

Equivalent Semigroup Properties for Curvature-Dimension Condition*

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Abstract

Some equivalent gradient and Harnack inequalities of a diffusion semigroup are presented for the curvature-dimension condition of the associated generator. As applications, the first eigenvalue, the log-Harnack inequality, the heat kernel estimates, and the HWI inequality are derived by using the curvature-dimension condition. The transportation inequality for diffusion semigroups is also investigated.

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1 Introduction

Let M be a d -dimensional complete connected Riemannian manifold without boundary or with a convex boundary ∂M . Let P_t be the (Neumann if $\partial M \neq \emptyset$) semigroup generated by $L = \Delta + Z$ for a C^1 -vector field Z on M . To describe analytic properties of P_t , the following curvature-dimension condition of Bakry-Emery [3] plays a very important role:

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$$(1.1) \quad \frac{1}{2}L|\nabla f|^2 - \langle \nabla Lf, \nabla f \rangle \geq -K|\nabla f|^2 + \frac{1}{n}(Lf)^2, \quad f \in C^\infty(M),$$

where $-K \in \mathbb{R}$ and $n \geq d$ provide a curvature lower bound and a dimension upper bound of L respectively. When $Z = 0$ this condition is equivalent to $\text{Ric} \geq -K$, where Ric is the Ricci curvature. In this case (1.1) holds for $n = d$. When $Z \neq 0$, n is essentially larger than d . Indeed, (1.1) is equivalent to

$$(1.2) \quad \text{Ric}(U, U) - \langle \nabla_U Z, U \rangle \geq -K|U|^2 - \frac{\langle Z, U \rangle^2}{n-d}, \quad U \in TM.$$

In particular, when $n = \infty$, (1.1) reduces to the curvature condition

$$(1.3) \quad \text{Ric}(U, U) - \langle \nabla_U \nabla Z, U \rangle \geq -K|U|^2, \quad U \in TM.$$

There are a number of equivalent semigroup inequalities for the curvature condition (1.3), including gradient inequalities, Poincaré/log-Sobolev inequalities, the dimension-free and logarithmic Harnack inequalities, and Wasserstein (or transportation-cost) inequalities, see e.g. [2, 10, 15, 19, 22] and references within for details.

When $n < \infty$, the curvature-dimension condition (1.1) has been used in the study of the Sobolev inequality, the first eigenvalue and the diameter estimates, and Li-Yau type Harnack inequalities. Besides the above mentioned references, we refer to [4, 5, 17] and references within for detailed applications of the curvature-dimension condition. On the other hand, however, unlike for (1.3), there is no any known equivalent semigroup inequalities for the curvature-dimension condition (1.1) with finite n . The purpose of this note is to find inequalities of P_t which are equivalent to (1.1), and to make further applications of these equivalent inequalities.

Let \mathcal{D}_0 be the set of all smooth functions on M with compact support and satisfying the Neumann boundary condition provided $\partial M \neq \emptyset$. Recall that throughout the paper ∂M is assumed to be convex if it exists. Let ρ be the Riemannian distance on M .

Theorem 1.1. *Each of the following statements is equivalent to (1.1):*

- (1) $|\nabla P_t f|^2 \leq e^{2Kt} P_t |\nabla f|^2 - \frac{2}{n} \int_0^t e^{2Ks} P_s (P_{t-s} Lf)^2 ds, \quad f \in \mathcal{D}_0, t \geq 0.$
- (2) $|\nabla P_t f|^2 \leq e^{2Kt} P_t |\nabla f|^2 - \frac{e^{2Kt}-1}{Kn} (P_t Lf)^2, \quad f \in \mathcal{D}_0, t \geq 0.$
- (3) $P_t f^2 - (P_t f)^2 \leq \frac{e^{2Kt}-1}{K} P_t |\nabla f|^2 - \frac{e^{2Kt}-1-2Kt}{K^2 n} (P_t Lf)^2, \quad f \in \mathcal{D}_0, t \geq 0.$
- (4) $P_t f^2 - (P_t f)^2 \geq \frac{1-e^{-2Kt}}{K} |\nabla P_t f|^2 + \frac{e^{-2Kt}-1+2Kt}{K^2 n} (P_t Lf)^2, \quad f \in \mathcal{D}_0, t \geq 0.$

$$(5) \quad e^{Kt} P_t |\nabla f| \geq |\nabla P_t f| + \frac{1}{n-d} \int_0^t e^{Ks} P_s \frac{\langle Z, \nabla P_{t-s} f \rangle^2}{|\nabla P_{t-s} f|} ds, \quad f \in \mathcal{D}_0, t \geq 0.$$

(6) For any $t > 0$ and increasing $\varphi \in C^1([0, t])$ with $\varphi(0) = 0$ and $\varphi'(0) = 1$, the log-Harnack inequality

$$P_{\varphi(t)} \log f(y) \leq \log P_t f(x) + \frac{\rho(x, y)^2}{4 \int_0^t e^{-2K\varphi(s)} ds} + \frac{Kn}{4} \int_0^t \frac{(\varphi'(s) - 1)^2 ds}{1 - e^{-2K\varphi(s)}}$$

holds for any positive function f with $\inf f > 0$ and all $x, y \in M$.

We remark that according to [22, Theorem 1.2], at least for compact manifolds and a class of non-compact manifolds, any of statements (1)-(6) implies that ∂M is convex if exists. Therefore, our assumption on the boundary is essential.

Now, we consider applications of the above equivalent inequalities. We first present some consequences of (6) for heat kernel bounds and HWI inequalities. According to Li-Yau's Harnack inequality [11, 4], if (1.1) holds then P_t can be dominated by P_{t+s} for $s, t > 0$. A nice point of (6) is that we are also able to dominate P_{t+s} by P_t with help of the logarithmic function. With concrete choices of φ we have the following explicit log-Harnack inequalities.

