = \inf\{J(u),\ ||u||_{L^2} = \rho, u \in H^1(\mathbb{R}^N)\}$$
$$c_r = \inf\{J(u),\ ||u||_{L^2} = \rho, u \in H^1_r(\mathbb{R}^N)\}$$

we have $c = c_r$. Clearly $c \le c_r$ and, if $u_n \ge 0$ is a minimizing sequence for c, we denote by u_n^* the Schwartz spherical rearrangement of u_n. Now, $u_n^* \in H^1_r(\mathbb{R}^N)$, $||u_n^*||_{L^2} = \rho$, and $J(u_n^*) \le J(u_n)$, thus u_n^* is a minimizing sequence. Therefore, $c = c_r$.

Proposition 11 *Let (F_0) and (F_p) hold; and Let u_n be a minimizing P-S sequence such that $J(u_n) \to c$, where $c = \inf_{||u||_{L^2}=\rho} J(u) < 0$. Then there exists a sequence λ_n of Lagrange multipliers such that*

$$-\Delta u_n + F'(u_n) - \lambda_n u_n = \sigma_n \to 0. \tag{21}$$

We have that λ_n is bounded in $\mathbb{R}$.

Proof. By (21) and Proposition 7, we have

$$\left|\int_{\mathbb{R}^N} |\nabla u_n|^2 + F'(u_n)u_n - \lambda_n u_n^2 dx\right| \le ||\sigma_n||_{H^{-1}}||u_n||_{H^1} \to 0.$$

We have

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 + F'(u_n)u_n - \lambda_n u_n^2 dx =$$
$$\int_{\mathbb{R}^N} |\nabla u_n|^2 + 2F(u_n) - 2F(u_n) + F'(u_n)u_n - \lambda_n u_n^2 dx =$$
$$2J(u_n) - \lambda_n\rho^2 + \int_{\mathbb{R}^N} F'(u_n)u_n - 2F(u_n)dx \to 0.$$

Furthermore, we have that $J(u_n)$ is bounded; and also by (F_p),

$$\left|\int_{\mathbb{R}^N} F'(u_n)u_n - 2F(u_n)dx\right| \leq C\left(||u_n||_{H^1}^q + ||u_n||_{H^1}^p\right) < +\infty.$$

Then λ_n is bounded and statement is proved. ■

Proof of Theorem 2. For the Palais principle the critical point of the functional on the manifold $\{||u||_{L^2} = \rho,\, u \in H_r^1\}$ are still critical points on the manifold $\{||u||_{L^2} = \rho,\, u \in H^1\}$. Remark 10 assures that the minimizers of the functional on the manifold $\{||u||_{L^2} = \rho,\, u \in H_r^1\}$ are still minimizers of the functional in H^1. So, we study the existence of minimizers in H_r^1.

Let u_n be a minimizing sequence such that $||u_n||_{L^2} = \rho$; $F(s)$ is an even function, we can even take $u_n \geq 0$. By Lemma 5, we can take ρ sufficiently large such that $J(u_n) \to c < 0$. By the Ekeland principle, we can assume that u_n is a Palais-Smale sequence for the functional J restricted on the manifold $||u_n||_{L^2} = \rho$.

By Lemma 7 and Proposition 11, u_n is bounded in H_r^1 and λ_n is bounded in $\mathbb{R}$. We have

$$\begin{aligned}
&\lambda_n \to \bar{\lambda} \\
&u_n \rightharpoonup u \quad \text{weakly in } H_r^1(\mathbb{R}^N) \\
&u_n \to u \quad \text{strongly in } \ L^p(\mathbb{R}^N),\ 2 < p < 2^* \\
&u_n \to u \quad \text{strongly in } \ L^p(B),\ B \text{ compact } 2 \leq p < 2^*.
\end{aligned}$$

Moreover, for any radially symmetric function we have the following decay when $|x| \to \infty$:

$$|u_n(x)| \leq \alpha \frac{||u_n||_{H^1}}{|x|^{\frac{N-1}{2}}} \quad \text{for } |x| > \beta \tag{22}$$

where α, β depend only on N (see for instance [2]).

