

# A CLASS OF METRIZABLE LOCALLY QUASI-CONVEX GROUPS WHICH ARE NOT MACKEY

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ABSTRACT. A topological group  $(G, \mu)$  from a class  $\mathcal{G}$  of MAP topological abelian groups will be called a *Mackey group* in  $\mathcal{G}$  if it has the following property: if  $\nu$  is a group topology in  $G$  such that  $(G, \nu) \in \mathcal{G}$  and  $(G, \nu)$  has the same continuous characters, say  $(G, \nu)^\wedge = (G, \mu)^\wedge$ , then  $\nu \leq \mu$ .

If LCS is the class of Hausdorff topological abelian groups which admit a structure of a locally convex topological vector space over  $\mathbb{R}$ , it is well-known that every metrizable  $(G, \mu) \in \text{LCS}$  is a Mackey group in LCS. For the class LQC of locally quasi-convex Hausdorff topological abelian groups, it was proved in 1999 that every **complete** metrizable  $(G, \mu) \in \text{LQC}$  is a Mackey group in LQC ([9]). The completeness cannot be dropped within the class LQC as we prove in this paper. In fact, we provide a large family of metrizable precompact (noncompact) groups which **are not** Mackey groups in LQC (Theorem 7.5). Those examples are constructed from groups of the form  $c_0(X)$ , whose elements are the null sequences of a topological abelian group  $X$ , and whose topology is the uniform topology. We first show that for a compact metrizable group  $X \neq \{0\}$  the topological group  $c_0(X)$  is a non-compact complete metrizable locally quasi-convex group, which has **countable** topological dual iff  $X$  is connected. Then we prove that for a connected compact metrizable group  $X \neq \{0\}$  the group  $c_0(X)$  endowed with the product topology induced from the product  $X^{\mathbb{N}}$  is metrizable precompact but not a Mackey group in LQC.

## 1. INTRODUCTION

For a locally convex space  $E$  there always exist a finest topology in the class of locally convex topologies giving rise to the same dual space  $E^*$ . This topology was introduced by Mackey and it was named after him.

A similar setting, as we describe below, can be considered for locally quasi-convex groups, a class which properly contains that of locally convex Hausdorff spaces. However the good results known to hold for the class of locally convex spaces are no longer valid in this new class.

In this paper we change the point of view: starting from sufficiently large classes of abelian groups, we define the corresponding Mackey groups. As will be seen, the theory is much richer and properties like connectedness, local compactness, etc. play an important role, which obviously was not the case in the framework of locally convex spaces.

Let  $X, Y$  be groups. We denote by  $\text{Hom}(X, Y)$  the set of all group homomorphisms from  $X$  to  $Y$ . If  $X, Y$  are topological groups,  $\text{CHom}(X, Y)$  stands for the continuous elements of  $\text{Hom}(X, Y)$ .

A set  $\Gamma \subset \text{Hom}(X, Y)$  will be called separating, if

$$(x_1, x_2) \in X \times X, x_1 \neq x_2 \implies \exists \gamma \in \Gamma, \gamma(x_1) \neq \gamma(x_2).$$

For a topological group  $X$ , a Hausdorff group  $Y$  and a non-empty  $\Gamma \subset \text{Hom}(X, Y)$  we denote by  $\sigma(X, \Gamma)$  the coarsest topology in  $X$  with respect to which all members of  $\Gamma$  are continuous. Note that  $\sigma(X, \Gamma)$  is a group topology in  $X$ ; moreover,  $\sigma(X, \Gamma)$  is Hausdorff iff  $\Gamma$  is separating.

The set

$$\mathbb{S} := \{s \in \mathbb{C} : |s| = 1\}.$$

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is an abelian group with respect to multiplication of complex numbers; it is endowed with the usual topology induced from  $\mathbb{C}$ .

From now on all considered groups will be abelian.

For a group  $G$  an element of  $\text{Hom}(G, \mathbb{S})$  is called a (multiplicative) *character*.

Let  $G$  be a topological group.

We write:

$$G^\wedge := \text{CHom}(G, \mathbb{S}).$$

An element of  $G^\wedge$  is called a *continuous character*. Always  $1 \in G^\wedge$ , where  $1(x) = 1 \in \mathbb{S}$ ,  $\forall x \in G$ . The set  $G^\wedge$  with respect to pointwise multiplication of characters is an abelian group with a neutral element 1.

The group  $G^\wedge$  is called *the topological dual* of  $G$ . We shall not fix in advance a topology in  $G^\wedge$ .

A topological group  $G$  is called *maximally almost periodic*, for short a MAP-group, if  $G^\wedge$  is separating. We denote by MAP also the class of all MAP-groups.

For a (not necessarily discrete) topological group  $G$  the topology  $\sigma(G, G^\wedge)$  is called *the Bohr topology* of  $G$ .

For a group  $G$  and for a group topology  $\tau$  in  $G$  we write  $\tau^+$  for the Bohr topology of  $(G, \tau)$ . Clearly  $\tau^+ \leq \tau$  and  $\tau^+$  is a Hausdorff topology iff  $G$  is MAP.

**Proposition 1.1.** ([10]; cf. also [13, Theorem 2.3.4] and [25]) *For a MAP-group  $(G, \tau)$  the following statements hold:*

- (a)  $\tau^+$  is a precompact group topology.
- (b)  $(G, \tau)^\wedge = (G, \tau^+)^\wedge$ .
- (c)  $(G, \tau)$  is precompact iff  $\tau = \tau^+$ .

**Definition 1.2.** *Let  $(G, \tau)$  be a topological group. A group topology  $\eta$  in  $G$  is said to be compatible for  $(G, \tau)$  if  $(G, \tau)^\wedge = (G, \eta)^\wedge$ .*

The concept of *compatible group topology* is due to Varopoulos [25]. This remarkable paper offers, among other results, a description of *locally precompact* group topologies compatible with a MAP-group  $(G, \tau)$ . It is also proved there that every MAP-group  $(G, \tau)$  admits *at most one locally compact group topology in  $G$  compatible with  $(G, \tau)$*  [25, p. 485]. It is natural to consider the maximum (provided it exists) of all compatible topologies for  $(G, \tau)$ , a topological group in a certain class of groups. The following definition is in the spirit of [4], where a categorical treatment of Mackey groups is given.

**Definition 1.3.** *Let  $\mathcal{G}$  be a class of MAP-groups. A topological group  $(G, \mu)$  will be called a Mackey group in  $\mathcal{G}$  if  $(G, \mu) \in \mathcal{G}$  and if  $\nu$  is a compatible group topology for  $(G, \mu)$ , with  $(G, \nu) \in \mathcal{G}$ , then  $\nu \leq \mu$ .*

**Definition 1.4.** *Let  $\mathcal{G}$  be a class of MAP-groups and  $(G, \nu) \in \mathcal{G}$ . If there exists a group topology  $\mu$  in  $G$  compatible for  $(G, \nu)$  such that  $(G, \mu) \in \mathcal{G}$  and  $(G, \mu)$  is a Mackey group in  $\mathcal{G}$ , then  $\mu$  is called the  $\mathcal{G}$ -Mackey topology in  $G$  associated with  $\nu$ .*

In this paper we will not discuss the problem of existence of  $\mathcal{G}$ -Mackey topologies (cf. [12]).

Let LPC be the class of Hausdorff locally precompact topological abelian groups.

A precompact  $(G, \tau) \in \text{LPC}$  may not be a Mackey group in LPC (for instance, if  $(G, \tau)$  is an infinite discrete group, then  $(G, \tau^+) \in \text{LPC}$  is not a Mackey group in LPC. Clearly,  $(G, \tau) \in \text{LPC}$ ,  $\tau$  is compatible for  $(G, \tau^+)$ , but  $\tau$  is strictly finer than  $\tau^+$ ). However, the metrizable changes the picture as the following statement shows.

**Theorem 1.5.** (cf. [25, Corollary 2 (p.484)]) *Every metrizable  $(G, \mu) \in \text{LPC}$  is a Mackey group in LPC.*

Next we consider another class of groups in which the metrizable groups are again Mackey.

We say that a topological group  $G$  admits a structure of a (locally convex) topological vector space over  $\mathbb{R}$  if there exists a map  $\mathbb{R} \times G \rightarrow G$  making  $G$  a (locally convex) topological vector space over  $\mathbb{R}$ . It is known that whenever a Hausdorff topological abelian group admits a topological vector space structure over  $\mathbb{R}$ , it must be unique.

Let LCS be the class of Hausdorff topological abelian groups which admit a structure of *locally convex* topological vector space over  $\mathbb{R}$ . It is an important consequence of Hahn-Banach theorem that  $\text{LCS} \subset \text{MAP}$ . The next theorem is proved in Section 2.

**Theorem 1.6.** *Every metrizable  $(G, \mu) \in \text{LCS}$  is a Mackey group in LCS.*

**Remark 1.7.** Let MAPVS be the class of MAP-groups which admit a structure of topological vector space over  $\mathbb{R}$ . It is known that  $\text{LCS} \subset \text{MAPVS}$  and that this inclusion is strict. An analogue of Theorem 1.6 fails for MAPVS: *A metrizable  $(G, \mu) \in \text{MAPVS}$  is a Mackey group in MAPVS iff  $(G, \mu)$  is locally compact (cf. [9, Proposition 2.1]).*

A natural class of groups containing  $\text{LPC} \cup \text{LCS}$  is provided by LQC, the class of locally quasi-convex Hausdorff groups (see Definition 2.1). In fact, it is known that

$$(1.1) \quad \text{LPC} \cup \text{LCS} \subset \text{LQC} \subset \text{MAP}$$

where the inclusions are strict (cf. [2, 3, 9]). It was proved in [9] that every **Cech-complete**  $(G, \mu) \in \text{LQC}$  is a Mackey group in LQC. In particular, every locally compact  $(G, \mu) \in \text{LQC}$  is a Mackey group in LQC. We prefer to isolate here another particular case more relevant for the specific purposes of this paper:

**Theorem 1.8.** ([9]) *Every complete metrizable  $(G, \mu) \in \text{LQC}$  is a Mackey group in LQC.*

In view of Theorems 1.5, 1.6, and 1.8, the following question arises:

**Question 1.9.** Is every metrizable  $(G, \mu) \in \text{LQC}$  a Mackey group in LQC ?

We provide a negative answer to this question, in fact we have:

**Theorem 1.10.** *Let  $(G, \tau) \in \text{LQC}$  be a non-precompact group with **countable dual**  $(G, \tau)^\wedge$ . Then  $(G, \tau^+)$  is a metrizable precompact group which is not a Mackey group in LQC.*

In order to use Theorem 1.10 and produce a large scale of counter-example to Question 1.9, we need a general construction of non-precompact locally quasi-convex groups with countable dual (to which the theorem can be applied). This goal is achieved by means of a class of groups which roughly speaking are 'groups of sequences'. Let us denote by  $c_0(X)$  the subgroup of  $X^{\mathbb{N}}$  of all null sequences of  $X$ . The following holds:

**Theorem.** *Let  $X$  be an infinite compact metrizable abelian group,  $u_0$  the uniform topology induced from  $X^{\mathbb{N}}$  on  $c_0(X)$ . Then  $G := (c_0(X), u_0)$  is a nonprecompact locally quasi-convex Polish group. Further, the following assertions are equivalent:*

- (i)  $X$  is connected.
- (ii)  $G^\wedge = (X^\wedge)^{(\mathbb{N})}$ .
- (iii)  $\text{Card } G^\wedge = \aleph_0$ .
- (iv)  $\text{Card } G^\wedge < \mathfrak{c}$ .

To the proof of this theorem (together with other more precise results) are dedicated Sections 3–6. The key point of the proof is the introduction of a class  $\mathfrak{B}$  of groups  $X$  (see Definition 6.1), such that  $c_0(X)^\wedge = (X^\wedge)^{(\mathbb{N})}$  for every precompact group  $X \in \mathfrak{B}$  (Theorem 6.3). It turns out, that this new class  $\mathfrak{B}$  can be described in some cases through well-known properties. Namely, in Corollary 6.13 (resp., Theorem 6.15) we show that in the class of all locally compact (resp., metrizable) groups,  $X \in \mathfrak{B}$  iff  $X$  is connected (resp., locally generated in the sense of Enflo [15], see Definition 6.10). This is the backbone of the proof of the theorem.

In the last section we offer some open questions and conjectures.

**Notation and terminology.** Let  $\mathbb{S}$  denote the circle group and  $\mathbb{S}_+ = \{s \in \mathbb{S} : \text{Re}(s) \geq 0\}$ .  $\mathfrak{c}$  will stand for the cardinality of continuum.

We denote by  $e$  the neutral element of a group. We also use the symbols 0 and 1 instead of  $e$  if the group is known to be additive or multiplicative respectively. For a subset  $A$  of a group  $X$  denote by  $\langle A \rangle$  the subgroup of  $X$  generated by  $A$ .

Let  $X$  be a set. As usual,  $X^{\mathbb{N}}$  will denote the set of all sequences  $\mathbf{x} = (x_n)_{n \in \mathbb{N}}$  of elements of  $X$  and  $(p_n)_{n \in \mathbb{N}}$  the sequence of projections  $X^{\mathbb{N}} \rightarrow X$ .

For a group  $X$ ,  $X^{(\mathbb{N})}$  will be the subgroup of  $X^{\mathbb{N}}$  consisting of all sequences eventually equal to  $e$ . For  $n \in \mathbb{N}$  define an injective homomorphism

$$\nu_n : X \rightarrow X^{(\mathbb{N})}, \quad \text{by } \nu_n(x) = (e, \dots, e, x, e, \dots),$$

where  $x \in X$  is placed in position  $n$ .