Corollary 1.2. *If (1.1) holds, then for any $s \geq 0, t > 0$,*

$$(1.4) \quad P_{t+s} \log f(y) \leq \log P_t f(x) + \frac{K(t+2s)\rho(x, y)^2}{2t(1 - e^{-2K(t+s)})} + \frac{nKs^2}{2t(1 - e^{-Kt})},$$

and

$$(1.5) \quad P_t \log f(y) \leq \log P_{t+s} f(x) + \frac{K\rho(x, y)^2}{2(1 - e^{-2Kt}) + 4Kse^{-2Kt}} + \frac{Kns}{4(1 - e^{-2Kt})}$$

hold for $x, y \in M$ and bounded measurable function f with $\inf f > 0$.

As shown in the proof of [22, Proposition 2.4(2)], it is easy to see that for any $t > 0, s \geq 0$ and $x, y \in M$, (1.4) and (1.5) are equivalent to the following heat kernel inequalities (1.6) and (1.7) respectively, where ν is a measure equivalent to dx and p_t^ν is the heat kernel of P_t w.r.t. ν :

$$(1.6) \quad \int_M p_{t+s}^\nu(y, z) \log \frac{p_{t+s}^\nu(y, z)}{p_t^\nu(x, z)} \nu(dz) \leq \frac{K(t+2s)\rho(x, y)^2}{2t(1 - e^{-2K(t+s)})} + \frac{nKs^2}{2t(1 - e^{-Kt})},$$

$$(1.7) \quad \int_M p_t^\nu(y, z) \log \frac{p_t^\nu(y, z)}{p_{t+s}^\nu(x, z)} \nu(dz) \leq \frac{K\rho(x, y)^2}{2(1 - e^{-2Kt}) + 4Kse^{-2Kt}} + \frac{Kns}{4(1 - e^{-2Kt})}.$$

In particular, when P_t is symmetric w.r.t a probability measure μ , we have the following heat kernel lower bound.

Corollary 1.3. *Let $Z = \nabla V$ such that $\mu(dx) := e^{V(x)}dx$ is a probability measure, and let $p_t(x, y)$ be the heat kernel of P_t w.r.t. μ . Then (1.3) and hence (1.1) implies*

$$(1.8) \quad p_t(x, y) \geq \exp \left[-\frac{K\rho(x, y)^2}{2(1 - e^{-Kt})} \right], \quad x, y \in M, t > 0.$$

We remark that (1.8) is new. Known heat kernel lower bounds derived from Li-Yau's Harnack inequality are dimension-dependent, and decay to zero as the dimension goes to infinity provided $K > 0$, see e.g. [18, Corollary 3.9] and [4, (13)].

Moreover, following the line of [6], we use the log-Harnack inequality (1.5) to establish the HWI inequality. Again let $Z = \nabla V$ such that $\mu(dx) := e^{V(x)}dx$ is a probability measure. Recall that for any non-negative measurable function \mathbf{c} on $M \times M$, and for any $p \geq 1$, the L^p -transportation cost induced by cost function \mathbf{c} is

$$W_p^{\mathbf{c}}(\mu_1, \mu_2) = \inf_{\pi \in \mathcal{C}(\mu_1, \mu_2)} \pi(\mathbf{c}^p)^{1/p}, \quad \mu_1, \mu_2 \in \mathcal{P}(M),$$

where $\mathcal{P}(M)$ is the set of all probability measures on M and $\mathcal{C}(\mu_1, \mu_2)$ is the set of all couplings for μ_1 and μ_2 .

Corollary 1.4. *Let $Z = \nabla V$ such that $\mu(dx) := e^{V(x)}dx$ is a probability measure. If (1.1) holds, then for any $f \in C^1(M)$ with $\mu(f^2) = 1$,*

$$(1.9) \quad \begin{aligned} \mu(f^2 \log f^2) &\leq r\mu(|\nabla f|^2) + \frac{(Kr + 2)W_2^\rho(f^2\mu, \mu)}{2r} \left(W_2^\rho(f^2\mu, \mu) \wedge \frac{\sqrt{rn}}{2\sqrt{2}} \right) \\ &\quad + \frac{\sqrt{n}(Kr + 2)}{4\sqrt{2r}} \left(W_2^\rho(f^2\mu, \mu) - \frac{\sqrt{rn}}{2\sqrt{2}} \right)^+, \quad r \in (0, \infty) \cap \left(0, \frac{2}{K^-} \right], \end{aligned}$$

where $K^- := \max\{0, -K\}$. Consequently,

$$(1.10) \quad \begin{aligned} \mu(f^2 \log f^2) &\leq 2W_2^\rho(f^2\mu, \mu)\sqrt{\mu(|\nabla f|^2)} + \frac{K}{2}W_2^\rho(f^2\mu, \mu)^2 \\ &\quad - \frac{KW_2^\rho(f^2\mu, \mu) + 2\sqrt{\mu(|\nabla f|^2)}}{2\sqrt{W_2^\rho(f^2\mu, \mu)}} \left(\sqrt{W_2^\rho(f^2\mu, \mu)} - \frac{\sqrt{n}}{2\sqrt{2}\mu(|\nabla f|^2)^{1/4}} \right)^+. \end{aligned}$$

It was proved in [13] and [6] that (1.3) (i.e. (1.1) for $n = \infty$) implies

$$\mu(f^2 \log f^2) \leq 2W_2^\rho(f^2\mu, \mu)\sqrt{\mu(|\nabla f|^2)} + \frac{K}{2}W_2^\rho(f^2\mu, \mu)^2$$

for all $f \in C^1(M)$ with $\mu(f^2) = 1$. According to (1.10), the dimension n contributes to a negative term in the right-hand side since $KW_2^\rho(f^2\mu, \mu) + 2\sqrt{\mu(|\nabla f|^2)} \geq 0$ as explained in the proof of (1.10). But this inequality is incomparable with the Sobolev type WHI inequality derived in [21].

