The generalized connectivity of complete bipartite graphs*

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Abstract

Let G be a nontrivial connected graph of order n, and k an integer with $2 \leq k \leq n$. For a set S of k vertices of G, let $\kappa(S)$ denote the maximum number ℓ of edge-disjoint trees $T_1, T_2, \ldots, T_\ell$ in G such that $V(T_i) \cap V(T_j) = S$ for every pair i, j of distinct integers with $1 \leq i, j \leq \ell$. Chartrand et al. generalized the concept of connectivity as follows: The *k-connectivity*, denoted by $\kappa_k(G)$, of G is defined by $\kappa_k(G)$ =min$\{\kappa(S)\}$, where the minimum is taken over all k-subsets S of $V(G)$. Thus $\kappa_2(G) = \kappa(G)$, where $\kappa(G)$ is the connectivity of G. Moreover, $\kappa_n(G)$ is the maximum number of edge-disjoint spanning trees of G.

This paper mainly focus on the k-connectivity of complete bipartite graphs $K_{a,b}$. First, we obtain the number of edge-disjoint spanning trees of $K_{a,b}$, which is $\lfloor \frac{ab}{a+b-1} \rfloor$, and specifically give the $\lfloor \frac{ab}{a+b-1} \rfloor$ edge-disjoint spanning trees. Then based on this result, we get the k-connectivity of $K_{a,b}$ for all $2 \leq k \leq a+b$. Namely, if $k > b-a+2$ and $a-b+k$ is odd then $\kappa_k(K_{a,b}) = \frac{a+b-k+1}{2} + \lfloor \frac{(a-b+k-1)(b-a+k-1)}{4(k-1)} \rfloor$, if $k > b-a+2$ and $a-b+k$ is even then $\kappa_k(K_{a,b}) = \frac{a+b-k}{2} + \lfloor \frac{(a-b+k)(b-a+k)}{4(k-1)} \rfloor$, and if $k \leq b-a+2$ then $\kappa_k(K_{a,b}) = a$.

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1 Introduction

We follow the terminology and notation of [1]. As usual, denote by $K_{a,b}$ the complete bipartite graph with bipartition of sizes a and b. The *connectivity* $\kappa(G)$ of a graph G is defined as the minimum cardinality of a set Q of vertices of G such that $G-Q$ is disconnected or trivial. A well-known theorem of Whitney [4] provides an equivalent definition of the connectivity. For each 2-subset $S=\{u,v\}$ of vertices of G, let $\kappa(S)$ denote the maximum number of internally disjoint uv-paths in G. Then $\kappa(G)=\min\{\kappa(S)\}$, where the minimum is taken over all 2-subsets S of $V(G)$.

In [2], the authors generalized the concept of connectivity. Let G be a nontrivial connected graph of order n, and k an integer with $2\leq k\leq n$. For a set S of k vertices of G, let $\kappa(S)$ denote the maximum number ℓ of edge-disjoint trees $T_1,T_2,\ldots,T_\ell$ in G such that $V(T_i)\cap V(T_j)=S$ for every pair i,j of distinct integers with $1\leq i,j\leq \ell$ (Note that the trees are vertex-disjoint in $G\backslash S$). A collection $\{T_1,T_2,\ldots,T_\ell\}$ of trees in G with this property is called an *internally disjoint set of trees connecting* S. The *k-connectivity*, denoted by $\kappa_k(G)$, of G is then defined as $\kappa_k(G)=\min\{\kappa(S)\}$, where the minimum is taken over all k-subsets S of $V(G)$. Thus, $\kappa_2(G)=\kappa(G)$ and $\kappa_n(G)$ is the maximum number of edge-disjoint spanning trees of G.

In [3], the authors focused on the investigation of $\kappa_3(G)$ and mainly studied the relationship between the 2-connectivity and the 3-connectivity of a graph. They gave sharp upper and lower bounds for $\kappa_3(G)$ for general graphs G, and showed that if G is a connected planar graph, then $\kappa(G)-1\leq\kappa_3(G)\leq\kappa(G)$. Moreover, they studied the algorithmic aspects for $\kappa_3(G)$ and gave an algorithm to determine $\kappa_3(G)$ for a general graph G.

Chartrand et al. in [2] proved that if G is the complete 3-partite graph $K_{3,4,5}$, then $\kappa_3(G)=6$. They also gave a general result for the complete graph K_n:

Theorem 1.1. *For every two integers n and k with $2\leq k\leq n$,*

$$\kappa_k(K_n)=n-\lceil k/2\rceil.$$

In this paper, we turn to complete bipartite graphs $K_{a,b}$. First, we give the number of edge-disjoint spanning trees of $K_{a,b}$, namely $\kappa_{a+b}(K_{a,b})$.

Theorem 1.2. *For every two integers a and b,*

$$\kappa_{a+b}(K_{a,b})=\lfloor\frac{ab}{a+b-1}\rfloor.$$

Actually, we specifically give the $\lfloor\frac{ab}{a+b-1}\rfloor$ edge-disjoint spanning trees of $K_{a,b}$. Then based on Theorem 1.2, we obtain the k-connectivity of $K_{a,b}$ for all $2\leq k\leq a+b$.

2 Proof of Theorem 1.2

Since $K_{a,b}$ contains ab edges and a spanning tree needs $a+b-1$ edges, the number of edge-disjoint spanning trees of $K_{a,b}$ is at most $\lfloor \frac{ab}{a+b-1} \rfloor$, namely, $\kappa_{a+b}(K_{a,b}) \leq \lfloor \frac{ab}{a+b-1} \rfloor$. Thus, it suffices to prove that $\kappa_{a+b}(K_{a,b}) \geq \lfloor \frac{ab}{a+b-1} \rfloor$. To this end, we want to find all the $\lfloor \frac{ab}{a+b-1} \rfloor$ edge-disjoint spanning trees.

Let $X = \{x_1, x_2, \ldots, x_a\}$ and $Y = \{y_1, y_2, \ldots, y_b\}$ be the bipartition of $K_{a,b}$. Without loss of generality, we may assume that $a \leq b$.

We will express the spanning trees by adjacency-degree lists. To be specific, the fist spanning tree T_1 we find can be represented by an adjacency-degree list as follows:

vertex	neighbors	degree
x_1	$y_1,\ y_2,\ \ldots,\ y_{d_1}$	d_1
x_2	$y_{d_1},\ y_{d_1+1},\ \ldots,\ y_{d_1+d_2-1}$	d_2
x_3	$y_{d_1+d_2-1},\ y_{d_1+d_2},\ \ldots,\ y_{d_1+d_2+d_3-2}$	d_3
$\ldots$	$\ldots$	$\ldots$
x_j	$y_{d_1+d_2+\cdots+d_{j-1}-(j-2)},\ y_{d_1+d_2+\cdots+d_{j-1}-(j-2)+1},\ \ldots,\ y_{d_1+d_2+\cdots+d_j-(j-1)}$	d_j
$\ldots$	$\ldots$	$\ldots$
x_a	$y_{d_1+d_2+\cdots+d_{a-1}-(a-2)},\ y_{d_1+d_2+\cdots+d_{a-1}-(a-2)+1},\ \ldots,\ y_{d_1+d_2+\cdots+d_a-(a-1)}$	d_a

where d_j denotes the degree of x_j in T_1, and $d_1 + d_2 + \cdots + d_a = a + b - 1$.

