Exponential Convergence Rates of Second Quantization Semigroups and Applications *

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Abstract

Exponential convergence rates in the L^2-tail norm and entropy are characterized for the second quantization semigroups by using the corresponding base Dirichlet form. This supplements the well known result on the L^2-exponential convergence rate of second quantization semigroups. As applications, birth-death type processes on Poisson spaces and the path space of Lévy processes are investigated.

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1 Introduction

Let E be a Polish space with Borel σ-field $\mathscr{F}$. Let μ be a non-trivial σ-finite measure on $(E, \mathscr{F})$. Let $(\mathscr{E}_0, \mathscr{D}(\mathscr{E}_0))$ be a symmetric Dirichlet form on $L^2(\mu)$. Consider the configuration space

$$\Gamma := \Big\{\gamma = \sum_i \delta_{x_i} \text{ (at most countable)} : \ x_i \in E\Big\},$$

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where δ_x is the Dirac measure at x and $\sum_\emptyset$ is regarded as the zero measure 0 on E. Let $\mathscr{F}_\Gamma$ be the σ-field induced by $\{\gamma \mapsto \gamma(A) : A \in \mathscr{F}\}$. The Poisson measure with intensity μ, denoted by π_μ, is the unique probability measure on $(\Gamma, \mathscr{F}_\Gamma)$ such that for any disjoint sets $A_1, \cdots, A_n \in \mathscr{F}$ with $\mu(A_i) < \infty$, $1 \le i \le n$,

$$\pi_\mu\big(\{\gamma \in \Gamma : \ \gamma(A_i) = k_i, 1 \le i \le n\}\big) = \prod_{i=1}^n \mathrm{e}^{-\mu(A_i)} \frac{\mu(A_i)^{k_i}}{k_i!}, \quad k_i \in \mathbb{Z}_+, \ 1 \le i \le n.$$

This measure has the Laplace transform

$$\pi_\mu(\mathrm{e}^{\langle \cdot, f\rangle}) = \exp\left[\mu(\mathrm{e}^f - 1)\right], \quad f \in L^1(\mu) \cap L^\infty(\mu), \tag{1.1}$$

where $\langle \gamma, f\rangle := \gamma(f) = \int_E f \, \mathrm{d}\gamma$.

The second quantization of $(\mathscr{E}_0, \mathscr{D}(\mathscr{E}_0))$ is a symmetric conservative Dirichlet form on $L^2(\pi_\mu)$ given by (see e.g. [13, Lemma 6.3])

$$\begin{aligned} \mathscr{D}(\mathscr{E}) := \Big\{ F \in L^2(\pi_\mu) : \ D_\cdot F(\gamma) := F(\gamma + \delta_\cdot) - F(\gamma) \in \mathscr{D}_e(\mathscr{E}_0), \ \pi_\mu\text{-a.e. } \gamma, \\ \mathscr{E}_0(D_\cdot F, D_\cdot F) \in L^1(\pi_\mu) \Big\}, \\ \mathscr{E}(F, G) := \int_\Gamma \mathscr{E}_0(D_\cdot F(\gamma), D_\cdot G(\gamma)) \pi_\mu(\mathrm{d}\gamma), \quad F, G \in \mathscr{D}(\mathscr{E}), \end{aligned}$$

where $\mathscr{D}_e(\mathscr{E}_0)$ is the extended domain of $\mathscr{E}_0$ (see [1]).

Let P_t^0 and P_t be the semigroups associated to $(\mathscr{E}_0, \mathscr{D}(\mathscr{E}_0))$ on $L^2(\mu)$ and $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ on $L^2(\pi_\mu)$ respectively. We aim to investigate the convergence rate of P_t to π_μ as $t \to \infty$ by using properties of the base Dirichlet form.

We would like to consider the following three kinds of exponential convergence rates:

(1) **Exponential convergence in the L^2-norm:** let λ_L be the largest constant such that
$$\|P_t - \pi_\mu\|_{L^2(\pi_\mu) \to L^2(\pi_\mu)} \le \mathrm{e}^{-\lambda_L t}, \quad t \ge 0,$$
where π_μ is regarded as a linear operator from $L^2(\pi_\mu)$ to $\mathbb{R}$ by letting $\pi_\mu(F) = \int_\Gamma F \mathrm{d}\pi_\mu$.

(2) **Exponential convergence in the L^2-tail norm:** let λ_T be the largest constant such that
$$\|P_t\|_T := \lim_{n \to \infty} \sup_{\pi_\mu(F^2) \le 1} \|1_{\{|P_t F| \ge n\}} P_t F\|_{L^2(\pi_\mu)} \le \mathrm{e}^{-\lambda_T t}, \quad t \ge 0.$$

(3) **Exponential convergence in entropy:** let λ_E be the largest constant such that
$$\pi_\mu((P_t F) \log P_t F) \le \pi_\mu(F \log F) \mathrm{e}^{-\lambda_E t}, \quad t \ge 0, F \ge 0, \pi_\mu(F) = 1.$$

The exponential convergence rate in the L^2-norm is already well described by the exponential decay rate of P_t^0, i.e. (see [8])

$$\lambda_L = \lambda_{L,0} := \inf\big\{\mathscr{E}_0(f,f) :\ f\in\mathscr{D}(\mathscr{E}_0), \mu(f^2)=1\big\}. \tag{1.2}$$

It is well known that $\lambda_{L,0}$ is the largest number such that

$$\|P_t^0 f\|_{L^2(\mu)} \le \|f\|_{L^2(\mu)}\mathrm{e}^{-\lambda_{L,0}t}, \quad t\ge 0, f\in L^2(\mu)$$

holds. See [7] and [13] for a criterion of the weak Poincaré inequality for second quantization Dirichlet forms.

Due to the above fact, in this paper we will only consider λ_T and λ_E. To study these two quantities, we first describe them by using the Dirichlet form.

Since π_μ is a probability measure, by [10, Theorem 3.3] for $\phi\equiv 1$ we conclude that λ_T is the largest number such that for any $C_1 > \lambda_T^{-1}$ the defective Poincaré inequality

$$\pi_\mu(F^2)\le C_1\mathscr{E}(F,F) + C_2\pi_\mu(|F|)^2, \quad F\in\mathscr{D}(\mathscr{E})$$

holds for some constant $C_2>0$. Consequently,

$$\lambda_T = \lim_{n\to\infty}\inf\big\{\mathscr{E}(F,F)+n\pi_\mu(|F|)^2 :\ F\in\mathscr{D}(\mathscr{E}), \pi_\mu(F^2)=1\big\}. \tag{1.3}$$

The quantity λ_T is also related to the essential spectrum $\sigma_{\mathrm{ess}}(\mathscr{L})$ of the generator $\mathscr{L}$ associated to $(\mathscr{E},\mathscr{D}(\mathscr{E}))$. Precisely, we have

$$\lambda_T \ge \inf\sigma_{\mathrm{ess}}(-\mathscr{L})$$

and the equality holds provided for some $t>0$ the operator P_t has an asymptotic density w.r.t. π_μ (see [11, Theorem 3.2.2]).

Next, it is easy to check that λ_E is the largest number such that the L^1 log-Sobolev inequality

$$\begin{aligned}\mathrm{Ent}_{\pi_\mu}(F) &:= \pi_\mu(F\log F) - \pi_\mu(F)\log\pi_\mu(F)\\ &\le \frac{1}{\lambda_E}\mathscr{E}(F,\log F), \quad F\in\mathscr{D}(\mathscr{E}), \inf F>0\end{aligned} \tag{1.4}$$

holds. That is (see [14, Theorem 1.1]),

$$\lambda_E = \inf\left\{\frac{\mathscr{E}(F,\log F)}{\mathrm{Ent}_{\pi_\mu}(F)} :\ \inf F>0, F\in\mathscr{D}(\mathscr{E}), \mathrm{Ent}_{\pi_\mu}(F)>0\right\}. \tag{1.5}$$

We remark that for $F\in\mathscr{D}(\mathscr{E})$ with $\inf F>0$, one has $\log F\in\mathscr{D}(\mathscr{E})$ so that $\mathscr{E}(F,\log F)$ exists.