Next, we consider the first non-trivial eigenvalue (i.e. the spectral gap) of L . To this end, let $Z = \nabla V$ for some $V \in C^2(M)$ such that

$$\mu(dx) := e^{V(x)} dx$$

is a probability measure, where dx stands for the Riemannian volume measure on the manifold. In this case the Friedrich extension of (L, \mathcal{D}_0) gives rise to a negatively definite self-adjoint operator on $L^2(\mu)$, whose spectral gap can be characterized as

$$\lambda_1 = \inf\{\mu(|\nabla f|^2) : f \in C_0^1(M), \mu(f) = 0, \mu(f^2) = 1\}.$$

The following lower bound of λ_1 is a simple consequence of Theorem 1.1 (2). This estimate is well known as the Lichnerowicz estimate [12] for $Z = 0$, and was extended to $Z \neq 0$ by Bakry and Qian [5].

Corollary 1.5 ([12, 5]). *Let $Z = \nabla V$ such that $\mu(dx) := e^{V(x)} dx$ is a probability measure. If (1.1) holds for some $K < 0$ and $n > 1$, then*

$$\lambda_1 \geq \frac{n(-K)}{n-1}.$$

Finally, we consider the transportation inequality of P_t deduced from (1.1). According to [15], (1.3) implies

$$(1.11) \quad W_p^\rho(\mu_1 P_t, \mu_2 P_t) \leq e^{Kt} W_p^\rho(\mu_1, \mu_2), \quad t \geq 0, \mu_1, \mu_2 \in \mathcal{P}(M)$$

for any $p \geq 1$. Using (1.1) we prove the following inequalities (1.12) and (1.13). Comparing with (1.11), when $p = 1$ (1.13) has better long time behavior for $K < 0$ while (1.12) is stronger for $K > 0$. In fact, since

$$H(r) := \frac{2}{\sqrt{K/(n-1)}} \sinh \left[\frac{r}{2} \sqrt{K/(n-1)} \right], \quad r \geq 0$$

is convex with $H'(r) > 1$ for $r > 1$, due to the Jensen inequality, (1.12) implies that

$$\begin{aligned} W_1^\rho(\mu_1 P_t, \mu_2 P_t) &\leq H^{-1}(W_1^{\tilde{\rho}}(\mu_1 P_t, \mu_2 P_t)) \leq H^{-1}(e^{Kt} H(W_1^{\tilde{\rho}}(\mu_1, \mu_2))) \\ &< e^{Kt} H^{-1} \circ H(W_1^{\tilde{\rho}}(\mu_1, \mu_2)) = e^{Kt} W_1^{\tilde{\rho}}(\mu_1, \mu_2), \quad t > 0, \mu_1 \neq \mu_2. \end{aligned}$$

Proposition 1.6. *Assume that (1.1) holds and let*

$$\tilde{\rho}(x, y) = \begin{cases} \frac{2}{\sqrt{-K/(n-1)}} \sin \left[\frac{\rho(x, y)}{2} \sqrt{-K/(n-1)} \right], & \text{if } K < 0, \\ \rho(x, y), & \text{if } K = 0, \\ \frac{2}{\sqrt{K/(n-1)}} \sinh \left[\frac{\rho(x, y)}{2} \sqrt{K/(n-1)} \right], & \text{if } K > 0. \end{cases}$$

Then for any $p \geq 1$,

$$(1.12) \quad W_p^{\tilde{\rho}}(\mu_1 P_t, \mu_2 P_t) \leq e^{Kt} W_p^{\tilde{\rho}}(\mu_1, \mu_2), \quad t \geq 0, \mu_1, \mu_2 \in \mathcal{P}(M).$$

If $K < 0$ then

$$(1.13) \quad W_1^{\tilde{\rho}}(\mu_1 P_t, \mu_2 P_t) \leq \exp \left[\frac{nK}{n-1} t \right] W_1^{\tilde{\rho}}(\mu_1, \mu_2), \quad t \geq 0, \mu_1, \mu_2 \in \mathcal{P}(M).$$

2 Proofs

According to the proof of [22, Theorem 1.1], the reflection at a convex boundary does not make any trouble for our proofs. So, for simplicity, we shall only consider the case without boundary. In this case, the the proofs of (1)-(5) in Theorem 1.1 are more or less standard according to the semigroup argument of Bakry, Emery and Ledoux. Our proof of equivalence between (6) and (1.1) is however highly technical.

Proof of Theorem 1.1. By the Jensen inequality, (2) follows from (1) immediately. So, it suffices to show that (1.1) implies (1), (2) implies (3) and (4), each of (3) and (4) implies (1.1), (5) is equivalent to (1.2), (2) implies (6), and (6) implies (1.1). Below we prove these implications respectively.

(1.1) implies (1). By (1.1) we have

$$\begin{aligned} \frac{d}{ds} P_s |\nabla P_{t-s} f|^2 &= P_s \{ L |\nabla P_{t-s} f|^2 - 2 \langle \nabla P_{t-s} f, \nabla L P_{t-s} f \rangle \} \\ &\geq -2K P_s |\nabla P_{t-s} f|^2 + \frac{2}{n} P_s (P_{t-s} L f)^2, \quad s \in [0, t]. \end{aligned}$$

By the Gronwall lemma, this implies (1) immediately.