To simplify the subscript, we denote $i_0 = 1$, $i_1 = d_1$, $i_2 = d_1 + d_2 - 1$, $\ldots$, $i_j = d_1 + d_2 + \cdots + d_j - (j-1)$, $\ldots$, $i_a = d_1 + d_2 + \cdots + d_a - (a-1) = b$. Note that, $i_j - i_{j-1} = d_j - 1$. So the adjacency-degree list of T_1 can be simplified as follows:

	vertex	neighbors	degree
	x_1	$y_{i_0},\ y_{i_0+1},\ \ldots,\ y_{i_1}$	d_1
	x_2	$y_{i_1},\ y_{i_1+1},\ \ldots,\ y_{i_2}$	d_2
T_1	x_3	$y_{i_2},\ y_{i_2+1},\ \ldots,\ y_{i_3}$	d_3
	$\ldots$	$\ldots$	$\ldots$
	x_j	$y_{i_{j-1}},\ y_{i_{j-1}+1},\ \ldots,\ y_{i_j}$	d_j
	$\ldots$	$\ldots$	$\ldots$
	x_a	$y_{i_{a-1}},\ y_{i_{a-1}+1},\ \ldots,\ y_{i_a}$	d_a

Then we can list the second spanning trees we find. Here and in what follows, for a vertex y_j, if $j > b$, y_j denotes y_{j-b}, for a subscript i_j, if $j > a$, y_{i_j} denotes $y_{i_{j-a}}$, and for degree d_j, if $j > a$, d_j denotes d_{j-a}.

	vertex	neighbors	degree
T_2	x_1	$y_{i_1+1},\ y_{i_1+2},\ \dots,\ y_{i_2+1}$	d_2
	x_2	$y_{i_2+1},\ y_{i_2+2},\ \dots,\ y_{i_3+1}$	d_3
	x_3	$y_{i_3+1},\ y_{i_3+2},\ \dots,\ y_{i_4+1}$	d_4
	$\dots$	$\dots$	$\dots$
	x_j	$y_{i_j+1},\ y_{i_j+2},\ \dots,\ y_{i_{j+1}+1}$	d_{j+1}
	$\dots$	$\dots$	$\dots$
	x_a	$y_{i_a+1},\ y_{i_a+2},\ \dots,\ y_{i_{a+1}}$	d_1

From the lists, we can see that T_2 and T_1 are edge-disjoint, if and only if for every vertex x_j, $d_j + d_{j+1} \le b$. If T_2 and T_1 are edge-disjoint, then we continue to list T_3.

	vertex	neighbors	degree
T_3	x_1	$y_{i_2+2},\ y_{i_2+3},\ \dots,\ y_{i_3+2}$	d_3
	x_2	$y_{i_3+2},\ y_{i_3+3},\ \dots,\ y_{i_4+2}$	d_4
	x_3	$y_{i_4+2},\ y_{i_4+3},\ \dots,\ y_{i_5+2}$	d_5
	$\dots$	$\dots$	$\dots$
	x_j	$y_{i_{j+1}+2},\ y_{i_{j+1}+3},\ \dots,\ y_{i_{j+2}+2}$	d_{j+2}
	$\dots$	$\dots$	$\dots$
	x_a	$y_{i_{a+1}+2},\ y_{i_{a+1}+3},\ \dots,\ y_{i_{a+2}+1}$	d_2

From the lists, we can see that T_3 and T_1, T_2 are edge-disjoint, if and only if for every vertex x_j, $d_j + d_{j+1} + d_{j+2} \le b$. If T_3 and T_1, T_2 are edge-disjoint, then we continue to list T_4. Continuing the procedure, our goal is to find the maximum t, such that T_t and $T_1, T_2, \dots, T_{t-1}$ are edge-disjoint.

	vertex	neighbors	degree
T_t	x_1	$y_{i_{t-1}+(t-1)},\ y_{i_{t-1}+t},\ \dots,\ y_{i_t+(t-1)}$	d_t
	x_2	$y_{i_t+(t-1)},\ y_{i_t+t},\ \dots,\ y_{i_{t+1}+(t-1)}$	d_{t+1}
	x_3	$y_{i_{t+1}+(t-1)},\ y_{i_{t+1}+t},\ \dots,\ y_{i_{t+2}+(t-1)}$	d_{t+2}
	$\dots$	$\dots$	
	x_j	$y_{i_{j+t-2}+(t-1)},\ y_{i_{j+t-2}+t},\ \dots,\ y_{i_{j+t-1}+(t-1)}$	d_{t+j-1}
	$\dots$	$\dots$	
	x_a	$y_{i_{a+t-2}+(t-1)},\ y_{i_{a+t-2}+t},\ \dots,\ y_{i_{a+t-1}+(t-2)}$	d_{t-1}

That is, we want to find the maximum t, such that $d_j + d_{j+1} + \cdots + d_{j+t-1} \le b$, for any $1 \le j \le a$.

Let $D_j^t = d_j + d_{j+1} + \cdots + d_{j+t-1}$. It can be observed that $D_j^t = D_{j+1}^t$ if and only if $d_j = d_{j+t}$. Consider the numbers $1, t+1, 2t+1, \dots, (a-1)t+1$, where addition is carried out by modula a.

Case 1. $1, t+1, 2t+1, \dots, (a-1)t+1$ are pairwise distinct.

Then we can assign the values to d_j as follows:

Let $a+b-1=ka+c$, where k, c are integers, and $0 \le c \le a-1$. Then $a+b-1=(k+1)c+k(a-c)$. If $c=0$, let $d_j=k$, for all $1 \le j \le a$. If $c \ne 0$, let $d_{it+1}=k+1$, for all $0 \le i \le c-1$, and let other $d_j=k$.

Case 2. Some of the numbers $1, t+1, 2t+1, \dots, (a-1)t+1$ are equal.

Without loss of generality, suppose $jt+1$ is the first number that equals a number $it+1$ before it, namely, $jt+1=it+1\ (mod\ a)$, where $j>i$. Then $(j-i)t+1=1\ (mod\ a)$. Since $jt+1$ is the first number that equals a number before it, we can get $i=0$. Thus, $1, t+1, 2t+1, \dots, (j-1)t+1$ are pairwise distinct.

Claim 1. $it+1 \ne 2\ (mod\ a)$, for any integer i.

If $it+1=2\ (mod\ a)$, then we have $it=1\ (mod\ a)$. Thus we have

$$\begin{array}{rcccl} it & + & 1 & = & 2\ \ (mod\ a) \\ 2it & + & 1 & = & 3\ \ (mod\ a) \\ \multicolumn{5}{c}{\dots\dots\dots\dots\dots\dots\dots\dots} \\ (a-1)it & + & 1 & = & a\ \ (mod\ a) \end{array}$$

So there are a distinct numbers in $\{1, it+1, 2it+1, \dots, (a-1)it+1\}$. On the other hand, since $jt+1=1\ (mod\ a)$, there are at most $j \le a-1$ distinct numbers in $\{ut+1, u \text{ is an integer}\} \supset \{1, it+1, 2it+1, \dots, (a-1)it+1\}$, a contradiction. Thus, $it+1 \ne 2\ (mod\ a)$ for any integer i.

Claim 2. $2, t+2, 2t+2, \dots, (j-1)t+2$ are pairwise distinct.

If $j_1t+2=j_2t+2\ (mod\ a)$, where $0 \le j_1 < j_2 \le j-1$, then $j_1t+1=j_2t+1\ (mod\ a)$. But $1, t+1, 2t+1, \dots, (j-1)t+1$ are pairwise distinct, a contradiction. Thus, $2, t+2, 2t+2, \dots, (j-1)t+2$ are pairwise distinct.

Claim 3. $\{1, t+1, 2t+1, \dots, (j-1)t+1\} \cap \{2, t+2, 2t+2, \dots, (j-1)t+2\} = \emptyset$.

If $i_1t+1=i_2t+2\ (mod\ a)$, then $(i_1-i_2)t+1=2\ (mod\ a)$, but $it+1 \ne 2\ (mod\ a)$ for any integer i, a contradiction by Claim 1. Thus, $1, t+1, 2t+1, \dots, (j-1)t+1, 2, t+2, 2t+2, \dots, (j-1)t+2$ are pairwise distinct.

Now, if $2=\frac{a}{j}$, then we have already ordered all numbers of $\{1,\ \dots,\ a\}$. Else if $2<\frac{a}{j}$, we will prove that $1+it \ne 3\ (mod\ a)$ and $2+it \ne 3\ (mod\ a)$ for any integer i.

Claim 4. If $2<\frac{a}{j}$, then $1+it \ne 3\ (mod\ a)$ and $2+it \ne 3\ (mod\ a)$ for any integer i.

