Tropical varieties with polynomial weights and corner loci of piecewise polynomials.

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To S. M. Gusein-Zade on the occasion of his 60th birthday

1 Introduction.

Counting Euler characteristics of the discriminant of the quadratic equation in terms of Newton polytopes in two different ways, G. Gusev ([Gus]) found an unexpected relation for mixed volumes of two polytopes S_1 and $S_2 \subset \mathbb{R}^n$ and the convex hull S of their union. For instance, assuming $n = 2$ and denoting the mixed area of polygons P and Q by $\mathrm{Vol}(P, Q) = \mathrm{Vol}(P + Q) - \mathrm{Vol}(P) - \mathrm{Vol}(Q)$, this relation specializes to

$$\mathrm{Vol}(S, S) - \mathrm{Vol}(S, S_1) - \mathrm{Vol}(S, S_2) + \mathrm{Vol}(S_1, S_2) = 0.$$

We call it unexpected because it is not a priori invariant under parallel translations of S_1. We give an elementary proof and a multidimensional generalization of this equality as requested in [Gus] (see Corollary 1.4 below), deducing it from the following fact (Theorem 1.2): the mixed volume of polytopes only depends on the product of their support functions.

This dependence is a specialization of the isomorphism between two well known combinatorial models of the cohomology of toric varieties. In the second (independent) part of the paper (Sections 2 and 3), we provide a new description of this isomorphism, generalizing the well known construction of the corner locus of a piecewise-linear function. Although other descriptions of this isomorphism are known (see e.g. [KP] or [Maz]), the new one has an advantage of being directly applicable to non-rational polytopes, leads to a new proof and a new form of the answer for Theorem 1.2, and involves new objects (tropical varieties with polynomial weights and their corner loci, see Definition 2.4) that may be of independent interest and have other applications (see e.g. Theorem 4.1: every tropical subvariety in a tropical manifold M is locally the intersection of M with another tropical variety).

Gusev's equality. To simplify notation, we denote the mixed volume of polytopes $A_1, \ldots, A_k$ in $\mathbb{R}^k$ by the monomial $A_1 \cdot \ldots \cdot A_k$ (this mixed volume is by definition the coefficient of the monomial $x_1 \ldots x_k$ in the polynomial $\mathrm{Vol}(x_1 A_1 + \ldots + x_k A_k)$ of variables $x_1, \ldots, x_k$). In the same way, for a homogeneous polynomial $P(x_1, \ldots, x_m) = \sum c_{a_1,\ldots,a_m} x_1^{a_1} \ldots x_m^{a_m}$ of degree k, we define $P(A_1, \ldots, A_m)$ as $\sum c_{a_1,\ldots,a_m} A_1^{a_1} \ldots A_m^{a_m}$.

Theorem **1.1** ([Gus]). *For any two polytopes S_1 and $S_2 \subset \mathbb{R}^n$ and the convex hull S of their union, we have* $(2^n - 2)S^n = \sum_{i=1}^{n-1} 2^i \big(S_1^{n-i} S^i + S^{n-i} S_2^i - S_1^{n-i} S_2^i\big)$.

We deduce this from the following fact. Denote the support function of a polytope $A \subset \mathbb{R}^n$ by $A(\cdot) : (\mathbb{R}^n)^* \to \mathbb{R}$, so that $A(v) = \max_{a \in A} v \cdot a$.

Theorem **1.2.** *There exists a linear function D on the space of conewise-polynomial functions on $(\mathbb{R}^n)^*$, such that*

$$D\Big(A_1(\cdot) \ldots A_n(\cdot)\Big) = A_1 \ldots A_n$$

for every collection of polytopes $A_1, \ldots, A_n$ in $\mathbb{R}^n$.

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Recall that a function f on $\mathbb{R}^m$ is said to be conewise-polynomial, if it is polynomial on every piece of a finite subdivision of $\mathbb{R}^m$ into polyhedral cones with vertices at 0.