(2) implies (3) and (4). Obviously, we have

$$(2.1) \quad \frac{d}{ds} P_s (P_{t-s} f)^2 = P_s \{ L (P_{t-s} f)^2 - 2 (P_{t-s} f) L P_{t-s} f \} = 2 P_s |\nabla P_{t-s} f|^2.$$

Next, according to (2) and noting that $P_s(P_{t-s}Lf)^2 \geq (P_tLf)^2$, we have

$$\begin{aligned} P_s|\nabla P_{t-s}f|^2 &\leq e^{2K(t-s)}P_t|\nabla f|^2 - \frac{e^{2K(t-s)} - 1}{Kn}P_s(P_{t-s}Lf)^2, \\ P_s|\nabla P_{t-s}f|^2 &\geq e^{-2Ks}|\nabla P_t f|^2 + \frac{1 - e^{-2Ks}}{Kn}(P_tLf)^2. \end{aligned}$$

Combining these with (2.1) respectively and integrating w.r.t. ds over $[0, t]$, we prove (3) and (4).

(3) or (4) implies (1.1). For small $t > 0$ we have

$$\begin{aligned} P_t f^2 &= f^2 t L f^2 + \frac{t^2}{2} L^2 f^2 + o(t^2), \\ (P_t f)^2 &= \left(f + t L f + \frac{t^2}{2} L^2 f + o(t^2) \right)^2 = f^2 + t^2 (L f)^2 + 2t f L f + t^2 f L^2 f + o(t^2). \end{aligned}$$

So,

$$(2.2) \quad P_t f^2 - (P_t f)^2 = 2t |\nabla f|^2 + t^2 \{2\langle \nabla L f, \nabla f \rangle + L |\nabla f|^2\} + o(t^2).$$

On the other hand,

$$\begin{aligned} \frac{e^{2Kt} - 1}{K} P_t |\nabla f|^2 &= \{2t + 2Kt^2 + o(t^2)\} \cdot \{|\nabla f|^2 + tL|\nabla f|^2 + o(t)\} \\ &= 2t |\nabla f|^2 + 2t^2 \{L|\nabla f|^2 + K|\nabla f|^2\} + o(t^2). \end{aligned}$$

Moreover, it is easy to see that

$$\frac{e^{2Kt} - 2Kt - 1}{K^2 n} (P_t L f)^2 = \frac{2}{n} t^2 (L f)^2 + o(t^2).$$

Combining these with (2.9), we see that (3) implies

$$2t^2 \left\{ \frac{1}{2} L |\nabla f|^2 - \langle \nabla L f, \nabla f \rangle + K |\nabla f|^2 + \frac{(L f)^2}{n} \right\} + o(t^2) \geq 0.$$

Therefore, (1.1) holds.

Next, it is easy to see that

$$\begin{aligned}
& \frac{1 - e^{-2Kt}}{K} |\nabla P_t|^2 + \frac{e^{-2Kt} - 1 + 2Kt}{K^2 n} (P_t Lf)^2 \\
&= \{2t - 2Kt^2 + o(t^2)\} \cdot |\nabla f + t\nabla Lf + o(t)|^2 + \frac{2t^2}{n} (Lf)^2 + o(t^2) \\
&= 2t|\nabla f|^2 + 2t^2 \left\{ 2\langle \nabla f, \nabla Lf \rangle + \frac{(Lf)^2}{n} - K|\nabla f|^2 \right\} + o(t^2).
\end{aligned}$$

Combining this with (2.9) and (4) we prove (1.1).

(5) is equivalent to (1.2). Using $\sqrt{|\nabla P_{t-s}f|^2 + \varepsilon}$ to replace $|\nabla P_{t-s}f|$ and letting $\varepsilon \rightarrow 0$, in the following calculations we may assume that $|\nabla P_{t-s}f|$ is positive and smooth, so that

$$\begin{aligned}
(2.3) \quad \frac{d}{ds} P_s |\nabla P_{t-s}f| &= P_s \left\{ L|\nabla P_{t-s}f| - \frac{\langle \nabla L P_{t-s}f, \nabla P_{t-s}f \rangle}{|\nabla P_{t-s}f|} \right\} \\
&= P_s \left\{ \frac{\frac{1}{2}L|\nabla P_{t-s}f|^2 - \langle \nabla L P_{t-s}f, \nabla P_{t-s}f \rangle - |\nabla |\nabla P_{t-s}f||^2}{|\nabla P_{t-s}f|} \right\}.
\end{aligned}$$

Since

$$\begin{aligned}
(2.4) \quad \frac{1}{2}L|\nabla f|^2 - \langle \nabla Lf, \nabla f \rangle &= \text{Ric}(\nabla f, \nabla f) - \langle \nabla_{\nabla f} Z, \nabla f \rangle + \|\text{Hess}_f\|_{HS}^2, \\
|\nabla |\nabla f||^2 &= \left\| \text{Hess}_f \left(\frac{\nabla f}{|\nabla f|}, \cdot \right) \right\|^2 \leq \|\text{Hess}_f\|_{HS}^2,
\end{aligned}$$

it follows from (1.2) and (2.3) that

$$\frac{d}{ds} P_s |\nabla P_{t-s}f| \geq -K P_s |\nabla P_{t-s}f| + \frac{1}{n-d} P_s \frac{\langle Z, \nabla P_{t-s}f \rangle^2}{|\nabla P_{t-s}f|}.$$

This implies (5).