If $2+it=3\ (mod\ a)$, then $1+it=2\ (mod\ a)$, a contradiction by Claim 1. If

$1 + it = 3\ (mod\ a)$, then we have $it = 2\ (mod\ a)$. Thus we have

$$\begin{array}{cccccl} it & + & 1 & = & 3 & (mod\ a) \\ it & + & 2 & = & 4 & (mod\ a) \\ 2it & + & 1 & = & 5 & (mod\ a) \\ 2it & + & 2 & = & 6 & (mod\ a) \\ \multicolumn{6}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\ \frac{a-2}{2}it & + & 1 & = & a-1 & (mod\ a)\ (\text{for } a \text{ even}) \\ \frac{a-3}{2}it & + & 2 & = & a-1 & (mod\ a)\ (\text{for } a \text{ odd}) \\ \frac{a-2}{2}it & + & 2 & = & a & (mod\ a)\ (\text{for } a \text{ even}) \\ \frac{a-1}{2}it & + & 1 & = & a & (mod\ a)\ (\text{for } a \text{ odd}) \end{array}$$

So there are at least a distinct numbers in $\{1, it+1, 2it+1, \ldots, \lceil \frac{a}{2} \rceil it+1, 2, it+2, 2it+2, \ldots, \lceil \frac{a}{2} \rceil it+2\}$. On the other hand, since $jt+1 = 1\ (mod\ a)$ and $j \le a-1$, there are at most $2j < a$ distinct numbers in $\{ut+1, u \text{ is an integer}\} \cup \{vt+2, v \text{ is an integer}\} \supset \{1, it+1, 2it+1, \ldots, \lceil \frac{a}{2} \rceil it+1, 2, it+2, 2it+2, \ldots, \lceil \frac{a}{2} \rceil it+2\}$, a contradiction. Hence, if $2 < \frac{a}{j}$, then $1 + it \ne 3\ (mod\ a)$ and $2 + it \ne 3\ (mod\ a)$ for any integer i.

Similarly, we can prove that $r + it \ne s\ (mod\ a)$ for $1 \le r < s \le \frac{a}{j}$. Thus we can get the following claim:

Claim 5. $1, t+1, 2t+1, \ldots, (j-1)t+1, 2, t+2, 2t+2, \ldots, (j-1)t+2, \ldots, \frac{a}{j}, t+\frac{a}{j}, 2t+\frac{a}{j}, \ldots, (j-1)t+\frac{a}{j}$ are pairwise distinct. And hence $\{1, t+1, 2t+1, \ldots, (j-1)t+1\} \cup \{2, t+2, 2t+2, \ldots, (j-1)t+2\} \cup \cdots \cup \{\frac{a}{j}, t+\frac{a}{j}, 2t+\frac{a}{j}, \ldots, (j-1)t+\frac{a}{j}\} = \{1, 2, \ldots, a\}$.

The proof is similar to those of Claims 2, 3 and 4. We thus have ordered $\{1, 2, \ldots, a\}$ by $1, t+1, 2t+1, \ldots, (j-1)t+1, 2, t+2, 2t+2, \ldots, (j-1)t+2, \ldots, \frac{a}{j}, t+\frac{a}{j}, 2t+\frac{a}{j}, \ldots, (j-1)t+\frac{a}{j}$. Let $a + b - 1 = ka + c$, where k, c are integers, and $0 \le c \le a-1$. Then $a + b - 1 = (k+1)c + k(a-c)$.

Now, we can assign the values of d_j as follows: If $c = 0$, let $d_j = k$ for all $1 \le j \le a$. If $c \ne 0$, for the first c numbers of our ordering, if d_j uses one of them as subscript, then $d_j = k+1$, else $d_j = k$.

Next, we will show that, in either case, $\mid D_i^t - D_j^t \mid \le 1$ for any integers $1 \le i, j \le a$ and $t > 0$.

If $c = 0$, $d_j = k$ for all $1 \le j \le a$, then $D_i^t = D_j^t$ for any integers $1 \le i, j \le a$. The assertion is certainly true. So we may assume that $c \ne 0$. For Case 1, we construct a weighted cycle: $C = v_1 v_2 \ldots v_a v_1$ and $w(v_i) = d_{(i-1)t+1}$, where v_i corresponds to vertex $x_{(i-1)t+1}$, $1 \le i \le a$.

According to the assignment,

$$w(v_1) = w(v_2) = \cdots = w(v_c) = k+1,$$

and

$$w(v_{c+1}) = w(v_{c+2}) = \cdots = w(v_a) = k.$$

Since $D_i^t = D_{i+1}^t$ if and only if $d_i = d_{i+t}$, then $D_{(i-1)t+1}^t = D_{(i-1)t+1+1}^t$ if and only if $w(v_i) = w(v_{i+1})$. Similarly, $D_{(i-1)t+1}^t = D_{(i-1)t+1+1}^t + 1$ if and only if $w(v_i) = w(v_{i+1}) + 1$, and $D_{(i-1)t+1}^t = D_{(i-1)t+1+1}^t - 1$ if and only if $w(v_i) = w(v_{i+1}) - 1$. We know that $w(v_c) = w(v_{c+1}) + 1$ and $w(v_a) = w(v_1) - 1$. For simplicity, let $(c-1)t + 1 = \alpha \ (mod\ a)$, $(a-1)t + 1 = \beta \ (mod\ a)$, that is, v_c corresponds to x_α and v_a corresponds to x_β, and by the hypothesis, $\alpha \neq \beta$.

If $\alpha < \beta$, then

$$D_1^t = D_2^t = \cdots = D_\alpha^t = D_{\alpha+1}^t + 1 = D_{\alpha+2}^t + 1 = \cdots = D_\beta^t + 1 = D_{\beta+1}^t = D_{\beta+2}^t = \cdots = D_a^t.$$

If $\alpha > \beta$, then

$$D_1^t = D_2^t = \cdots = D_\beta^t = D_{\beta+1}^t - 1 = D_{\beta+2}^t - 1 = \cdots = D_\alpha^t - 1 = D_{\alpha+1}^t = D_{\alpha+2}^t = \cdots = D_a^t.$$

In any case, we have $\mid D_i^t - D_j^t \mid \leq 1$ for any integers $1 \leq i, j \leq a$ and $t > 0$.

For Case 2, we construct $\frac{a}{j}$ weighted cycles. $C_i = v_{i_1} v_{i_2} \ldots v_{i_j} v_{i_1}$, $1 \leq i \leq \frac{a}{j}$, and $w(v_{i_r}) = d_{(r-1)t+i}$, where v_{i_r} corresponds to vertex $x_{(r-1)t+i}$, $1 \leq r \leq j$. By the assignment, there is at most one cycle in which the vertices have two distinct weights. If such cycle does not exist, clearly, we have $D_1^t = D_2^t = \cdots = D_a^t$. So we may assume that for some cycle C_s, $w(v_{s_\gamma}) = w(v_{s_{\gamma+1}}) + 1$ and $w(v_{s_j}) = w(v_{s_1}) - 1$. Similar to the proof of Case 1, we can get that $\mid D_i^t - D_j^t \mid \leq 1$ for any integers $1 \leq i, j \leq a$ and $t > 0$.

Then, we can show that, with the assignment we can get $t \geq \lfloor \frac{ab}{a+b-1} \rfloor$.

