QUASI-SHAPE THEORY OF LOCALLY FINITE AND PARACOMPACT SPACES

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Abstract. Shape theory works nice for (Hausdorff) paracompact spaces, but for spaces with no separation axioms, it seems to be quite poor. However, for finite and locally finite spaces their weak homotopy type is rather rich, and is equivalent to the weak homotopy type of finite and locally finite polynedra, respectively. In the paper there is proposed a variant of shape theory called quasi-shape, which suits both paracompact and locally finite spaces, i.e. the quas-shape is isomorphic to the weak homotopy type for locally finite spaces, and is $\natural$-equivalent to the ordinary shape in the case of paracompact spaces.

1. Main construction

1.1. **The connected component functor π.** We need an appropriate definition of

$$\pi : TOP \longrightarrow SETS$$

where TOP and $SETS$ are the categories of topological spaces and sets, respectively. Neither the usual functor π_0 (the set of pathwise connected components) nor π_0' (the set of connected components) is suitable for our purposes. We will introduce instead the following functor

$$\pi : TOP \longrightarrow pro\text{-}SETS :$$

$$\pi(X)_{\mathcal{U}} := \mathcal{U}$$

for any **open** partition of X (i.e., a partition into open subsets). We say that $\mathcal{U} \leq \mathcal{V}$ if $\mathcal{V}$ refines $\mathcal{U}$. The set $Part(X)$ of all open partitions of X is clearly directed, and we obtain an inverse system of sets by defining

$$p_{\mathcal{U}\leq\mathcal{V}} : \pi(X)_{\mathcal{V}} \longrightarrow \pi(X)_{\mathcal{U}}$$

where

$$\begin{aligned} p_{\mathcal{U}\leq\mathcal{V}}(Y) &= Y', \\ Y &\in \mathcal{V}, \\ Y' &\in \mathcal{U}, \end{aligned}$$

and Y' is the unique element of $\mathcal{U}$, containing Y.

Let now

$$f : X \longrightarrow Y$$

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be a continuous mapping. Define

$$\pi(f) : \pi(X) \longrightarrow \pi(Y)$$

by the following. Let $\mathcal{U}$ be an open partition of Y, and let

$$\mathcal{V} = \xi(\mathcal{U}) = \xi_f(\mathcal{U}) := \left\{Y' = f^{-1}(Y) : Y \in \mathcal{U}, Y' \neq \varnothing\right\}.$$

$\mathcal{V}$ is clearly an open partition of X, and we have just defined a mapping

$$\xi = \xi_f : Part(Y) \longrightarrow Part(X).$$

There can be defined also a mapping

$$f_{\mathcal{U}} : \pi(X)_{\mathcal{V}} \longrightarrow \pi(Y)_{\mathcal{U}}$$

by

$$f_{\mathcal{U}}(Y') := Y$$

where

$$\varnothing \neq Y' = f^{-1}(Y).$$

It can be easily checked that the pair

$$\left(\xi_f : Part(Y) \longrightarrow Part(X), \left(f_{\mathcal{U}} : \pi(X)_{\xi_f(\mathcal{V})} \longrightarrow \pi(Y)_{\mathcal{U}} : \mathcal{U} \in Part(Y)\right)\right)$$

gives a well-defined morphism

$$\pi(f) : \pi(X) \longrightarrow \pi(Y)$$

in the category *pro-SETS*, and the correspondence $f \longmapsto \pi(f)$ defines a functor

$$\pi : TOP \longrightarrow pro\text{-}SETS.$$

Proposition 1.1. *Let X be a locally connected space. Then $\pi(X)$ is isomorphic in the category pro-SETS to the set $\pi_0'(X)$ of connected components of X.*

Proof. The set $\pi_0'(X)$ is an open partition of X which refines any other open partition. Therefore, $Part(X)$ has a maximal element $\pi_0'(X)$, and $\pi(X)$ is isomorphic to the trivial pro-set $\pi_0'(X)$ indexed by a one-point index set, i.e. to the **set** $\pi_0'(X)$. □

1.2. **Quasi-shape.** Let $Cov(X)$ be the set of open coverings on X, pre-ordered by the refinement relation. Analogously to $Part(X)$, $Cov(X)$ is a directed **pre-ordered** set, while $Part(X)$ is a directed **ordered** set. Let

$$U. = (U_*, d_*, s_*)$$

be a hypercovering on X (see [AM86], Definition 8.4), i.e. a simplicial space with an augmentation

$$\varepsilon : U. \longrightarrow X,$$

and the following properties:

Hyper$_0$:

$$\varepsilon_0 : U_0 \longrightarrow X$$

is an open covering;

Hyper$_n$:

$$U_{n+1} \longrightarrow (Cosk_n U.)_{n+1}$$

are open coverings, $n \geq 0$.