Finally, we would like to mention that the log-Sobolev inequality introduced in [2]

$$\mathrm{Ent}_{\pi_\mu}(F^2) \le C\mathscr{E}(F,F), \quad F \in \mathscr{D}(\mathscr{E}) \tag{1.6}$$

for some constant $C > 0$ implies that $\lambda_E \ge 4/C$ (see e.g. [14, Theorem 1.2]). But it is easy to see that the second quantization Dirichlet form does not satisfy the log-Sobolev inequality (see [9] and the first page of [12]). Indeed, given nonnegative function $f \in L^\infty(\mu) \cap L^1(\mu) \cap \mathscr{D}(\mathscr{E}_0)$, applying (1.6) to $F(\gamma) := \mathrm{e}^{\gamma(f)}$ and using (1.1) we obtain

$$\int_E (2f\mathrm{e}^{2f} - \mathrm{e}^{2f} + 1)\,\mathrm{d}\mu \le C\mathscr{E}_0\big(\mathrm{e}^f - 1, \mathrm{e}^f - 1\big).$$

Replacing f by $\log(nf+1)$ which is once again in $L^\infty(\mu) \cap L^1(\mu) \cap \mathscr{D}(\mathscr{E}_0)$, we obtain

$$\frac{1}{n^2 \log n}\int_E \big\{2(nf+1)^2\log(nf+1) - (nf+1)^2 + 1\big\}\,\mathrm{d}\mu \le \frac{C}{\log n}\mathscr{E}_0(f,f).$$

Letting $n \to \infty$ we arrive at $\mu(f^2) \le 0$ which is impossible if f is non-trivial.

It is now the place to state our main result of the paper where λ_E and λ_T are described by using the base Dirichlet form $(\mathscr{E}_0, \mathscr{D}(\mathscr{E}_0))$.

Theorem 1.1. *We have*

$$\lambda_E = \inf\left\{\frac{\mathscr{E}_0\big(\mathrm{e}^f - 1, f\big)}{\mu(f\mathrm{e}^f - \mathrm{e}^f + 1)} :\ f \in \mathscr{D}(\mathscr{E}_0) \cap L^\infty(\mu), \mu(f^2) > 0\right\} \tag{1.7}$$

and

$$\lambda_{L,0} \le \lambda_T \le \lambda_{T,0} := \lim_{n\to\infty} \inf\big\{\mathscr{E}_0(f,f) + n\mu(|f|)^2 :\ f \in \mathscr{D}(\mathscr{E}_0), \mu(f^2) = 1\big\}. \tag{1.8}$$

To derive the exact value of these two quantities, let us decompose the Dirichlet form $\mathscr{E}_0$ into three parts: the diffusion part, the jump part and the killing part. We will see in the next result that in many cases λ_E is determined merely by the killing term.

Let W be a nonnegative measurable function on E, $\mathscr{A} \subset L^1(W\mu) \cap L^\infty(\mu)$ be a linear subspace, $q \ge 0$ be a symmetric measurable function on $E \times E$, and $\Gamma_1 : \mathscr{A} \times \mathscr{A} \to L^1(\mu)$ be a nonnegative definite bilinear map such that

(i) $\mathscr{A}$ is dense in $L^2\big((1+W)\mu\big)$;

(ii) If $f \in \mathscr{A}$ and $\phi : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous with $\phi(0) = 0$, then $\phi(f) \in \mathscr{A}$;

(iii) For any $f \in \mathscr{A}$, $\int_{E\times E} |f(x) - f(y)|^2 q(x,y)\mu(\mathrm{d}x)\mu(\mathrm{d}y) < \infty$;

(iv) $\Gamma_1(f,\phi(g)) = \phi'(g)\Gamma_1(f,g)$ holds for any $\phi \in C^1(\mathbb{R})$ with $\phi(0)=0$ and any $f,g \in \mathscr{A}$.

Consider the following diffusion-jump type quadric form with potential:

$$\begin{aligned}\mathscr{E}_0(f,g) &:= \mu\big(\Gamma_1(f,g) + Wfg\big) \\ &\quad + \frac{1}{2}\int_{E\times E} (f(x)-f(y))(g(x)-g(y))q(x,y)\mu(\mathrm{d}x)\mu(\mathrm{d}y), \quad f,g\in\mathscr{A}.\end{aligned} \tag{1.9}$$

Assume that $(\mathscr{E}_0, \mathscr{A})$ is closable such that its closure $(\mathscr{E}_0, \mathscr{D}(\mathscr{E}_0))$ is a Dirichlet form on $L^2(\mu)$. When $\Gamma_1 = 0$, $q = 0$ and $W \equiv 1$, the framework goes back to [12] where the Poincaré inequality and the L^1 log-Sobolev inequality with constant 1 are proved. The contribution of our next result is to confirm that these inequalities are sharp under a more general framework.

Corollary 1.2. *Let $(\mathscr{E}_0, \mathscr{D}(\mathscr{E}_0))$ be given in (1.9) such that* (i)–(iv) *hold.*

(1) *If there exists a sequence of nonnegative functions $\{f_n\}_{n\geq 1} \subset \mathscr{A}$ such that $\{f_n > 0\} \uparrow E$ as $n \uparrow \infty$, then $\lambda_E = \mathrm{ess}_\mu \inf W$.*

(2) *Let $\Gamma_1 = 0$ and $q = 0$, and let μ be finite on bounded sets. If $\mathrm{supp}\mu \cap \{W < \varepsilon\}$ is uncountable whenever $\mu(W<\varepsilon) > 0$ (it is the case if μ does not have atom), then $\lambda_L = \lambda_T = \mathrm{ess}_\mu \inf W$.*

To conclude this section, we present below an example to illustrate Corollary 1.2(1).

Example 1.1. Let E be a connected (not necessarily complete) Riemannian manifold and V a locally bounded measurable function. Let $\mu(\mathrm{d}x) = \mathrm{e}^{V(x)}\mathrm{d}x$ with $\mathrm{d}x$ the volume measure. Then we take $\mathscr{A}$ to be the set of all Lipschitz continuous functions on E with compact supports. It is trivial that conditions (i) and (ii) hold and $\mathscr{A} \subset L^1(W\mu) \cap L^\infty(\mu)$ provided W is locally bounded. Define

$$\Gamma_1(f,g) = \langle \nabla f, \nabla g\rangle, \quad f,g \in \mathscr{A}.$$

Then condition (iv) holds. Finally, let $\rho(x,y)$ be the Riemannian distance between x and y. If $q(x,y)$ satisfies

$$\int_{K\times E} \big(\rho(x,y)^2 \wedge 1\big) q(x,y)\mu(\mathrm{d}x)\mu(\mathrm{d}y) < \infty \tag{1.10}$$

for any compact subset K of E, then (iii) is satisfied. Thus, by Corollary 1.2(1) where the required sequence $\{f_n\}_{n\geq 1}$ automatically exists according to the definition of $\mathscr{A}$, we have

$$\lambda_E = \mathrm{ess}_\mu \inf W.$$

In particular, let μ be the Lebesgue measure and E a bounded open domain in $\mathbb{R}^d$ (it is complete under a compatible metric), a typical choice of $q(x,y)$ such that (1.10) holds is $\frac{1}{|x-y|^{\alpha+d-1}}$ for $\alpha \in [0,2)$. Moreover, if $E=\mathbb{R}^d$ and $\mu(\mathrm{d}x)=\mathrm{d}x$, then (1.10) holds for this $q(x,y)$ with $\alpha \in (1,2)$.

The remainder of the paper is organized as follows. In Section 2 complete proofs of Theorem 1.1 and Corollary 1.2 are presented; In Section 3 the exponential convergence rates are considered for birth-death type Dirichlet forms on $L^2(\pi_\mu)$ with a weighted function on $\Gamma \times E$; and in Section 4 results derived in Section 3 are applied to the path space of Lévy processes by following the line of [12].