Let $t' = \lfloor \frac{ab}{a+b-1} \rfloor$. And let

$$\begin{array}{lcccccccc}
D_1^{t'} & = & d_1 & + & d_2 & + & \cdots & + & d_{t'} \\
D_2^{t'} & = & d_2 & + & d_3 & + & \cdots & + & d_{t'+1} \\
\multicolumn{9}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\
D_j^{t'} & = & d_j & + & d_{j+1} & + & \cdots & + & d_{j+t'-1} \\
\multicolumn{9}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\
D_a^{t'} & = & d_a & + & d_1 & + & \cdots & + & d_{t'-1}
\end{array}$$

we have $D_1^{t'} + D_2^{t'} + \cdots + D_a^{t'} = t'(d_1 + d_2 + \cdots + d_a) = t'(a + b - 1)$

It follows from $\mid D_i^t - D_j^t \mid \leq 1$, for any integers $1 \leq i, j \leq a$ and $t > 0$, that

$$D_j^{t'} \leq \lceil \frac{t'(a+b-1)}{a} \rceil < \frac{t'(a+b-1)}{a} + 1 \leq \frac{ab}{a+b-1} \frac{a+b-1}{a} + 1 = b + 1$$

The third inequality holds since $t' = \lfloor \frac{ab}{a+b-1} \rfloor \leq \frac{ab}{a+b-1}$. Since $D_j^{t'}$ is an integer, we have $D_j^{t'} \leq b$ for all $1 \leq j \leq a$. Since t is the maximum integer such that $D_j^t =$

$d_j + d_{j+1} + \cdots + d_{j+t-1} \leq b$ for any $1 \leq j \leq a$, then $t \geq t' = \lfloor \frac{ab}{a+b-1} \rfloor$. So we can find at least $\lfloor \frac{ab}{a+b-1} \rfloor$ edge-disjoint spanning trees of $K_{a,b}$. And hence $\kappa_{a+b}(K_{a,b}) \geq \lfloor \frac{ab}{a+b-1} \rfloor$. So we have proved that $\kappa_{a+b}(K_{a,b}) = \lfloor \frac{ab}{a+b-1} \rfloor$. ∎

3 The k-connectivity of complete bipartite graphs

Next, we will calculate $\kappa_k(K_{a,b})$, for $2 \leq k \leq a+b$.

Recall that $\kappa_k(G) = \min\{\kappa(S)\}$, where the minimum is taken over all k-element subsets S of $V(G)$. Denote by $K_{a,\ b}$ a complete bipartite graph with bipartition $X = \{x_1, x_2, \ldots, x_a\}$ and $Y = \{y_1, y_2, \ldots, y_b\}$, where $a \leq b$. Actually, all vertices in X are equivalent and all vertices in Y are equivalent. So instead of considering all k-element subsets S of $V(G)$, we can restrict our attention to the subsets S_i, for $0 \leq i \leq k$, where S_i is an k-element subsets of $V(G)$ such that $S_i \cap X = \{x_1, x_2, \ldots, x_i\}$, $S_i \cap Y = \{y_1, y_2, \ldots, y_{k-i}\}$, $1 \leq i \leq k$ and $S_0 \cap X = \emptyset$, $S_0 \cap Y = \{y_1, y_2, \ldots, y_k\}$. Notice that, if $i > a$ or $k - i > b$ then S_i does not exist, and if $k > b$ then S_0 does not exist. So, we need only to consider S_i for $\max\{0,\ k-b\} \leq i \leq \min\{a,\ k\}$.

Now, let A be a maximum set of internally disjoint trees connecting S_i. Let $\mathfrak{A}_0$ be the set of trees connecting S_i whose vertex set is S_i, let $\mathfrak{A}_1$ be the set of trees connecting S_i whose vertex set is $S_i \cup \{u\}$, where $u \notin S_i$ and let $\mathfrak{A}_2$ be the set of trees connecting S_i whose vertex set is $S_i \cup \{u, v\}$, where $u, v \notin S_i$ and they belong to distinct partitions.

Lemma 3.1. *Let A be a maximum set of internally disjoint trees connecting S_i. Then we can always find a set A' of internally disjoint trees connecting S_i, such that $\mid A \mid = \mid A' \mid$ and $A' \subset \mathfrak{A}_0 \cup \mathfrak{A}_1 \cup \mathfrak{A}_2$.*

Proof. If there is a tree T^0 in A whose vertex set $V(T^0) \supseteq \{u_1, u_2\}$, where $u_1, u_2 \notin S_i$ and u_1, u_2 belong to the same partition, then we can connect all neighbors of u_2 to u_1 by some new edges and delete u_2 and the multiple edges (if exist). Obviously, the new graph we obtain is still a tree T' that connect S_i. Since $V(T_m) \cap V(T_n) = S_i$ for every pair of trees in A, other trees in A will not contain u_1, including the edges incident with u_1. So for all trees T_n in A other than T^0, $V(T') \cap V(T_n) = S_i$ and $E(T') \cap E(T_n) = \emptyset$. Moreover, T' has less vertices which are not in S_i than T^0. Repeat this process, until we get a tree $T \in \mathfrak{A}_0 \cup \mathfrak{A}_1 \cup \mathfrak{A}_2$. Replace A by $A^1 = A \setminus \{T^0\} \cup \{T\}$, and then A^1 contains less trees that are not in $\mathfrak{A}_0 \cup \mathfrak{A}_1 \cup \mathfrak{A}_2$ than A. Repeating the process, we can get a series of sets $A^0, A^1, \ldots, A^t$, such that $A^0 = A$, $A^t = A'$, and A^j contains less trees not in $\mathfrak{A}_0 \cup \mathfrak{A}_1 \cup \mathfrak{A}_2$ than A^{j-1} for $1 \leq j \leq t$, where all A^s are sets of internally disjoint trees connecting S_i for $0 \leq s \leq t$, and $\mid A^0 \mid = \cdots = \mid A^t \mid$. So we finally get the set $A' \subset \mathfrak{A}_0 \cup \mathfrak{A}_1 \cup \mathfrak{A}_2$ which has the same cardinality as A. ∎

So, we can assume that the maximum set A of internally disjoint trees connecting S_i is contained in $\mathfrak{A}_0 \cup \mathfrak{A}_1 \cup \mathfrak{A}_2$.

Next, we will define the standard structure of trees in $\mathfrak{A}_0$, $\mathfrak{A}_1$ and $\mathfrak{A}_2$, respectively.

Every tree in $\mathfrak{A}_0$ is of standard structure. A tree T in $\mathfrak{A}_1$ with vertex set $V(T) = S_i \cup \{u\}$, where $u \in X \setminus S_i$, is of standard structure, if u is adjacent to every vertex in $S_i \cap Y$, and every vertex in $S_i \cap X$ has degree 1. A tree T in $\mathfrak{A}_1$ with vertex set $V(T) = S_i \cup \{v\}$, where $v \in Y \setminus S_i$, is of standard structure, if v is adjacent to every vertex in $S_i \cap X$, and every vertex in $S_i \cap Y$ has degree 1. A tree T in $\mathfrak{A}_2$ with vertex set $V(T) = S_i \cup \{u, v\}$, where $u \in X \setminus S_i$ and $v \in Y \setminus S_i$, is of standard structure, if u is adjacent to every vertex in $S_i \cap Y$ and v is adjacent to every vertex in $S_i \cap X$, particularly, we denote the tree by $T_{u,v}$. Denote the set of trees in $\mathfrak{A}_0$ with the standard structure by $\mathcal{A}_0$, clearly, $\mathcal{A}_0 = \mathfrak{A}_0$. Similarly, denote the set of trees in $\mathfrak{A}_1$ and $\mathfrak{A}_2$ with the standard structure by $\mathcal{A}_0$ and $\mathcal{A}_2$, respectively.

Lemma 3.2. *Let A be a maximum set of internally disjoint trees connecting S_i, $A \subset \mathfrak{A}_0 \cup \mathfrak{A}_1 \cup \mathfrak{A}_2$. Then we can always find a set A'' of internally disjoint trees connecting S_i, such that $\mid A \mid = \mid A'' \mid$ and $A'' \subset \mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{A}_2$.*