This theorem follows from a stronger fact about the product of support functions of rational polytopes: Theorem 5.1 in [KP]. Note that the results of [KP] remain valid for non-rational polytopes as well; to prove them in the non-rational setting, one should replace the reference to Brion's formula in [KP] with the reference to the combinatorial Riemann-Roch formula of [KhP] (i.e. to replace summation over lattice points of a polyhedron with integration over a polyhedron). The relation of Theorem 1.2 to the equivalence of certain models of cohomology of toric varieties gives one more proof for free at the end of this section.

In general (for non-rational cones and polytopes) there is a following explicit formula for D. For an (ordered) basis $v_1, \ldots, v_n$ in $\mathbb{R}^n$, denote the cone generated by $v_1, \ldots, v_n$ by $\langle v_1, \ldots, v_n \rangle$, and denote the Gram-Schmidt orthogonalization of $v_1, \ldots, v_n$ by $v_1^\perp, \ldots, v_n^\perp$ (so that $v_1^\perp, \ldots, v_n^\perp$ is orthonormal, $v_1^\perp, \ldots, v_i^\perp$ generate the same subspace as $v_1, \ldots, v_i$ do for $i = 1, \ldots, n$, and $v_i \cdot v_i^\perp > 0$). For a continuous conewise polynomial function $f : \mathbb{R}^n \to \mathbb{R}$, consider a simple complete fan Γ, on whose cones $C \in \Gamma$ the function f coincide with polynomials f_C. Then Theorem 1.2 can be formulated as follows.

Proposition **1.3.** *Define $D(f)$ as*

$$\frac{1}{n!} \sum_{\langle v_1, \ldots, v_n \rangle \in \Gamma} \frac{\partial^n f_{\langle v_1, \ldots, v_n \rangle}}{\partial v_1^\perp \ldots \partial v_n^\perp}, \tag{$*$}$$

where the sum is taken over all ordered bases of unit vectors $v_1, \ldots, v_n$, generating cones from Γ. Then $D(f)$ does not depend on the choice of the fan Γ, linearly depends on f, and $D\Big(A_1(\cdot) \ldots A_n(\cdot)\Big)$ equals the mixed volume of polytopes $A_1, \ldots, A_n$.

Sketch of the proof. Independence of subdivisions of Γ and linearity follow by definition. Since $D\Big(A_1(\cdot) \ldots A_n(\cdot)\Big)$ is a multilinear function of $A_1, \ldots, A_n$, it is enough to check that $D\Big(A^n(\cdot)\Big)$ equals the volume of the polytope A. Let $\Gamma_1 \subset \mathbb{R}^n$ be the set of all external normal vectors to the faces of A of positive dimension, and let Γ_2 be the union of all rays from the origin, passing through the points of faces of A of codimension greater than 1. If we assume for simplicity that the orthogonal complement to the affine span of every (relatively open) face $B \subset A$ intersects B, then $\Gamma_1 \cup \Gamma_2$ subdivides A into n-dimensional simplices that are in one to one correspondence with the terms of the sum $(*)$, and these terms are equal to the volumes of the corresponding simplices. If we drop the additional assumption, we can still in the same way construct certain simplices, whose volumes are the terms of $(*)$ up to the sign, and whose characteristic functions, endowed with the corresponding signs, sum up to the characteristic function of A. $\square$

Instead of making this explanation precise, we prefer to deduce the statement from a certain more general machinery (see Theorem 3.2), developed in the next section.

Note that the existence of a function D in Theorem 1.2 (aside from its linearity) is not obvious a priori, because the collection of polytopes is not uniquely determined by the product of their support functions: the two different pairs of polygons on the following picture have the same product of support functions (and, thus, the same mixed volume, which is equal to 4).

Picture 1.

Also note that the function D is not monotonous: if A, B and C are the segments in the plane from the origin to the points $(1,0)$, $(0,1)$ and $(1,1)$ respectively, then $A(\cdot)B(\cdot) < C(\cdot)^2$, although $A \cdot B = 1 > C \cdot C = 0$. It would be interesting to find out whether the function D is continuous, and to extend Theorem 1.2 to convex bodies.