2 Proofs of Theorem 1.1 and Corollary 1.2

Proof of (1.7). We first remark that for any $f \in \mathscr{D}(\mathscr{E}_0)\cap L^\infty(\mu)$ one has $\mathrm{e}^f-1\in\mathscr{D}(\mathscr{E}_0)$, since the function $\phi(r):=\mathrm{e}^r-1$ is locally Lipschitz continuous and $\phi(0)=0$. Therefore, it suffices to show that for any $\lambda>0$, the L^1 log-Sobolev inequality

$$\text{(2.1)} \qquad \mathrm{Ent}_{\pi_\mu}(F) \le \frac{1}{\lambda}\mathscr{E}(F,\log F), \quad F\in\mathscr{D}(\mathscr{E}), \inf F>0$$

is equivalent to

$$\text{(2.2)} \qquad \mu\big(f\mathrm{e}^f-\mathrm{e}^f+1\big) \le \frac{1}{\lambda}\mathscr{E}_0\big(\mathrm{e}^f-1,f\big), \quad f\in\mathscr{D}(\mathscr{E}_0)\cap L^\infty(\mu).$$

(a) (2.2) implies (2.1). It suffices to prove (2.1) for $F\in\mathscr{D}(\mathscr{E})\cap L^\infty(\pi_\mu)$ with $\inf F>0$. In this case we have $g_\gamma := \frac{F(\gamma+\delta_\cdot)}{F(\gamma)}-1 \in \mathscr{D}_e(\mathscr{E}_0)$ for π_μ-a.e. $\gamma\in\Gamma$. Since $\frac{\sup F}{\inf F}\ge g_\gamma+1>0$, it follows that

$$f_\gamma := \log(g_\gamma+1)\in\mathscr{D}_e(\mathscr{E}_0)\cap L^\infty(\mu)$$

for π_μ-a.e. $\gamma\in\Gamma$. By (2.2) which holds also for $f\in\mathscr{D}_e(\mathscr{E}_0)\cap L^\infty(\mu)$, we have

$$\text{(2.3)} \qquad \lambda\int_E (f_\gamma \mathrm{e}^{f_\gamma}-\mathrm{e}^{f_\gamma}+1)\,\mathrm{d}\mu \le \mathscr{E}_0\big(\mathrm{e}^{f_\gamma}-1,f_\gamma\big)=\mathscr{E}_0\big(g_\gamma,\log(g_\gamma+1)\big).$$

On the other hand, by the modified log-Sobolev inequality presented in [12, Theorem 1.1] (note that $\Phi(r)=r\log r$ therein), it holds that

$$\text{(2.4)} \qquad \mathrm{Ent}_{\pi_\mu}(F)\le\int_\Gamma \pi_\mu(\mathrm{d}\gamma)\int_E\big\{D_z(F\log F)(\gamma)-\big(1+\log F(\gamma)\big)D_zF(\gamma)\big\}\mu(\mathrm{d}z).$$

Since

$$D_zF(\gamma)=F(\gamma)\big(\mathrm{e}^{f_\gamma}(z)-1\big), \quad \log\frac{F(\gamma+\delta_z)}{F(\gamma)}=f_\gamma(z),$$

it is not hard to verify that

$$\begin{aligned}
&D_z(F\log F)(\gamma) - \big(1+\log F(\gamma)\big)D_zF(\gamma) = F(\gamma+\delta_z)\log\frac{F(\gamma+\delta_z)}{F(\gamma)} - D_zF(\gamma)\\
&= (D_zF(\gamma))\Big(\log\frac{F(\gamma+\delta_z)}{F(\gamma)} - 1\Big) + F(\gamma)\log\frac{F(\gamma+\delta_z)}{F(\gamma)}\\
&= F(\gamma)\big\{\big(\mathrm{e}^{f_\gamma}-1\big)(f_\gamma-1)+f_\gamma\big\}(z) = F(\gamma)\big(f_\gamma\mathrm{e}^{f_\gamma}-\mathrm{e}^{f_\gamma}+1\big)(z).
\end{aligned}$$

Combining this with (2.3) and (2.4), we obtain

$$\begin{aligned}
\lambda\,\mathrm{Ent}_{\pi_\mu}(F) &\le \lambda\int_\Gamma F(\gamma)\pi_\mu(\mathrm{d}\gamma)\int_E (f_\gamma\mathrm{e}^{f_\gamma}-\mathrm{e}^{f_\gamma}+1)\mathrm{d}\mu\\
&\le \int_\Gamma F(\gamma)\mathscr{E}_0(g_\gamma,\log(g_\gamma+1))\pi_\mu(\mathrm{d}\gamma)\\
&= \int_\Gamma \mathscr{E}_0(D.F, D.\log F)\mathrm{d}\pi_\mu = \mathscr{E}(F,\log F).
\end{aligned}$$

(b) (2.1) implies (2.2). We first consider $f\in\mathscr{D}(\mathscr{E}_0)\cap L^\infty(\mu)\cap L^1(\mu)$. Let $F(\gamma)=\mathrm{e}^{\gamma(f)}$. By (1.1) we have $F\in L^2(\pi_\mu)$ and

$$\mathrm{Ent}_{\pi_\mu}(F) = \pi_\mu(F)\int_E (f\mathrm{e}^f - \mathrm{e}^f+1)\,\mathrm{d}\mu. \tag{2.5}$$

Moreover, for any $\varepsilon>0$ one has $F+\varepsilon\in\mathscr{D}(\mathscr{E})$, $\inf(F+\varepsilon)>0$ and

$$\begin{aligned}
&\mathscr{E}\big(F+\varepsilon,\log(F+\varepsilon)\big)\\
&\quad= \int_\Gamma F(\gamma)\left\{\mathscr{E}_0\big(\mathrm{e}^f-1,f\big)+\mathscr{E}_0\Big(\mathrm{e}^f-1,\log\frac{\mathrm{e}^{\gamma(f)}+\varepsilon\mathrm{e}^{-f}}{\mathrm{e}^{\gamma(f)}+\varepsilon}\Big)\right\}\pi_\mu(\mathrm{d}\gamma).
\end{aligned}\tag{2.6}$$

Since $\phi(s):=\log\frac{\mathrm{e}^{\gamma(f)}+\varepsilon\mathrm{e}^{-s}}{\mathrm{e}^{\gamma(f)}+\varepsilon}$ satisfies $\phi(0)=0$ and $|\phi'(s)|\le 1$, we get

$$\begin{aligned}
\left|\mathscr{E}_0\Big(\mathrm{e}^f-1,\log\frac{\mathrm{e}^{\gamma(f)}+\varepsilon\mathrm{e}^{-f}}{\mathrm{e}^{\gamma(f)}+\varepsilon}\Big)\right| &\le \sqrt{\mathscr{E}_0\big(\mathrm{e}^f-1,\mathrm{e}^f-1\big)\mathscr{E}_0\big(\phi(f),\phi(f)\big)}\\
&\le \sqrt{\mathscr{E}_0\big(\mathrm{e}^f-1,\mathrm{e}^f-1\big)\mathscr{E}_0(f,f)}<\infty.
\end{aligned}$$

Thus, by (2.6) and the dominated convergence theorem we arrive at

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\mathscr{E}\big(F+\varepsilon,\log(F+\varepsilon)\big)\\
&= \int_\Gamma F(\gamma)\mathscr{E}_0\big(\mathrm{e}^f-1,f\big)\pi_\mu(\mathrm{d}\gamma)+\int_\Gamma F(\gamma)\lim_{\varepsilon\downarrow 0}\mathscr{E}_0\Big(\mathrm{e}^f-1,\log\frac{\mathrm{e}^{\gamma(f)}+\varepsilon\mathrm{e}^{-f}}{\mathrm{e}^{\gamma(f)}+\varepsilon}\Big)\pi_\mu(\mathrm{d}\gamma)\\
&= \pi_\mu(F)\mathscr{E}_0\big(\mathrm{e}^f-1,f\big).
\end{aligned}\tag{2.7}$$

Therefore, first applying (2.1) to $F+\varepsilon$ then letting $\varepsilon \downarrow 0$, we obtain (2.2) from (2.5) and (2.7).

In general, for any $f \in \mathscr{D}(\mathscr{E}_0) \cap L^\infty(\mu)$, let

$$f_n = \left(f - \frac{1}{n}\right)^+ - \left(f + \frac{1}{n}\right)^-, \quad n \geq 1.$$

Then it is easy to see that $f_n \in \mathscr{D}(\mathscr{E}_0) \cap L^\infty(\mu) \cap L^1(\mu)$ and $f_n \to f$ in $\mathscr{D}(\mathscr{E}_0) \cap L^\infty(\mu)$. Therefore, (2.2) holds. □

Proof of (1.8). Since it is well known that

$$\lambda_L = \inf\{\mathscr{E}(F,F) : F \in \mathscr{D}(\mathscr{E}), \pi_\mu(F^2) - \pi_\mu(F)^2 = 1\},$$

(1.2) and (1.3) imply $\lambda_T \geq \lambda_{L,0}$. So, it remains to prove $\lambda_T \leq \lambda_{T,0}$. If $0 < \lambda < \lambda_T$, then there exists $C > 0$ such that

$$\pi_\mu(F^2) \leq \frac{1}{\lambda}\mathscr{E}(F,F) + C\pi_\mu(F)^2, \quad F \in \mathscr{D}(\mathscr{E}), F \geq 0. \tag{2.8}$$

For any $f \in \mathscr{D}(\mathscr{E}_0)$, letting $F(\gamma) = \gamma(|f|)$ we have $\mathscr{E}(F,F) = \mathscr{E}_0(|f|,|f|) \leq \mathscr{E}_0(f,f)$ and (see e.g. [7, Proof of Lemma 7.2])

$$\pi_\mu(F^2) = \mu(f^2) + \mu(|f|)^2, \ \ \pi_\mu(F) = \mu(|f|).$$

Therefore, it follwos from (2.8) that

$$\mu(f^2) \leq \frac{1}{\lambda}\mathscr{E}_0(f,f) + (C-1)\mu(|f|)^2, \quad f \in \mathscr{D}(\mathscr{E}_0).$$

This implies that $\lambda_{T,0} \geq \lambda$ holds for any $\lambda < \lambda_T$. Hence, $\lambda_T \leq \lambda_{T,0}$. □

To prove Corollary 1.2, we need the following fundamental lemma. We include a simple proof for completeness.