If  $X$  is a topological group, let

$$c_0(X) := \{(x_n)_{n \in \mathbb{N}} \in X^{\mathbb{N}} : \lim_n x_n = e\}.$$

Clearly  $c_0(X)$  is a subgroup of  $X^{\mathbb{N}}$  containing  $X^{(\mathbb{N})}$ ; moreover,  $c_0(X) = X^{(\mathbb{N})}$  iff  $X$  has only trivial convergent sequences.

For a topological group  $X$ ,  $\mathcal{N}(X)$  is the set of all neighborhoods of  $e \in X$ . Clearly,  $\mathbb{S}_+ \in \mathcal{N}(\mathbb{S})$ . We write  $X_{\text{co}}^\wedge$  for the group  $X^\wedge$  endowed with the compact-open topology. For a subset  $V$  of  $X$  let:

$$V^\triangleright := \{\xi \in X^\wedge : \xi(V) \subset \mathbb{S}_+\}.$$

If  $X, Y$  are topological abelian groups and  $\varphi \in CHom(X, Y)$ , the mapping  $\varphi^\wedge : Y^\wedge \rightarrow X^\wedge$ , defined by  $\varphi^\wedge(\eta) = \eta \circ \varphi$  for  $\eta \in Y^\wedge$ , is a group homomorphism called *the dual homomorphism*.

The *von Neumann's kernel* of a topological abelian group  $X$  is defined by

$$\mathbf{n}(X) = \bigcap \{\ker \xi : \xi \in X^\wedge\}.$$

Clearly,  $\mathbf{n}(X)$  is a subgroup of  $X$ , and  $X$  is MAP iff  $\mathbf{n}(X) = \{0\}$ . If  $\mathbf{n}(X) = X$  (i.e.,  $X^\wedge = \{1\}$ ),  $X$  is called *minimally almost periodic*.

## 2. LOCALLY QUASI CONVEX GROUPS

Let us recall the definition of a locally quasi convex group.

**Definition 2.1.** [27] *A subset  $A$  of a topological group  $G$  is called **quasi-convex** if for every  $x \in G \setminus A$  there exists  $\chi \in G^\wedge$  such that*

$$\chi(A) \subset \mathbb{S}_+, \quad \text{but } \chi(x) \notin \mathbb{S}_+.$$

*A topological group  $G$  is called **locally quasi-convex** if  $\mathcal{N}(G)$  admits a basis consisting of quasi-convex subsets of  $G$ .*

Similar concepts were defined later in [23], where the terms *polar set* and *locally polar group* are used instead of 'quasi-convex set' and 'locally quasi-convex group'. The author might not have been aware of [27].

The locally precompact groups are a prominent class of locally quasi-convex groups. The following statement characterizes the groups  $X \in \text{LPC}$  with countable duals.

**Proposition 2.2.** *For an infinite locally precompact Hausdorff topological abelian group  $X$  TFAE:*

- (i)  $X$  is precompact metrizable.
- (ii)  $X^\wedge$  is countable.

*Proof.* (i)  $\implies$  (ii). This follows from [18, (24.14)].

(ii)  $\implies$  (i). Let  $Y$  be the completion of  $X$ . It is known that the groups  $X^\wedge$  and  $Y^\wedge$  are algebraically isomorphic, hence,  $Y^\wedge$  is countable. On the other hand,  $Y$  is a locally compact Hausdorff topological abelian group therefore  $Y_{\text{co}}^\wedge$  is LCA. Since a second category countable Hausdorff topological group is discrete, we have that  $Y_{\text{co}}^\wedge$  is a discrete countable group. Hence  $(Y_{\text{co}}^\wedge)_{\text{co}}^\wedge$  is a compact metrizable group. By Pontryagin's theorem,  $Y$  and  $(Y_{\text{co}}^\wedge)_{\text{co}}^\wedge$  are topologically isomorphic. Thus  $Y$  is compact metrizable and its topological subgroup  $X$  is precompact metrizable.  $\square$

**Remark 2.3.** It follows from Proposition 2.2 that if  $X$  is a compact non-metrizable group, then  $\text{Card}(X^\wedge) > \aleph_0$ .

We shall see below that implication (ii)  $\implies$  (i) of Proposition 2.2 may fail if  $X$  is a locally quasi-convex Hausdorff group.

**Proposition 2.4.** *Let  $X$  be a precompact Hausdorff topological group and  $V \in \mathcal{N}(X)$ . Then  $V^\triangleright$  is a finite subset of  $X^\wedge$ .*

*Proof.* It is known that  $V^\triangleright$  is a compact subset of  $X_{c_0}^\wedge$ .

Suppose first that  $X$  is compact. Then  $X_{c_0}^\wedge$  is discrete and in this case  $V^\triangleright$  is finite.

If  $X$  is not compact, then it can be viewed as a dense subgroup of a compact Hausdorff topological group  $K$ . Let  $U$  denote the closure of  $V$  in  $K$ ; clearly  $U \in \mathcal{N}(K)$ . Since  $\mathbb{S}_+$  is closed in  $\mathbb{S}$ , the density of  $V$  in  $U$  implies that

$$V^\triangleright = \{\xi|_X : \xi \in U^\triangleright\}.$$

Now,  $U^\triangleright$  finite, implies that  $V^\triangleright$  is finite as well.  $\square$

Let us conclude this section with the proofs of Theorems 1.6 and 1.10.

**Proof of Theorem 1.6.** Let  $(G, \tau) \in \text{LCS}$ . Then  $(G, \tau)'$  will stand for the set of  $\tau$ -continuous linear forms from  $G$  to  $\mathbb{R}$ . For every  $l \in (G, \tau)'$  the mapping  $\rho_l : G \rightarrow \mathbb{S}$  defined by  $\rho_l(x) = \exp\{2\pi il(x)\}$  for all  $x \in G$  is a continuous character. It is easy to see that the mapping  $\rho : (G, \tau)' \rightarrow (G, \tau)^\wedge$ , given by  $l \mapsto \rho_l$  is an injective group homomorphism. It is surjective as well [18, (23.32)]. Therefore, we have:

$$(2.1) \quad (G, \tau)^\wedge = \{\rho_l : l \in (G, \tau)'\}.$$

Take a metrizable  $(G, \mu) \in \text{LCS}$  and let  $(G, \tau) \in \text{LCS}$  be such that  $(G, \tau)^\wedge = (G, \mu)^\wedge$ . Then from (2.1) we get  $(G, \tau)' = (G, \mu)'$ . Since  $\mu$  is a metrizable locally convex vector topology in  $G$ , the last equality according to [22, IV.3.4] implies that  $\tau \leq \mu$  and we are done.  $\square$

**Proof of Theorem 1.10.** Since  $(G, \tau)^\wedge$  is countable,  $\tau^+$  is metrizable. The topology  $\tau^+$  is precompact and compatible for  $(G, \tau)$  by Proposition 1.1. The group  $(G, \tau^+) \in \text{LQC}$  because precompact groups are locally quasi-convex. Since  $(G, \tau)$  is not precompact we have that  $\tau^+ < \tau$  being  $\tau^+ \neq \tau$  again by Proposition 1.1. Hence  $(G, \tau^+)$  is a metrizable precompact group which is not a Mackey group in LQC.  $\square$

### 3. GROUPS OF SEQUENCES

**3.1. The uniform topology in  $X^\mathbb{N}$ .** In what follows  $X$  will be a fixed Hausdorff topological abelian group. We denote by  $\mathfrak{p}_X$  the product topology in  $X^\mathbb{N}$  and by  $\mathfrak{b}_X$  the box topology in  $X^\mathbb{N}$ .

It is easily verified that the collection

$$\{V^\mathbb{N} : V \in \mathcal{N}(X)\}$$

is a basis at  $e$  for a group topology in  $X^\mathbb{N}$  which we denote by  $\mathfrak{u}_X$  and call *the uniform topology*. In all three cases we shall omit the subscript  $X$  when no confusion is possible.

The topology  $\mathfrak{u}$  in  $X^\mathbb{N}$  is nothing else but the topology of uniform convergence on  $\mathbb{N}$  when the elements of  $X^\mathbb{N}$  are viewed as functions from  $\mathbb{N}$  to  $X$  and  $X$  is considered as a uniform space with respect to its left (=right) uniformity. Since it plays an important role in the sequel, we give in the next proposition an account of its main properties.

We write:

$$\mathfrak{p}_0 := \mathfrak{p}|_{c_0(X)}, \quad \mathfrak{b}_0 := \mathfrak{b}|_{c_0(X)} \quad \text{and} \quad \mathfrak{u}_0 := \mathfrak{u}|_{c_0(X)}.$$

**Proposition 3.2.** *Let  $(X, +)$  be a Hausdorff topological abelian group.*

- (a) *The uniform topology  $\mathfrak{u}$  is a Hausdorff group topology in  $X^\mathbb{N}$  with  $\mathfrak{p} \leq \mathfrak{u} \leq \mathfrak{b}$ . Moreover,*
  - (a<sub>1</sub>)  $\mathfrak{p}|_{X^{(\mathbb{N})}} = \mathfrak{u}|_{X^{(\mathbb{N})}} \iff X = \{0\}$ .
  - (a<sub>2</sub>)  $\mathfrak{u}|_{X^{(\mathbb{N})}} = \mathfrak{b}|_{X^{(\mathbb{N})}} \implies X$  is a P-group  $\implies \mathfrak{u} = \mathfrak{b}$ ; in particular, if  $X$  is metrizable and  $\mathfrak{u}|_{X^{(\mathbb{N})}} = \mathfrak{b}|_{X^{(\mathbb{N})}}$ , then  $X$  is discrete.
- (b) *The passage from  $X$  to  $(X^\mathbb{N}, \mathfrak{u})$  preserves (sequential) completeness, metrizability, MAP and local quasi-convexity.*
- (c) *If  $X \neq \{0\}$ , then:*
  - (c<sub>1</sub>)  $(X^\mathbb{N}, \mathfrak{u})$  is not separable.
  - (c<sub>2</sub>)  $(X^\mathbb{N}, \mathfrak{u}|_{X^{(\mathbb{N})}})$  is not precompact and hence,  $(c_0(X), \mathfrak{u}_0)$  and  $(X^\mathbb{N}, \mathfrak{u})$  are neither precompact.

*Proof.* (a) The first assertion has a straightforward proof.

(a<sub>1</sub>) Suppose that  $X \neq \{0\}$ . Take  $x \in X \setminus \{0\}$ . Then  $\nu_k(x) \in X^{(\mathbb{N})}$ ,  $k = 1, 2, \dots$  and the sequence  $(\nu_k(x))_{k \in \mathbb{N}}$  tends to 0 in  $\mathfrak{p}$ . Since  $X$  is Hausdorff, there is a  $V \in \mathcal{N}(X)$  such that  $x \notin V$ . Then  $\nu_k(x) \notin V^{\mathbb{N}}$ ,  $k = 1, 2, \dots$ . Hence, the sequence  $(\nu_k(x))_{k \in \mathbb{N}}$  does not tend to 0 in  $\mathfrak{u}$ .

(a<sub>2</sub>) Suppose that  $\mathfrak{u}|_{X^{(\mathbb{N})}} \geq \mathfrak{b}|_{X^{(\mathbb{N})}}$ . Take arbitrarily  $U_n \in \mathcal{N}(X)$ ,  $n = 1, 2, \dots$ . Then  $(\prod_{n \in \mathbb{N}} U_n) \cap X^{(\mathbb{N})}$  is a neighborhood of zero in  $\mathfrak{b}|_{X^{(\mathbb{N})}}$ . As  $\mathfrak{u}|_{X^{(\mathbb{N})}} \geq \mathfrak{b}|_{X^{(\mathbb{N})}}$ , there is a  $V \in \mathcal{N}(X)$  such that

$$V^{\mathbb{N}} \cap X^{(\mathbb{N})} \subset \left( \prod_{n \in \mathbb{N}} U_n \right) \cap X^{(\mathbb{N})}.$$

From  $\nu_k(V) \subseteq V^{\mathbb{N}} \cap X^{(\mathbb{N})}$ ,  $k = 1, 2, \dots$ , we get:  $\nu_k(V) \subseteq (\prod_{n \in \mathbb{N}} U_n) \cap X^{(\mathbb{N})}$ ,  $k = 1, 2, \dots$ . So,  $V \subseteq U_n$ ,  $n = 1, 2, \dots$ , therefore  $V \subset \bigcap_{n \in \mathbb{N}} U_n$ . Thus, for each sequence  $U_n \in \mathcal{N}(X)$ ,  $n = 1, 2, \dots$ ,  $\bigcap_{n \in \mathbb{N}} U_n \in \mathcal{N}(X)$ . Consequently,  $X$  is a P-group.

The implication ' $X$  is a P-group  $\implies \mathfrak{u} = \mathfrak{b}$ ' is easy to verify.

The last assertion in (a<sub>2</sub>) follows from the well-known fact that a metrizable P-space is discrete.

(b) We omit the standard proofs of the first two cases.

Assume that  $X$  is MAP and let  $\mathbf{x} = (x_n)_{n \in \mathbb{N}} \in X^{\mathbb{N}} \setminus \{\mathbf{0}\}$ . Take  $n \in \mathbb{N}$  such that  $p_n(\mathbf{x}) \neq 0$ . Since  $X$  is MAP, there is  $\xi \in X^\wedge$  such that  $\xi(p_n(\mathbf{x})) \neq 1$ . Clearly  $p_n$  is  $\mathfrak{u}$ -continuous, therefore  $\varphi := \xi \circ p_n \in (X^{\mathbb{N}}, \mathfrak{u})^\wedge$  and  $\varphi(\mathbf{x}) \neq 1$ . Hence,  $(X^{\mathbb{N}}, \mathfrak{u})$  is MAP.