Corollary **1.4.** *For any polytopes $B_1, \ldots, B_n$ in $\mathbb{R}^n$ and the convex hull B of their union, we have $(B - B_1) \ldots (B - B_n) = 0$.*

Proof. Since $B(v) = \max_i B_i(v)$ for every $v \in (\mathbb{R}^n)^*$, we have $\Bigl(B(v) - B_1(v)\Bigr) \ldots \Bigl(B(v) - B_n(v)\Bigr) = 0$, and the desired equality follows by Theorem 1.2. □

Proof of Theorem 1.1. Sum up the equality $2^i(S^{n-i} - S_1^{n-i})(S^i - S_2^i) = 0$ (which is a special case of Corollary 1) over $i = 1, \ldots, n-1$. □

We now show that Theorem 1.2 is a special case of the isomorphism between two well known models for cohomology of toric varieties, which leads to an alternative proof of Theorem 1.2 for rational polytopes and cones at the end of Section 1, and to the proof of Proposition 1.3 in Section 3 (Theorem 3.2).

Cohomology ring of toric varieties and its Brion-Stanley description.

The set of all complete rational fans in $\mathbb{R}^n$ admits the following partial order relation: $\Gamma_1 \leqslant \Gamma_2$ if every cone of the fan Γ_2 is contained in a cone of the fan Γ_1. Denoting the toric variety of a fan Γ by $\mathbb{T}^\Gamma$, the natural mapping $\mathbb{T}^{\Gamma_2} \to \mathbb{T}^{\Gamma_1}$ induces a homomorphism of cohomology rings $h_{\Gamma_1,\Gamma_2} : H^\cdot(\mathbb{T}^{\Gamma_1}) \to H^\cdot(\mathbb{T}^{\Gamma_2})$. The direct system of these rings and homomorphisms gives rise to the direct limit

$$\mathcal{H} = \lim_{\to} H^\cdot(\mathbb{T}^\Gamma).$$

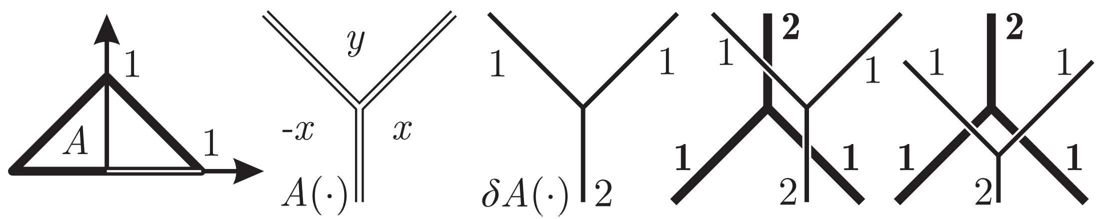
Note that we get the same ring $\mathcal{H}$, independently of which version of cohomology theory we consider (e.g. singular cohomology, Chow cohomology or intersection cohomology; see e.g. [Pa] for a good overview of this kind of results). There are two well-known ways to describe this ring combinatorially.

Brion's description of Chow rings [Br1] and Stanley's description [St] of intersection cohomology of toric varieties lead to the following one. Let $\mathcal{P}_\mathbb{Q}$ be the ring of continuous piecewise-polynomial functions on $\mathbb{R}^n$, whose domains of polynomiality are rational convex polyhedral cones with the vertex 0. Denote its ideal, generated by linear functions, by $\mathcal{L}_\mathbb{Q}$. Then $\mathcal{H} = \mathcal{P}_\mathbb{Q}/\mathcal{L}_\mathbb{Q}$.

Fulton-Kazarnovskii-McMullen-Sturmfels description.