On the other hand, since when $t = 0$ the equality in (5) holds, one may take derivatives at $t = 0$ for both sides of (5) to derive at points such that $|\nabla f| > 0$

$$K|\nabla f| + L|\nabla f| \geq \frac{\langle \nabla Lf, \nabla f \rangle}{|\nabla f|} + \frac{\langle Z, \nabla f \rangle^2}{(n-d)|\nabla f|}.$$

This implies

$$\frac{1}{2}L|\nabla f|^2 - \langle \nabla Lf, \nabla f \rangle \geq -K|\nabla f|^2 + \frac{\langle Z, \nabla f \rangle^2}{n-d}.$$

Combining this with (2.4) we obtain

$$\text{Ric}(\nabla f, \nabla f) - \langle \nabla_{\nabla f} Z, \nabla f \rangle \geq -K|\nabla f|^2 + \frac{\langle Z, \nabla f \rangle^2}{n-d}, \quad f \in C^\infty(M),$$

which is equivalent to (1.2).

(2) implies (6). By the monotone class theorem, we may assume that $f \in C^2(M)$ which is constant outside a compact set. Let $\gamma : [0, 1] \rightarrow M$ be the minimal geodesic from x to y , and let

$$h(s) = \frac{\int_0^s e^{-2K\varphi(r)} dr}{\int_0^t e^{-2K\varphi(r)} dr}, \quad s \in [0, t].$$

By (2) we have

$$\begin{aligned} & \frac{d}{ds} P_{\varphi(s)} \log P_{t-s} f(\gamma_{h(s)}) \\ &= P_{\varphi(s)} \left\{ \varphi'(s) L \log P_{t-s} f - \frac{L P_{t-s} f}{P_{t-s} f} \right\} (\gamma_{h(s)}) + h'(s) \langle \dot{\gamma}_{h(s)}, \nabla P_{\varphi(s)} \log P_{t-s} f(\gamma_{h(s)}) \rangle \\ &\leq P_{\varphi(s)} \left\{ (\varphi'(s) - 1) L \log P_{t-s} f - |\nabla \log P_{t-s} f|^2 \right\} (\gamma_{h(s)}) \\ &\quad + \{ |h'(s)| \rho(x, y) \} |\nabla P_{\varphi(s)} \log P_{t-s} f| (\gamma_{h(s)}) \\ &\leq \left\{ |\varphi'(s) - 1| \cdot |P_{\varphi(s)} L \log P_{t-s} f| - \frac{1 - e^{-2K\varphi(s)}}{Kn} (P_{\varphi(s)} L \log P_{t-s} f)^2 \right\} (\gamma_{h(s)}) \\ &\quad + \left\{ \rho(x, y) h'(s) |\nabla P_{\varphi(s)} \log P_{t-s} f| - e^{-2K\varphi(s)} |\nabla P_{\varphi(s)} \log P_{t-s} f|^2 \right\} (\gamma_{h(s)}) \\ &\leq \frac{e^{2K\varphi(s)} \rho(x, y)^2 h'(s)^2}{4} + \frac{Kn(\varphi'(s) - 1)^2}{4(1 - e^{-2K\varphi(s)})}. \end{aligned}$$

This completes the proof by integrating w.r.t. ds over $[0, t]$.

(6) implies (1.1). For fixed $x \in M$ and strictly positive $f \in C^\infty(M)$ which is constant outside a compact set. Let

$$\varphi(s) = s + \frac{2L(\log f)(x)}{n} s^2, \quad \gamma_s = \exp[-2s \nabla \log f(x)], \quad s \geq 0.$$

According to (6), for small $t > 0$ we have

$$(2.5) \quad P_{\varphi(t)}(\log f)(x) \leq \log P_t f(\gamma_t) + \frac{t^2 |\nabla \log f|^2(x)}{\int_0^t e^{-2K\varphi(s)} ds} + \frac{Kn}{4} \int_0^t \frac{(\varphi'(s) - 1)^2}{1 - e^{-2K\varphi(s)}} ds.$$

According to (3.3) in [22] and noting that $\varphi(t)^2 = t^2 + o(t^2)$, we have

$$(2.6) \quad \begin{aligned} P_{\varphi(t)}(\log f)(x) &= \log f(x) + \varphi(t)L \log f(x) + o(t^2) \\ &+ \frac{t^2}{2} \left\{ \frac{L^2 f}{f} - \frac{(Lf)^2}{f^2} - \frac{2\langle \nabla Lf, \nabla f \rangle}{f^2} - \frac{L|\nabla f|^2}{f^2} + \frac{4|\nabla f|^2 Lf}{f^3} - \frac{6|\nabla f|^4}{f^4} \right\}. \end{aligned}$$

Moreover, according to line 10 on page 310 in [22] and noting that we do not assume $\text{Hess}_f(x) = 0$,

$$(2.7) \quad \begin{aligned} \log P_t f(\gamma_t) &= \log f(x) + t \{ L \log f(x) - |\nabla \log f|^2 \}(x) + o(t^2) \\ &+ \frac{t^2}{2} \left\{ \frac{L^2 f}{f} - \frac{(Lf)^2}{f^2} - \frac{4\langle \nabla Lf, \nabla f \rangle}{f^2} + \frac{4|\nabla f|^2 Lf}{f^3} - \frac{4|\nabla f|^4}{f^4} + \frac{4\text{Hess}_f(\nabla f, \nabla f)}{f^3} \right\}. \end{aligned}$$

Finally, since it is easy to see that

$$\lim_{t \rightarrow 0} \frac{Kn}{4t^2} \int_0^t \frac{(\varphi'(s) - 1)^2}{1 - e^{-2K\varphi(s)}} ds = (L \log f)^2(x),$$

we have

$$(2.8) \quad \frac{Kn}{4} \int_0^t \frac{(\varphi'(s) - 1)^2}{1 - e^{-2K\varphi(s)}} ds = t^2 (L \log f)^2(x) + o(t^2).$$