Lemma 2.1. *Let ν be a measure on E such that ν is finite on bounded sets. If there exists a constant $c > 0$ such that $\nu(f^2) \leq c\nu(|f|)^2$ holds for all $f \in L^2(\nu)$, then* suppν *is at most countable. If moreover $\nu(E) < \infty$ then* suppν *is finite.*

Proof. Since ν is finite on bounded sets and E is separable, there exists a sequence of open sets $\{G_n\}_{n\geq 1}$ such that $\cup_{n\geq 1} G_n = E$ and $\nu(G_n) < \infty$ for $n \geq 1$. Now we fix $n \geq 1$. Suppose there are m many different points $\{x_i\}_{i=1}^m$ in supp$\nu \cap G_n$, where $m \geq 1$. For each i there exists $r_i > 0$ such that $B_i := \{x : d(x,x_i) < r_i\} \subset G_n$ and $\{B_i\}_{i=1}^m$ are disjoint.

Since x_i is in the support of ν, we have $\nu(B_i) > 0$ for each $i \in \{1, \cdots, m\}$. Moreover, since

$$\sum_{i=1}^{m} \nu(B_i) = \nu\left(\bigcup_{i=1}^{m} B_i\right) \le \nu(G_n) < \infty,$$

there exists $i_0 \in \{1, \cdots, m\}$ such that

$$0 < \nu(B_{i_0}) \le \frac{1}{m}\nu(G_n).$$

But applying $\nu(f^2) \le c\nu(|f|)^2$ to $f = 1_{B_{i_0}}$ we obtain $\nu(B_{i_0}) \ge 1/c$. Therefore, $m \le c\nu(G_n)$. This means that for each fixed $n \ge 1$ the set $\mathrm{supp}\nu \cap G_n$ is finite, so that $\mathrm{supp}\nu$ is at most countable. The second assertion follows from the same argument by taking $G_n = E$. □

Proof of Corollary 1.2 (1). Since for any $r \in \mathbb{R}$ one has

$$r(\mathrm{e}^r - 1) \ge r\mathrm{e}^r - \mathrm{e}^r + 1,$$

it holds that

$$\begin{aligned}\mathscr{E}_0\big(\mathrm{e}^f - 1, f\big) &\ge \int_E W f(\mathrm{e}^f - 1)\,\mathrm{d}\mu \ge (\mathrm{ess}_\mu \inf W) \int_E f(\mathrm{e}^f - 1)\,\mathrm{d}\mu \\ &\ge (\mathrm{ess}_\mu \inf W) \int_E (f\mathrm{e}^f - \mathrm{e}^f + 1)\,\mathrm{d}\mu.\end{aligned}$$

Therefore, it follows from (1.7) that $\lambda_E \ge \mathrm{ess}_\mu \inf W$.

On the other hand, let $g \in \mathscr{A}$ be a fixed nonnegative function. For any $n \ge 1$, applying (1.7) to $f := 2\log(ng+1) \in \mathscr{A} \subset \mathscr{D}(\mathscr{E}_0) \cap L^\infty(\mu)$ and noting that by (iv)

$$\Gamma_1\big((ng+1)^2 - 1, 2\log(ng+1)\big) = 4n^2\Gamma_1(g,g),$$

we obtain

$$\begin{aligned}&\lambda_E \int_E \big\{(ng+1)^2 \log\big[(ng+1)^2\big] - (ng+1)^2 + 1\big\}\,\mathrm{d}\mu \\ &\le \mathscr{E}_0\big((ng+1)^2 - 1, 2\log(ng+1)\big) \\ &= \int_E \big\{4n^2\Gamma_1(g,g) + W(n^2g^2 + 2ng)\log\big[(ng+1)^2\big]\big\}\,\mathrm{d}\mu \\ &\quad + \int_{E\times E} \big\{(ng(x)+1)^2 - (ng(y)+1)^2\big\}\Big(\log\frac{ng(x)+1}{ng(y)+1}\Big) q(x,y)\mu(\mathrm{d}x)\mu(\mathrm{d}y).\end{aligned}$$

Multiplying both sides by $\frac{1}{n^2 \log n}$ and letting $n \to \infty$, by the dominated convergence theorem we arrive at

$$2\mu(g^2(\lambda_E - W)) \le \limsup_{n\to\infty} \int_{E\times E} G_n(x,y)q(x,y)\mu(\mathrm{d}x)\mu(\mathrm{d}y), \tag{2.9}$$

where

$$\begin{aligned} 0 \le G_n(x,y) &:= \frac{(ng(x)+1)^2 - (ng(y)+1)^2}{n^2\log n} \log\frac{ng(x)+1}{ng(y)+1} \\ &\le \frac{(ng(x)+ng(y)+2)\log(n[g(x)\vee g(y)]+1)}{n\log n}|g(x)-g(y)| \\ &\le c|g(x)-g(y)| \end{aligned}$$

for μ-a.e. $x, y \in E$ and some constant $c > 0$ since $g \in L^\infty(\mu)$. Thus, by (iii) and the dominated convergence theorem it follows that

$$\begin{aligned} &\lim_{n\to\infty} \int_{E\times E} G_n(x,y)q(x,y)\mu(\mathrm{d}x)\mu(\mathrm{d}y) \\ &= \int_{E\times E} \lim_{n\to\infty} G_n(x,y)q(x,y)\mu(\mathrm{d}x)\mu(\mathrm{d}y) \\ &= \int_{E\times E} \big(g(x)^2 - g(y)^2\big)\big(1_{\{g>0\}}(x) - 1_{\{g>0\}}(y)\big)q(x,y)\mu(\mathrm{d}x)\mu(\mathrm{d}y). \end{aligned}$$

Combining this with (2.9) and using the symmetry of $q(x,y)$ we get

$$\begin{aligned} &\mu(g^2(\lambda_E - W)) \\ &\le \frac{1}{2}\int_{E\times E} \big(g(x)^2 - g(y)^2\big)\big(1_{\{g>0\}}(x) - 1_{\{g>0\}}(y)\big)q(x,y)\mu(\mathrm{d}x)\mu(\mathrm{d}y) \\ &= \int_{\{g>0\}\times\{g=0\}} g(x)^2 q(x,y)\mu(\mathrm{d}x)\mu(\mathrm{d}y), \quad g \in \mathscr{A}, g \ge 0. \end{aligned} \tag{2.10}$$

Next, let $E_n = \{f_n > 0\}$. For any $n, m \ge 1$, applying (2.10) to $g_{nm} := g + f_n/m$ we have

$$\begin{aligned} &\mu(g_{nm}^2(\lambda_E - W)) \\ &\le \int_{(E_n\cup\{g>0\})\times(E_n^c\cap\{g=0\})} g_{nm}(x)^2 q(x,y)\mu(\mathrm{d}x)\mu(\mathrm{d}y) \\ &\le \Big(\|g\|_\infty + \frac{\|f_n\|_\infty}{m}\Big)\int_{\{g>0\}\times(E_n^c\cap\{g=0\})} \Big\{|g(x)-g(y)| \\ &\qquad\qquad + \frac{1}{m}|f_n(x)-f_n(y)|\Big\} q(x,y)\mu(\mathrm{d}x)\mu(\mathrm{d}y) \\ &\quad + \frac{1}{m^2}\|f_n\|_\infty \int_{(E_n\setminus\{g>0\})\times(E_n^c\cap\{g=0\})} |f_n(x)-f_n(y)|q(x,y)\mu(\mathrm{d}x)\mu(\mathrm{d}y). \end{aligned}$$