In order to prove that  $(X^{\mathbb{N}}, \mathfrak{u})$  is locally quasi-convex provided that  $X$  has the same property, just observe that for any quasi-convex  $V \in \mathcal{N}(X)$ ,  $V^{\mathbb{N}} = \bigcap_{n \in \mathbb{N}} p_n^{-1}(V)$ , and quasi-convexity is preserved under inverse images by continuous homomorphisms and under arbitrary intersections.

(c<sub>1</sub>) Let us fix  $x \in X \setminus \{0\}$  and  $U \in \mathcal{N}(X)$  such that  $U \cap \{-x, x\} = \emptyset$ . Write  $C = \{0, x\}^{\mathbb{N}}$ . Then

$$(3.1) \quad \text{card}(C) = \mathfrak{c}, \quad \mathbf{y}_1 \in C, \mathbf{y}_2 \in C, \mathbf{y}_1 \neq \mathbf{y}_2 \implies \mathbf{y}_1 - \mathbf{y}_2 \notin U^{\mathbb{N}}.$$

Take a symmetric  $V \in \mathcal{N}(X)$  such that  $V + V \subset U$ . Let  $D$  be a dense subset of  $(X^{\mathbb{N}}, \mathfrak{u})$ . Then for every  $\mathbf{y} \in C$  we can find a  $\mathbf{d}_\mathbf{y} \in D$  such that  $\mathbf{d}_\mathbf{y} \in V + \mathbf{y}$ . From (3.1) we get:

$$\mathbf{y}_1 \in C, \mathbf{y}_2 \in C, \mathbf{y}_1 \neq \mathbf{y}_2 \implies \mathbf{d}_{\mathbf{y}_1} \neq \mathbf{d}_{\mathbf{y}_2}.$$

Thus, for any dense subset  $D$ ,  $\text{card}(D) \geq \mathfrak{c}$ , and therefore  $(X^{\mathbb{N}}, \mathfrak{u})$  is nonseparable.

(c<sub>2</sub>) Fix  $x \in X$ ,  $x \neq 0$  and a symmetric  $V \in \mathcal{N}(X)$  such that  $x \notin V$ . Then

$$(3.2) \quad m, n \in \mathbb{N}, m \neq n \implies \nu_m(x) - \nu_n(x) \notin V^{\mathbb{N}}.$$

Now (3.2) together with  $\nu_m(x) - \nu_n(x) \in X^{(\mathbb{N})}$ ,  $\forall m, n \in \mathbb{N}$ , yield that  $(X^{(\mathbb{N})}, \mathfrak{u}|_{X^{(\mathbb{N})}})$  is not precompact.  $\square$

**Remark 3.3.** If  $X$  is a compact metrizable group and  $\rho$  is an invariant metric for  $X$ , then the equality

$$d_\infty(\mathbf{x}, \mathbf{y}) = \sup_{n \in \mathbb{N}} \rho(x_n, y_n), \quad \mathbf{x}, \mathbf{y} \in X^{\mathbb{N}}$$

defines an invariant metric for  $(X^{\mathbb{N}}, \mathfrak{u})$ . In particular, the topology of  $(\mathbb{S}^{\mathbb{N}}, \mathfrak{u})$  can be induced by the following metric:

$$d_\infty(\mathbf{x}, \mathbf{y}) = \sup_{n \in \mathbb{N}} |x_n - y_n|, \quad \mathbf{x}, \mathbf{y} \in \mathbb{S}^{\mathbb{N}}. \quad (\dagger)$$

According to [11, Example 4.2] the metric group  $(\mathbb{S}^{\mathbb{N}}, d_\infty)$  has the following remarkable property: it is not precompact, but every uniformly continuous real-valued function defined on it, is bounded.

**3.4. The group of null sequences  $c_0(X)$ .** In this section we study the group of all null sequences of a topological abelian group  $X$  as a subgroup of  $(X^{\mathbb{N}}, \mathfrak{u})$ .

**Lemma 3.5.** *Let  $X$  be a topological group. Then:*

- (a)  $c_0(X)$  is closed in  $(X^{\mathbb{N}}, \mathfrak{u})$ .
- (b) If  $\mathbf{x} = (x_n)_{n \in \mathbb{N}} \in c_0(X)$ , then the sequence  $(\sum_{k=1}^n \nu_k(x_k))_{n \in \mathbb{N}}$  converges to  $\mathbf{x}$  in the topology  $\mathfrak{u}$ ; in particular,  $X^{(\mathbb{N})}$  is a  $\mathfrak{u}$ -dense subset of  $c_0(X)$ .

*Proof.* In order to prove (a), pick  $\mathbf{x} = (x_n)_{n \in \mathbb{N}} \in c_0(X)$ . There exists then  $V \in \mathcal{N}(X)$  and a subsequence  $x_{n_k} \notin V$ ,  $k \in \mathbb{N}$ . Take a symmetric  $V_1 \in \mathcal{N}(X)$  such that  $V_1 + V_1 \subset V$ . Now we have:  $(\mathbf{x} + (V_1)^{\mathbb{N}}) \cap c_0(X) = \emptyset$ . In fact, for any  $\mathbf{z} \in (V_1)^{\mathbb{N}}$ ,  $x_{n_k} + z_{n_k} \notin V_1$  for otherwise  $x_{n_k} \in V_1 + V_1 \subset V$ . Therefore  $\mathbf{x} + \mathbf{z} \notin c_0(X)$ .

In order to prove (b), let  $\mathbf{y}_n = \sum_{k=1}^n \nu_k(x_k)$  for  $n \in \mathbb{N}$ . Fix  $U \in \mathcal{N}(X)$  and pick a symmetric  $V \in \mathcal{N}(X)$  with  $V \subset U$ . Since  $\mathbf{x} = (x_n)_{n \in \mathbb{N}} \in c_0(X)$ , for some  $k_0 \in \mathbb{N}$  we have  $x_k \in V$ ,  $\forall k \geq k_0$ , i.e.,  $\mathbf{y}_k - \mathbf{x} \in V^{\mathbb{N}} \subset U^{\mathbb{N}}$ . Hence  $\mathbf{y}_k \in \mathbf{x} + U^{\mathbb{N}}$   $\forall k \geq k_0$  and therefore  $\mathbf{y}_n$  converges to  $\mathbf{x}$ . This proves also the last assertion of (b).  $\square$

Thus, the situation described in the previous Lemma is:

$$X^{(\mathbb{N})} \underset{c}{\overset{u\text{-densely}}{\subset}} c_0(X) \underset{c}{\overset{u\text{-closed}}{\subset}} X^{\mathbb{N}}$$

Since the groups of the form  $(c_0(X), \mathbf{u}_0)$  are the main object of our future considerations, we summarize now those properties inherited from the corresponding ones in  $(X^{\mathbb{N}}, \mathbf{u})$ , or lifted from properties of  $X^{(\mathbb{N})}$ .

**Proposition 3.6.** *Let  $X$  be a Hausdorff topological abelian group.*

- (a)  $(c_0(X), \mathbf{u}_0)$  is a Hausdorff topological group having as a basis at zero the collection  $\{V^{\mathbb{N}} \cap c_0(X) : V \in \mathcal{N}(X)\}$ .
- (b)  $\mathfrak{p}_0 \leq \mathbf{u}_0 \leq \mathfrak{b}_0$ . Moreover,  $\mathfrak{p}_0 = \mathbf{u}_0 \iff X = \{0\}$ ; if  $X$  is metrizable and  $\mathbf{u}_0 = \mathfrak{b}_0$ , then  $X$  is discrete.
- (c) The passage from  $X$  to  $(c_0(X), \mathbf{u}_0)$  preserves (sequential) completeness, metrizability, separability, MAP, local quasi-convexity, non-discreteness, and connectedness.

*Proof.* (c) Assume  $X$  is separable. The density of  $X^{(\mathbb{N})}$  in  $c_0(X)$  yields that  $c_0(X)$  is also separable. If  $(c_0(X), \mathbf{u}_0)$  is discrete, for some  $V \in \mathcal{N}(X)$  we have that  $V^{\mathbb{N}} \cap c_0(X) = \{0\}$ . Thus  $V = \{0\}$  and  $X$  is discrete.

The rest of (c) (except connectedness), as well as (a,b) follows from Proposition 3.2.

Assume now that  $X$  is connected. Consequently, the product spaces  $X^n$  are also connected, for all  $n \in \mathbb{N}$ . Let

$$G_n := \{\mathbf{x} \in c_0(X) : x_k = 0, k = n + 1, n + 2, \dots\}$$

Then  $(G_n, \mathbf{u}|_{G_n})$  is topologically isomorphic to  $(X^n, \mathfrak{p}|_{X^n})$ , therefore connected. Since  $X^{(\mathbb{N})} = \bigcup_{n \in \mathbb{N}} G_n$  and  $\bigcap_{n \in \mathbb{N}} G_n \neq \emptyset$ , we obtain that  $(X^{(\mathbb{N})}, \mathbf{u}|_{X^{(\mathbb{N})})$  and its closure  $(c_0(X), \mathbf{u}_0)$  are connected.  $\square$

**Remark 3.7.** The metric group  $(c_0(\mathbb{S}), d_\infty)$  was introduced by Rolewicz in [21], where he proves that it is a monothetic group. As he underlines, a monothetic and completely metrizable group need not be compact or discrete, a fact which breaks the dichotomy existing in the class of LCA-groups: namely, a monothetic LCA-group must be either compact or discrete ([26, Lemme 26.2 (p. 96)]; see also [1, Remark 5], where a construction of a different example of a complete metrizable monothetic non-locally compact group is indicated).

A proof of the fact that  $(c_0(\mathbb{S}), \mathbf{u}_0)$  is monothetic is contained in [13, pp. 20–21] (cf. also [16], where it is shown further that  $(c_0(\mathbb{S}), \mathbf{u}_0)$  is Pontryagin reflexive).

**Remark 3.8.** Let  $X$  be the group  $\mathbb{R}$  with the usual topology.

- (1) By Proposition 3.2  $(\mathbb{R}^{\mathbb{N}}, \mathbf{u})$  is a complete metrizable topological abelian group. The group  $(\mathbb{R}^{\mathbb{N}}, \mathbf{u})$  is not connected; the connected component of the null element coincides with  $l_\infty$  and the topology  $\mathbf{u}|_{l_\infty}$  is the usual Banach-space topology of  $l_\infty$ . It follows that although  $\mathbb{R}^{\mathbb{N}}$  is a vector space over  $\mathbb{R}$ , the topological group  $(\mathbb{R}^{\mathbb{N}}, \mathbf{u})$  is **not** a topological vector space over  $\mathbb{R}$ .
- (2) By Proposition 3.6 (c),  $(c_0(\mathbb{R}), \mathbf{u}_0)$  is a complete separable metrizable connected topological abelian group. Note that  $c_0(\mathbb{R})$  is a vector space over  $\mathbb{R}$  and  $(c_0(\mathbb{R}), \mathbf{u}_0)$  is a topological vector space over  $\mathbb{R}$ . The topology  $\mathbf{u}_0$  is the usual Banach-space topology of  $c_0$ .
- (3) It is easy to see that  $\mathbb{Z}^{(\mathbb{N})}$  is a closed subgroup of  $(c_0(\mathbb{R}), \mathbf{u}_0)$  and the quotient group

$$(c_0(\mathbb{R}), \mathbf{u}_0) / \mathbb{Z}^{(\mathbb{N})}$$

is topologically isomorphic with  $(c_0(\mathbb{S}), \mathbf{u}_0)$ .

For an additive topological abelian group  $X$  we introduce the following three subgroups of  $X^{\mathbb{N}}$  included between  $X^{(\mathbb{N})}$  and  $c_0(X)$ :

$$\begin{aligned} \text{cs}(X) &= \{\mathbf{x} = (x_n)_{n \in \mathbb{N}} \in X^{\mathbb{N}} : \left( \sum_{k=1}^n x_k \right)_{n \in \mathbb{N}} \text{ is a Cauchy sequence in } X\}, \\ \text{ss}(X) &= \{\mathbf{x} = (x_n)_{n \in \mathbb{N}} \in X^{\mathbb{N}} : \left( \sum_{k=1}^n x_k \right)_{n \in \mathbb{N}} \text{ is a convergent sequence in } X\}. \end{aligned}$$

and

$$l(X) = \{\mathbf{x} = (x_n)_{n \in \mathbb{N}} \in X^{\mathbb{N}} : (x_{\sigma(n)})_{n \in \mathbb{N}} \in \text{ss}(X) \text{ for every bijection } \sigma : \mathbb{N} \rightarrow \mathbb{N}\}.$$

The same notation will be used if  $X$  is a multiplicative topological abelian group: in fact, these three groups are defined similarly.

Clearly,

$$X^{(\mathbb{N})} \subset l(X) \subset \text{ss}(X) \subset \text{cs}(X) \subset c_0(X).$$

It is easy to observe that for a Hausdorff topological abelian group  $X$  the equality  $\text{ss}(X) = \text{cs}(X)$  holds iff  $X$  is sequentially complete.