Substituting (2.6), (2.7) and (2.8) into (2.5), and noting that

$$(\varphi(t) - t)L(\log f)(x) = \frac{2t^2}{n} (L \log f)^2(x),$$

we arrive at

$$\begin{aligned} &\frac{1}{t} \left(1 - \frac{t}{\int_0^t e^{-2K\varphi(s)} ds} \right) |\nabla \log f|^2(x) + \frac{(L \log f)^2(x)}{n} \\ &\leq \frac{1}{2} \left(\frac{L|\nabla f|^2 - 2\langle \nabla Lf, \nabla f \rangle}{f^2} + \frac{2|\nabla f|^4}{f^4} + \frac{4|\text{Hess}_f(\nabla f, \nabla f)|}{f^3} \right)(x) + o(1). \end{aligned}$$

Letting $t \rightarrow 0$ and multiplying both sides by f^2 , we obtain

$$\begin{aligned} &-K|\nabla f|^2(x) + \frac{(Lf - |\nabla f|^2/f)^2(x)}{n} \\ &\leq \left(\frac{1}{2}L|\nabla f|^2 - \langle \nabla Lf, \nabla f \rangle + \frac{|\nabla f|^4}{f^2} + \frac{2|\text{Hess}_f(\nabla f, \nabla f)|}{f} \right)(x). \end{aligned}$$

Replacing f by $f + m$ and letting $m \rightarrow \infty$, this implies that

$$-K|\nabla f|^2(x) + \frac{(Lf)^2(x)}{n} \leq \frac{1}{2}L|\nabla f|^2(x) - \langle \nabla Lf, \nabla f \rangle(x).$$

Therefore, (1.1) holds. \square

Proof of Corollary 1.2. Let $t_0 \in (0, t)$. Taking

$$\varphi(r) = r \wedge \frac{t}{2} + \frac{t+2s}{t} \left(r - \frac{t}{2}\right)^+, \quad r \in [0, t],$$

we have

$$\begin{aligned} \int_0^t e^{-2K\varphi(r)} dr &= \frac{1 - e^{-Kt}}{2K} + \frac{t(e^{-Kt} - e^{-2K(t+s)})}{2K(t+s)} \\ &\geq \frac{t(1 - e^{-2K(t+s)})}{2K(t+2s)}, \end{aligned}$$

and

$$K \int_0^t \frac{(\varphi'(r) - 1)^2}{1 - e^{-2K\varphi(r)}} dr = \frac{4Ks^2}{t^2} \int_{t/2}^t \frac{dr}{1 - \exp[-\frac{2K(t+2s)}{t}(r - \frac{t}{2}) - Kt]} \leq \frac{2Ks^2}{t(1 - e^{-Kt})}.$$

Thus, (1.4) follows from (6).

Next, applying Lemma (6) for $t+s$ in place of t and taking $\varphi(r) = r \wedge t$, we prove (1.5). \square

Proof of Corollary 1.3. When $s = 0$, (1.4) and (1.5) hold for $n = \infty$ (see [22]). Applying e.g. (1.4) to $s = 0$ and $f(z) := p_t(y, z) \wedge m + \varepsilon$ for $m, \varepsilon > 0$ and letting $m \rightarrow \infty, \varepsilon \rightarrow 0$, we obtain

$$\int_M p_t(y, z) \log p_t(y, z) \mu(dz) \leq \log p_{2t}(x, y) + \frac{K\rho(x, y)^2}{2(1 - e^{-2K(t)})}.$$

Since μ is a probability measure and $\int_M p_{t+s}(y, z) \mu(dz) = 1$, by the Jensen inequality this implies

$$p_{2t}(x, y) \geq \exp \left[- \frac{K\rho(x, y)^2}{2(1 - e^{-2Kt})} \right].$$

Replacing t by $\frac{t}{2}$, we prove the desired heat kernel lower bound. \square

Proof of Corollary 1.4. Applying (1.5) for $P_t f^2 + \varepsilon$ in place of f and letting $\varepsilon \rightarrow 0$, we obtain

$$(P_t \log P_t f^2)(x) \leq \log P_{2t+s} f^2(y) + \frac{\rho(x, y)^2 K}{2(1 - e^{-2Kt}) + 4sK e^{-2Kt}} + \frac{Kns}{4(1 - e^{-2Kt})}, \quad s \geq 0.$$

Let $\pi \in \mathcal{C}(f^2 \mu, \mu)$ be the optimal coupling for $W_2^\rho(f^2 \mu, \mu)$, integrating both sides w.r.t. π and noting that due to the Jensen inequality and $\mu(f^2) = 1$ it follows that $\mu(\log P_{2t+s} f^2) \leq 0$, we arrive at

$$(2.9) \quad \mu((P_t f^2) \log P_t f^2) \leq \frac{W_2^2 K}{2(1 - e^{-2Kt}) + 4se^{-2Kt}K} + \frac{Kns}{4(1 - e^{-2Kt})},$$

where and in the remainder of the proof, W_2 stands for $W_2^\rho(f^2\mu, \mu)$ for simplicity. On the other hand, it is well known that (1.1) (indeed, (1.3) implies

$$P_t f^2 \log f^2 \leq (P_t f^2) \log P_t f^2 + \frac{e^{2Kt} - 1}{K} P_t |\nabla f|^2.$$