If $\mathcal{U}$ is an open covering, one can define the corresponding Čech hypercovering by

$$U_n = \coprod_{U_i \in \mathcal{U}} (U_0 \cap U_1 \cap ... \cap U_n)$$

with the evident face (d_*) and degeneracy (s_*) mappings, where $\amalg$ is the coproduct in the category of topological spaces. For the Čech hypercovering, the mappings

$$U_{n+1} \longrightarrow (Cosk_n U.)_{n+1}, n \geq 0,$$

are homeomorphisms.

Remark 1.2. *The Čech hypercoverings are used in the definition of ordinary shape of a topological space, see* [Mar00].

Definition 1.3. *Let X be a topological space. The* ***shape*** *of X is the following pro-space. Given a* ***normal*** *(i.e. admitting a partition of unity) covering $\mathcal{U}$, let $N\mathcal{U}$ (the Čech nerve of $\mathcal{U}$) be a simplicial set with*

$$(N\mathcal{U})_n = \{(U_0, U_1, ..., U_n) : (U_i \in \mathcal{U}) \,\&\, (U_0 \cap U_1 \cap ... \cap U_n \neq \varnothing)\}$$

with the evident face (d_*) and degeneracy (s_*) mappings. If $\mathcal{V}$ refines $\mathcal{U}$, there exists a unique (up to homotopy) mapping

$$p_{\mathcal{U} \leq \mathcal{V}} : N\mathcal{V} \longrightarrow N\mathcal{U}.$$

The correspondence

$$\mathcal{U} \longmapsto |N\mathcal{U}|$$

where $\mathcal{U}$ runs over all normal coverings on X, and $|N\mathcal{U}|$ is the geometric realization of $N\mathcal{U}$, defines an object $SH(X)$ in pro-$H(TOP)$ which is called the shape of X.

Let $HCov(X)$ be the following category: the objects are hypercoverings on X, and the morphisms from $U.$ to $V.$ are homotopy classes of simplicial mappings

$$U. \longrightarrow V.$$

This category is co-filtering. Given a hypercovering $U.$, let

$$\Gamma(U., \pi) = |\pi(U.)|$$

where $|\pi(U.)|$ is the geometric realization of the simplicial pro-set $\pi(U.)$. Varying $U.$, one gets an object

$$U. \longmapsto |\pi(U.)|$$

in *pro-H* (*pro-TOP*). Finally, applying the canonical functor

$$\text{pro-}H(\text{pro-}TOP) \longrightarrow \text{pro-}(\text{pro-}H(TOP)) \longrightarrow \text{pro-}H(TOP),$$

one gets an object $QSH(X)$ in *pro-H* (TOP) which will be called the **quasi-shape** of X.

Theorem 1.4. *The correspondence above gives a well-defined functor*

$$QSH : TOP \longrightarrow \text{pro-}H(TOP),$$

which factors through the homotopy category $H(TOP)$:

$$QSH : TOP \longrightarrow H(TOP) \longrightarrow \text{pro-}H(TOP).$$

Remark 1.5. *The functor from $H(TOP)$ to pro-$H(TOP)$ will be denoted QSH as well*

2. Comparison

Let X be a locally finite space (see [McC66], p. 466). It means that every point has a finite neighborhood. Due to [McC66], Theorem 2, there exists a simplicial set $\mathcal{K}(X)$, functorially dependent on X, and a weak homotopy equivalence

$$|\mathcal{K}(X)| \longrightarrow X.$$

Let us consider the functor above as a functor to $pro\text{-}H(TOP)$:

$$X \longmapsto |\mathcal{K}(X)| : LF\text{-}TOP \longrightarrow TOP \subseteq pro\text{-}H(TOP)$$

where $LF\text{-}TOP$ is the full subcategory of locally finite spaces.

Example 2.1. *Let X be a so called **4-point circle**, i.e. a space with four points $\{a, b, c, d\}$ and the following topology*

$$\tau = \{X, \varnothing, \{a\}, \{c\}, \{a, b, c\}, \{a, d, c\}, \{a, c\}\}.$$

Then $|\mathcal{K}(X)|$ is homeomorphic to an ordinary circle S^1.