The notation  $\text{cs}(X)$  and  $\text{ss}(X)$  are not standard, while  $l(X)$  can be justified as follows: usually  $l$  stands for the set of all real absolutely summable sequences and by Riemann-Dirichlet theorem,

$$(3.3) \quad l(\mathbb{R}) = \{\mathbf{x} = (x_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : (|x_n|)_{n \in \mathbb{N}} \in \text{ss}(\mathbb{R})\} = l.$$

Observe that we also have the following analogue of (3.3) for  $\mathbb{S}$  (cf. [8, Ch. VIII.2, Theorem 1 (p.116)]):

$$(3.4) \quad l(\mathbb{S}) = \{\mathbf{x} = (x_n)_{n \in \mathbb{N}} \in \mathbb{S}^{\mathbb{N}} : (|1 - x_n|)_{n \in \mathbb{N}} \in \text{ss}(\mathbb{R})\}.$$

It is well known that  $l(\mathbb{R}) \neq \text{ss}(\mathbb{R}) \neq c_0(\mathbb{R})$ .

Let us consider now the situation in the general case. First of all we note that if  $X$  has only trivial convergent sequences, then

$$X^{(\mathbb{N})} = l(X) = \text{ss}(X) = \text{cs}(X) = c_0(X).$$

However, for a group  $X$  the equality  $\text{cs}(X) = c_0(X)$  need not imply the equality  $X^{(\mathbb{N})} = c_0(X)$  as the following proposition shows.

**Proposition 3.9.** *Let  $X$  be a topological abelian group.*

- (a) *If  $\mathcal{N}(X)$  admits a base consisting of subgroups of  $X$ , then  $\text{cs}(X) = c_0(X)$ .*
- (b) *If  $X$  is sequentially complete and  $\mathcal{N}(X)$  admits a base consisting of subgroups of  $X$ , then  $l(X) = \text{ss}(X) = c_0(X)$ .*
- (c) *If  $X$  is totally disconnected and locally compact, then  $l(X) = \text{ss}(X) = c_0(X)$ .*

*Proof.* (a) is easy to verify and (b) follows from (a). Finally, (c) follows from (b), since our hypothesis implies that  $\mathcal{N}(X)$  admits a basis consisting of subgroups of  $X$  ([18, Theorem II.7.7 (p. 62)]).  $\square$

We shall see (Remark 6.17) that if a non-trivial group  $X$  is either connected and metrizable or connected and locally compact, then  $\text{ss}(X) \neq c_0(X)$ . *It is not clear whether for a complete metrizable abelian group  $X$  the equality  $l(X) = \text{ss}(X)$  implies the equality  $\text{ss}(X) = c_0(X)$ .*

#### 4. THE $\beta$ -DUAL OF A GROUP OF SEQUENCES

For topological abelian groups  $X, Y$  and a non-empty  $A \subset X^{\mathbb{N}}$  we write:

$$A^\beta(Y) = \{\mathbf{h} = (\xi_n)_{n \in \mathbb{N}} \in (CHom(X, Y))^{\mathbb{N}} : (\xi_n(x_n))_{n \in \mathbb{N}} \in \text{ss}(Y) \quad \forall \mathbf{x} = (x_n)_{n \in \mathbb{N}} \in A\}.$$

The notation  $A^\beta(Y)$  is taken from the theory of sequence spaces, where it is used for  $X = Y = \mathbb{R}$  (see, e.g., [5]).

Instead of  $A^\beta(\mathbb{S})$  we will use the shorter notation  $A^\beta$ . In other words,

$$A^\beta = \{\mathbf{h} = (\xi_n)_{n \in \mathbb{N}} \in (X^\wedge)^{\mathbb{N}} : (\xi_n(x_n))_{n \in \mathbb{N}} \in \text{ss}(\mathbb{S}) \quad \forall \mathbf{x} = (x_n)_{n \in \mathbb{N}} \in A\}.$$

Clearly,  $A^\beta$  is a subgroup of  $(X^\wedge)^{\mathbb{N}}$  containing  $(X^\wedge)^{(\mathbb{N})}$ . If  $A$  is a subgroup, we will call  $A^\beta$  the  $\beta$ -dual of  $A$ .



Let  $A \subset X^{\mathbb{N}}$  be a subgroup and  $\tau$  a group topology in  $A$ . The main motivation to introduce the  $\beta$ -dual comes from the fact that, under appropriate hypotheses,  $(A, \tau)^\wedge$  is canonically isomorphic with  $A^\beta$ . Now we describe several cases when this occurs, being  $A := c_0(X)$  the most important of them (See next section).

**Lemma 4.1.** *Let  $X$  be a topological abelian group. Then*

$$(4.1) \quad \left(X^{(\mathbb{N})}\right)^\beta = (X^\wedge)^\mathbb{N} \quad \text{and} \quad (X^\mathbb{N})^\beta = (X^\wedge)^{(\mathbb{N})}.$$

*Proof.* The first equality in (4.1) is trivial. To prove the second one it is sufficient to show that  $(X^\mathbb{N})^\beta \subset (X^\wedge)^{(\mathbb{N})}$ . Take

$$(4.2) \quad \mathbf{h} = (\xi_n)_{n \in \mathbb{N}} \in (X^\mathbb{N})^\beta$$

and let us see that the assumption

$$(4.3) \quad \mathbf{h} = (\xi_n)_{n \in \mathbb{N}} \notin (X^\wedge)^{(\mathbb{N})}$$

leads to a contradiction with (4.2).

From (4.3) we get the existence of a strictly increasing sequence  $(k_n)_{n \in \mathbb{N}}$  of natural numbers such that

$$(4.4) \quad \xi_{k_n}(X) \neq \{1\}, \quad n = 1, 2, \dots$$

Since  $\mathbb{S}_+$  contains no nontrivial subgroup of  $\mathbb{S}$ , from (4.4) we get the existence of a sequence  $(x_{k_n})_{n \in \mathbb{N}}$  of elements of  $X$  such that

$$(4.5) \quad \xi_{k_n}(x_{k_n}) \notin \mathbb{S}_+, \quad n = 1, 2, \dots$$

Put  $x_j = 0$  for all  $j \in \mathbb{N} \setminus \{k_1, k_2, \dots\}$ . For the sequence  $\mathbf{x} = (x_n)_{n \in \mathbb{N}} \in X^\mathbb{N}$  obtained in this way, the sequence  $(\prod_{k=1}^n \xi_k(x_k))_{n \in \mathbb{N}}$  converges in  $\mathbb{S}$  by virtue of (4.2). This implies that  $\lim_n \xi_n(x_n) = 1$ . Hence,  $\lim_n \xi_{k_n}(x_{k_n}) = 1$ . However, as  $\mathbb{S}_+ \in \mathcal{N}(\mathbb{S})$ , the equality  $\lim_n \xi_{k_n}(x_{k_n}) = 1$  contradicts (4.5).  $\square$

In the following assertion an additive abelian group  $A \subset X^\mathbb{N}$  is treated as a  $\mathbb{Z}^\mathbb{N}$ -module.

**Lemma 4.2.** *Let  $X$  be a topological abelian group,  $A \subset X^\mathbb{N}$  a non-empty subset having the property:*

$$(4.6) \quad \{0, 1\}^\mathbb{N} A \subset A,$$

*$(\xi_n)_{n \in \mathbb{N}} \in A^\beta$  and  $(k_n)_{n \in \mathbb{N}}$  a strictly increasing sequence of natural numbers. Then  $(\xi_{k_n})_{n \in \mathbb{N}} \in A^\beta$ .*

*Proof.* Take an arbitrary  $\mathbf{x} = (x_n)_{n \in \mathbb{N}} \in A$  and define  $\mathbf{y} = (y_n)_{n \in \mathbb{N}} \in X^\mathbb{N}$  as follows:  $y_{k_n} = x_n$ ,  $n = 1, 2, \dots$  and  $y_j = 0$ ,  $\forall j \in \mathbb{N} \setminus \{k_1, k_2, \dots\}$ . Then  $\mathbf{y} = (y_n)_{n \in \mathbb{N}} \in A$  by (4.6).

Since  $\mathbf{y} = (y_n)_{n \in \mathbb{N}} \in A$  and  $(\xi_n)_{n \in \mathbb{N}} \in A^\beta$ , the sequence

$$\left( \prod_{k=1}^n \xi_k(y_k) \right)_{n \in \mathbb{N}}$$

converges in  $\mathbb{S}$ . Observe that

$$(4.7) \quad \prod_{j=1}^{k_n} \xi_j(y_j) = \prod_{j=1}^n \xi_{k_j}(x_j), \quad n = 1, 2, \dots$$

From (4.7) we conclude that the sequence

$$\left( \prod_{j=1}^n \xi_{k_j}(x_j) \right)_{n \in \mathbb{N}}$$

converges in  $\mathbb{S}$  too. Since this is true for an arbitrary  $\mathbf{x} = (x_n)_{n \in \mathbb{N}} \in A$ , we get that  $(\xi_{k_n})_{n \in \mathbb{N}} \in A^\beta$ .  $\square$

For a topological abelian group  $X$ , a subset  $A \subset X^{\mathbb{N}}$  and for a fixed  $\mathbf{h} = (\xi_n)_{n \in \mathbb{N}} \in A^\beta$  we define a mapping  $\chi_{\mathbf{h}} : A \rightarrow \mathbb{S}$  by the equality:

$$\chi_{\mathbf{h}}(\mathbf{x}) = \prod_{n=1}^{\infty} \xi_n(x_n) := \lim_n \prod_{k=1}^n \xi_k(x_k), \quad \mathbf{x} = (x_n)_{n \in \mathbb{N}} \in A.$$

It is easy to observe that

- if  $\mathbf{h} = (\xi_n)_{n \in \mathbb{N}} \in (X^\wedge)^{(\mathbb{N})}$ , then  $\chi_{\mathbf{h}}$  is defined on the whole  $X^{\mathbb{N}}$ .
- If  $A$  is a subgroup of  $X^{\mathbb{N}}$ , then

$$\chi_{\mathbf{h}} \in \text{Hom}(A, \mathbb{S}), \quad \forall \mathbf{h} \in A^\beta.$$

**Notation 4.3.** For a subgroup  $A \subset X^{\mathbb{N}}$ , the letter  $\chi$  will denote in the sequel the mapping

$$\chi : A^\beta \rightarrow \text{Hom}(A, \mathbb{S})$$

defined by the equality:

$$\chi(\mathbf{h}) = \chi_{\mathbf{h}} \quad \forall \mathbf{h} = (\xi_n)_{n \in \mathbb{N}} \in A^\beta.$$

Of course, the mapping  $\chi$  depends on  $A$ , but, to simplify notation, we shall not indicate this dependence.

**Lemma 4.4.** *Let  $X$  be a topological abelian group and  $A \subset X^{\mathbb{N}}$  a subgroup such that  $X^{(\mathbb{N})} \subset A$ . Then, the mapping  $\chi : A^\beta \rightarrow \text{Hom}(A, \mathbb{S})$  is an injective group homomorphism.*

*Proof.* It is easy to see that  $\chi$  is a group homomorphism. Let  $\mathbf{h} = (\xi_n)_{n \in \mathbb{N}} \in \ker(\chi)$ . Then

$$\chi_{\mathbf{h}}(\mathbf{x}) = 1, \quad \forall \mathbf{x} = (x_n)_{n \in \mathbb{N}} \in A.$$

Fix  $n \in \mathbb{N}$  and  $x \in X$ . As  $\nu_n(x) \in X^{(\mathbb{N})} \subset A$ , we get:  $\xi_n(x) = \chi_{\mathbf{h}}(\nu_n(x)) = 1$ . Since  $x \in X$  is arbitrary,  $\xi_n$  must be the null character. Therefore,  $\mathbf{h} = 1 := (1, 1, \dots)$  and  $\ker(\chi) = \{1\}$ . Hence,  $\chi$  is injective.  $\square$

The following example illustrate the usefulness of the non-topological Lemma 4.1.

**Example 4.5.** It is a well-known fact, that if  $X$  is a topological abelian group, then

$$(4.8) \quad \chi_{\mathbf{h}} \in (X^{\mathbb{N}}, \mathfrak{p})^\wedge \quad \forall \mathbf{h} = (\xi_n)_{n \in \mathbb{N}} \in (X^\wedge)^{(\mathbb{N})} = (X^{\mathbb{N}})^\beta$$

and the mapping  $\chi : (X^\wedge)^{(\mathbb{N})} \rightarrow (X^{\mathbb{N}}, \mathfrak{p})^\wedge$  is a group isomorphism.

Indeed, (4.8) is easy to verify. The injectivity of  $\chi$  derives from Lemma 4.4. It remains to show that  $\chi$  is surjective too.

Write  $G = (X^{\mathbb{N}}, \mathfrak{p})$  and fix an arbitrary  $\kappa \in G^\wedge$ . We need to find  $\mathbf{h} = (\xi_n)_{n \in \mathbb{N}} \in (X^\wedge)^{(\mathbb{N})}$  such that  $\kappa = \chi_{\mathbf{h}}$ . For every  $n \in \mathbb{N}$  the homomorphism  $\nu_n : X \rightarrow G$  is continuous. Hence  $\xi_n := \kappa \circ \nu_n \in X^\wedge$ . Let us see that  $\mathbf{h} := (\xi_n)_{n \in \mathbb{N}}$  meets the requirements.