Proof. Suppose there is a tree T^0 in A such that $T^0 \in \mathfrak{A}_1$ but $T^0 \notin \mathcal{A}_1$, and $V(T^0) = S_i \cup \{u_0\}$, where $u_0 \in X \setminus S_i$. Note that the case $u_0 \in Y \setminus S_i$ is similar. Since $T^0 \notin \mathcal{A}_1$, there are some vertices in $S_i \cap Y$, say $y_{i_1}, \ldots, y_{i_t}$, not adjacent to u_0. Then we can connect y_{i_1} to u_0 by a new edge. It will produce a unique cycle. Delete the other edge incident with y_{i_1} on the cycle. The graph remains a tree. Do the operation to $y_{i_2}, \ldots, y_{i_t}$ in turn. Finally we get a tree T whose vertex set is $S_i \cup \{u_0\}$ and u_0 is adjacent to every vertex in $S_i \cap Y$, that is, T is of standard structure. For each tree $T_n \in A \setminus \{T^0\}$, clearly T_n does not contain u_0, including the edges incident with u_0. So $V(T) \cap V(T_n) = S_i$ and $E(T) \cap E(T_n) = \emptyset$. Replace A by $A^1 = A \setminus \{T^0\} \cup \{T\}$, and then A^1 contains less trees not in $\mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{A}_2$ than A. Suppose that there is a tree T^1 in A such that $T^1 \in \mathfrak{A}_2$ but $T^1 \notin \mathcal{A}_2$ and $V(T^1) = S_i \cup \{u_1, v_1\}$, where $u_1 \in X \setminus S_i$ and $v_1 \in Y \setminus S_i$. T_{u_1,v_1} is the tree in $\mathcal{A}_2$ whose vertex set is $S_i \cup \{u_1, v_1\}$. Then for each tree $T_n \in A \setminus \{T^1\}$, $V(T_{u_1,v_1}) \cap V(T_n) = S_i$ and $E(T_{u_1,v_1}) \cap E(T_n) = \emptyset$. Replace A by $A^1 = A \setminus \{T^1\} \cup \{T_{u_1,v_1}\}$. Then A^1 contains less trees not in $\mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{A}_2$ than A. Repeating the process, we can get a series of sets $A^0, A^1, \ldots, A^t$, such that $A^0 = A$, $A^t = A''$, and A^j contains less trees not in $\mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{A}_2$ than A^{j-1}, for $1 \leq j \leq t$, where all A^s are sets of internally disjoint trees connecting S_i, $A^s \subset \mathfrak{A}_0 \cup \mathfrak{A}_1 \cup \mathfrak{A}_2$, for $0 \leq s \leq t$, and $\mid A^0 \mid = \cdots = \mid A^t \mid$. So we finally get the set $A'' \subset \mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{A}_2$ which has the same cardinality as A. ∎

So, we can assume that the maximum set A of internally disjoint trees connecting S_i is contained in $\mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{A}_2$. Namely, all trees in A are of standard structure.

For simplicity, we denote the union of the vertex sets of all trees in set A by $V(A)$ and the union of the edge sets of all trees in set A by $E(A)$. Let A be a set of internally disjoint trees connecting S_i. Let $A_0 := A \cap \mathcal{A}_0$, $A_1 := A \cap \mathcal{A}_1$ and $A_2 := A \cap \mathcal{A}_2$. Then $A = A_0 \cup A_1 \cup A_2$. Let $U(A) := V(G) \setminus V(A)$.

Lemma 3.3. *Let $A \subset \mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{A}_2$ be a maximum set of internally disjoint trees connecting S_i, $A = A_0 \cup A_1 \cup A_2$ and $U(A) := V(G) \setminus V(A)$. Then either $U(A) \cap X = \emptyset$ or $U(A) \cap Y = \emptyset$.*

Proof. If $U(A) \cap X \neq \emptyset$ and $U(A) \cap Y \neq \emptyset$, let $x \in U(A) \cap X$ and $y \in U(A) \cap Y$. Then the tree $T_{x,y} \in \mathcal{A}_2$ with vertex set $S_i \cup \{x, y\}$ is a tree that connects S_i. Moreover, $V(T) \cap V(A) = S_i$ and for any tree $T' \in A$, T and T' are edge-disjoint. So, $A \cup \{T\}$ is also a set of internally disjoint trees connecting S_i, contradicting to the maximality of A. ∎

So we conclude that if A is a maximum set of internally disjoint trees connecting S_i, then $U(A) \subset X$ or $U(A) \subset Y$.

Lemma 3.4. *Let $A \subset \mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{A}_2$ be a maximum set of internally disjoint trees connecting S_i, $A = A_0 \cup A_1 \cup A_2$ and $U(A) := V(G) \setminus V(A)$. If $U(A) \neq \emptyset$ and $A_0 \neq \emptyset$, then we can find a set $A' = A'_0 \cup A'_1 \cup A'_2$ of internally disjoint trees connecting S_i, such that $|A'_0| = |A_0| - 1$, $|A'_1| = |A_1| + 1$, $A'_2 = A_2$ and $|U(A')| = |U(A)| - 1$.*

Proof. Let $u \in U(A)$ and $T \in A_0$. Without loss of generality, suppose $u \in X$. Then we can connect u to y_1 by a new edge, and the new graph becomes a tree $T' \in \mathfrak{A}_1$. Using the method in Lemma 3.2, we can transform T' into a tree T'' with the standard structure. Then $T'' \in \mathcal{A}_1$. Let $A'_0 := A_0 \setminus T$, $A'_1 := A_1 \cup \{T''\}$ and $A'_2 = A_2$. It is easy to see that $A' = A'_0 \cup A'_1 \cup A'_2$ is a set of internally disjoint trees connecting S_i. Since $|A'_0| = |A_0| - 1$, $|A'_1| = |A_1| + 1$, and $A'_2 = A_2$, A' is a maximum set of internally disjoint trees connecting S_i and $|U(A')| = |U(A)| - 1$. ∎

So, we can assume that for the maximum set A of internally disjoint trees connecting S_i, either $U(A) = \emptyset$ or $A_0 = \emptyset$. Moreover, if A' is a set of internally disjoint trees connecting S_i which we find currently, $U(A') \neq \emptyset$ and the edges in $E(G[S_i]) \setminus E(A')$ can form a tree T in $\mathcal{A}_0$, then we will add to A' the tree T'' in Lemma 3.4 rather than the tree T.

Lemma 3.5. *Let $A \subset \mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{A}_2$ be a maximum set of internally disjoint trees connecting S_i, $A = A_0 \cup A_1 \cup A_2$ and $U(A) := V(G) \setminus V(A)$. If there is a vertex $x \in U(A) \subset X$ and a tree $T \in A_1$ with vertex set $S_i \cup \{y\}$, where $y \in Y \setminus S_i$. Then we can find a set $A' = A'_0 \cup A'_1 \cup A'_2$ of internally disjoint trees connecting S_i, such that $A'_0 = A_0$, $|A'_1| = |A_1| - 1$, $|A'_2| = |A_2| + 1$ and $|U(A')| = |U(A)| - 1$.*

Proof. Let $T_{x,y}$ be the tree in $\mathcal{A}_2$ whose vertex set is $S_i \cup \{x, y\}$. Then $A' = A \setminus T \cup \{T_{x,y}\}$ is just the set we want. ∎

The case that there is a vertex $y \in U(A) \subset Y$ and a tree $T \in A_1$ with vertex set $S_i \cup \{x\}$, where $x \in X \setminus S_i$, is similar. So we can assume that, for the maximum set A of internally disjoint trees connecting S_i, A satisfies one of the following properties:

(1) $U(A) = \emptyset$

(2) $\emptyset \neq U(A) \subset X$ and $V(A_1) \setminus S_i \subset X$

(3) $\emptyset \neq U(A) \subset Y$ and $V(A_1) \setminus S_i \subset Y$

Now, we can see that if $U(A) \neq \emptyset$, then all vertices in $V(A_1) \setminus S_i$ belong to the same partition. Next, we will show that we can always find a set A of internally disjoint trees connecting S_i, such that no matter whether $U(A)$ is empty, all vertices in $V(A_1) \setminus S_i$ belong to the same partition. To show this, we need the following lemma.