One more combinatorial model for the cohomology ring $\mathcal{H}$ is given independently by many authors, and is known as McMullen's polytope weights [McM], Fulton–Sturmfels Minkowski weights [FS], and Kazarnovskii's c-fans [Kaz]. A k-dimensional *weighted piecewise-linear set* is a pair (P, p), where the *support set* $P \subset \mathbb{R}^n$ is a union of finitely many rational k-dimensional polyhedra (closed and not necessary bounded), and the *weight* $p : P \to \mathbb{R}$ is a locally constant function on the smooth locus of P. It is said to be *homogeneous*, if P is a union of polyhedral cones with the vertex 0. For a smooth point x of P, let $N_xP \subset \mathbb{R}^n$ be the codimension k subspace, orthogonal to the tangent space of P at x. The *tropical intersection number* $\circ_i(P_i, p_i)$ of transversal weighted piecewise-linear sets (P_i, p_i) with $\sum_i \operatorname{codim} P_i = n$ is the sum of the products $\Bigl|\mathbb{Z}^n/(\mathbb{Z}^n \cap \bigoplus_i N_xP_i)\Bigr| \cdot \prod_i p_i(x)$ over all points $x \in \cap_i P_i$ (transversality means that all P_i are smooth at every point of their intersection, and the tangent planes are transversal).

A weighted piecewise-linear set (P, p) is called a *tropical variety*, if, for every rational subspace $L \in \mathbb{R}^n$ of the complementary dimension, the tropical intersection number $(P, p) \circ (L + x, 1)$ does not depend on the point $x \in \mathbb{R}^n$ (note that the intersection number makes sense for almost all x). Arbitrary tropical varieties (P_i, p_i) with $\sum_i \operatorname{codim} P_i = n$ in $\mathbb{R}^n$ intersect transversally when shifted by generic vectors $x_i \in \mathbb{R}^n$, and this intersection number $\circ_i\Bigl(P_i+x_i,\ p_i(\cdot - x_i)\Bigr)$ does not depend on the choice of x_i. This allows to call it the intersection number of the varieties (P_i, p_i) and to denote it by $\circ_i(P_i, p_i)$. See, for example, the two ways to count the intersection number of a pair of tropical curves on the right of the following picture; both ways lead to the same answer 4.



Picture 2.

The *product* (R, r) of tropical varieties (P, p) and (Q, q) is uniquely characterized by the equality of the intersection numbers $(R, r) \circ (S, s) = (P, p) \circ (Q, q) \circ (S, s)$ for every tropical variety (S, s) of the complementary dimension (the existence of such (R, r) is not clear, see a more constructive definition in Section 2). In particular, if (P, p) and (Q, q) are homogeneous tropical varieties of complimentary dimension, then their product is the 0-dimensional tropical variety $\Bigl(\{0\}, (P, p) \circ (Q, q)\Bigr)$. With respect to this multiplication, the natural addition $(P, p) + (Q, q) = (P \cup Q, p + q)$, and the equivalence relation $(P, 0) = (\varnothing, 0)$ for every set P, homogeneous tropical varieties form a ring $\mathcal{C}_\mathbb{Q}$, and we have $\mathcal{C}_\mathbb{Q} = \mathcal{H}$.

The isomorphism.

The isomorphisms $\mathcal{P}_\mathbb{Q}/\mathcal{L}_\mathbb{Q} = \mathcal{H} = \mathcal{C}_\mathbb{Q}$ induce the isomorphism $I_\mathbb{Q} : \mathcal{P}_\mathbb{Q}/\mathcal{L}_\mathbb{Q} \to \mathcal{C}_\mathbb{Q}$ of the two combinatorial models for cohomology of toric varieties. There is one more well known combinatorial model for $\mathcal{H}$ by Khovanskii and Pukhlikov, whose isomorphism with $\mathcal{C}_\mathbb{Q}$ is combinatorially described in [KKh], but we do not need this construction here.

Explicit combinatorial constructions for the isomorphism $I_\mathbb{Q}$ are given in [KP] and [Maz]. Its degree 1 component, sending conewise linear functions to homogeneous tropical hypersurfaces, is much simpler and admits the following well known description.