Fix  $\mathbf{x} = (x_n)_{n \in \mathbb{N}} \in X^{\mathbb{N}}$ . Evidently, the sequence  $(\sum_{k=1}^n \nu_k(x_k))_{n \in \mathbb{N}}$  converges in  $G$  to  $\mathbf{x} = (x_n)_{n \in \mathbb{N}}$ . Hence,

$$(4.9) \quad \kappa(\mathbf{x}) = \lim_n \kappa\left(\sum_{k=1}^n \nu_k(x_k)\right) = \lim_n \prod_{k=1}^n \xi_k(x_k).$$

Since  $\mathbf{x} = (x_n)_{n \in \mathbb{N}} \in X^{\mathbb{N}}$  is arbitrary, (4.9) implies that  $\mathbf{h} = (\xi_n)_{n \in \mathbb{N}} \in (X^{\mathbb{N}})^\beta$  and  $\kappa = \chi_{\mathbf{h}}$ . By Lemma 4.1,  $(X^{\mathbb{N}})^\beta = (X^\wedge)^{(\mathbb{N})}$ . Consequently we have found  $\mathbf{h} = (\xi_n)_{n \in \mathbb{N}} \in (X^\wedge)^{(\mathbb{N})}$  such that  $\kappa = \chi_{\mathbf{h}}$ , and the surjectivity of  $\chi$  is thus proved.

**Remark 4.6.** Let  $X$  be a topological abelian group.

- Example 4.5 asserts only that the group  $(X^{\mathbb{N}}, \mathfrak{p})^\wedge$  can be algebraically identified with the group  $(X^\wedge)^{(\mathbb{N})}$  by means of the group isomorphism  $\chi$ . In fact more is known:  $\chi$  is also a homeomorphism between  $(X^{\mathbb{N}}, \mathfrak{p})_{\text{co}}^\wedge$  and  $\left((X^\wedge)^{(\mathbb{N})}, \hat{\mathfrak{b}}\right)$ , where  $\hat{\mathfrak{b}}$  stands for the topology induced from the box product  $\left((X_{\text{co}}^\wedge)^{\mathbb{N}}, \mathfrak{b}\right)$  in  $(X^\wedge)^{(\mathbb{N})}$ .
- An application of (a) for  $X = \mathbb{Z}$  gives that  $(\mathbb{S}^{(\mathbb{N})}, \mathfrak{b}_{|\mathbb{S}^{(\mathbb{N})}})$  is a complete non-metrizable group. In particular, we get that  $\mathbb{S}^{(\mathbb{N})}$  is a closed subgroup of  $(\mathbb{S}^{\mathbb{N}}, \mathfrak{b})$ .
- It is known also that  $(X^{\mathbb{N}}, \mathfrak{b}|_{X^{\mathbb{N}}})_{\text{co}}^\wedge$  is topologically isomorphic with  $\left((X_{\text{co}}^\wedge)^{\mathbb{N}}, \mathfrak{p}\right)$ .

- (d) An application of (c) for  $X = \mathbb{S}$  gives that the group  $(\mathbb{S}^{(\mathbb{N})}, \mathfrak{b}_{|\mathbb{S}^{(\mathbb{N})}})^\wedge$  has cardinality  $\mathfrak{c}$ . It follows that  $\mathfrak{b}_{|\mathbb{S}^{(\mathbb{N})}}$  is not a compatible topology for  $(\mathbb{S}^{(\mathbb{N})}, \mathfrak{p}_{|\mathbb{S}^{(\mathbb{N})}})$  (cf. Proposition 5.3).

## 5. THE TOPOLOGICAL DUAL OF $(c_0(X), \mathbf{u}_0)$ . COINCIDENCE WITH THE $\beta$ -DUAL

In this section we will prove that for a complete metrizable group  $X$ , the dual of the topological group  $(c_0(X), \mathbf{u}_0)$  algebraically coincides with the  $\beta$ -dual. We start calculating the  $\beta$ -dual for the particular group  $c_0(\mathbb{S})$ , which has interest in itself: in fact, from it we derive the first example of a metrizable locally quasi-convex group which is not LQC-Mackey (See Proposition 5.3).

The following statement is a bit more delicate than Lemma 4.1.

**Proposition 5.1.** *For  $c_0(\mathbb{S})$  we have:*

$$(5.1) \quad c_0(\mathbb{S})^\beta = (\mathbb{S}^\wedge)^{(\mathbb{N})}.$$

*Proof.* For a fixed  $m \in \mathbb{Z}$  let  $\varphi_m : \mathbb{S} \rightarrow \mathbb{S}$  be the mapping  $t \mapsto t^m$ . It is known that

$$\mathbb{S}^\wedge = \{\varphi_m : m \in \mathbb{Z}\}.$$

So, fix a sequence  $(m_n)_{n \in \mathbb{N}} \in \mathbb{Z}^{\mathbb{N}}$  such that  $(\varphi_{m_n})_{n \in \mathbb{N}} \in c_0(\mathbb{S})^\beta$  and let us see that in fact  $(m_n)_{n \in \mathbb{N}} \in \mathbb{Z}^{(\mathbb{N})}$ .

Suppose that  $(m_n)_{n \in \mathbb{N}} \notin \mathbb{Z}^{(\mathbb{N})}$ . Then for some strictly increasing sequence  $(k_n)_{n \in \mathbb{N}}$  of natural numbers we shall have:  $m_{k_n} \neq 0$ ,  $n = 1, 2, \dots$ . As  $(\varphi_{m_n})_{n \in \mathbb{N}} \in c_0(\mathbb{S})^\beta$ , by Lemma 4.2 we have:

$$(5.2) \quad (\varphi_{m_{k_n}})_{n \in \mathbb{N}} \in c_0(\mathbb{S})^\beta.$$

Let  $x_1 = x_2 = 1$ ; then for a natural number  $j > 2$  find the unique natural number  $n$  with  $2^n < j \leq 2^{n+1}$  and write

$$x_j = \exp\left(2\pi i \frac{1}{m_{k_j} 2^{n+1}}\right).$$

Clearly,  $\mathbf{x} = (x_j)_{j \in \mathbb{N}} \in c_0(\mathbb{S})$  and

$$(5.3) \quad \prod_{j=2^{n+1}}^{2^{n+1}} \varphi_{m_{k_j}}(x_j) = \exp\left(2\pi i \sum_{k=2^{n+1}}^{2^{n+1}} \frac{1}{2^{n+1}}\right) = -1, \quad n = 1, 2, \dots$$

It follows from (5.3) that

$$\left(\prod_{j=1}^n \varphi_{m_{k_j}}(x_j)\right)_{n \in \mathbb{N}}$$

is not a Cauchy sequence in  $\mathbb{S}$ , hence it is not convergent in  $\mathbb{S}$  in contradiction with (5.2).  $\square$

The next proposition plays a pivotal role in the computation of the dual of  $(c_0(X), \mathbf{u}_0)$ .

**Proposition 5.2.** *Let  $X$  be a topological abelian group and  $G := (c_0(X), \mathbf{u}_0)$ . The following assertions hold:*

- (a)  $\chi_{\mathbf{h}} \in G^\wedge$  for every  $\mathbf{h} = (\xi_n)_{n \in \mathbb{N}} \in (X^\wedge)^{(\mathbb{N})} \subset c_0(X)^\beta$  and the mapping  $\chi : (X^\wedge)^{(\mathbb{N})} \rightarrow G^\wedge$  is an injective group homomorphism.  
 (b) We have:

$$G^\wedge \subset \{\chi_{\mathbf{h}} : \mathbf{h} = (\xi_n)_{n \in \mathbb{N}} \in c_0(X)^\beta\}.$$

- (c) If  $X$  is complete metrizable (or, more generally,  $(c_0(X), \mathbf{u}_0)$  is a Baire space), then we have:

$$G^\wedge = \{\chi_{\mathbf{h}} : \mathbf{h} = (\xi_n)_{n \in \mathbb{N}} \in c_0(X)^\beta\}$$

and the mapping  $\chi : c_0(X)^\beta \rightarrow G^\wedge$  is a group isomorphism.

*Proof.* (a) As  $\mathbf{u}_0 \geq \mathbf{p}_0$  we have  $\chi_{\mathbf{h}} \in G^\wedge$ ,  $\forall \mathbf{h} = (\xi_n)_{n \in \mathbb{N}} \in (X^\wedge)^{(\mathbb{N})}$ . The rest follows from Lemma 4.4.

(b) Fix  $\kappa \in G^\wedge$ . We need to find  $\mathbf{h} = (\xi_n)_{n \in \mathbb{N}} \in c_0(X)^\beta$  such that  $\kappa = \chi_{\mathbf{h}}$ . For every  $n \in \mathbb{N}$  the homomorphism  $\nu_n : X \rightarrow G$  is continuous, so  $\xi_n := \kappa \circ \nu_n \in X^\wedge$ . Let us see that  $\mathbf{h} := (\xi_n)_{n \in \mathbb{N}}$  meets the requirements.

Fix  $\mathbf{x} = (x_n)_{n \in \mathbb{N}} \in c_0(X)$ . By Lemma 3.5 the sequence  $(\sum_{k=1}^n \nu_k(x_k))_{n \in \mathbb{N}}$  converges in  $G$  to  $\mathbf{x} = (x_n)_{n \in \mathbb{N}}$ . Hence,

$$(5.4) \quad \kappa(\mathbf{x}) = \lim_n \kappa\left(\sum_{k=1}^n \nu_k(x_k)\right) = \lim_n \prod_{k=1}^n \xi_k(x_k).$$

Since  $\mathbf{x} = (x_n)_{n \in \mathbb{N}} \in c_0(X)$  is arbitrary, (5.4) implies that  $\mathbf{h} = (\xi_n)_{n \in \mathbb{N}} \in c_0(X)^\beta$  and  $\kappa = \chi_{\mathbf{h}}$ .

(c) Taking into account (b), we only need to see that

$$G^\wedge \supset \{\chi_{\mathbf{h}} : \mathbf{h} = (\xi_n)_{n \in \mathbb{N}} \in c_0(X)^\beta\}.$$

So, fix  $\mathbf{h} = (\xi_n)_{n \in \mathbb{N}} \in c_0(X)^\beta$ . As we have already noted,  $\chi_{\mathbf{h}} : c_0(X) \rightarrow \mathbb{S}$  is a group homomorphism. For  $n \in \mathbb{N}$ , set  $\mathbf{h}_n = (\xi_1, \dots, \xi_n, 1, 1, \dots)$ . Then  $\mathbf{h}_n \in (X^\wedge)^{(\mathbb{N})}$ . Hence,

$$(5.5) \quad \chi_{\mathbf{h}_n} \in G^\wedge, \quad n = 1, 2, \dots$$

Clearly,

$$(5.6) \quad \lim_n \chi_{\mathbf{h}_n}(\mathbf{x}) = \chi_{\mathbf{h}}(\mathbf{x}), \quad \forall \mathbf{x} = (x_n)_{n \in \mathbb{N}} \in c_0(X).$$

Since  $X$  is complete metrizable, by Proposition 3.6(c), the group  $G = (c_0(X), \mathbf{u}_0)$  is complete metrizable too. In particular,  $G = (c_0(X), \mathbf{u}_0)$  is a Baire space. This and relations (5.5) and (5.6), according to Osgood's theorem [19, Theorem 9.5 (pp. 86–87)] imply that the function  $\chi_{\mathbf{h}}$  has a  $\mathbf{u}_0$ -continuity point  $\mathbf{x} = (x_n)_{n \in \mathbb{N}} \in c_0(X)$ . Since  $\chi_{\mathbf{h}}$  is a group homomorphism, we get that  $\chi_{\mathbf{h}}$  is  $\mathbf{u}_0$ -continuous. Therefore,  $\chi_{\mathbf{h}} \in G^\wedge$ .  $\square$

Now we are ready to give the first example of a precompact metrizable group which is not a Mackey group in LQC:

**Proposition 5.3.** *For  $X = \mathbb{S}$  the following assertions hold:*

- (a) (cf. [20, Lemma])  $(c_0(\mathbb{S}), \mathbf{p}_0)^\wedge = (c_0(\mathbb{S}), \mathbf{u}_0)^\wedge$ . In particular, the set  $(c_0(\mathbb{S}), \mathbf{u}_0)^\wedge$  is countable.
- (b)  $\mathbf{u}_0$  is a compatible locally quasi-convex Polish group topology for  $(c_0(\mathbb{S}), \mathbf{p}_0)$ .
- (c)  $(c_0(\mathbb{S}), \mathbf{p}_0)$  is a precompact metrizable group which is not a Mackey group in LQC. Further, it is connected and monothetic.

*Proof.* (a) Since  $\mathbf{p}_0 \leq \mathbf{u}$ , we have:  $(c_0(\mathbb{S}), \mathbf{p}_0)^\wedge \subset (c_0(\mathbb{S}), \mathbf{u}_0)^\wedge$ . To prove the converse inclusion, fix an arbitrary  $\kappa \in (c_0(\mathbb{S}), \mathbf{u}_0)^\wedge$ . By Proposition 5.2(b) there exists  $\mathbf{h} = (\xi_n)_{n \in \mathbb{N}} \in (c_0(\mathbb{S}))^\beta$  such that  $\kappa = \chi_{\mathbf{h}}$ . By Proposition 5.1  $c_0(\mathbb{S})^\beta = (\mathbb{S}^\wedge)^{(\mathbb{N})}$ . Therefore we have:  $\kappa = \chi_{\mathbf{h}}$ , where  $\mathbf{h} = (\xi_n)_{n \in \mathbb{N}} \in (\mathbb{S}^\wedge)^{(\mathbb{N})}$ . Consequently,  $\kappa \in (c_0(\mathbb{S}), \mathbf{p}_0)^\wedge$  and the first part of (a) is proved. The second part of (a) follows from the first one because  $(c_0(\mathbb{S}), \mathbf{p}_0)^\wedge$  is algebraically isomorphic to  $(\mathbb{S}^\mathbb{N}, \mathbf{p})^\wedge$ .