By (F_p), it is easy to see that

$$-\Delta u + F'(u) = \bar{\lambda} u. \tag{23}$$

Indeed, we have for any $\varphi \in C_0^\infty(\mathbb{R}^N)$ radially symmetric

$$\int_{\mathbb{R}^N} \nabla u_n \nabla \varphi dx - \int_{\mathbb{R}^N} F'(u_n)\varphi dx - \lambda_n \int_{\mathbb{R}^N} u_n \varphi dx \to 0 \tag{24}$$

as $n \to +\infty$. By (F_p), we have that

$$\int_{\mathbb{R}^N} F'(u_n)\varphi \to \int_{\mathbb{R}^N} F'(u)\varphi.$$

Then, as $n \to +\infty$,

$$\begin{aligned}
&\int_{\mathbb{R}^N} \nabla u_n \nabla \varphi dx - \int_{\mathbb{R}^N} F'(u_n)\varphi dx - \lambda_n \int_{\mathbb{R}^N} u_n \varphi dx \to \\
&\to \int_{\mathbb{R}^N} \nabla u \nabla \varphi dx - \int_{\mathbb{R}^N} F'(u)\varphi dx - \bar{\lambda} \int_{\mathbb{R}^N} u \varphi dx.
\end{aligned} \tag{25}$$

We show that $u \neq 0$.

Indeed, the Neminski operator

$$F : L^t(\mathbb{R}^N) \to L^1(\mathbb{R}^N)\ 2 < t < 2^*$$

is continuous by (F_p) and $u_n \to u$ in $L^t(\mathbb{R}^N),\ 2 < t < 2^*$. Hence we have

$$\frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \int_{\mathbb{R}^N} F(u)dx \leq \lim_{n\to\infty} J(u_n) = c < 0, \tag{26}$$

which proves that $u \neq 0$.

At this point, we have that $u \neq 0$ is a weak solution of

$$-\Delta u + F'(u) = \bar{\lambda} u \tag{27}$$

and that

$$J(u) < 0. \tag{28}$$

Thus the hypotheses of Lemma 8 are fulfilled, and $\bar{\lambda} < 0$.

Considering two functions u_n and u_m in the minimizing P-S sequence, we have

$$\begin{aligned} -\Delta u_n + F'(u_n) - \lambda_n u_n = \sigma_n \to 0 \\ -\Delta u_m + F'(u_m) - \lambda_m u_m = \sigma_m \to 0. \end{aligned}$$

By subtraction we get

$$-\Delta(u_n - u_m) + F'(u_n) - F'(u_m) - \bar{\lambda}(u_n - u_m) \to 0, \tag{29}$$

and we obtain

$$\int_{\mathbb{R}^N} |\nabla(u_n - u_m)|^2 dx + \int_{\mathbb{R}^N} (F'(u_n) - F'(u_m))(u_n - u_m) - \bar{\lambda}(u_n - u_m)^2 dx \to 0. \tag{30}$$

On any compact ball B we have, by standard arguments, that

$$\begin{aligned} &\int_B (F'(u_n) - F'(u_m))(u_n - u_m) \to 0 \\ &\int_B \bar{\lambda}(u_n - u_m)^2 dx \to 0. \end{aligned}$$

Then,

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla(u_n - u_m)|^2 dx \quad &+ \quad \int_{B^c} (F'(u_n) - F'(u_m))(u_n - u_m) - \\ &- \quad \int_{B^c} \bar{\lambda}(u_n - u_m)^2 dx \to 0. \end{aligned} \tag{31}$$

By lemma 8, we have $\bar{\lambda} < 0$ and

$$\begin{aligned} &\int_{B^c} (F'(u_n) - F'(u_m))(u_n - u_m) - \bar{\lambda}(u_n - u_m)^2 dx = \\ = \quad &\int_{B^c} (F''(\theta u_n + (1-\theta)u_m) - \bar{\lambda})(u_n - u_m)^2 dx. \end{aligned} \tag{32}$$

Thus, remembering that $F''(0) = 0$ and due to (22), we have $F''(\theta u_n + (1-\theta)u_m) << 1$, and

$$F''(\theta u_n + (1-\theta)u_m) - \bar{\lambda} > 0 \tag{33}$$

for B sufficiently large. By equation (30) we get

$$\int_{\mathbb{R}^N} |\nabla(u_n - u_m)|^2 dx \to 0 \tag{34}$$

$$\int_{\mathbb{R}^N} |(u_n - u_m)|^2 dx \to 0. \tag{35}$$

Then the sequence $\{u_n\}_n$ is a Cauchy sequence in $H^1_r(\mathbb{R}^N)$; thus $u_n \to u$ strongly in $H^1(\mathbb{R}^N)$ and $||u||_{L^2} = \rho$.