Theorem 2.2. *On the category*

$$LF\text{-}TOP \subseteq TOP,$$

there exists a natural isomorphism

$$|\mathcal{K}(X)| \approx QSH(X).$$

Remark 2.3. *The shape of a locally finite (even a finite) space differs significantly from $|\mathcal{K}(X)|$. Say, the space from Example 2.1 has the shape of a point.*

Let now X be a Hausdorff paracompact space. We will simply call such spaces **paracompact**. Remind that a $\natural$-equivalence between pro-spaces is a mapping

$$f : \mathbf{X} \longrightarrow \mathbf{Y}$$

in $pro\text{-}H(TOP)$ inducing an isomorphism of pro-sets

$$\pi_0(f) : \pi_0(\mathbf{X}) \longrightarrow \pi_0(\mathbf{Y}),$$

and isomorphisms of pro-groups

$$\pi_n(f) : \pi_n\left(\mathbf{X}, f^{-1}(y)\right) \longrightarrow \pi_n(\mathbf{Y}, y), n \geq 1,$$

for any point $y \longrightarrow \mathbf{Y}$. It is known [AM86] that the canonical morphism

$$X_\alpha \longrightarrow (Cosk_n X_\alpha)$$

is a $\natural$-equivalence between pro-spaces

$$\mathbf{X} \longrightarrow Cosk(\mathbf{X}).$$

Theorem 2.4. *Let X be a paracompact space. Then $QSH(X)$ is naturally $\natural$-equivalent to the ordinary shape $SH(X)$ of X.*

3. Proofs

3.1. **Proof of Theorem 1.4.**

Proof. The crucial step is the following. Given two homotopic mappings

$$f, g : X \rightrightarrows Y,$$

the corresponding morphisms:

$$QSH(f) = QSH(g) : QSH(X) \longrightarrow QSH(Y)$$

are equal in the category *pro*-$H(TOP)$. This, in turn is proved using compactness of the unit interval and the technique of Proposition (8.11) from [AM86]: given a hypercovering $U.$ on Y, one constructs a sequence of truncated hypercoverings on X, resulting in a hypercovering $V.$ on X, which refines both $f^{-1}(U.)$ and $g^{-1}(U.)$, and such that the corresponding morphisms

$$\Gamma(V., \pi) \longrightarrow \Gamma(U., \pi)$$

are equal in the category *pro*-$H(TOP)$. □

3.2. **Proof of Theorem 2.2.**

Proof. Introduce the following pre-order on X (see [McC66], p. 468):

$$x \leq y \Longleftrightarrow V_y \subseteq V_x$$

where V_x is the minimal (finite) open neighborhood of x. Let now $U.$ be the following hypercovering:

$$U_n = \coprod_{x_0 \leq x_1 \leq \ldots \leq x_n} V_{x_n}$$

with the evident face and degeneracy mappings. This hypercovering is clearly an initial object in the category $HCov(X)$. All spaces V_x are connected, therefore, for each n, $\pi(U_n)$ is a **set** (i.e. a trivial pro-set). Finally, $QSH(X)$ is a **space** (i.e. a trivial pro-space) $|K(X)|$ where $K(X)$ is the following simplicial set:

$$K(X)_n = \{x_0 \leq x_1 \leq \ldots \leq x_n\}.$$

The latter simplicial set is exactly the simplicial set $\mathcal{K}(X)$ from [McC66], Theorem 2. It follows that

$$QSH(X) \approx |\mathcal{K}(X)|$$

(homotopy equivalent) while

$$|\mathcal{K}(X)| \overset{weak}{\approx} X$$

(weak homotopy equivalent). □

3.3. **Proof of Theorem 2.4.**

Proof. There exists [AM86] a natural ♮-equivalence

$$QSH(X) \longrightarrow Cosk(QSH(X)).$$

Let now construct a homotopy equivalence

$$Sh(X) \longrightarrow Cosk(QSH(X)).$$

Let $U. \in HCov(X)$, let $n \in \mathbb{N}$ and let

$$V. = Cosk_n(QSH(X)) = Cosk_n(\Gamma(U., \pi)).$$

Consider the following open covering $\mathcal{U}$ on X:

$$\mathcal{U} = (d_0)^n : U_n \longrightarrow X.$$

Let us now consider the open partitions $\mathcal{W}_0$, $\mathcal{W}_1$, ... , $\mathcal{W}_n$, of U_0, U_1, ... , U_n, involved in the construction of pro-sets $\pi(U_0)$, $\pi(U_1)$, ... , $\pi(U_n)$. Finally, since X is paracompact, there exists a normal open covering $\mathcal{V}$ on X, refining $\mathcal{U}$ and all coverings

$$(d_0)^i \mathcal{W}_i, i = 0, 1, ..., n.$$

Denote the correspondence

$$(U_., n, \mathcal{W}_i) \longmapsto \mathcal{V}$$

by

$$\xi(U_., n, \mathcal{W}_i) = \mathcal{V}.$$

Given $V \in \mathcal{V}$, there exist unique elements W_i from $\mathcal{W}_i$ such that

$$V \subseteq (d_0)^i W_i.$$

This gives a well-defined mapping from the Čech nerve

$$\varphi_{(U_., n, \mathcal{W}_i)} : N\mathcal{V} \longrightarrow Cosk_n(\Gamma(U_., \pi)).$$

Finally, the pair (ξ, φ) gives the desired equivalence

$$SH(X) \longrightarrow Cosk(QSH(X))$$

in *pro*-$H(TOP)$. □

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