(b)  $\mathbf{u}_0$  is a locally quasi-convex Polish group topology by Proposition 3.6. By (a) it is compatible for  $(c_0(\mathbb{S}), \mathbf{p}_0)$ .

(c) Observe that  $(c_0(\mathbb{S}), \mathbf{p}_0)$  is a topological subgroup of the compact metrizable group  $(\mathbb{S}^\mathbb{N}, \mathbf{p})$ . Therefore it is metrizable and precompact. By (b),  $\mathbf{u}_0$  is a locally quasi-convex group topology compatible for  $(c_0(\mathbb{S}), \mathbf{p}_0)$  and strictly finer than  $\mathbf{p}_0$  (by Proposition 3.6). This proves that  $(c_0(\mathbb{S}), \mathbf{p}_0)$  is not a Mackey group in LQC.

The last two assertions follow respectively from Proposition 3.6 (c) and from Remark 3.7.  $\square$

**Remark 5.4.** It follows from Proposition 3.6 and Proposition 5.3(a), that  $(c_0(\mathbb{S}), \mathbf{u}_0)$  is a non-precompact locally quasi-convex group with countable dual  $(c_0(\mathbb{S}), \mathbf{u}_0)^\wedge$ ; consequently by Theorem 1.10  $(c_0(\mathbb{S}), (\mathbf{u}_0)^+)$  is a metrizable precompact group which is not a Mackey group in LQC. Observe, however that (again by Proposition 5.3(a)) we have:  $(\mathbf{u}_0)^+ = \mathbf{p}_0$  and we get a second proof of Proposition 5.3(c).

We shall see below that the group  $\mathbb{S}$  in Proposition 5.3 can be replaced by an arbitrary non-trivial compact connected metrizable group (see Theorem 7.5). However the proof of this fact will require a subtle preparation, to which the rest of the paper is devoted.

6. THE CLASS  $\mathfrak{B}$ 

In this section we introduce a large class of compact metrizable groups  $X$  that can be used as input in Proposition 5.3.

**Definition 6.1.** For a topological abelian group  $X$ , let

$$\Gamma_{\text{abs}}(X) := \{\xi \in X^\wedge : (\xi(x_n))_{n \in \mathbb{N}} \in \text{ss}(\mathbb{S}) \quad \forall \mathbf{x} = (x_n)_{n \in \mathbb{N}} \in c_0(X)\}.$$

Clearly,  $\Gamma_{\text{abs}}(X)$  is a subgroup of  $X^\wedge$ . Denote by  $\mathfrak{B}$  the class of groups  $X$  such that  $\Gamma_{\text{abs}}(X) = \{1\}$ .

**Remark 6.2.** Let  $X$  be a topological abelian group.

- (a) The notation  $\Gamma_{\text{abs}}(X)$  is justified by the following facts easy to prove:

$$\Gamma_{\text{abs}}(X) = \{\xi \in X^\wedge : (\xi(x_n))_{n \in \mathbb{N}} \in l(\mathbb{S}), \quad \forall \mathbf{x} = (x_n)_{n \in \mathbb{N}} \in c_0(X)\}.$$

Taking into account the equality (3.4), for any character  $\xi \in X^\wedge$ , we have:

$$\xi \in \Gamma_{\text{abs}}(X) \text{ iff } \sum_{k=1}^{\infty} |1 - \xi(x_k)| < \infty, \quad \forall \mathbf{x} = (x_n)_{n \in \mathbb{N}} \in c_0(X).$$

- (b)  $\Gamma_{\text{abs}}(X)$  can be described also by means of the diagonal homomorphism  $\Delta : X^\wedge \rightarrow (X^\wedge)^\mathbb{N}$ . Clearly, if  $\xi \in X^\wedge$ , then  $\xi \in \Gamma_{\text{abs}}(X)$  iff  $(\xi, \xi, \dots, \xi, \dots) \in c_0(X)^\beta$ . Therefore:

$$\Gamma_{\text{abs}}(X) = \Delta^{-1}(c_0(X)^\beta).$$

Whenever  $c_0(X)^\beta = (X^\wedge)^\mathbb{N}$ ,  $\Gamma_{\text{abs}}(X) = \{1\}$ , and  $X \in \mathfrak{B}$ . In particular, by Proposition 5.1,  $\mathbb{S} \in \mathfrak{B}$ .

- (c) We shall show below (Theorem 6.3), that if  $X \in \mathfrak{B}$  is a precompact group, then  $(c_0(X), \mathbf{u}_0)^\wedge$  can be canonically identified with  $(X^\wedge)^\mathbb{N}$ . Thus, summarizing:

$$c_0(X)^\beta = (X^\wedge)^\mathbb{N} \implies X \in \mathfrak{B}, \quad \text{whilst} \quad X \in \mathfrak{B} \ \& \ X \text{ is precompact} \implies c_0(X)^\wedge = (X^\wedge)^\mathbb{N}.$$

Our ultimate aim to have the equality  $c_0(X)^\wedge = (X^\wedge)^\mathbb{N}$  has motivated the introduction of the class  $\mathfrak{B}$ .

In what follows assume that  $c_0(X)$  is endowed with the topology  $\mathbf{u}_0$ . The dual  $c_0(X)^\wedge$  is a subgroup of the group  $\text{Hom}(c_0(X), \mathbb{S})$ , and the density of  $X^\mathbb{N}$  in  $c_0(X)$  allows the algebraic identification of  $c_0(X)^\wedge$  with a subgroup of  $(X^\wedge)^\mathbb{N}$  contained in  $c_0(X)^\beta$  (see also the assignment  $\kappa \mapsto (\xi_n)$  from the proof of item (b) of Proposition 5.2). In the sequel we denote by  $j : G^\wedge \rightarrow (X^\wedge)^\mathbb{N}$  this assignment  $\kappa \mapsto (\xi_n)$ , which actually is the inverse of  $\chi$  defined in the previous section. Then  $j(G^\wedge) \subseteq c_0(X)^\beta$ .

**Theorem 6.3.** *If  $X \in \mathfrak{B}$  is precompact, and  $G := (c_0(X), \mathbf{u}_0)$ , then  $j(G^\wedge) = (X^\wedge)^\mathbb{N}$ . Hence the mapping  $\chi : (X^\wedge)^\mathbb{N} \rightarrow G^\wedge$  is a group isomorphism.*

*Proof.* By Proposition 5.2(a), we only need to see that  $\chi$  is surjective. To this end fix

$$\mathbf{h} = (\xi_n)_{n \in \mathbb{N}} \in j(G^\wedge) \subseteq c_0(X)^\beta \subseteq (X^\wedge)^\mathbb{N}.$$

We have to see that, in fact,  $\mathbf{h} \in (X^\wedge)^\mathbb{N}$ .

By using the  $\mathbf{u}_0$  continuity of  $\chi_{\mathbf{h}}$  we can find some  $V \in \mathcal{N}(X)$  such that  $\chi_{\mathbf{h}}(V^\mathbb{N} \cap c_0(X)) \subset \mathbb{S}_+$ . This implies:

$$(6.1) \quad \xi_n \in V^\triangleright, \quad \forall n \in \mathbb{N}.$$

By Proposition 2.4 the precompactness of  $X$  implies that the set  $V^\triangleright$  is finite. Thus, by (6.1), the set

$$W_{\mathbf{h}} := \{\xi \in V^\triangleright : \exists n \in \mathbb{N}, \xi = \xi_n\}$$

is finite. Fix  $\xi \in W_{\mathbf{h}}$  and write

$$N_\xi := \{n \in \mathbb{N} : \xi = \xi_n\}.$$

Clearly,  $N_\xi \neq \emptyset$ ,  $\forall \xi \in W_{\mathbf{h}}$ . We need to prove the following

**Claim.** *If the set  $\mathbb{N}_\xi$  is infinite for some  $\xi \in W_{\mathbf{h}}$ , then  $\xi = 1$ .*

*Proof of the Claim.* Let  $\mathbb{N}_\xi$  is infinite for  $\xi \in W_{\mathbf{h}}$ . Write  $\mathbb{N}_\xi$  as a strictly increasing sequence:  $\mathbb{N}_\xi = \{k_1, k_2, \dots\}$ .

From  $(\xi_n)_{n \in \mathbb{N}} \in c_0(X)^\beta$  by Lemma 4.2 we conclude that  $(\xi_{k_n})_{n \in \mathbb{N}} \in c_0(X)^\beta$ . Hence,  $(\xi, \xi, \dots, \xi, \dots) \in c_0(X)^\beta$ , and so,  $\xi \in \Gamma_{\text{abs}}(X)$ . As  $X \in \mathfrak{B}$ , we obtain that  $\xi = 1$ . This proves the claim.

As the set  $W_{\mathbf{h}}$  is finite, we deduce from the claim that for some  $n_0 \in \mathbb{N}$  we have:  $\xi_n = 1, \forall n \geq n_0$ . Consequently,  $\mathbf{h} = (\xi_n)_{n \in \mathbb{N}} \in (X^\wedge)^{(\mathbb{N})}$  and the surjectivity of  $\chi$  is proved.  $\square$

**Corollary 6.4.** *If  $X \in \mathfrak{B}$  is a non-trivial precompact group and  $G := (c_0(X), \mathbf{u}_0)$ , then  $\text{Card } G^\wedge = \text{Card } X^\wedge$ .*

*Proof.* According to Theorem 6.3, the mapping  $\chi : (X^\wedge)^{(\mathbb{N})} \rightarrow G^\wedge$  is a group isomorphism. Hence,  $\text{Card } G^\wedge = \text{Card } (X^\wedge)^{(\mathbb{N})}$ . Clearly, for any precompact nontrivial group  $X \in \mathfrak{B}$ ,  $X^\wedge$  is infinite. Otherwise  $X$  would also be finite and  $c_0(X)$  would coincide with  $X^{(\mathbb{N})}$ . Then  $\Gamma_{\text{abs}}(X) \neq 1$ , which contradicts the fact that  $X \in \mathfrak{B}$ . Thus,  $\text{Card } G^\wedge = \text{Card } (X^\wedge)^{(\mathbb{N})} = \text{Card } X^\wedge$ .  $\square$

**6.5. Properties of the class  $\mathfrak{B}$ .** It is clear that  $\mathfrak{B}$  contains all minimally almost periodic groups. We will consider soon other interesting examples.

**Proposition 6.6.** *Let  $X$  be a topological abelian group.*

- (a) *If  $\text{cs}(X) = c_0(X)$ , then  $\Gamma_{\text{abs}}(X) = X^\wedge$ .*
- (b) *If  $\text{cs}(X) = c_0(X)$  and  $X^\wedge \neq \{1\}$ , then  $X \notin \mathfrak{B}$ .*
- (c) *If  $X \neq \{0\}$  is locally compact and totally disconnected, then  $X \notin \mathfrak{B}$ .*

*Proof.* (a) is easy to verify and (b) follows from (a). Finally, (c) follows from (b) and Proposition 3.9 (b) because  $X^\wedge \neq \{1\}$ .  $\square$

The class  $\mathfrak{B}$  is stable through continuous homomorphisms (and in particular, through quotients), as proved in the next lemma.

**Lemma 6.7.** *Let  $X, Y$  be topological abelian groups and  $\varphi \in \text{CHom}(X, Y)$ . We have:*

- (a)  $\varphi^\wedge(\Gamma_{\text{abs}}(Y)) \subset \Gamma_{\text{abs}}(X)$ .
- (b) *If  $X \in \mathfrak{B}$  and  $\varphi(X)$  is dense in  $Y$ , then  $Y \in \mathfrak{B}$ .*
- (c) *If  $\mathbf{n}(X)$  is the von-Neumann's kernel of  $X$ , then  $X \in \mathfrak{B}$  iff  $X/\mathbf{n}(X) \in \mathfrak{B}$ .*

*Proof.* (a) is easy to verify.

(b) By the assumption  $\Gamma_{\text{abs}}(X) = \{1\}$ , and from (a) we get that  $\varphi^\wedge(\Gamma_{\text{abs}}(Y)) = \{1\}$ . Now it remains to note that  $\varphi^\wedge$  is injective by the density of  $\varphi(X)$  in  $Y$ .

(c) The implication  $X \in \mathfrak{B} \implies X/\mathbf{n}(X) \in \mathfrak{B}$  follows from (b). The implication  $X/\mathbf{n}(X) \in \mathfrak{B} \implies X \in \mathfrak{B}$  follows from the fact that the canonical homomorphism  $\varphi : X \rightarrow X/\mathbf{n}(X)$  induces an isomorphism  $\varphi^\wedge : (X/\mathbf{n}(X))^\wedge \rightarrow X^\wedge$ .  $\square$

Now we prove that the class  $\mathfrak{B}$  is stable also under arbitrary direct products.

**Proposition 6.8.** *Let  $I$  be a non-empty index set,  $(X_i)_{i \in I}$  a family of topological groups. Then the cartesian product  $\prod_{i \in I} X_i$  belongs to  $\mathfrak{B}$  iff  $X_i \in \mathfrak{B}$  for every  $i \in I$ .*

*Proof.* Assume that  $X_i \in \mathfrak{B}$  for every  $i \in I$ . Fix  $\varphi \in (\prod_{i \in I} X_i)^\wedge$ . It is known (see, e.g., [3] or [13, Exercise 2.10.4(g,h)]) that there is a family  $(\xi_i)_{i \in I} \in \prod_{i \in I} X_i^\wedge$  such that  $\text{Card}\{i \in I : \xi_i \neq 1\} < \infty$  and

$$\varphi(\mathbf{x}) = \prod_{i \in I} \xi_i(x_i), \quad \forall \mathbf{x} = (x_i)_{i \in I} \in \prod_{i \in I} X_i.$$

Suppose now that  $\varphi \in \Gamma_{\text{abs}}(\prod_{i \in I} X_i)$ . It is easy to see that  $\xi_i \in \Gamma_{\text{abs}}(X_i), \forall i \in I$ . From the assumption  $\Gamma_{\text{abs}}(X_i) = \{1\}, \forall i \in I$ , we get that  $\xi_i = 1, \forall i \in I$ . Hence,  $\varphi = 1$  and so,  $\Gamma_{\text{abs}}(\prod_{i \in I} X_i) = \{1\}$ .