■

The proofs of Corollary 3 and Corollary 4 are straightforward. Moreover, we prove a non existence result when $F(s) = s^p$ with $2 < p < 2^*$.

Remark 12 *Let F satisfy (F_p) and (F_0). If*

$$0 \leq 2F(s) \leq F'(s)s \text{ for all } s \tag{36}$$

then (*) has no nontrivial solution in $H^1(\mathbb{R}^N)$ for all λ. The proof is a consequence of the Derrick-Pohozaev identity.

Let us suppose that there exists $u \in H^1(\mathbb{R}^N)$, $u \neq 0$, and λ such that (*) holds: by bootstrap arguments we have that $u \in H^{2,q}$ for all q and we can apply the Derrick-Pohozaev identity. Therefore, by (16) no solution of (*) can satisfy (36).

3 Stability of the nonlinear Schrödinger equation

We consider the nonlinear Schrödinger equation

$$\begin{cases} i\frac{\partial \psi}{\partial t} + \Delta\psi - F'(\psi) = 0 \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N \\ \\ \psi(0,x) = \psi_0(x) \end{cases} \tag{†}$$

where $F : \mathbb{C} \to \mathbb{R}$ is a an even radial function such that $F(|\xi|)$ satisfies (F_0),(F_p) and (F_1). It is well known that there exists a unique solution $\psi \in C([0,+\infty), H^1(\mathbb{R}^N))$, see [11, 5, 6]. We notice that problem (1) reduces to (†) in the case $\Omega = 0$. It is easy to see that this hypothesis is not restrictive. Setting $\psi = u(t,x)e^{iS(t,x)}$ we have that any solution of (†) verifies

$$\begin{cases} \frac{1}{2}\frac{d}{dt}u^2 + \nabla \cdot (u^2 \nabla S) = 0 \\ \\ u\partial_t S - \Delta u + u|\nabla S|^2 + F'(u) = 0. \end{cases} \tag{37}$$

It is well known that (37) are the Euler-Lagrange equations of the action functional given by

$$A(u,S) = \iint \frac{1}{2}u^2\partial_t S + \frac{1}{2}|\nabla u|^2 + \frac{1}{2}u^2|\nabla S|^2 + F(u)dtdx. \tag{38}$$

Since the energy is given by

$$E(\psi) = E(u,S) = \int \frac{1}{2}|\nabla u|^2 + \frac{1}{2}u^2|\nabla S|^2 + F(u)dx, \tag{39}$$

any solution of (37) satisfies

$$\frac{d}{dt}\int u(t,x)^2 dx = 0 \tag{40}$$

$$\frac{d}{dt}E(u,S) = 0. \tag{41}$$

Hence, for any solution of (†), equations (40) and (41) become

$$||\psi(t,\cdot)||_{L^2(\mathbb{R}^N)} = ||\psi_0(\cdot)||_{L^2(\mathbb{R}^N)} \tag{42}$$

$$E(\psi(t,x)) = E(\psi_0) \tag{43}$$

for all t.

A solution of (†) is called stationary solution if $\psi = v(x)e^{-i\omega t}$. Such a solution satisfies the nonlinear eigenvalue problem

$$-\Delta v + F'(v) = \omega v. \tag{*}$$

By Theorem 2, if F satisfies (F_0), (F_1) and (F_p), there exist $(\bar{u},\lambda)$ that satisfies (*) such that

$$J(\bar{u}) = \inf_{\{v\in H^1,\ ||v||_{L^2}=\rho\}} J(v),$$

with $||\bar{u}||_{L^2} = \rho$, for some ρ, and λ is a Lagrange multiplier. So we have that $\psi = \bar{u}(x)e^{-i\lambda t}$ is a stationary solution of (†) with initial condition $\psi(0,x) = \bar{u}(x)$. Notice that for stationary solution, $E(\psi) = J(u)$. Indeed, we have

$$E(u(t,x)e^{iS(t,x)}) = J(u(t,x)) + \frac{1}{2}\int u^2|\nabla S|^2 dx \tag{44}$$

for all t. Now we prove the orbital stability of the stationary solution found in the previous section.