Integrating both sides w.r.t. μ and using (2.9) we obtain

$$\mu(f^2 \log f^2) \leq \frac{e^{2Kt} - 1}{K} \mu(|\nabla f|^2) + \frac{W_2^2 K}{2(1 - e^{-2Kt}) + 4se^{-2Kt}K} + \frac{Kns}{4(1 - e^{-2Kt})}.$$

Letting $r = 2(e^{2Kt} - 1)/K$ which runs over all $(0, \frac{2}{K^-})$ as t varies in $(0, \infty)$, and using r s to replace s , we get

$$\mu(f^2 \log f^2) \leq r\mu(|\nabla f|^2) + (Kr + 2) \left\{ \frac{W_2^2}{2(1 + 4s)r} + \frac{ns}{4} \right\}, \quad 0 < r \leq \frac{2}{K^-}, s > 0.$$

Taking

$$s = \frac{1}{4} \left(\frac{2\sqrt{2}W_2}{\sqrt{rn}} - 1 \right)^+,$$

we prove (1.9).

To prove (1.10), let

$$\delta = \mu(|\nabla f|^2), \quad r = \frac{W_2}{\sqrt{\delta}}.$$

Since according to [3, 13] one has

$$\frac{K^-}{2} W_2^\rho(f^2\mu, \mu)^2 \leq \mu(f^2 \log f^2) \leq \frac{2}{K^-} \mu(|\nabla f|^2),$$

it is clear that $r \leq \frac{2}{K^-}$. Thus, (1.9) applies to this specific r . Therefore, (1.10) follows by noting that

$$\begin{aligned} & r\delta + \frac{(Kr + 2)W_2}{2r} \left(W_2 \wedge \frac{\sqrt{rn}}{2\sqrt{2}} \right) + \frac{\sqrt{n}(Kr + 2)}{4\sqrt{2r}} \left(W_2 - \frac{\sqrt{rn}}{2\sqrt{2}} \right)^+ \\ &= \delta r + \frac{(Kr + 2)W_2^2}{2r} - \frac{(Kr + 2)W_2}{2r} \left(W_2 - \frac{\sqrt{rn}}{2\sqrt{2}} \right)^+ + \frac{\sqrt{n}(Kr + 2)}{4\sqrt{2r}} \left(W_2 - \frac{\sqrt{rn}}{2\sqrt{2}} \right)^+ \\ &= \delta r + \left(\frac{K}{2} + \frac{1}{r} \right) W_2^2 - \frac{Kr + 2}{2r} \left(W_2 - \frac{\sqrt{rn}}{2\sqrt{2}} \right)^+ \\ &= 2W_2\sqrt{\delta} + \frac{K}{2} W_2^2 - \frac{KW_2 + 2\sqrt{\delta}}{2\sqrt{W_2}} \left(\sqrt{W_2} - \frac{\sqrt{n}}{2\sqrt{2}\delta^{1/4}} \right)^+. \end{aligned}$$

□

Proof of Corollary 1.5. Since $K < 0$, the manifold is compact (cf. [10]). In this case the spectrum of L is discrete so that $\lambda_1 > 0$ and there exists an eigenfunction f with $\mu(f^2) = 1$ and $Lf = -\lambda_1 f$. By Theorem 1.1(2) we have

$$\mu(|\nabla P_t f|^2) \leq e^{2Kt} \mu(|\nabla f|^2) - \frac{e^{2Kt} - 1}{Kn} \mu((P_t Lf)^2), \quad t > 0.$$

For f being the above mentioned eigenfunction, this implies

$$\lambda_1 e^{-2\lambda_1 t} \leq \lambda_1 e^{2Kt} - \lambda_1^2 e^{-2\lambda_1 t} \frac{e^{2Kt} - 1}{Kn}, \quad t > 0.$$

Equivalently,

$$\frac{e^{2(K+\lambda_1)t} - 1}{t} \geq \lambda_1 \frac{e^{2Kt} - 1}{Knt}, \quad t > 0.$$

Letting $t \rightarrow 0$ we obtain the desired lower bound of λ_1 . □

Proof of Proposition 1.6. Since the assertion for $K = 0$ follows from that for $K > 0$ by letting $K \rightarrow 0$, below we only prove the desired inequality for $K < 0$ and $K > 0$ respectively.

(a) Let $K < 0$. Take $\pi \in \mathcal{C}(\mu_1, \mu_2)$ such that $W_1^{\tilde{\rho}}(\mu_1, \mu_2) = \pi(\tilde{\rho})$, and let (X_0, Y_0) be an $M \times M$ -valued random variable with distribution π . Let (X_t, Y_t) be the coupling by reflection of the L -diffusion process with initial data (X_0, Y_0) . This coupling was initiated by Kendall [9] and Cranston [8] (see [20, §2.1] for a complete construction). We have (see [7] or [20, Theorem 2.1.1])

$$(2.10) \quad d\rho(X_t, Y_t) \leq 2\sqrt{2} db_t + I_Z(X_t, Y_t) dt$$

for a one-dimensional Brownian motion b_t and

$$(2.11) \quad I_Z(x, y) := I(x, y) + \langle Z, \nabla \rho(\cdot, y) \rangle(x) + \langle Z, \nabla \rho(x, \cdot) \rangle(y),$$

where letting $\gamma : [0, \rho(x, y)] \rightarrow M$ be the minimal geodesic from x to y and $\{J_i\}_{i=1}^{d-1}$ the Jacobi fields along γ such that at points x, y they together with $\dot{\gamma}$ consist of an orthonormal basis of the tangent space, we have

$$I(x, y) = \sum_{i=1}^{d-1} \int_0^{\rho(x, y)} (|\nabla_{\dot{\gamma}} J_i|^2 - \langle \mathcal{R}(\dot{\gamma}, J_i) \dot{\gamma}, J_i \rangle_s) ds,$$

where \mathcal{R} is the curvature tensor on M .