It follows by letting $m \to \infty$ that

$$\mu(g^2(\lambda_E - W)) \le \|g\|_\infty \int_{\{g>0\}\times E_n^c} |g(x) - g(y)| q(x,y)\mu(\mathrm{d}x)\mu(\mathrm{d}y).$$

Finally, letting $n \to \infty$ we conclude that $\mu(g^2(\lambda_E - W)) \le 0$ for any nonnegative $g \in \mathscr{A}$. Since $\phi(x) = |x|$ is Lipschitz continuous with $\phi(0) = 0$, it holds that $\mu(g^2(\lambda_E - W)) \le 0$ for any $g \in \mathscr{A}$. Noting that $\mathscr{A}$ is dense in $L^2\big((1+W)\mu\big)$, then it is trivial to see that $\lambda_E \le \mathrm{ess}_\mu \inf W$. This completes the proof. □

Proof of Corollary 1.2 (2). Let $\Gamma_1 = 0$ and $q = 0$. Then $\mathscr{E}_0(f,g) = \mu(Wfg)$. In this case, we have

$$\lambda_{L,0} = \inf_{f\in L^2(\mu),\mu(f^2)>0} \frac{\mu(Wf^2)}{\mu(f^2)} = \mathrm{ess}_\mu \inf W.$$

So, by Theorem 1.1, it suffices to show that $\lambda_{T,0} \le \mathrm{ess}_\mu \inf W$. If $\lambda_{T,0} > \mathrm{ess}_\mu \inf W$ then there exist $0 < r < \{\mathrm{ess}_\mu \inf W\}^{-1}$ and $c > 0$ such that

$$\mu(f^2) \le r\mathscr{E}_0(f,f) + c\mu(|f|)^2 = r\mu(Wf^2) + c\mu(|f|)^2, \quad f \in L^2(\mu) \tag{2.11}$$

holds. Take $\varepsilon \in (0, r^{-1})$ such that $\mu(W < \varepsilon) > 0$. Let $\mu_\varepsilon = 1_{\{W<\varepsilon\}}\mu$. Using $f1_{\{W<\varepsilon\}}$ to replace f, we obtain from (2.11) that

$$\mu_\varepsilon(f^2) \le \frac{c}{1 - r\varepsilon}\mu_\varepsilon(|f|)^2, \quad f \in L^2(\mu_\varepsilon).$$

Thus, according to Lemma 2.1 $\mathrm{supp}\mu_\varepsilon$ is at most countable. This is contradictive to the assumption that $\mathrm{supp}\mu \cap \{W < \varepsilon\}$ is uncountable. □

3 Birth-death type Dirichlet forms on $L^2(\pi_\mu)$

Let ψ be a nonnegative measurable function on $\Gamma \times E$ such that

$$\psi_\mu(z) := \int_\Gamma \psi(\gamma, z)\pi_\mu(\mathrm{d}\gamma) < \infty, \quad \mu\text{-a.e. } z \in E.$$

Consider the quadric form

$$\mathscr{E}^\psi(F,G) := \int_{\Gamma\times E} \big(F(\gamma+\delta_z) - F(\gamma)\big)\big(G(\gamma+\delta_z) - G(\gamma)\big)\psi(\gamma,z)\pi_\mu(\mathrm{d}\gamma)\mu(\mathrm{d}z),$$
$$\mathscr{D}(\mathscr{E}^\psi) := \{F \in L^2(\pi_\mu) : \ \mathscr{E}^\psi(F,F) < \infty\}.$$

According to Propositions 3.3 and 3.4 below, $(\mathscr{E}^\psi, \mathscr{D}(\mathscr{E}^\psi))$ is a conservative symmetric Dirichlet form on $L^2(\pi_\mu)$, which is regular provided $\mu(\psi_\mu) < \infty$. Obviously, if $\psi(\gamma,z)$ does not depend on γ, then $\mathscr{E}^\psi$ goes back to the second quantization Dirichlet form for $\mathscr{E}_0(f,g) := \mu(\psi fg)$ with $\mathscr{D}(\mathscr{E}_0) = L^2((1+\psi)\mu)$.

Theorem 3.1. *Let $\lambda_L(\psi), \lambda_T(\psi)$ and $\lambda_E(\psi)$ be, respectively, the exponential convergence rates in the L^2-norm, the L^2-tail norm and entropy for the semigroup associated to $(\mathscr{E}^\psi, \mathscr{D}(\mathscr{E}^\psi))$.*

(1) *In general, we have $\mathrm{ess}_{\pi_\mu\times\mu}\inf\psi \le \lambda_L(\psi), \lambda_E(\psi) \le \mathrm{ess}_\mu \inf \psi_\mu$. If $\psi(\gamma, z)$ is independent of γ, then $\lambda_L(\psi) = \lambda_E(\psi) = \mathrm{ess}_\mu\inf\psi$.*

(2) *Let μ do not have atom and be finite on bounded sets. Then $\mathrm{ess}_{\pi_\mu\times\mu}\inf\psi \le \lambda_T(\psi) \le \mathrm{ess}_\mu\inf\psi_\mu$. If moreover $\psi(\gamma, z)$ does not depend on γ, then $\lambda_T(\psi) = \mathrm{ess}_\mu\inf\psi$.*

Proof. (1) Let $\mathscr{E}$ be the second quantization Dirichlet form for $\mathscr{E}_0(f,g) := \big(\mathrm{ess}_{\pi_\mu\times\mu}\inf\psi\big)\mu(fg)$. Obviously, we have $\mathscr{E}^\psi \ge \mathscr{E}$. Combining this with Corollary 1.2 and (1.2) we conclude that

$$\lambda_L(\psi)\wedge\lambda_E(\psi) \ge \mathrm{ess}_{\pi_\mu\times\mu}\inf\psi.$$

Consequently, it suffices to prove the desired upper bound estimate.

Taking $F(\gamma) = \gamma(f)$ for nonnegative $f \in L^1(\mu)\cap L^\infty(\mu)$, we see that the defective Poincaré inequality

$$\pi_\mu(F^2) \le C_1\mathscr{E}^\psi(F,F) + C_2\pi_\mu(F)^2 \tag{3.1}$$

implies that

$$\mu(f^2) \le C_1\mu(\psi_\mu f^2) + (C_2 - 1)\mu(f)^2. \tag{3.2}$$

Thus, (3.1) for $C_2 = 1$ (i.e. the Poincaré inequality) implies that $C_1 \ge (\mathrm{ess}_\mu\inf\psi_\mu)^{-1}$. That is, $\lambda_L(\psi) \le \mathrm{ess}_\mu\inf\psi_\mu$.

On the other hand, according to (b) in the proof of (1.7), the L^1 log-Sobolev inequality

$$\pi_\mu(F\log F) \le \lambda\mathscr{E}^\psi(F, \log F) + \pi_\mu(F)\log\pi_\mu(F) \tag{3.3}$$

for $F(\gamma) := \mathrm{e}^{\gamma(f)}$ implies that

$$\mu(f\mathrm{e}^f - \mathrm{e}^f + 1) \le \lambda\mu\big(\psi_\mu(\mathrm{e}^f - 1)f\big), \quad f \in L^\infty(\mu)\cap L^1(\mu).$$

Hence, by the proof of Corollary 1.2 for $W = \psi_\mu$, $\Gamma_1 = 0$ and $q = 0$, we conclude that (3.3) implies $\lambda \ge (\mathrm{ess}_\mu\inf\psi_\mu)^{-1}$. This means that $\lambda_E(\psi) \le \mathrm{ess}_\mu\inf\psi_\mu$.

(2) Assume that μ does not have atom and is finite on bounded sets. According to Theorem 1.1, we obtain

$$\lambda_T \ge \lambda_{L,0} = \mathrm{ess}_{\pi_\mu\times\mu}\inf\psi.$$

Finally, by Lemma 2.1, (3.2) for any $C_2 > 0$ implies that $C_1 \ge (\mathrm{ess}_\mu\inf\psi_\mu)^{-1}$. Now we conclude that $\lambda_T(\psi) \le \mathrm{ess}_\mu\inf\psi_\mu$ and the proof is completed. □

The remainder of this section devotes to characterizing the form $(\mathscr{E}^\psi, \mathscr{D}(\mathscr{E}^\psi))$. To see that it is a Dirichlet form on $L^2(\pi_\mu)$, we need the following quasi-invariant property of the map $\gamma \mapsto \gamma + \delta_z$.