We define

$$S = \left\{ u(x)e^{i\theta};\ \theta \in S^1, ||u||_{L^2} = \rho, J(u) = \inf_{\{v\in H^1,\ ||v||_{L^2}=\rho\}} J(v) \right\}. \tag{45}$$

Clearly, for any $q \in \mathbb{R}^N$ we have that $\bar{u}(x+q) \in S$.

Definition 13 *S is orbitally stable if*

$$\forall \varepsilon,\ \exists \delta > 0 \ s.t.\ \forall \psi_0 \in H^1(\mathbb{R}^N),\ \inf_{u\in S} ||\psi_0 - u||_{H^1} < \delta \ \textit{implies}$$

$$\forall t \geq 0 \quad \inf_{u\in S} ||\psi(t,\cdot) - u||_{H^1} < \varepsilon$$

where $\psi(t,x)$ is the solution of (†) with initial data ψ_0.

Let us suppose that S is not orbitally stable, i.e that

$$\exists \varepsilon,\ \exists \psi_n(0,x) \in H^1(\mathbb{R}^N),\ \inf_{u\in S} ||\psi_n(0,x)-u||_{H^1} \to 0 \text{ implies}$$

$$\exists t_n \geq 0 \quad \inf_{u\in S} ||\psi_n(t,\cdot)-u||_{H^1} > \varepsilon.$$

We can suppose that $||\psi_n(t_n,x)||_{L^2} = ||\psi_n(0,x)||_{L^2} = \rho$. Indeed, if

$$||u_n(t,x)||_{L^2} \to \rho$$

there exists a sequence $\alpha_n = \frac{\rho}{||u_n||_{L^2}}$ such that $||\alpha_n u_n||_{L^2} = \rho$ and $J(\alpha_n u_n) - J(u_n) \to 0$. We notice that, denoting $\psi_n(t,x) = u_n(t,x)e^{iS_n(t,x)}$,

$$J(u_n(0,x)) \to J(\bar{u}), \tag{46}$$

i.e $u_n(0,x)$ is a minimizing sequence of $J(u)$ on $||u||_{L^2} = \rho$. Moreover,

$$E(\psi_n(t_n,x)) = E(\psi_n(0,x)) \to E(\bar{u}(0,x)) = J(\bar{u}). \tag{47}$$

Hence we have that

$$u_n(t_n,x) \textit{ is a minimizing sequence on } ||u||_{L^2} = \rho.$$

Now we prove that any minimizing sequence for $J(u)$ on $||u||_{L^2} = \rho$ does converge in H^1. This proves clearly that S is orbitally stable. As a matter of fact this result can be proved, as claimed in [3], as a consequence of the concentration-compacteness principle of P.L. Lions [7]-[8]. Here, in order to give a self contained and simpler formulation, we prove a "Splitting Lemma" which describes the behaviour the Palais-Smale sequences. This lemma is a well known result of Struwe [12]. To prove this Lemma we make the following growth assumption

$$|F''(s)| \leq c_1|s|^{q-2} + c_2|s|^{p-2} \text{ for some } 2 < q \leq p < 2^*. \tag{F_p'}$$

We know that every critical point of J on $||u||_{L^2} = \rho$ is still a critical point of a corresponding functional

$$J_\lambda(u) = J(u) - \lambda \int_{\mathbb{R}^N} u^2 dx \tag{48}$$

where λ is the suitable Lagrange multiplier.

Lemma 14 *Let (F_0) and (F_p') hold, and let u_n be a Palais-Smale sequence for J_λ with $\lambda < 0$ and $||u_n||_{L^2} \to \rho$. Then there exist k sequence of points $\{y_n^j\}_{n\in\mathbb{N}}$ $(1 \leq j \leq k)$ with $|y_n^j| \to +\infty$ such that, up to a subsequence:*

1. $u_n = u^0 + \sum_j u^j(x+y_n^j) + w_n$ *with* $w_n \to 0$ *in* H^1
2. $||u_n||^2_{L^2} \to ||u^0||^2_{L^2} + \sum_j ||u^j||^2_{L^2}$
3. $J_\lambda(u_n) \to J_\lambda(u^0) + \sum_j J_\lambda(u^j)$

where u^0 and u^j are weak solutions of (*).