The converse follows from Lemma 6.7, (b).  $\square$

**6.9. A description of the class  $\mathfrak{B}$ .** Here we offer a complete description of the metrizable groups in  $\mathfrak{B}$  (see Theorem 6.12 and Corollary 6.13). The following notion due to Enflo [15] is a cornerstone in the sequel.

**Definition 6.10.** [15, p. 236] A topological group  $X$  is called *locally generated* if

$$\langle V \rangle = X \quad \forall V \in \mathcal{N}(X).$$

It is easy to observe that a topological group  $X$  is locally generated iff

$$X = \bigcup_{k=1}^{\infty} (V + \dots + V) \quad \forall V \in \mathcal{N}(X).$$

Some easy properties and known facts of the locally generated groups are collected in the next Remark.

- Remark 6.11.** (a) All connected topological groups are locally generated ([18, Theorem II.7.4]). On the other hand, every locally generated *locally compact* group is connected [18, Corollary II.7.9].
- (b) Obviously, a group  $X$  is locally generated iff  $X$  has no proper open subgroups. Consequently, if  $H$  is a dense subgroup of a topological group  $G$ , then  $H$  is locally generated iff  $G$  is locally generated.
- (c) From (a) and (b) one can deduce that a locally precompact group is locally generated iff its completion is connected (i.e., the locally generated locally precompact groups are precisely the dense subgroups of the connected locally compact groups).
- (d) The additive group of rational numbers  $\mathbb{Q}$  with the usual topology is a metrizable locally generated group which is totally disconnected. A complete metrizable locally generated topological abelian group also can be totally disconnected [15, Example 2.2.1].

**Theorem 6.12.** *Let  $X$  be a topological abelian group.*

- (a) *If  $X \in \mathfrak{B}$ , then  $X$  is locally generated.*  
 (b) *If  $X$  is locally generated and metrizable, then  $X \in \mathfrak{B}$ .*

*Proof.* (a) Take a symmetric open  $V \in \mathcal{N}(X)$  and let  $H := \langle V \rangle$ . Then  $H$  is open in  $X$  and so, the quotient group  $X/H$  is discrete. Now from Lemma 6.7 (b) we get that the discrete group  $X/H \in \mathfrak{B}$ . This implies that  $X/H$  is a singleton. Consequently,  $H = X$  and so,  $X$  is locally generated.

(b) Take a character  $\xi \in X^\wedge \setminus \{1\}$ . In order to prove that  $\xi \notin \Gamma_{abs}(X)$  we must find a sequence  $(x_n)_{n \in \mathbb{N}} \in c_0(X)$  such that

$$\left( \prod_{k=1}^n \xi(x_k) \right)_{n \in \mathbb{N}}$$

is not convergent in  $\mathbb{S}$ .

As  $\xi \in X^\wedge \setminus \{1\}$ , there is  $x \in X$  such that  $\xi(x) \neq 1$ . Let  $\{V_1, V_2, \dots\}$  be a basis for  $\mathcal{N}(X)$  such that  $V_n \supset V_{n+1}, n = 1, 2, \dots$ . Since  $X$  is locally generated,

$$X = \bigcup_{k=1}^{\infty} (V_n + \dots + V_n) \quad \forall n \in \mathbb{N}.$$

Thus for a given  $n \in \mathbb{N}$  we can find  $k_n \in \mathbb{N}$  such that

$$x \in V_n + \dots + V_n.$$

Therefore, we can also find a finite sequence  $x_{n,1}, \dots, x_{n,k_n}$  such that

$$x_{n,i} \in V_n, i = 1, \dots, k_n$$

and

$$x = \sum_{i=1}^{k_n} x_{n,i}.$$

Let

$$m_0 = 0, \quad m_n := \sum_{i=1}^n k_i, \quad n = 1, 2, \dots$$

Define now a sequence  $(x_j)_{j \in \mathbb{N}}$  as follows: find for  $j \in \mathbb{N}$  the unique  $n \in \mathbb{N}$  with  $m_{n-1} < j \leq m_n$  and put

$$x_j := x_{n, j-m_{n-1}}.$$

Clearly,

$$x_1 = x_{1,1}, \dots, x_{m_1} = x_{1,m_1}, x_{m_1+1} = x_{2,1}, \dots, x_{m_2} = x_{2,k_2}, x_{m_2+1} = x_{3,1}, \dots, x_{m_3} = x_{3,k_3}, \dots$$

and

$$x_j \in V_n, \quad j = m_{n-1} + 1, \dots, m_n; \quad n = 1, 2, 3, \dots$$

The last relation, since  $m_n \rightarrow \infty$  and  $(V_n)_{n \in \mathbb{N}}$  is a decreasing basis for  $\mathcal{N}(X)$ , implies that the sequence  $(x_j)_{j \in \mathbb{N}}$  converges to zero in  $X$ . Now,

$$\prod_{j=m_{n-1}+1}^{m_n} \xi(x_j) = \prod_{i=1}^{k_n} \xi(x_{n,i}) = \xi\left(\sum_{i=1}^{k_n} x_{n,i}\right) = \xi(x) \neq 1, \quad n = 1, 2, 3, \dots$$

Consequently,  $\left(\prod_{j=1}^n \xi(x_j)\right)_{n \in \mathbb{N}}$  is not a Cauchy sequence in  $\mathbb{S}$  and hence it is not convergent.  $\square$

Since connected groups are locally generated, we obtain:

**Corollary 6.13.** *A metrizable abelian group  $X \in \mathfrak{B}$  iff  $X$  is locally generated. In particular,  $\mathfrak{B}$  contains all connected metrizable groups.*

**Remark 6.14.** Since  $\mathfrak{B}$  contains all minimally almost periodic groups (see Lemma 6.7 (c)), from Theorem 6.12(a) we conclude that a minimally almost periodic group is necessarily locally generated. In [13, p. 21] this observation is used for producing a Hausdorff group topology  $\tau$  in  $\mathbb{Z}^{(\mathbb{N})}$  such that  $(\mathbb{Z}^{(\mathbb{N})}, \tau)$  is minimally almost periodic.

The following theorem implies, in particular, that the metrizability assumption can be removed from Theorem 6.12 (b) in the locally compact case (however this cannot be done in general, see Remark 6.16)).

**Theorem 6.15.** *For a locally compact abelian group  $X$  TFAE:*

- (i)  $X \in \mathfrak{B}$ .
- (ii)  $X$  is locally generated.
- (iii)  $X$  is connected.

*Proof.* (i)  $\implies$  (ii). By Theorem 6.12 (a), (i) implies that  $X$  is locally generated.

(ii)  $\implies$  (iii). Follows from [18, Corollary (7.9)].

(iii)  $\implies$  (i) Take  $\xi \in \Gamma_{\text{abs}}(X)$  and let us verify that  $\xi = 1$ .

Consider the set

$$A = \bigcup_{\varphi \in CHom(\mathbb{R}, X)} \varphi(\mathbb{R}).$$

Let us see first that

$$(6.2) \quad \xi|_A = 1.$$

Fix  $\varphi \in CHom(\mathbb{R}, X)$  and set  $H = \varphi(\mathbb{R})$ . Clearly  $\xi|_H \in \Gamma_{\text{abs}}(H)$ . By Theorem 6.12 (b),  $\mathbb{R} \in \mathfrak{B}$ . By Lemma 6.7(b),  $H = \varphi(\mathbb{R}) \in \mathfrak{B}$  too. Therefore,  $\xi|_H \in \Gamma_{\text{abs}}(H) = \{1\}$  and  $\xi|_H = 1$ . Consequently (6.2) is proved.

Clearly (6.2) implies:

$$(6.3) \quad \xi|_{\langle A \rangle} = 1.$$

Now, according to [18, Theorem 25.20 (p.410)] the connectedness of  $X$  implies that  $\langle A \rangle$  is a dense subgroup of  $X$ . From this and (6.3) we obtain that  $\xi = 1$ .  $\square$



**Remark 6.16.** Local compactness is essential for the implication  $(iii) \implies (i)$  of Theorem 6.15. Indeed, that implication may fail in general even for a pseudocompact group  $X$ . In fact, according to [17, Corollary 2.10] (see also [24, Remark 3.4]) there exists an infinite connected pseudocompact abelian group  $X$  which contains no non-trivial convergent sequence. For such a group we have that  $c_0(X) = X^{(\mathbb{N})}$ ; hence  $\Gamma_{\text{abs}}(X) = X^\wedge$  and so,  $X \notin \mathfrak{B}$ . Consistent examples of connected countably compact groups without infinite compact subsets can be found in [14, Corollary 2.21] (note that in both cases the groups are sequentially complete).

**Remark 6.17.** From the proof of Theorem 6.12(b) and implication  $(ii) \implies (i)$  of Theorem 6.15 it follows that if a locally generated Hausdorff topological abelian group  $X$  is either complete metrizable or locally compact, then  $c_0(X) \neq \text{ss}(X)$ . According to [7, Ch. III. 5, Exercise 6 (b)(p. 315)] if  $X \neq \{0\}$  is an arbitrary locally generated complete Hausdorff topological abelian group, then  $l(X) \neq \text{ss}(X)$  and hence,  $c_0(X) \neq \text{ss}(X)$  as well. Remark 6.16 implies that similar conclusion may fail for a connected (hence locally generated) sequentially complete Hausdorff pseudocompact abelian group.

Another class of non-metrizable groups in  $\mathfrak{B}$  can be obtained from Proposition 6.8.

## 7. APPLICATIONS OF THE CLASS $\mathfrak{B}$

### 7.1. Groups with countable dual.

**Proposition 7.2.** *Let  $X \neq \{0\}$  be a compact abelian group and  $G := (c_0(X), \mathbf{u}_0)$ . We have:*

- (a) *If  $X$  is connected, then  $\text{Card } G^\wedge = \text{Card } X^\wedge$ .*
- (b) *If  $X$  is connected and metrizable, then  $\text{Card } G^\wedge = \aleph_0$ .*
- (c) *If  $X$  is metrizable and disconnected, then  $\text{Card } G^\wedge = \mathfrak{c}$ .*

*Proof.* (a) follows from Corollary 6.4 via implication  $(iii) \implies (i)$  of Theorem 6.15.

(b) follows from Theorem 6.12 (b), and the equality  $\text{Card } X^\wedge = \aleph_0$  ( Proposition 2.2).

(c) It is easy to verify that

$$(7.1) \quad \xi \in \Gamma_{\text{abs}}(X) \implies \{1, \xi\}^{\mathbb{N}} \subset c_0(X)^\beta.$$

Since  $X$  is compact metrizable, by Proposition 5.2 (c) we get:

$$(7.2) \quad \xi \in \Gamma_{\text{abs}}(X), \mathbf{h} \in \{1, \xi\}^{\mathbb{N}} \implies \chi_{\mathbf{h}} \in G^\wedge.$$

Now by Theorem 6.15,  $X \notin \mathfrak{B}$  and consequently there exists  $\xi \in \Gamma_{\text{abs}}(X)$ , with  $\xi \neq 1$ . Then  $\text{Card } \{1, \xi\}^{\mathbb{N}} = \mathfrak{c}$ . Therefore,

$$\{\chi_{\mathbf{h}} : \mathbf{h} \in \{1, \xi\}^{\mathbb{N}}\} \subset G^\wedge$$

and since the correspondence  $\mathbf{h} \mapsto \chi_{\mathbf{h}}$  is injective, we get that  $\text{Card } G^\wedge \geq \mathfrak{c}$ . □

The following statement shows that Proposition 7.2(b) is the best possible in the class of locally compact groups.

**Proposition 7.3.** *For an infinite locally compact Hausdorff topological abelian group  $X$  TFAE:*

- (i)  *$X$  is compact connected and metrizable.*
- (ii)  *$\text{Card } (c_0(X), \mathbf{u}_0)^\wedge = \aleph_0$ .*

*Proof.* (i)  $\implies$  (ii) by Proposition 7.2(b).

(ii)  $\implies$  (i). Let us see first that  $X$  is compact and metrizable. Write:  $G := (c_0(X), \mathbf{u}_0)$ . Let  $\varphi : G \rightarrow X$  be the first projection, i.e., the mapping which sends  $(x_n)_{n \in \mathbb{N}}$  to  $x_1$ . Then  $\varphi^\wedge : X^\wedge \rightarrow G^\wedge$  is injective. So,  $\text{Card } G^\wedge = \aleph_0$  implies that  $\text{Card } X^\wedge \leq \aleph_0$ . From this by Proposition 2.2 we get that  $X$  is compact metrizable. Then by Proposition 7.2(c) the equality  $\text{Card } (c_0(X), \mathbf{u}_0)^\wedge = \aleph_0$  implies that  $X$  is connected. □

The connected groups in the class of all compact metrizable abelian groups can be characterized also as follows:

**Proposition 7.4.** *For an infinite compact metrizable abelian group  $X$  TFAE:*

- (i)  *$X$  is connected.*

- (ii)  $(c_0(X), \mathbf{u}_0)^\wedge = (X^\wedge)^{(\mathbb{N})}$ ;
- (iii)  $\text{Card } (c_0(X), \mathbf{u}_0)^\wedge = \aleph_0$ .
- (iv)  $\text{Card } (c_0(X), \mathbf{u}_0)^\wedge < \mathfrak{c}$ .