Lemma 3.2. *If $A \in \mathscr{F}_\Gamma$ is a π_μ-null set, then*

$$\tilde{A} := \{(\gamma, z) \in \Gamma \times E : \ \gamma + \delta_z \in A\}$$

is a $(\pi_\mu \times \mu)$-null set.

Proof. We shall make use of the Mecke identity [5] (see also [6]), i.e.

$$\int_{\Gamma\times E} H(\gamma+\delta_z, z)\pi_\mu(\mathrm{d}\gamma)\mu(\mathrm{d}z) = \int_{\Gamma\times E} H(\gamma, z)\gamma(\mathrm{d}z)\pi_\mu(\mathrm{d}\gamma) \tag{3.4}$$

holds for any measurable function H on $\Gamma \times E$ such that one of the above integrals exists. Applying (3.4) to $H(\gamma, z) = 1_A(\gamma)$ and noting that $\pi_\mu(A) = 0$, we obtain

$$\begin{aligned}(\pi_\mu \times \mu)(\tilde{A}) &= \int_{\Gamma\times E} 1_A(\gamma+\delta_z)\pi_\mu(\mathrm{d}\gamma)\mu(\mathrm{d}z) \\ &= \int_{\Gamma\times E} 1_A(\gamma)\gamma(\mathrm{d}z)\pi_\mu(\mathrm{d}\gamma) \\ &= \int_A \gamma(E)\pi_\mu(\mathrm{d}\gamma) = 0.\end{aligned}$$

□

Proposition 3.3. *$(\mathscr{E}^\psi, \mathscr{D}(\mathscr{E}^\psi))$ is a conservative symmetric Dirichlet form on $L^2(\pi_\mu)$ with $\mathscr{D}(\mathscr{E}^\psi)$ including the family of cylindrical functions*

$$\begin{aligned}\mathscr{F}_\mu^C := \Big\{\gamma \mapsto f\big(\gamma(h_1), \cdots, \gamma(h_m)\big) : \ m \geq 1, f \in C_b^1(\mathbb{R}^m), \\ h_i \in L^1(\mu) \cap L^\infty(\mu), \|\psi_\mu 1_{h_i \neq 0}\|_\infty < \infty\Big\},\end{aligned}$$

where $\|\cdot\|_\infty$ is the $L^\infty(\mu)$-norm.

Proof. According to Lemma 3.2, for $F, G \in \mathscr{D}(\mathscr{E}^\psi)$, $\mathscr{E}^\psi(F, G)$ is finite and does not depend on π_μ-versions of F and G. Thus, $(\mathscr{E}^\psi, \mathscr{D}(\mathscr{E}^\psi))$ is a well defined positive bilinear form on $L^2(\pi_\mu)$. Since $\mathscr{F}_\mu^C$ is dense in $L^2(\pi_\mu)$ and the normal contractivity property is trivial by the definition of $\mathscr{E}^\psi$, it remains to show $\mathscr{D}(\mathscr{E}^\psi) \supset \mathscr{F}_\mu^C$ and the closed property of the form. We prove these two points separately.

(a) Let $F \in \mathscr{F}_\mu^C$ with

$$F(\gamma) = f\big(\gamma(h_1), \cdots, \gamma(h_m)\big), \quad \gamma \in \Gamma,$$

which is well defined in $L^2(\pi_\mu)$ since $\gamma(K) < \infty$ for π_μ-a.e. $\gamma \in \Gamma$ and any compact subset K of E. We intend to show that $\mathscr{E}^\psi(F, F) < \infty$. Since $f \in C_0^1(\mathbb{R}^m)$, $h_i \in L^1(\mu) \cap L^\infty(\mu)$, and there exists $n \geq 1$ such that

$$\mu\big(h_i \neq 0, \psi_\mu > n\big) = 0, \quad i = 1, \cdots, m,$$

we obtain

$$\begin{aligned}
\mathscr{E}^\psi(F, F) = \int_{\Gamma \times (\bigcup_{i=1}^m \{h_i \neq 0\})} \Big[& f\big(\gamma(h_1) + h_1(z), \cdots, \gamma(h_m) + h_m(z)\big) \\
& - f\big(\gamma(h_1), \cdots, \gamma(h_m)\big)\Big]^2 \psi(\gamma, z)\pi_\mu(\mathrm{d}\gamma)\mu(\mathrm{d}z) \\
\leq \|\nabla f\|_\infty^2 & \int_{\Gamma \times \{\psi_\mu \leq n\}} \sum_{i=1}^m h_i(z)^2 \psi(\gamma, z)\pi_\mu(\mathrm{d}\gamma)\mu(\mathrm{d}z) \\
= \|\nabla f\|_\infty^2 & \sum_{i=1}^m \int_{\{\psi_\mu \leq n\}} h_i(z)^2 \psi_\mu(z)\mu(\mathrm{d}z) \\
\leq n\|\nabla f\|_\infty^2 & \sum_{i=1}^m \mu(h_i^2) \leq n\|\nabla f\|_\infty^2 \sum_{i=1}^m \|h_i\|_\infty \mu(|h_i|) < \infty.
\end{aligned}$$

(b) Let $\{F_n\}_{n\geq 1}$ be an $\mathscr{E}_1^\psi$-Cauchy sequence. We shall find $F \in \mathscr{D}(\mathscr{E}^\psi)$ such that $\mathscr{E}_1^\psi(F_n - F, F_n - F) := \mathscr{E}^\psi(F_n - F, F_n - F) + \pi_\mu(|F_n - F|^2) \to 0$ as $n \to \infty$. Since $\{F_n\}_{n\geq 1}$ is a Cauchy sequence in $L^2(\pi_\mu)$ (which is complete), there exists $F \in L^2(\pi_\mu)$ such that $F_n \to F$ in $L^2(\pi_\mu)$. Now we can choose a subsequence $\{F_{n_k}\}_{k\geq 1}$ such that $F_{n_k} \to F$ π_μ-a.e. By Lemma 3.2 we have $F_{n_k}(\gamma + \delta_z) \to F(\gamma + \delta_z)$ for $(\pi_\mu \times \mu)$-a.e. $(\gamma, z) \in \Gamma \times E$. Therefore, it follows from the Fatou lemma that

$$\begin{aligned}
&\mathscr{E}^\psi(F_n - F, F_n - F) \\
&= \int_{\Gamma \times E} \liminf_{n_k \to \infty} \big[(F_n - F_{n_k})(\gamma + \delta_z) - (F_n - F_{n_k})(\gamma)\big]^2 \psi(\gamma, z)\pi_\mu(\mathrm{d}\gamma)\mu(\mathrm{d}z) \\
&\leq \liminf_{n_k \to \infty} \mathscr{E}^\psi(F_n - F_{n_k}, F_n - F_{n_k}).
\end{aligned}$$

Since $\{F_n\}_{n\geq 1}$ is an $\mathscr{E}^\psi$-Cauchy sequence and $F_n \to F$ in $L^2(\pi_\mu)$, this implies that

$$\lim_{n\to\infty} \mathscr{E}_1^\psi(F_n - F, F_n - F) = 0.$$

Combining this with the fact that

$$\mathscr{E}^\psi(F, F) \leq 2\mathscr{E}^\psi(F_n - F, F_n - F) + 2\mathscr{E}^\psi(F_n, F_n), \quad n \geq 1,$$

we conclude that $F \in \mathscr{D}(\mathscr{E}^\psi)$ and $F_n \to F$ in $\mathscr{D}(\mathscr{E}^\psi)$ as $n \to \infty$. □

The next result provides a criterion for the regularity of the Dirichlet form, which ensures the existence of the associated Markov process according to the Dirichlet form theory (see [1, 4]). To this end, we first reduce Γ to a locally compact subspace Γ_μ. Since Γ is a Polish space such that the set $\{\pi_\mu\}$ of single probability measure is tight, we can choose an increasing sequence $\{K_n\}_{n\geq 1}$ consisting of compact subsets of Γ such that $\pi_\mu(K_n^c) \leq 1/n$ for any $n \geq 1$. Then π_μ has full measure on $\Gamma_\mu := \cup_{n=1}^\infty K_n$, which is a locally compact separable metric space.