The Proof of Lemma 14 is given in Appendix.

Proposition 15 *Suppose that for any ρ there exists $I_{\rho^2} := \min_{||u||^2_{L^2}=\rho^2} J(u)$. Then, for any $\mu \in (0,\rho)$ we have*

$$I_{\rho^2} < I_{\mu^2} + I_{\rho^2-\mu^2}. \tag{49}$$

Proof. We prove that $I_{\theta\rho^2} < \theta I_{\rho^2}$ for any $\rho > 0$ and for any $\theta > 1$. We take u such that $J(u) = I_{\rho^2}$. Thus, $||u(\frac{x}{\theta^{1/N}})||^2_{L^2} = \theta\rho^2$. We have

$$\begin{aligned} I_{\theta\rho^2} \leq J\left(u\left(\frac{x}{\theta^{1/N}}\right)\right) &= \theta\left(\int_{R^N} \frac{1}{2}\left(\frac{1}{\theta}\right)^{2/N} |\nabla u|^2 + F(u)dx\right) \\ &< \theta\left(\int_{\mathbb{R}^N} \frac{1}{2}|\nabla u|^2 + F(u)dx\right) = \theta I_{\rho^2}. \end{aligned} \tag{50}$$

By simple arguments we obtain (49). In fact, if $h(x)$ is a real function such that, for all $x > 0$ and for all $\theta > 1$

$$h(\theta x) < \theta h(x) \tag{51}$$

then we have, for all $y \in (0,x)$

$$h(x) < h(y) + h(x-y). \tag{52}$$

■

With Lemma 14 and Proposition 15 we can prove the following Theorem.

Theorem 16 *Let (F_0), (F_2) and (F_p') hold. Then S is orbitally stable.*

Proof. Let, as before, $u_n(t_n,x)$ be a minimizing sequence of J on $||u||_{L^2} = \rho$. By the Ekeland principle we can suppose that it is a Palais-Smale sequence for J and, thus, a Palais-Smale sequence for J_λ with $\lambda < 0$, as proved in Theorem 2.

By Lemma 14 and Proposition 15 we have two cases:

1. $u_n = u^0 + w_n$ with $w_n \to 0$ in H^1.
2. there exists a sequence y_n such that $u_n = u^1(x+y_n) + w_n$ with $w_n \to 0$ in H^1.

We have that $u^0, u^1 \in S$. In both cases Theorem 16 holds. ■

We give two examples of functions F which satisfy the assumption (F_0), (F_1), (F_p'):

1. $F(s) = -\frac{1}{4}|s|^4 + \frac{1}{5}|s|^5$ with $N = 3$,
2. $F(s) = -\frac{|s|^q}{1+|s|^{q-p}}$ with $2 < p < q < 2 + \frac{4}{N}$.

With this potential we have that problem (*) has a solution and that the nonlinear Schrödinger equation

$$\begin{cases} i\frac{\partial\psi}{\partial t} + \Delta\psi - F'(\psi) = 0 & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N \\ \psi(0,x) = \psi_0(x) \end{cases} \tag{†}$$

admits a stationary solution which is orbitally stable.

4 Appendix

Proof of the splitting lemma. We do it by steps.

Step I. There exists $u^0 \in H^1$ such that $u_n \rightharpoonup u^0$ in H^1 and u_0 is a weak solution of (*).

In fact, we have that u_n is bounded in L^2 by hypothesis. Furthermore, using (F_0), we also have that u_n is bounded in H^1. So there exist u^0 in H^1 such that $u_n \rightharpoonup u^0$.

Now, because u_n is a P-S sequence for J_λ, we have that for all $\varphi \in C_0^\infty(\mathbb{R}^N)$,

$$\int \nabla u_n \varphi + \int F'(u_n)\varphi - \lambda \int u_n \varphi \to 0. \tag{53}$$

We have that, for any compact set B, $u_n \to u^0$ strongly in $L^p(B)$ for $2 \le p < 2^*$. Thus using (F_p') and the fact that $u_n \rightharpoonup u^0$, we can conclude that u^0 is a weak solution of (*).