Lemma 3.6. *Let p, q be two nonnegative integers. If $p(k-1)+qi \leq i(k-i)$, and there are q vertices $u_1, u_2, \ldots, u_q \in X \setminus S_i$, then we can always find p trees $T_1, T_2, \ldots, T_p$ in $\mathcal{A}_0$ and q trees $T_{p+1}, T_{p+2}, \ldots, T_{p+q}$ in $\mathcal{A}_1$, such that $V(T_j) = S_i$ for $1 \leq j \leq p$, $V(T_{p+m}) = S_i \cup \{u_m\}$ for $1 \leq m \leq q$, and T_r and T_s are edge-disjoint for $1 \leq r < s \leq p+q$. Similarly, if $p(k-1) + q(k-i) \leq i(k-i)$, and there are q vertices $v_1, v_2, \ldots, v_q \in Y \setminus S_i$, then we can always find p trees $T_1, T_2, \ldots, T_p$ in $\mathcal{A}_0$ and q trees $T_{p+1}, T_{p+2}, \ldots, T_{p+q}$ in $\mathcal{A}_1$, such that $V(T_j) = S_i$ for $1 \leq j \leq p$, $V(T_{p+m}) = S_i \cup \{v_m\}$ for $1 \leq m \leq q$, and T_r and T_s are edge-disjoint for $1 \leq r < s \leq p+q$.*

Proof. If $p(k-1)+qi \leq i(k-i)$, then $p(k-1) \leq i(k-i)$, namely $p \leq \lfloor \frac{i(k-i)}{k-1} \rfloor$. Then with the method which we used to find edge-disjoint spanning trees in the proof of Theorem 1.2, we can find p edge-disjoint trees $T_1, T_2, \ldots, T_p$ in $\mathcal{A}_0$, just by taking $a = i$, $b = k-i$ and $t = p$. Moreover, let D_s^p denote the number of edges incident with x_s in all of the p trees, then according to the method, $|D_s^p - D_t^p| \leq 1$ for $1 \leq s, t \leq i$. Now, denote by B_s^p the number of edges incident with x_s which we have not used in the p trees. Then $|B_s^p - B_t^p| \leq 1$ for $1 \leq s, t \leq i$. Since $B_1^p + B_2^p + \cdots + B_i^p = i(k-i) - p(k-1) \geq qi$, $B_s^p \geq q$. Because for each tree in $\mathcal{A}_1$ with vertex set $S_i \cup \{u\}$, where $u \in X \setminus S_i$, the vertices in $S_i \cap X$ all have degree 1, we can find q edge-disjoint trees $T_{p+1}, T_{p+2}, \ldots, T_{p+q}$ in $\mathcal{A}_1$. Since the edges in $T_{p+1}, T_{p+2}, \ldots, T_{p+q}$ are not used in $T_1, T_2, \ldots, T_p$ for $1 \leq r < s \leq p+q$, T_r and T_s are edge-disjoint. The proof of the second half of the lemma is similar. ∎

Lemma 3.7. *Let $A \subset \mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{A}_2$ be a maximum set of internally disjoint trees connecting S_i, $A = A_0 \cup A_1 \cup A_2$ and $U(A) := V(G) \setminus V(A)$. If there are s trees $T_1, T_2, \ldots, T_s \in A_1$ with vertex set $S_i \cup \{u^1\}$, $S_i \cup \{u^2\}$, $\ldots$, $S_i \cup \{u^s\}$ respectively, where $u^j \in X \setminus S_i$ for*

$1 \le j \le s$, *and* t *trees* $T_{s+1}, T_{s+2}, \ldots, T_{s+t} \in A_1$ *with vertex set* $S_i \cup \{v^1\}$, $S_i \cup \{v^2\}$, $\ldots$, $S_i \cup \{v^t\}$ *respectively, where* $v^j \in Y \setminus S_i$ *for* $1 \le j \le t$. *Then we can find a set* $A' = A'_0 \cup A'_1 \cup A'_2$ *of internally disjoint trees connecting* S_i, *such that* $|A| = |A'|$ *and all vertices in* $V(A'_1) \setminus S_i$ *belong to the same partition.*

Proof. Let $|A_0| = p$. Since A is a set of internally disjoint trees connecting S_i, we have $p(k-1) + si + t(k-i) \le i(k-i)$, where si denote the si edges incident with $x_1, \ldots, x_i$ in $T_1, T_2, \ldots, T_s$, and $t(k-i)$ denote the $t(k-i)$ edges incident with $y_1, \ldots, y_{k-i}$ in $T_{s+1}, T_{s+2}, \ldots, T_{s+t}$. If $s \le t$, then $p(k-1) + si + s(k-i) + (t-s)(k-i) \le i(k-i)$, and hence $(p+s)(k-1) + (t-s)(k-i) \le i(k-i)$. Obviously, there are $t-s$ vertices $v^{s+1}, v^{s+2}, \ldots, v^t \in Y \setminus S_i$, and therefore by Lemma 3.6, we can find $p+s$ trees in $\mathcal{A}_0$ and $t-s$ trees in $\mathcal{A}_1$, such that all these trees are internally disjoint trees connecting S_i. Now let A'_0 be the set of the $p+s$ trees in $\mathcal{A}_0$, A'_1 be the set of the $t-s$ trees in $\mathcal{A}_1$ and $A'_2 := A_2 \cup \{T_{u^j, v^j}, 1 \le j \le s\}$. Then $A' = A'_0 \cup A'_1 \cup A'_2$ is just the set we want. The case that $s > t$ is similar. ∎

From Lemmas 3.5 and 3.7, we can see that, if A' is a set of internally disjoint trees connecting S_i which we find currently, $U(A') \cap X \neq \emptyset$ and $U(A') \cap Y \neq \emptyset$, then no matter how many edges are there in $E(G[S_i]) \setminus E(A')$, we always add to A' the trees in $\mathcal{A}_2$ rather than the trees in $\mathcal{A}_1$.

Next, let us state and prove our main result.

Theorem 3.1. *Given any two positive integers* a *and* b, *let* $K_{a,b}$ *denote a complete bipartite graph with a bipartition of sizes* a *and* b, *respectively. Then we have the following results: if* $k > b-a+2$ *and* $a-b+k$ *is odd then*

$$\kappa_k(K_{a,b}) = \frac{a+b-k+1}{2} + \lfloor \frac{(a-b+k-1)(b-a+k-1)}{4(k-1)} \rfloor;$$

if $k > b-a+2$ *and* $a-b+k$ *is even then*

$$\kappa_k(K_{a,b}) = \frac{a+b-k}{2} + \lfloor \frac{(a-b+k)(b-a+k)}{4(k-1)} \rfloor;$$

and if $k \le b-a+2$ *then*

$$\kappa_k(K_{a,b}) = a.$$

Proof. Recall that $\kappa_k(G) = \min\{\kappa(S)\}$, where the minimum is taken over all k-element subsets S of $V(G)$. Let $X = \{x_1, x_2, \ldots, x_a\}$ and $Y = \{y_1, y_2, \ldots, y_b\}$ be the bipartition of $K_{a,b}$, where $a \le b$. As we have mentioned, all vertices in X are equivalent and all vertices in Y are equivalent. So instead of considering all k-element subsets S of $V(G)$, we can restrict our attention to the subsets S_i, for $0 \le i \le k$, where S_i is an k-element subsets of $V(G)$ such that $S_i \cap X = \{x_1, x_2, \dots, x_i\}$, $S_i \cap Y = \{y_1, y_2, \dots, y_{k-i}\}$, $1 \le i \le k$ and $S_0 \cap X = \emptyset$, $S_0 \cap Y = \{y_1, y_2, \dots, y_k\}$. Notice that, if $i > a$ or $k - i > b$ then S_i does not exist, and if $k > b$ then S_0 does not exist. So, we need only to consider S_i for $\max\{0,\ k-b\} \le i \le \min\{a,\ k\}$.

From the above lemmas, we can decide our principle to find the maximum set of internally disjoint trees connecting S_i. Namely, first we find as many trees in $\mathcal{A}_2$ as possible, next we find as many trees in $\mathcal{A}_1$ as possible, and finally we find as many trees in $\mathcal{A}_0$ as possible.

For a set $S_i = \{x_1, x_2, \dots, x_i, y_1, y_2, \dots, y_{k-i}\}$, let A be the maximum set of internally disjoint trees connecting S_i we find with our principle. We now compute $|A|$.

Case 1. $k \le b - a + 2$

Obviously, $\kappa(S_0) = a$.

For S_1, since $k \le b - a + 2$, then

$$b - (k-1) = b - k + 1 \ge a - 2 + 1 = a - 1.$$

So, $|A_2| = a - 1$. If $b - k + 1 = a - 1$, then $|A_1| = 0$, $|A_0| = 1$. If $b - k + 1 > a - 1$, then $|A_1| = 1$, $|A_0| = 0$. No matter which case happens, we have $\kappa(S_1) = |A_2| + |A_1| + |A_0| = a$.