Proposition 3.4. *If $\psi \in L^1(\pi_\mu \times \mu)$, then $(\mathscr{E}^\psi, \mathscr{D}(\mathscr{E}^\psi))$ is a regular Dirichlet form on $L^2(\Gamma_\mu; \pi_\mu)$.*

Proof. Since $\psi \in L^1(\pi_\mu \times \mu)$, we have $\mathscr{B}_b(\Gamma_\mu) \subset \mathscr{D}(\mathscr{E}^\psi)$, where $\mathscr{B}_b(\Gamma_\mu)$ is the set of all bounded measurable functions on Γ_μ. In particular, $C_0(\Gamma_\mu) \subset \mathscr{D}(\mathscr{E}^\psi)$. Thus, it suffices to prove that $C_0(\Gamma_\mu)$ is dense in $\mathscr{D}(\mathscr{E}^\psi)$ w.r.t. the $\mathscr{E}_1^\psi$-norm, i.e. for any $F \in \mathscr{D}(\mathscr{E}^\psi)$, one may find a sequence $\{F_n\}_{n\geq 1} \subset C_0(\Gamma_\mu)$ such that $\mathscr{E}_1^\psi(F_n - F, F_n - F) \to 0$ as $n \to \infty$.

Since $\mathscr{B}_b(\Gamma_\mu) \cap \mathscr{D}(\mathscr{E}^\psi)$ is dense in $\mathscr{D}(\mathscr{E}^\psi)$ (see e.g. [4, Proposition I.4.17]), we may assume that $F \in \mathscr{B}_b(\Gamma_\mu)$. Moreover, since $C_0(\Gamma_\mu)$ is dense in $L^2(\Gamma_\mu; \pi_\mu)$, we may find a sequence $\{F_n\}_{n\geq 1} \subset C_0(\Gamma_\mu)$ such that $\sup_{n\in\mathbb{N}} \|F_n\|_\infty \leq \|F\|_\infty$ and $\pi_\mu(|F_n - F|^2) \to 0$ as $n \to \infty$. Without loss of generality, we assume furthermore that $F_n \to F$ π_μ-a.e. By Lemma 3.2, $F_n(\gamma + \delta_z) \to F(\gamma + \delta_z)$ and $(F_n - F)^2(\gamma + \delta_z) \leq (\|F_n\|_\infty + \|F\|_\infty)^2 \leq 4\|F\|_\infty^2$ for $(\pi_\mu \times \mu)$-a.e. $(\gamma, z) \in \Gamma \times E$.

Note that (we do not have to distinguish integrals on Γ_μ and Γ since $\pi_\mu(\Gamma_\mu^c) = 0$)

$$\begin{aligned}
&\mathscr{E}^\psi(F_n - F, F_n - F) \\
&\leq 2 \int_{\Gamma\times E} (F_n - F)^2(\gamma + \delta_z)\psi(\gamma, z)\pi_\mu(\mathrm{d}\gamma)\mu(\mathrm{d}z) \\
&\quad + 2 \int_{\Gamma\times E} (F_n - F)^2(\gamma)\psi(\gamma, z)\pi_\mu(\mathrm{d}\gamma)\mu(\mathrm{d}z).
\end{aligned}$$

Since $\psi \in L^1(\pi_\mu \times \mu)$, by the dominated convergence theorem we obtain

$$\lim_{n\to\infty} \mathscr{E}^\psi(F_n - F, F_n - F) = 0.$$

Combining this with $\pi_\mu(|F_n - F|^2) \to 0$, we conclude that

$$\lim_{n\to\infty} \mathscr{E}_1^\psi(F_n - F, F_n - F) = 0,$$

which completes the proof. □

Finally, we consider the generator $(\mathscr{L}^\psi, \mathscr{D}(\mathscr{L}^\psi))$ of the Dirichlet form $(\mathscr{E}^\psi, \mathscr{D}(\mathscr{E}^\psi))$. For a measurable function F on Γ, let

$$\begin{aligned}
\mathscr{L}_b^\psi F(\gamma) &= \int_E \left(F(\gamma + \delta_z) - F(\gamma)\right)\psi(\gamma, z)\mu(\mathrm{d}z), \\
\mathscr{L}_d^\psi F(\gamma) &= \int_E 1_{\{\gamma \geq \delta_z\}}\big(F(\gamma - \delta_z) - F(\gamma)\big)\psi(\gamma - \delta_z, z)\gamma(\mathrm{d}z), \quad \gamma \in \Gamma
\end{aligned}$$

provided the integrals above exist.

Proposition 3.5. *Suppose* $F \in \mathscr{D}(\mathscr{E}^\psi)$ *such that* $\mathscr{L}_b^\psi F, \mathscr{L}_d^\psi F \in L^2(\pi_\mu)$. *Then* $F \in \mathscr{D}(\mathscr{L}^\psi)$ *and* $\mathscr{L}^\psi F = \mathscr{L}_d^\psi F + \mathscr{L}_b^\psi F$. *In particular, if* μ *is locally finite and*

$$\int_\Gamma \psi(\gamma,\cdot)^2 \pi_\mu(\mathrm{d}\gamma) \in L^1_{loc}(\mu), \tag{3.5}$$

then

$$\mathscr{D}(\mathscr{L}^\psi) \supset \Big\{\gamma \mapsto f\big(\gamma(h_1),\cdots,\gamma(h_m)\big): \ m\ge 1, f\in C_b^1(\mathbb{R}^m), h_i \in C_0(E)\Big\}.$$

Proof. (1) For any $F \in \mathscr{D}(\mathscr{E}^\psi)$ such that $\mathscr{L}_b^\psi F, \mathscr{L}_d^\psi F \in L^2(\pi_\mu)$, by the Mecke identity (3.4) for

$$H(\gamma,z) = F(\gamma)1_{\{\gamma\ge\delta_z\}}\big(F(\gamma-\delta_z)-F(\gamma)\big)\psi(\gamma-\delta_z,z),$$

we obtain

$$\begin{aligned}
&-\mathscr{E}^\psi(F,F)\\
&=\int_{\Gamma\times E} F(\gamma)\big(F(\gamma+\delta_z)-F(\gamma)\big)\psi(\gamma,z)\pi_\mu(\mathrm{d}\gamma)\mu(\mathrm{d}z)\\
&\quad+\int_{\Gamma\times E} F(\gamma+\delta_z)\big(F(\gamma)-F(\gamma+\delta_z)\big)\psi(\gamma,z)\pi_\mu(\mathrm{d}\gamma)\mu(\mathrm{d}z)\\
&=\int_{\Gamma\times E} F(\gamma)\big(F(\gamma+\delta_z)-F(\gamma)\big)\psi(\gamma,z)\pi_\mu(\mathrm{d}\gamma)\mu(\mathrm{d}z)\\
&\quad+\int_{\Gamma\times E} F(\gamma)1_{\{\gamma\ge\delta_z\}}\big(F(\gamma-\delta_z)-F(\gamma)\big)\psi(\gamma-\delta_z,\gamma)\gamma(\mathrm{d}z)\pi_\mu(\mathrm{d}\gamma)\\
&=\int_\Gamma F(\gamma)\big(\mathscr{L}_b^\psi F+\mathscr{L}_d^\psi F\big)(\gamma)\pi_\mu(\mathrm{d}\gamma).
\end{aligned}$$

Hence, the first assertion follows.

(2) Let

$$F(\gamma) = f\big(\gamma(h_1),\cdots,\gamma(h_m)\big), \quad \gamma\in\Gamma,$$

where $f\in C_b^1(\mathbb{R}^m)$, $h_i \in C_0(E)$ and $m\ge 1$. By the Schwartz inequality we have

$$\begin{aligned}
&\int_{\Gamma\times E}\big(F(\gamma+\delta_z)-F(\gamma)\big)^2\psi(\gamma,z)^2\pi_\mu(\mathrm{d}\gamma)\mu(\mathrm{d}z)\\
&=\int_{\Gamma\times(\bigcup_{i=1}^m \mathrm{supp} h_i)}\Big[f\big(\gamma(h_1)+h_1(z),\cdots,\gamma(h_m)+h_m(z)\big)\\
&\qquad\qquad -f\big(\gamma(h_1),\cdots,\gamma(h_m)\big)\Big]^2\psi(\gamma,z)^2\pi_\mu(\mathrm{d}\gamma)\mu(\mathrm{d}z)
\end{aligned}$$

$$\le \|\nabla f\|_\infty^2 \sum_{i=1}^m \int_{\Gamma\times(\bigcup_{i=1}^m \mathrm{supp} h_i)} h_i(z)^2\psi(\gamma,z)^2\pi_\mu(\mathrm{d}\gamma)\mu(\mathrm{d}z)$$
$$\le \|\nabla f\|_\infty^2 \left(\sum_{i=1}^m \|h_i\|_\infty^2\right) \int_{\Gamma\times(\bigcup_{i=1}^m \mathrm{supp} h_i)} \psi(\gamma,z)^2\pi_\mu(\mathrm{d}\gamma)\mu(\mathrm{d}z)$$
$$< \infty,$$

where the last step is due to (3.5). Then $\mathscr{L}_b^\psi F \in L^2(\pi_\mu)$ since

$$\|\mathscr{L}_b^\psi F\|_{L^2(\pi_\mu)}^2 \le \int_{\Gamma\times E} \big(F(\gamma+\delta_z)-F(\gamma)\big)^2\psi(\gamma,z)^2\pi_\mu(\mathrm{d}\gamma)\mu(\mathrm{d}z) < \infty.$$