Step II. Setting $\psi_n = u_n - u^0$, we have that

$$||\psi_n||^2_{H^1} = ||u_n||^2_{H^1} - ||u^0||^2_{H^1} + o(1) \tag{54}$$

$$J(\psi_n) = J(u_n) - J(u^0) + o(1). \tag{55}$$

We have that $\psi_n \rightharpoonup 0$ in H^1. Thus, obviously,

$$\begin{aligned}\int |u_n|^2 &= \int (u^0 + \psi_n)^2 = \int |u^0|^2 + \int |\psi_n|^2 + 2\int u^0\psi_n = \\ &= \int |u^0|^2 + \int |\psi_n|^2 + o(1)\end{aligned}$$

In the same way, we can proceed with $\int |\nabla u_n|^2$, obtaining (54)

To obtain (55), we prove that

$$\int F(u_n) - F(u^0) - F(\psi_n) \to 0. \tag{56}$$

For all $R > 0$, we can write this integral as follows:

$$\begin{aligned}\int_{\mathbb{R}^N} F(u_n) - F(u^0) - F(\psi_n) &= \int_{B_R} [F(u^0 + \psi_n) - F(u^0)] - \int_{B_R^C} F(u^0) + \\ &\quad + \int_{B_R^C} [F(u^0 + \psi_n) - F(\psi_n)] - \int_{B_R} F(\psi_n).\end{aligned}$$

On every compact set, we have that $\psi_n \to 0$ in L^p for all $2 \le p < 2^*$ and that, by (F_p'),

$$\int_{B_R} [F(u^0 + \psi_n) - F(u^0)] - \int_{B_R} F(\psi_n) \to 0 \text{ when } n \to 0.$$

Easily, we have also that

$$\int_{B_R^C} F(u_0) \to 0 \text{ when } R \to \infty.$$

Finally, for some $0 < \theta < 1$,

$$\int_{B_R^C} [F(u^0 + \psi_n) - F(\psi_n)] = \int_{B_R^C} F'(\theta u^0 + \psi_n) u^0,$$

with $||u^0||_{L^p(B_R^C)} \to 0$ strongly in $L^p(B_R^C)$ when $R \to \infty$ and $\theta u^0 + \psi_n$ is bounded in $L^p(B_R^C)$. By (F_p'), we have also that this term vanishes when R is sufficiently large, and this proves Step II.

Step III. Set $\psi_n = u_n - u^0$. If $\psi_n \nrightarrow 0$ in H^1 then there exists a sequence of points $y_n \in \mathbb{R}^N$, with $|y_n| \to \infty$, and a function $u^1 \in H^1$, $u^1 \neq 0$, such that

$$\psi_n(x + y_n) \rightharpoonup u^1 \ \in H^1. \tag{57}$$

Notice that, if $\psi_n \to 0$ strongly in H^1 the splitting lemma is proved. Otherwise, we start to prove that, when $n \to \infty$,

$$\int_{\mathbb{R}^N} F'(u_n)u_n - F'(u^0)u^0 - F'(\psi_n)\psi_n \to 0. \tag{58}$$

As usual, for a fixed $R > 0$, we have that both $u_n \to u^0$ and $\psi_n \to 0$ in $L^p(B_R)$ as $n \to \infty$. So, by (F_p')

$$\int_{B_R} F'(u_n)u_n - F'(u^0)u^0 - F'(\psi_n)\psi_n \to 0. \tag{59}$$

Moreover, there exist θ, η, $0 < \theta, \eta < 1$ such that

$$\int_{B_R^C} F'(u_n)u_n - F'(u^0)u^0 - F'(\psi_n)\psi_n =$$

$$= \int_{B_R^C} [F'(u^0 + \psi_n) - F(u^0)]u^0 + \int_{B_R^C} [F'(u^0 + \psi_n) - F'(\psi_n)]\psi_n =$$

$$= \int_{B_R^C} [F''(u^0 + \theta\psi_n) - F''(\eta u^0 + \psi_n)]u^0\psi_n$$

and we can conclude as above that for R sufficiently large this term vanishes.