For S_i, $i \ge 2$, since $k \le b - a + 2$, then

$$b - (k-i) = b - k + i \ge a - 2 + i > a - i.$$

So, $|A_2| = a - i$. Since $b - k + i - (a - i) = b - a - k + 2i \ge -2 + 2i \ge i$, then $|A_1| = i$ and $|A_0| = 0$. Thus $\kappa(S_i) = |A_2| + |A_1| + |A_0| = a$.

In summary, if $k \le b - a + 2$, then $\kappa_k(G) = a$.

Case 2. $k > b - a + 2$

First, let us compare $\kappa(S_i)$ with $\kappa(S_{k-i})$, for $0 \le i \le \lfloor \frac{k}{2} \rfloor$. If $a = b$, clearly, $\kappa(S_i) = \kappa(S_{k-i})$. So we may assume that $a < b$.

For $i = 0$, $\kappa(S_0) = a < b = \kappa(S_k)$.

For $1 \le i \le \lfloor \frac{k}{2} \rfloor$, we will give the expressions of $\kappa(S_i)$ and $\kappa(S_{k-i})$.

First for S_i, since every pair of vertices $u \in X \setminus S_i$ and $v \in Y \setminus S_i$ can form a tree $T_{u,v}$, then $|A_2| = \min\{a - i, b - (k - i)\}$. Namely,

$$|A_2| = \begin{cases} a - i & \text{if } \ i \ge \frac{a-b+k}{2} \ ; \\ b - k + i & \text{if } \ i < \frac{a-b+k}{2} \ . \end{cases}$$

Next, since every tree T in A_1 has a vertex in $V \setminus (S_i \cup V(A_2))$, we have

$$|A_1| \leq \begin{cases} b-k+i-(a-i) & \text{if } \ i \geq \frac{a-b+k}{2} \ ; \\ a-i-(b-k+i) & \text{if } \ i < \frac{a-b+k}{2} \ . \end{cases}$$

On the other hand, if the tree T has vertex set $S_i \cup \{u\}$, where $u \in X \setminus S_i$, then every vertex in $S_i \cap X$ is incident with one edge in $E(S_i)$, where $E(S_i)$ denotes the set of edges whose ends are both in S_i. And if the tree T has vertex set $S_i \cup \{v\}$, where $v \in Y \setminus S_i$, then every vertex in $S_i \cap Y$ is incident with one edge in $E(S_i)$. Since every vertex in $S_i \cap X$ is incident with $k-i$ edges in $E(S_i)$ and every vertex in $S_i \cap Y$ is incident with i edges in $E(S_i)$, we have

$$|A_1| \leq \begin{cases} i & \text{if } \ i \geq \frac{a-b+k}{2} \ ; \\ k-i & \text{if } \ i < \frac{a-b+k}{2} \ . \end{cases}$$

Combining the two inequalities, we get

$$|A_1| = \begin{cases} \min\{b-a-k+2i, i\} & \text{if } \ i \geq \frac{a-b+k}{2} \ ; \\ \min\{a-b+k-2i, k-i\} & \text{if } \ i < \frac{a-b+k}{2} \ . \end{cases}$$

Thus

$$|A_1| = \begin{cases} i & \text{if } \ i \geq a-b+k \ ; \\ b-a-k+2i & \text{if } \ \frac{a-b+k}{2} \leq i < a-b+k \ ; \\ a-b+k-2i & \text{if } \ i < \frac{a-b+k}{2} \ . \end{cases}$$

Finally, by Lemma 3.6 we have

$$|A_0| = \begin{cases} \lfloor \frac{i(k-i)-|A_1|(k-i)}{k-1} \rfloor & \text{if } \ i \geq \frac{a-b+k}{2} \ ; \\ \lfloor \frac{i(k-i)-|A_1|i}{k-1} \rfloor & \text{if } \ i < \frac{a-b+k}{2} \ . \end{cases}$$

Thus

$$|A_0| = \begin{cases} 0 & \text{if } \ i \geq a-b+k \ ; \\ \lfloor \frac{[i-(b-a-k+2i)](k-i)}{k-1} \rfloor & \text{if } \ \frac{a-b+k}{2} \leq i < a-b+k \ ; \\ \lfloor \frac{[k-i-(a-b+k-2i)]i}{k-1} \rfloor & \text{if } \ i < \frac{a-b+k}{2} \ . \end{cases}$$

And hence

$$\kappa(S_i) = \begin{cases} a & \text{if } \ i \geq a-b+k \ ; \\ b-k+i+\lfloor \frac{[i-(b-a-k+2i)](k-i)}{k-1} \rfloor & \text{if } \ \frac{a-b+k}{2} \leq i < a-b+k \ ; \\ a-i+\lfloor \frac{[k-i-(a-b+k-2i)]i}{k-1} \rfloor & \text{if } \ i < \frac{a-b+k}{2} \ . \end{cases}$$

Notice that $i \geq 1$, and hence $k-i \leq k-1$.

If $\frac{a-b+k}{2} \leq i < a-b+k$, then

$$\lfloor \frac{[i-(b-a-k+2i)](k-i)}{k-1} \rfloor \leq i-(b-a-k+2i) = a-b+k-i.$$

So, $\kappa(S_i) \leq b-k+i+a-b+k-i = a$.

If $i < \frac{a-b+k}{2}$, then $a-b+k-2i > 0$, $k-i-(a-b+k-2i) < k-i \leq k-1$, and hence

$$\lfloor \frac{[k-i-(a-b+k-2i)]i}{k-1} \rfloor \leq i.$$

So, $\kappa(S_i) \leq a-i+i = a$

Thus $\kappa(S_i) \leq a$, for $i \geq 1$.

Next, considering S_{k-i}, similarly, we have

$$|A_2| = \min\{a-(k-i), b-i\}.$$

Since $a < b$ and $i \leq \lfloor \frac{k}{2} \rfloor \leq \lceil \frac{k}{2} \rceil \leq k-i$, then $b-i > a-(k-i)$. So $|A_2| = a-k+i$ and $|A_1| = \min\{b-i-(a-k+i), k-i\}$. Hence

$$|A_1| = \begin{cases} k-i & \text{if } \ i \leq b-a \ ; \\ b-a+k-2i & \text{if } \ i > b-a \ . \end{cases}$$

Moreover,

$$|A_0| = \begin{cases} 0 & \text{if } \ i \leq b-a \ ; \\ \lfloor \frac{[k-i-(b-a+k-2i)]i}{k-1} \rfloor & \text{if } \ i > b-a \ . \end{cases}$$

So,

$$\kappa(S_{k-i}) = \begin{cases} a & \text{if } \ i \leq b-a \ ; \\ b-i+\lfloor \frac{[k-i-(b-a+k-2i)]i}{k-1} \rfloor & \text{if } \ i > b-a \ . \end{cases}$$

Now, we can compare $\kappa(S_i)$ with $\kappa(S_{k-i})$. For $i \leq b-a$, $\kappa(S_{k-i}) = a \geq \kappa(S_i)$. For $i > b-a$, there must be $b-a < k-i$, that is, $i < a-b+k$.

If $\frac{a-b+k}{2} \leq i < a-b+k$, then

$$\begin{aligned} \kappa(S_{k-i}) - \kappa(S_i) &= b-i+\lfloor \frac{[k-i-(b-a+k-2i)]i}{k-1} \rfloor \\ &\quad -\{b-k+i+\lfloor \frac{[i-(b-a-k+2i)](k-i)}{k-1} \rfloor\} \\ &\geq (k-2i)+\lfloor \frac{(k-2i)(b-a-k)}{k-1} \rfloor \\ &\geq (k-2i)+\lfloor \frac{(k-2i)(1-k)}{k-1} \rfloor \\ &\geq (k-2i)-(k-2i) = 0. \end{aligned}$$

So, $\kappa(S_{k-i}) \geq \kappa(S_i)$.