On the other hand, using the Mecke identity (3.4) for

$$H(\gamma,z) = 1_{\{\gamma\ge\delta_z\}}\big(F(\gamma-\delta_z)-F(\gamma)\big)^2\psi(\gamma-\delta_z,z)^2,$$

we arrive at

$$\|\mathscr{L}_d^\psi F\|_{L^2(\pi_\mu)}^2 \le \int_{\Gamma\times E} 1_{\{\gamma\ge\delta_z\}}\big(F(\gamma-\delta_z)-F(\gamma)\big)^2\psi(\gamma-\delta_z,z)^2\gamma(\mathrm{d}z)\pi_\mu(\mathrm{d}\gamma)$$
$$= \int_{\Gamma\times E} \big(F(\gamma+\delta_z)-F(\gamma)\big)^2\psi(\gamma,z)^2\pi_\mu(\mathrm{d}\gamma)\mu(\mathrm{d}z) < \infty.$$

Consequently, $\mathscr{L}_d^\psi F \in L^2(\pi_\mu)$ and the proof is now completed according to the first assertion. □

4 The path space of Lévy processes

Let $X = \{X_t : t \ge 0\}$ be the Lévy process on $\mathbb{R}^d$ starting from 0 with a constant drift $b \in \mathbb{R}^d$ and the Lévy measure ν, which satisfies $\nu(\{0\}) = 0$ and

$$\int_{\mathbb{R}^d} \big(|z|^2 \wedge 1\big)\,\nu(\mathrm{d}z) < \infty.$$

So, X_t is generated by

$$\mathscr{L}f = \langle b, \nabla f\rangle + \int_{\mathbb{R}^d} \big\{f(z+\cdot) - f - \langle \nabla f, z\rangle 1_{\{|z|\le 1\}}\big\}\nu(\mathrm{d}z),$$

which is well defined for $f \in C_b^2(\mathbb{R}^d)$.

Let Λ be the distribution of X, which is a probability measure on the path space

$$W := \{w : [0,\infty) \to \mathbb{R}^d \,|\, w \text{ is right continuous having left limits}\}.$$

It is well known that W is a Polish space under the Skorokhod metric

$$\begin{aligned}\mathrm{dist}(v,w) := \inf\Big\{ &\delta>0: \text{there exist } n\geq 1,\ 0=s_0<s_1<\cdots<s_n, \text{ and } 0=t_0<t_1 \\ &<\cdots<t_n \quad \text{such that} \quad |t_i-s_i|\leq\delta \quad \text{and} \\ &\sup_{s\in[s_{i-1},s_i),t\in[t_{i-1},t_i)} 1\wedge|v_s-w_t|\leq\delta \quad \text{hold for all} \quad 1\leq i\leq n\Big\}.\end{aligned}$$

Let $\tilde\psi \in L^1(\Lambda\times\nu\times\mathrm{d}t)$ be a nonnegative measurable function on $W\times(\mathbb{R}^d\setminus\{0\})\times[0,\infty)$ such that

$$\tilde\psi_{\nu\times\mathrm{d}t}(x,t) := \int_W \tilde\psi(w,x,t)\Lambda(\mathrm{d}w)<\infty, \qquad (\nu\times\mathrm{d}t)\text{-a.e. } (x,t)\in(\mathbb{R}^d\setminus\{0\})\times[0,\infty).$$

Consider

$$\begin{aligned}\tilde{\mathscr{E}}^{\tilde\psi}(F,G) := \int_{W\times\mathbb{R}^d\times[0,\infty)} &\big(F(w+x1_{[t,\infty]})-F(w)\big)\big(G(w+x1_{[t,\infty]})-G(w)\big) \\ &\times\tilde\psi(w,x,t)\,\Lambda(\mathrm{d}w)\nu(\mathrm{d}x)\mathrm{d}t\end{aligned}$$

for

$$F,\,G\in\mathscr{D}(\tilde{\mathscr{E}}^{\tilde\psi}) := \{F\in L^2(\Lambda): \tilde{\mathscr{E}}^{\tilde\psi}(F,F)<\infty\}.$$

To apply the known Poincaré inequality on Poisson space, we follow the line of [12] by constructing the Lévy process using Poisson point processes. Let $E=(\mathbb{R}^d\setminus\{0\})\times[0,\infty)$, which is a Polish space by taking the following complete metric on $\mathbb{R}^d\setminus\{0\}$:

$$\begin{aligned}\rho(x,y) := \sup\Big\{&|f(x)-f(y)|: |\nabla f(z)|\leq\frac{1}{|z|}\vee 1, \\ &z\in\mathbb{R}^d\setminus\{0\}, f\in C^1\big(\mathbb{R}^d\setminus\{0\}\big)\Big\}.\end{aligned}$$

Next, let $\mu=\nu\times\mathrm{d}t$, which is finite on bounded subsets of E and does not have atom. Let π_μ be the Poisson measure with intensity μ, which is a probability measure on the configuration space

$$\Gamma := \left\{\sum_{i=1}^n \delta_{(x_i,t_i)} : x_i\in\mathbb{R}^d\setminus\{0\}, t_i\in[0,\infty), 1\leq i\leq n, n\in\mathbb{Z}_+\cup\{\infty\}\right\}.$$

Then on the probability space $(\Gamma,\mathscr{F}_\Gamma,\pi_\mu)$, the Lévy process X_t can be formulated as (see [3])

$$X_t(\gamma)=bt+\int_{\{|z|>1\}\times[0,t]} z\,\gamma(\mathrm{d}z,\mathrm{d}s)+\int_{\{|z|\leq 1\}\times[0,t]} z\,(\gamma-\mu)(\mathrm{d}z,\mathrm{d}s), \quad t\geq 0,$$

where the second term in the right hand side above is the Stieltjes integral, and the last term is the Itô integral. Therefore,

$$\Lambda = \pi_\mu \circ X^{-1}. \tag{4.1}$$

Combining this with the Mecke identity (3.4), we obtain

$$\begin{aligned}\int_W \sum_{\triangle w_t \neq 0} h(w, \triangle w_t, t)\Lambda(\mathrm{d}w) \\ = \int_{W\times(\mathbb{R}^d\setminus\{0\})\times[0,\infty)} h(w + x1_{[t,T]}, x, t)\Lambda(\mathrm{d}w)\nu(\mathrm{d}x)\mathrm{d}t\end{aligned} \tag{4.2}$$

for any non-negative measurable function h on $W \times \mathbb{R}^d \times [0, \infty)$. Due to (4.1) and (4.2), arguments used in Section 3 also work for $(\tilde{\mathscr{E}}^{\tilde\psi}, \mathscr{D}(\tilde{\mathscr{E}}^{\tilde\psi}))$, Λ and $\tilde\psi$ in place of $(\mathscr{E}^\psi, \mathscr{D}(\mathscr{E}^\psi))$, π_μ and ψ respectively. In particular, letting $\tilde\lambda_L(\tilde\psi)$, $\tilde\lambda_T(\tilde\psi)$ and $\tilde\lambda_E(\tilde\psi)$ be, respectively, the exponential convergence rates in the L^2-norm, the L^2-tail norm and entropy for the semigroup associated to $(\tilde{\mathscr{E}}^{\tilde\psi}, \mathscr{D}(\tilde{\mathscr{E}}^{\tilde\psi}))$, we obtain the following result.

Theorem 4.1. *We have*

$$\mathrm{ess}_{\Lambda\times\mu} \inf \tilde\psi \le \tilde\lambda_L(\tilde\psi), \tilde\lambda_T(\tilde\psi), \tilde\lambda_E(\tilde\psi) \le \mathrm{ess}_\mu \inf \tilde\psi_\mu,$$

and the equalities hold provided $\tilde\psi(w, x, t)$ *does not depend on* w.

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