Using that u_n is a P-S sequence and that u^0 is a weak solution of (*), we have that

$$\begin{aligned}
||\psi_n||_{H^1}^2 &= \int |\nabla u_n|^2 - \int |\nabla u^0|^2 + \int |u_n|^2 - \int |u^0|^2 + o(1) = \\
&= -\int [F'(u_n)u_n - F'(u^0)u^0] + \\
&+ (\lambda + 1) \int [|u_n|^2 - |u^0|^2] + o(1) = \\
&= -\int F'(\psi_n)\psi_n + (\lambda + 1) \int |\psi_n|^2 + o(1).
\end{aligned}$$

Now, for a fixed $L > 0$, we decompose $\mathbb{R}^N$ into a numerable union of N-dimensional hypercubes Q_i, having edge L.

By (F'_p), we have

$$||\psi_n||^2_{H^1} + o(1) = -\int F'(\psi_n)\psi_n + (\lambda+1)\int |\psi_n|^2 \leq$$

$$\leq C_1 \sum_i \left[||\psi_n||^q_{L^q(Q_i)} + ||\psi_n||^p_{L^p(Q_i)} + (\lambda+1)||\psi_n||^2_{L^2(Q_i)} \right]$$

$$\leq C_1 \sum_i \left[L^{N\left(\frac{p-q}{p}\right)} ||\psi_n||^q_{L^p(Q_i)} + ||\psi_n||^p_{L^p(Q_i)} + (\lambda+1) L^{N\left(\frac{p-2}{p}\right)} ||\psi_n||^2_{L^p(Q_i)} \right]$$

$$\leq C_2 ||\psi_n||^2_{H^1} \left[L^N \left(L^{\left(\frac{p-q}{p}\right)} ||\psi_n||^{q-2}_{L^p(\mathbb{R}^N)} + (\lambda+1) L^{\left(\frac{p-2}{p}\right)} \right) + d_n \right]$$

where $d_n = \sup_i ||\psi_n||^{p-2}_{L^p(Q_i)}$ and C_1, C_2 are positive constants. We can choose L small enough such that for a suitable C_3 we have

$$C_3 ||\psi_n||^2_{H^1} \leq d_n ||\psi_n||^2_{H^1} + o(1).$$

Because $\psi_n \nrightarrow 0$ in H^1, we must have $\inf d_n > 0$. Thus there exists an $\alpha > 0$ and a sequence of index i_n such that

$$||\psi_n||_{L^p(Q_{i_n})} > \alpha \text{ for all } n. \tag{60}$$

We call y_n the center of the hypercube Q_{i_n}. Because $\psi_n \to 0$ in $L^p(B)$ for any compact set B, we have that $|y_n| \to \infty$ when $n \to \infty$.

Finally, we know that there exists a u^1 in H^1 such that

$$\psi_n(\cdot + y_n) \rightharpoonup u^1 \tag{61}$$

weakly in u^1 because the sequence $\psi_n(\cdot + y_n)$ is bounded in H^1, and by (60) we conclude that $u^1 \neq 0$.

Step IV. The function u^1 is a weak solution of (*)

We know that $\psi_n(x + y_n) \to u^1$ weakly in H^1 and strongly in $L^p(B)$ for all B compact, $2 \leq p < 2^*$. So it is sufficient to prove that, for all $\varphi \in C_0^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} \nabla\psi_n(x+y_n)\nabla\varphi(x) + F'(\psi_n(x+y_n))\varphi(x) - \lambda\psi_n(x+y_n)\varphi(x)dx \to 0. \tag{62}$$

After a change of variables we obtain

$$\int_{\mathbb{R}^N} \nabla\psi_n(x)\nabla\varphi(x-y_n) + F'(\psi_n(x))\varphi(x-y_n) - \lambda\psi_n(x)\varphi(x-y_n)dx,$$

and, using that u^0 is a weak solution of (*) and that u_n is a P-S sequence, we have that

$$\int_{\mathbb{R}^N} \nabla\psi_n(x)\nabla\varphi(x-y_n) - \lambda\psi_n(x)\varphi(x-y_n)dx =$$

$$= -\int_{\mathbb{R}^N} F'(u_n(x))\varphi(x-y_n) + \int_{\mathbb{R}^N} F'(u^0(x))\varphi(x-y_n) + o(1).$$

Thus we prove that

$$\int_{\mathbb{R}^N} [F'(\psi_n(x)) - F'(u_n(x)) + F'(u^0(x))]\varphi(x - y_n) \to 0 \text{ as } n \to \infty. \tag{63}$$