Definition **1.5.** Assume that a continuous conewise linear function $L : \mathbb{R}^n \to \mathbb{R}$ equals linear functions L_+ and L_- on complementary half-spaces H_+ and H_-, separated by a rational hyperplane P (such a function is called a *book*). Choose a vector $v \in H_+$ that generates the 1-dimensional lattice $\mathbb{Z}^n/P$, and define the (constant) function

$$p(x) = \partial_v L_+(x) - \partial_v L_-(x) \text{ for every } x \in P.$$

The *corner locus* of L is defined as the pair (P, p) for $p \neq 0$ and $(\varnothing, 0)$ otherwise (i.e. for linear L). It does not depend on the choice of v and is denoted by δL. For an arbitrary continuous piecewise linear function L, whose domains of linearity are rational polyhedra, its corner locus is the weighted piecewise-linear set δL, such that whenever L equals a book B near some point, we have $\delta L = \delta B$ near that point.

LEMMA **1.6.** *1) Corner loci, and only they, are tropical hypersurfaces.*
2) The isomorphism $I_{\mathbb{Q}}$ sends every conewise linear function to its corner locus.

For instance, the corner locus (P, p) of the support function of an integer polytope A admits the following simple description: the set P contains all external normal covectors to the edges of A, and the value of p at such a covector equals the integer length of the corresponding edge. In this case, A is called the *Newton polytope* of the tropical hypersurface (P, p), and the following tropical version of the Kouchnirenko-Bernstein theorem is well known (note the absence of assumptions of general position):

THEOREM **1.7** (Tropical Bernstein theorem)**.** *The intersection number of n tropical hypersurfaces in $\mathbb{R}^n$ equals the mixed volume of their Newton polytopes, i.e. we have*

$$\delta A_1(\cdot) \cdot \ldots \cdot \delta A_n(\cdot) = (\{0\}, A_1 \cdot \ldots \cdot A_n).$$

EXAMPLE**.** The support function of a triangle and its corner locus are shown on the left of Picture 2. Thus, the pair of triangles on Picture 1 are the Newton polygons of the tropical curves on the right of Picture 2, so the mixed area of the triangles equals the intersection number of the curves.

Proof of Theorem 1.2 for rational polytopes.

The isomorphism $I_{\mathbb{Q}}$ maps a conewise polynomial F of degree n to a 0-dimensional tropical variety $(\{0\}, c_F)$, where $0 \in \mathbb{R}^n$ is the origin and c_F is a real number, depending on F. We prove that the map, sending every conewise polynomial F to the constant c_F, is the desired function D, i.e.

$$I_{\mathbb{Q}}\Big(A_1(\cdot) \cdot \ldots \cdot A_n(\cdot)\Big) = (\{0\}, A_1 \cdot \ldots \cdot A_n). \qquad (*)$$

For this, we firstly note that

$$I_{\mathbb{Q}}\Big(A_1(\cdot) \cdot \ldots \cdot A_n(\cdot)\Big) = I_{\mathbb{Q}}\Big(A_1(\cdot)\Big) \cdot \ldots \cdot I_{\mathbb{Q}}\Big(A_n(\cdot)\Big),$$

for every collection of integer polytopes $A_1, \ldots, A_n$, because $I_{\mathbb{Q}}$ is a ring isomorphism. Secondly, by Lemma 1.6(2) we have

$$I_{\mathbb{Q}}\Big(A_i(\cdot)\Big) = \delta A_i(\cdot).$$

The two latter equalities together with Theorem 1.7 imply the desired equality $(*)$.

2 Tropical varieties with polynomial weights.

It turns out that $I_{\mathbb{Q}}$ acts on a conewise polynomial of arbitrary degree d as the d-th degree of a certain corner locus operator, generalizing Definition 1.5 (see Definition 2.4 below), in the same way as it is shown above for $d = 1$. To make this precise and applicable to non-rational polytopes and cones, we need the notion of a tropical variety with polynomial weights, which may be of independent interest. We introduce this notion here, and apply it to the study of the isomorphism $I_{\mathbb{Q}}$ in the next section.