*Proof.* (i)  $\implies$  (ii)  $\implies$  (iii) by Proposition 7.2(a). (iii)  $\implies$  (iv) is clear.  
 (iv)  $\implies$  (i) by Proposition 7.2(c). □

The following theorem provides a wide class of precompact metrizable groups which are not Mackey in LQC, extending thus the result of Proposition 5.3.

**Theorem 7.5.** *Let  $X$  be an infinite compact connected metrizable group. We have:*

- (a)  $(c_0(X), \mathfrak{p}_0)^\wedge = (c_0(X), \mathbf{u}_0)^\wedge$ .
- (b)  $\mathbf{u}_0$  is a locally quasi-convex Polish group topology compatible for  $(c_0(X), \mathfrak{p}_0)$ .
- (c)  $(c_0(X), \mathfrak{p}_0)$  is a precompact metrizable group which is not a Mackey group in LQC. Further, it is connected.

*Proof.* (a) Since  $\mathfrak{p}_0 \leq \mathbf{u}_0$ , we have:  $(c_0(X), \mathfrak{p}_0)^\wedge \subset (c_0(X), \mathbf{u}_0)^\wedge$ . To prove the converse inclusion, fix an arbitrary  $\kappa \in (c_0(X), \mathbf{u}_0)^\wedge$ . By Proposition 7.4 there exists  $\mathbf{h} = (\xi_n)_{n \in \mathbb{N}} \in (X^\wedge)^{(\mathbb{N})}$  such that  $\kappa = \chi_{\mathbf{h}}$ . Consequently,  $\kappa \in (c_0(X), \mathfrak{p}_0)^\wedge$  and (a) is proved.

(b)  $\mathbf{u}_0$  is a locally quasi-convex Polish group topology by Proposition 3.6. By (a) it is compatible for  $(c_0(X), \mathfrak{p}_0)$ .

(c)  $(c_0(X), \mathfrak{p}_0)$  is a precompact metrizable group because it is a topological subgroup of the compact metrizable group  $(X^\mathbb{N}, \mathfrak{p})$ . It is not a Mackey group in LQC because by (b)  $\mathbf{u}_0$  is a locally quasi-convex group topology, compatible for  $(c_0(X), \mathfrak{p}_0)$  and strictly finer than  $\mathfrak{p}_0$  (Proposition 3.2 (a<sub>1</sub>)). Connectedness follows from Proposition 3.6 (c). □

**7.6. Groups with uncountable dual.** The following statement shows that the topological group  $(\mathbb{S}^\mathbb{N}, \mathbf{u})$  has a dual much "bigger" than  $(c_0(\mathbb{S}), \mathbf{u}_0)$ .

**Proposition 7.7.** *Let  $X \neq \{0\}$  be a compact group,  $G = (X^\mathbb{N}, \mathbf{u})$ . Then  $\text{Card } CHom(G, X) \geq 2^{\mathfrak{c}}$ .*

*In particular, if  $G = \mathbb{S}$ , then  $\text{Card } G^\wedge = 2^{\mathfrak{c}}$ .*

*Proof.* Denote by  $\mathfrak{F}$  the set of all ultrafilters on  $\mathbb{N}$ . It is known that

$$(7.3) \quad \text{Card } \mathfrak{F} = 2^{\mathfrak{c}}.$$

For a filter  $\mathcal{F}$  on  $\mathbb{N}$ ,  $(x_n)_{n \in \mathbb{N}} \in X^\mathbb{N}$  and  $x \in X$  we write:

$$\lim_{n, \mathcal{F}} x_n = x$$

if for every  $W \in \mathcal{N}(X)$  one has  $\{n \in \mathbb{N} : x_n - x \in W\} \in \mathcal{F}$ . Since  $X$  is compact Hausdorff, it follows that for every  $\mathcal{F} \in \mathfrak{F}$  and  $(x_n)_{n \in \mathbb{N}} \in X^\mathbb{N}$  there exists a unique  $x \in X$  such that  $\lim_{n, \mathcal{F}} x_n = x$ .

For a filter  $\mathcal{F} \in \mathfrak{F}$  define the mapping  $\chi_{\mathcal{F}} : X^\mathbb{N} \rightarrow X$  by the equality:

$$\chi_{\mathcal{F}}(\mathbf{x}) = \lim_{n, \mathcal{F}} x_n, \quad \forall \mathbf{x} = (x_n)_{n \in \mathbb{N}} \in X^\mathbb{N}.$$

Then

$$(7.4) \quad \chi_{\mathcal{F}} \in CHom(X^\mathbb{N}, X) \quad \forall \mathcal{F} \in \mathfrak{F}.$$

To verify (7.4), fix  $\mathcal{F} \in \mathfrak{F}$ . As

$$\chi_{\mathcal{F}}(\mathbf{x} + \mathbf{y}) = \lim_{n, \mathcal{F}} (x_n + y_n) = \lim_{n, \mathcal{F}} x_n + \lim_{n, \mathcal{F}} y_n = \chi_{\mathcal{F}}(\mathbf{x}) + \chi_{\mathcal{F}}(\mathbf{y}), \quad \forall \mathbf{x}, \mathbf{y} \in X^\mathbb{N},$$

we conclude that  $\chi_{\mathcal{F}} \in Hom(X^\mathbb{N}, X)$ . To see that  $\chi$  is continuous on  $(X^\mathbb{N}, \mathbf{u})$ , fix a closed  $W \in \mathcal{N}(X)$ . Since  $W$  is closed, for  $\mathbf{x} = (x_n)_{n \in \mathbb{N}} \in W^\mathbb{N}$  we shall have

$$\chi_{\mathcal{F}}(\mathbf{x}) = \lim_{n, \mathcal{F}} x_n \in W.$$

Consequently,  $\chi_{\mathcal{F}}(W^\mathbb{N}) \subset W$ . From this relation, as  $W^\mathbb{N} \in \mathcal{N}(X^\mathbb{N}, \mathbf{u})$ , we get that  $\chi_{\mathcal{F}}$  is continuous on  $(X^\mathbb{N}, \mathbf{u})$  and (7.4) is proved.

We have also:

$$(7.5) \quad \mathcal{F}_1 \in \mathfrak{F}, \mathcal{F}_2 \in \mathfrak{F}, \mathcal{F}_1 \neq \mathcal{F}_2 \implies \chi_{\mathcal{F}_1} \neq \chi_{\mathcal{F}_2}$$

In fact, as  $\mathcal{F}_1$  and  $\mathcal{F}_2$  are *distinct* ultrafilters, there is  $F \in \mathcal{F}_1$  such that  $F \notin \mathcal{F}_2$ . Let  $\mathbf{x} = (x_n)_{n \in \mathbb{N}} \in X^{\mathbb{N}}$  be defined by conditions:  $x_n = 0$  if  $n \in F$  and  $x_n = a \neq 0$  if  $n \in \mathbb{N} \setminus F$ . Then  $\chi_{\mathcal{F}_1}(\mathbf{x}) = 0$  and  $\chi_{\mathcal{F}_2}(\mathbf{x}) = a$ . Therefore,  $\chi_{\mathcal{F}_1} \neq \chi_{\mathcal{F}_2}$  and (7.5) is proved.

Clearly (7.3),(7.4) and (7.5) imply that  $\text{Card } CHom(X^{\mathbb{N}}, X) \geq 2^c$ .  $\square$

## 8. OPEN QUESTIONS

It follows from [9, Proposition 5.4] that every **non-meager**  $(G, \mu) \in \text{LCS}$  is a Mackey group in LQC.

**Conjecture 8.1.** Every metrizable  $(G, \mu) \in \text{LCS}$  is a Mackey group in LQC.

We do not even know if  $\mathbb{R}^{(\mathbb{N})}$  with the topology induced from the product space  $\mathbb{R}^{\mathbb{N}}$  is a Mackey group in LQC.

**Remark 8.2.** We conjecture that Proposition 5.2(c) remains true for a (not necessarily complete metrizable) topological abelian group.

It is clear that if  $G$  is a discrete group, then  $G$  is a Mackey group in MAP.

**Conjecture 8.3.** If  $G \in \text{MAP}$  is a Mackey group in MAP, then  $G$  is a discrete group.

The conjecture 8.3 in terms of the Mackey topology can be reformulated as follows.

**Conjecture 8.4.** If for a precompact topological group  $(G, \nu)$  there exists the MAP-Mackey topology in  $G$  associated with  $\nu$ , then  $\nu$  is the finest precompact group topology in  $G$ .

**Remark 8.5.** In [6] it is shown that a non-complete precompact metrizable  $(G, \mu)$  can be a Mackey group in LQC. It is also known that every locally pseudocompact  $(G, \mu)$  is a Mackey group in LQC (cf. [12]). An internal description of groups  $(G, \tau) \in \text{LQC}$  which are Mackey in LQC is unknown.

## REFERENCES

- [1] H. Anzai, and S. Kakutani, *Bohr compactifications of a locally compact Abelian group. II.* Proc. Imp. Acad. Tokyo 19, (1943), pp. 533–539.
- [2] Aussenhofer, L. *Contributions to the duality theory of abelian topological groups and to the theory of nuclear groups*, Diss. Math. CCCLXXXIV. Warsaw, 1999.
- [3] W. Banaszczyk, *Additive Subgroups of Topological Vector Spaces*, Lecture Notes in Mathematics, 1466. Springer-Verlag, Berlin, 1991.
- [4] M. Barr and H. Kleisli, *On Mackey topologies in topological abelian groups.* Theory Appl. Categ. 8 (2001), pp. 54–62
- [5] G. Bennett and N. J. Kalton, *Inclusion theorems for  $K$ -spaces.* Canad. J. Math. 25 (1973), pp. 511–524. 46A45
- [6] F. G. Bonales, F. J. Trigos-Arrieta and R. Vera Mendoza, *A Mackey-Arens theorem for topological abelian groups*, Bol. Soc. Mat. Mexicana (3) 9 (2003), no. 1, pp. 79–88.
- [7] N. Bourbaki, *General topology*, I, Hermann, Paris VI, 1966.
- [8] N. Bourbaki, *General topology*, II, Hermann, Paris VI, 1966.
- [9] M. J. Chasco, E. Martín-Peinador and V. Tarieladze, *On Mackey Topology for groups*, Stud. Math. 132, No.3, pp. 257–284 (1999).
- [10] W. W. Comfort and K. A. Ross, *Topologies induced by groups of characters.* Fund. Math. 55 (1964), pp. 283–291.
- [11] W. W. Comfort and K. A. Ross, *Pseudocompactness and uniform continuity in topological groups.* Pacific J. Math. 16 (1966) pp. 483–496.
- [12] L. De Leo, D. Dikranjan, E. Martín-Peinador and V. Tarieladze, *Duality theory for groups revisited:  $g$ -barrelled, Mackey and Arens groups.* Preprint, 2009.
- [13] D. Dikranjan, Iv. Prodanov and L. Stoyanov, *Topological Groups: Characters, Dualities and Minimal Group Topologies*, Pure and Applied Mathematics, vol. 130, Marcel Dekker Inc., New York-Basel, 1989.
- [14] D. Dikranjan and D. Shakhmatov, *Forcing hereditarily separable compact-like group topologies on abelian groups.* Topology Appl. 151 (2005), no. 1-3, pp. 2–54.
- [15] P. Enflo, *Uniform structures and square roots in topological groups.* Israel J. Math. 8 (1970), pp. 230–252.
- [16] S. S. Gabrielyan, *Groups of quasi-invariance and the Pontryagin duality.* Arxiv preprint arXiv:0812.1671v6[Math.GN] 1oct. 2009
- [17] J. Galindo and S. Garcia-Ferreira, *Compact groups containing dense pseudocompact subgroups without non-trivial convergent sequences.* Topology Appl. 154 (2007), no. 2, 476–490.

- [18] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis I*. Die Grundlehren der Mathematischen Wissenschaften 115. Springer-Verlag 1963.
- [19] J.L. Kelley and I. Namioka, *Linear topological spaces*, Springer-Verlag, 1963.
- [20] J.W. Nienhuys, *A solenoidal and monothetic minimally almost periodic group*. Fund. Math. 73 (1971/72), no. 2, pp. 167–169.
- [21] S. Rolewicz, *Some remarks on monothetic groups*, Colloq. Math. **XIII**, ( 1964), pp. 28–29.
- [22] H. H. Schaefer, *Topological vector spaces*. The Macmillan Co., New York; Collier-Macmillan Ltd., London 1966
- [23] L. J. Sulley, *On countable inductive limits of locally compact abelian groups*. J. London Math. Soc. (2) 5 (1972), pp. 629–637.
- [24] M. Tkachenko, *Self-duality in the class of precompact groups*. Topology Appl. 156 (2009), no. 12, pp. 2158–2165.
- [25] N. Th. Varopoulos, *Studies in harmonic analysis*, Proc. Cambridge Philos. Soc. 60 1964 pp. 465–516.
- [26] A. Weil, *L'intégration dans les groupes topologiques et ses applications*. (French) Actual. Sci. Ind., no. 869. Hermann et Cie., Paris, 1940. 158 pp
- [27] N. Ya. Vilenkin, *The theory of characters of topological Abelian groups with boundedness given*, Izvestiya Akad. Nauk SSSR. Ser. Mat. **15**, (1951), pp. 439–162.

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