As usual, for fixed B, we can split this integral as follows

$$\begin{aligned}
&\int_{\mathbb{R}^N} [F'(\psi_n(x)) - F'(u_n(x)) + F'(u^0(x))]\varphi(x - y_n) = \\
&\int_{B_R} [F'(u^0(x)) - F'(u_n(x))]\varphi(x - y_n) + \int_{B_R^C} F'(u^0(x))\varphi(x - y_n) + \\
&\int_{B_R^C} [F'(\psi_n(x)) - F'(u_n(x))]\varphi(x - y_n) + \int_{B_R} F'(\psi_n(x))\varphi(x - y_n).
\end{aligned}$$

All the integrals over B_R are definitively 0 because φ has compact support and $|y_n| \to \infty$. Moreover, we observe that $\int_{B_R^C} |F'(u^0)|^{\frac{p}{p-1}} \to 0$ as $R \to \infty$.

Finally, for some $0 < \theta < 1$,

$$\begin{aligned}
&\int_{B_R^C} [F'(\psi_n(x)) - F'(u_n(x))]\varphi(x - y_n) = \\
&= - \int_{B_R^C} F''(\psi_n(x) + \theta u^0(x))\varphi(x - y_n)u^0(x) \to 0,
\end{aligned}$$

as usual.

Step V. Conclusion.

We can now iterate this procedure by defining a function

$$\psi_n^1(x) = \psi_n(x + y_n) - u^1(x).$$

We have that $||u_n||_{H^1}^2 = ||u^0||_{H^1}^2 + ||u^1||_{H^1}^2 + ||\psi_n^1||$. If $\psi_n^1 \to 0$ strongly in H^1, the lemma is proved. Otherwise, we have that $\psi_n^1 \rightharpoonup 0$ in H^1 and there exist a sequence of point y_n^1 with $|y_n^1| \to \infty$ and a function u^2 in H^1 such that $\psi_n^1(x + y_n^1) \rightharpoonup u^2$ in H^1. Furthermore u^2 is a weak solution of (*), and so on.

We can have a finite number of iterative steps. Indeed, there exists an $\alpha > 0$ such that

$$||u^j||_{H^1} > \alpha \text{ for all } j. \tag{64}$$

Hence, by (54) and (55) we get the claim. Now we prove (64). We know that every u^j is a weak solution of (*), so it belongs to the set

$$\mathcal{N} := \left\{ u \in H^1,\ u \neq 0\ :\ \int |\nabla u|^2 + \int F'(u)u - \lambda \int u^2 = 0 \right\}. \tag{65}$$

We want to prove that

$$\inf_{u \in \mathcal{N}} ||u||_{H^1} = \alpha > 0 \tag{66}$$

Notice that, because $\lambda < 0$, then we can endow H^1 with the following equivalent norm:

$$|||u|||_{H^1} = \int |\nabla u|^2 - \lambda \int |u|^2. \tag{67}$$

We suppose, by contradiction, that there exists a sequence $w_n \in \mathcal{N}$ with $|||w_n|||_{H^1} \to 0$. We can set $w_n = t_n v_n$ with $|||v_n|||_{H^1} = 1$, thus $t_n \to 0$. We have

$$\begin{aligned} 0 &= \int |\nabla w_n|^2 + \int F'(w_n)w_n - \lambda \int w_n^2 = |||w_n|||_{H^1} + \int F'(w_n)w_n = \\ &= t_n^2 + t_n \int F'(w_n)v_n. \end{aligned}$$

Thus,

$$\begin{aligned} t_n &= -\int F'(w_n)v_n \leq c_1 \int |t_n v_n|^{p-1} v_n + c_2 \int |t_n v_n|^{q-1} v_n \leq \\ &\leq c_1 t_n^{p-1} \int |v_n|^p + c_2 t_n^{q-1} \int |v_n|^q; \\ 1 &\leq c_1 t_n^{p-2} \int |v_n|^p + c_2 t_n^{q-2} \int |v_n|^q, \end{aligned}$$

and this lead to a contradiction. Indeed, if v_n is bounded in H^1 then it is bounded in L^p for all $2 \leq p < 2^*$, and by hypothesis $t_n \to 0$. ■

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