Pseudovectors.

For an m-dimensional vector space M over $\mathbb{R}$, an *n-pseudovector* V on M is a function

$$\{\text{orientations of } M\} \to \bigwedge^n M,$$

that assigns two opposite n-vectors V_α and V_β, $V_\alpha + V_\beta = 0$, to the orientations α and β of the space M. A 0-pseudovector on a 0-dimensional vector space is, by definition, a real number. We denote the vector space of n-pseudovectors on M by $\mathcal{V}_n(M)$.

For a vector space $L \supset M$ of dimension $m+1$, a vector $v \in L \setminus M$, a pseudovector $V \in \mathcal{V}_n(M)$, and an orientation α on M, we define the following objects on N.

The orientation $\alpha \wedge v$ on L is defined by the basis $v_1, \ldots, v_m, v$, where $v_1, \ldots, v_m$ is an α-oriented basis in M.

The pseudovector $W \in \mathcal{V}_n(L)$, defined by the equality $W_{\alpha\wedge v} = V_\alpha$, is denoted by V^v.

The pseudovector $U \in \mathcal{V}_{n+1}(L)$, defined as $U_{\alpha\wedge v} = V_\alpha \wedge v$, is denoted by $V \wedge v$.

For orientations α and β of vectors spaces M and N, we define the orientation $\alpha \wedge \beta$ of $M \oplus N$ by the basis $u_1, \ldots, u_m, v_1, \ldots, v_n$, where $(u_1, \ldots, u_m)$ and $(v_1, \ldots, v_n)$ are α and β-oriented bases of M and N respectively. For pseudovectors $X \in \mathcal{V}_k(M)$ and $Y \in \mathcal{V}_l(N)$, we define the wedge product $X \wedge Y \in \mathcal{V}_{k+l}(M \oplus N)$ by the equality $(X \wedge Y)_{\alpha\wedge\beta} = X_\alpha \wedge Y_\beta$.

Remark. If M is endowed with a metric (or with an m-dimensional lattice), then every m-pseudovector V, in contrast to an m-vector, can be identified with the constant $\frac{V_\alpha}{v_1\wedge\ldots\wedge v_m}$, such that the basis $v_1, \ldots, v_m$ is α-oriented and orthonormal (or, respectively, generates the lattice). This constant does not depend on the choice of the orientation α and the basis. In the same way, every $(m-1)$-pseudovector can be identified with a vector. Whenever we introduce a metric in what follows, we always regard pseudovectors of the two highest degrees as constants and vectors respectively.

Weighted fans.

A *convex polyhedral cone* in an m-dimensional vector space M is an intersection of its subspace and finitely many open half-spaces. A union C of finitely many convex polyhedral cones in M is called a *smooth cone* of codimension k, if every its point x has a neighborhood, where C coincides with an $(m-k)$-dimensional plane. The orthogonal complement of this plane is denoted by N_xC (it is a k-dimensional plane in M^*).

An *n-weighted pre-fan* of codimension k in M is a pair (P, p), such that the *support set* P is a smooth fan of codimension k, and the *weight* p is a locally polynomial pseudovector-valued function with values $p(x) \in \mathcal{V}_{m-n}(N_xP)$; note that dependence of N_xP on x is locally constant.

Definition **2.1.** For n-weighted pre-fans (P,p) and (Q,q) of codimension k in M, we define the sum $(P,p)+(Q,q)$ as the pre-fan $(R_1 \sqcup R_2 \sqcup R, r)$, where

$R_1 = P \setminus \overline{Q}$, and $r = p$ on R_1;

$R_2 = Q \setminus \overline{P}$, and $r = q$ on R_2;

$R = \{x \in P \cap Q \mid N_x P = N_x Q\}$, and $r = p + q$ on R.