To calculate $I(x, y)$, let us fix points $x \neq y$ and simply denote $\rho = \rho(x, y)$. Let $\{U_i\}_{i=1}^{d-1}$ be constant vector fields along γ such that $\{\dot{\gamma}, U_i : 1 \leq i \leq d-1\}$ is an orthonormal basis. By the index lemma, for any $f \in C^1([0, \rho])$ with $f(0) = f(\rho) = 1$, we have

$$(2.12) \quad \begin{aligned} I(x, y) &\leq \sum_{i=1}^{d-1} \int_0^\rho (|\nabla_{\dot{\gamma}} f U_i|^2 - f^2 \langle \mathcal{R}(U_i, \dot{\gamma}) \dot{\gamma}, U_i \rangle_s) ds \\ &= \int_0^\rho \{(d-1)f'(s)^2 - f(s)^2 \text{Ric}(\dot{\gamma}, \dot{\gamma})_s\} ds. \end{aligned}$$

On the other hand, since $f(0) = f(\rho) = 1$,

$$\begin{aligned} \langle Z, \nabla \rho(\cdot, y) \rangle(x) + \langle Z, \nabla \rho(x, \cdot) \rangle(y) &= \int_0^\rho \frac{d}{ds} \{f(s)^2 \langle \dot{\gamma}, Z \circ \gamma \rangle_s\} ds \\ &= \int_0^\rho \{2(f f')(s) \langle \dot{\gamma}, Z \circ \gamma \rangle_s + f(s)^2 \langle \nabla_{\dot{\gamma}} Z \circ \gamma, \dot{\gamma} \rangle_s\} ds \\ &\leq \int_0^\rho \left\{ \frac{f(s)^2 \langle \dot{\gamma}, Z \circ \gamma \rangle_s^2}{n-d} + (n-d)f'(s)^2 + f(s)^2 \langle \nabla_{\dot{\gamma}} Z \circ \gamma, \dot{\gamma} \rangle_s \right\} ds. \end{aligned}$$

Combining this with (2.12), (2.11) and (1.2), we obtain

$$(2.13) \quad I_Z(x, y) \leq \int_0^\rho [(n-1)f'(s)^2 + K f(s)^2] ds.$$

Taking

$$f(s) = \tan\left(\frac{\rho}{2} \sqrt{-K/(n-1)}\right) \sin\left(\sqrt{-K/(n-1)} s\right) + \cos\left(\sqrt{-K/(n-1)} s\right)$$

for $s \in [0, \rho]$, we obtain

$$(2.14) \quad I_Z(x, y) \leq -2\sqrt{-K(n-1)} \tan\left(\frac{\rho}{2} \sqrt{-K/(n-1)}\right).$$

Therefore, it follows from (2.10) and the Itô formula that

$$d\tilde{\rho}(X_t, Y_t) \leq dM_t + \frac{nK}{n-1} \tilde{\rho}(X_t, Y_t) dt$$

holds for some martingale M_t . Thus,

$$W_1^{\tilde{\rho}}(\mu_1 P_t, \mu_2 P_t) \leq \mathbb{E} \tilde{\rho}(X_t, Y_t) \leq \exp\left[\frac{nK}{n-1} t\right] \mathbb{E} \tilde{\rho}(X_0, Y_0) = \exp\left[\frac{nK}{n-1} t\right] W_1^{\tilde{\rho}}(\mu_1, \mu_2).$$

(b) When $K > 0$, we take

$$\begin{aligned} f(s) &= \cosh\left(\frac{\rho}{2} \sqrt{K/(n-1)}\right) \sinh\left(\sqrt{K/(n-1)} s\right) \\ &\quad + \frac{1 - \cosh(\rho \sqrt{K/(n-1)})}{\sinh(\rho \sqrt{K/(n-1)})} \sinh\left(s \sqrt{K/(n-1)}\right), \quad s \in [0, \rho]. \end{aligned}$$

It follows from (2.13) that

$$I_Z(x, y) \leq 2\sqrt{K(n-1)} \tanh\left(\frac{\rho(x, y)}{2} \sqrt{K/(n-1)}\right).$$

Combining this with (2.14), we obtain

$$(2.15) \quad I_Z(x, y) = \begin{cases} 2\sqrt{K(n-1)} \tanh\left(\frac{\rho(x, y)}{2} \sqrt{K/(n-1)}\right), & \text{if } K > 0; \\ -2\sqrt{-K(n-1)} \tan\left(\frac{\rho(x, y)}{2} \sqrt{-K/(n-1)}\right), & \text{if } K < 0. \end{cases}$$

Now, let (X_0, Y_0) have distribution π such that $\pi(\tilde{\rho}^p) = W_p^{\tilde{\rho}}(\mu_1, \mu_2)^p$. Using the coupling by parallel displacement rather than by reflection, we have (see [20, Proof of Proposition 2.5.1] or [1])

$$d\rho(X_t, Y_t) \leq I_Z(X_t, Y_t)dt.$$

Combining this with (2.15) we conclude that

$$d\tilde{\rho}(X_t, Y_t) \leq e^{Kt}\tilde{\rho}(X_t, Y_t).$$

Therefore,

$$W_p^{\tilde{\rho}}(\mu_1 P_t, \mu_2 P_t) \leq (\mathbb{E}\tilde{\rho}(X_t, Y_t)^p)^{1/p} \leq e^{Kt}(\mathbb{E}\tilde{\rho}(X_0, Y_0)^p)^{1/p} = e^{Kt}W_p^{\tilde{\rho}}(\mu_1, \mu_2).$$

□

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