If $i < \frac{a-b+k}{2}$, then

$$\begin{aligned} \kappa(S_{k-i}) - \kappa(S_i) &= b-i+\lfloor \frac{[k-i-(b-a+k-2i)]i}{k-1} \rfloor \\ &\quad -\{a-i+\lfloor \frac{[k-i-(a-b+k-2i)]i}{k-1} \rfloor\} \\ &\geq (b-a)+\lfloor \frac{(2i)(a-b)}{k-1} \rfloor. \end{aligned}$$

Since $i < \frac{a-b+k}{2}$, then $2i \leq k-1$, and hence $\frac{(2i)(a-b)}{k-1} \geq a-b$. So, $\kappa(S_{k-i}) - \kappa(S_i) \geq b-a+a-b = 0$. Thus, $\kappa(S_{k-i}) \geq \kappa(S_i)$.

In summary, $\kappa(S_{k-i}) \geq \kappa(S_i)$, for $0 \leq i \leq \lfloor \frac{k}{2} \rfloor$. So, in order to get $\kappa_k(G)$, it is enough to consider $\kappa(S_i)$, for $0 \leq i \leq \lfloor \frac{k}{2} \rfloor$.

Next, let us compare $\kappa(S_i)$ with $\kappa(S_{i+1})$, for $0 \leq i \leq \lfloor \frac{k}{2} \rfloor - 1$. For $i = 0$, $\kappa(S_i) = a \geq \kappa(S_{i+1})$. For $1 \leq i \leq \lfloor \frac{k}{2} \rfloor - 1$,

$$\kappa(S_i) = \begin{cases} a & \text{if } \ i \geq a-b+k \ ; \\ b-k+i+\lfloor \frac{[i-(b-a-k+2i)](k-i)}{k-1} \rfloor & \text{if } \ \frac{a-b+k}{2} \leq i < a-b+k \ ; \\ a-i+\lfloor \frac{[k-i-(a-b+k-2i)]i}{k-1} \rfloor & \text{if } \ i < \frac{a-b+k}{2} \ . \end{cases}$$

and

$$\kappa(S_{i+1}) = \begin{cases} a & \text{if } \ i \geq a-b+k-1 \ ; \\ b-k+i+1+\lfloor \frac{[i+1-(b-a-k+2i+2)](k-i-1)}{k-1} \rfloor & \text{if } \ \frac{a-b+k}{2} - 1 \leq i < a-b+k-1 \ ; \\ a-i-1+\lfloor \frac{[k-i-1-(a-b+k-2i-2)](i+1)}{k-1} \rfloor & \text{if } \ i < \frac{a-b+k}{2} - 1 \ . \end{cases}$$

So, $\kappa(S_{a-b+k}) = \kappa(S_{a-b+k+1}) = \cdots = \kappa(S_{\min\{a,k\}}) = a$.

If $i < \frac{a-b+k}{2} - 1$, then

$$\begin{aligned} \kappa(S_i) - \kappa(S_{i+1}) &= a-i+\lfloor \frac{[k-i-(a-b+k-2i)]i}{k-1} \rfloor \\ &\quad -\{a-i-1+\lfloor \frac{[k-i-1-(a-b+k-2i-2)]i+1}{k-1} \rfloor\} \\ &\geq 1+\lfloor \frac{(a-b-2i-1)}{k-1} \rfloor \\ &\geq 1+\lfloor \frac{1-k}{k-1} \rfloor \\ &\geq 1-1 = 0. \end{aligned}$$

So, $\kappa(S_i) \geq \kappa(S_{i+1})$. Namely, if $a-b+k$ is odd, we have

$$\kappa(S_0) \geq \kappa(S_1) \geq \cdots \geq \kappa(S_{\frac{a-b+k-3}{2}}) \geq \kappa(S_{\frac{a-b+k-1}{2}}).$$

and if $a-b+k$ is even, we have

$$\kappa(S_0) \geq \kappa(S_1) \geq \cdots \geq \kappa(S_{\frac{a-b+k-4}{2}}) \geq \kappa(S_{\frac{a-b+k-2}{2}}).$$

If $i = \frac{a-b+k}{2} - 1$, $\kappa(S_i) = \frac{a+b-k}{2} + 1 + \lfloor \frac{(b-a+k-2)(a-b+k-2)}{4(k-1)} \rfloor$.

If $i = \frac{a-b+k-1}{2}$, $\kappa(S_i) = \frac{a+b-k+1}{2} + \lfloor \frac{(b-a+k-1)(a-b+k-1)}{4(k-1)} \rfloor$.

If $i = \frac{a-b+k}{2}$, $\kappa(S_i) = \frac{a+b-k}{2} + \lfloor \frac{(b-a+k)(a-b+k)}{4(k-1)} \rfloor$.

If $i = \frac{a-b+k+1}{2}$, $\kappa(S_i) = \frac{a+b-k+1}{2} + \lfloor \frac{(b-a+k-1)(a-b+k-1)}{4(k-1)} \rfloor$.

If $a-b+k$ is even, since

$$\begin{aligned}
& (a-b+k)(b-a+k)-(b-a+k-2)(a-b+k-2) \\
= & (a-b+k)(b-a+k)-[(a-b+k)(b-a+k)-2(b-a+k)-2(a-b+k-2)] \\
= & 4(k-1),
\end{aligned}$$

then we have $\kappa(S_{\frac{a-b+k}{2}-1}) = \kappa(S_{\frac{a-b+k}{2}})$. If $a-b+k$ is odd, we have $\kappa(S_{\frac{a-b+k-1}{2}}) = \kappa(S_{\frac{a-b+k+1}{2}})$.

If $\frac{a-b+k}{2} \leq i \leq a-b+k-1$, then

$$\begin{aligned}
\kappa(S_{i+1})-\kappa(S_i) &= b-k+i+1+\lfloor\frac{[i+1-(b-a-k+2i+2)](k-i-1)}{k-1}\rfloor \\
& -\{b-k+i+\lfloor\frac{[i-(b-a-k+2i)](k-i)}{k-1}\rfloor\} \\
&\geq 1+\lfloor\frac{(b-a-2k+2i+1)}{k-1}\rfloor \\
&\geq 1+\lfloor\frac{1-k}{k-1}\rfloor \\
&\geq 1-1=0.
\end{aligned}$$

So, $\kappa(S_{i+1}) \geq \kappa(S_i)$. Namely, if $a-b+k$ is odd, we have

$$\kappa(S_{\frac{a-b+k+1}{2}}) \leq \kappa(S_{\frac{a-b+k+3}{2}}) \leq \cdots \leq \kappa(S_{a-b+k-1}) \leq \kappa(S_{a-b+k}) = a,$$

and if $a-b+k$ is even, we have

$$\kappa(S_{\frac{a-b+k}{2}}) \leq \kappa(S_{\frac{a-b+k+2}{2}}) \leq \cdots \leq \kappa(S_{a-b+k-1}) \leq \kappa(S_{a-b+k}) = a.$$

Thus, if $k > b-a+2$ and $a-b+k$ is odd,

$$\kappa_k(K_{a,b}) = \kappa(S_{\frac{a-b+k-1}{2}}) = \frac{a+b-k+1}{2}+\lfloor\frac{(a-b+k-1)(b-a+k-1)}{4(k-1)}\rfloor,$$

and if $k > b-a+2$ and $a-b+k$ is even,

$$\kappa_k(K_{a,b}) = \kappa(S_{\frac{a-b+k}{2}}) = \frac{a+b-k}{2}+\lfloor\frac{(a-b+k)(b-a+k)}{4(k-1)}\rfloor.$$

The proof is complete. ∎

Notice that, when $k=a+b$, the result coincides with Theorem 1.2.

References

[1] J.A. Bondy and U.S.R. Murty, Graph Theory, GTM 244, Springer, 2008.

[2] G. Chartrand, F. Okamoto, P. Zhang, Rainbow trees in graphs and generalized connectivity, Networks 55(4)(2010), 360–367 .

[3] S. Li, X. Li, W. Zhou, Sharp bounds for the generalized connectivity $\kappa_3(G)$, Discrete Math. 310(2010), 2147C2163.

[4] H. Whitney, Congruent graphs and the connectivity of graphs and the connectivity of graphs, Amer. J. Math. 54(1932), 150–168.