Definition **2.2.** An *n-weighted fan* of codimension k in M is an equivalence class of n-weighted pre-fans of codimension k with respect to the following equivalence relation:

$$(P,p) \sim (Q,q) \;\Leftrightarrow\; (P,p) + (R,0) = (Q,q) + (R,0) \text{ for some } R \subset M.$$

By a *weighted fan* of codimension k we always mean a k-weighted fan of codimension k.

Denote the set of all weighted fans of codimension k in M, whose weights are local polynomials of degree at most d, by $\mathcal{F}_k^d(M)$. This is an $\mathbb{R}$-vector space with respect to the summation of Definition 2.1 and the scalar multiplication $c \cdot (P,\, p) = (P,\, c \cdot p)$.

Example. A 0-dimensional weighted fan in M is a pair $(\{0\}, p)$, where p is a pseudo-volume form on M (i.e. an m-pseudovector on M^*).

Example. A weighted fan of codimension 0 in M is represented by a pair (P,p), where P is a complement of a union of hyperplanes, and $p: P \to \mathbb{R}$ is locally polynomial.

Balance differential ∂ and corner locus differential δ.

A *book* in an m-dimensional vector space M is a preimage of an 1-dimensional smooth cone (which is a union of finitely many open rays) in a vector space N under a projection $M \to N$. A point $x \in \overline{C} \setminus C$ is said to be in the *stable boundary* of a smooth k-dimensional cone $C \subset M$, if it admits a neighborhood, where C coincides with a book. We denote the stable boundary of C by ∂C, it is a smooth $(k-1)$-dimensional cone.

For a weighted fan (P,p) of codimension k in M, consider a point x in the stable boundary ∂P. In a small neighborhood of x, the cone P splits into the union of connected components P_i. For every i, we choose a small vector $v_i \in M$ and a covector $\omega_i \in N_x \partial P$, such that $x + v_i \in P_i$, the linear span of the vectors v_i is transversal to $T_x \partial P$, and $\omega_i \cdot v_i = 1$. For every i, we denote the limits of $p(y)$ and $\partial_{v_i} p(y)$, as $y \in P_i$ tends to x, by q_i and r_i respectively. Finally, we denote the sums of $q_i^{\omega_i}$ and $r_i \wedge \omega_i$ over all i by $q(x)$ and $r(x)$ respectively (see Subsection "Pseudovectors" for this notation).

Lemma **2.3.** *The pseudovector $q(x)$ does not depend on the choice of the vectors v_i. If $q = 0$ in a neighborhood of x, then $r(x)$ does not depend on the choice of the vectors v_i (otherwise it does).*

We omit the proof, because it follows by definition.

Definition **2.4.** The k-weighted fan $(\partial P, q)$ and the $(k-1)$-weighted fan $(\partial P, r)$ are denoted by $\partial(P,p)$ and $\delta(P,p)$, and are called the *balance* and the *corner locus* of the fan (P,p). If $\partial(P,p) = 0$, then (P,p) is said to be a *polynomially weighted tropical variety*, and $\delta(P,p)$ is well defined (by Lemma 2.3).

Remark. If M is endowed with a metric, then the weight q can be considered real-valued, r can be considered vector-valued, and the overall Definition 2.3 becomes more elementary. However, we prefer not to do so, because we have to expose the same tropical variety to many different metrics for different purposes in what follows (the standard metric in the proof of Theorem 2.6, a not so standard one in the proof of Theorem 2.13, and the "integer metric" to identify our corner locus with the classical one of Definition 1.5).

Remark. Although we only admit piecewise polynomial weights for weighted fans, everything will work with piecewise smooth weights as well. One example of where piecewise smooth weights are relevant is kindly provided by D. Siersma. If $F(x)$ is the distance from a point $x \in \mathbb{R}^n$ to a finite set $A \subset \mathbb{R}^n$, then the function $F : \mathbb{R}^n \to \mathbb{R}$ is piecewise smooth, and its k-th corner locus $\delta^k F$ is a well defined tropical variety (P, p). One can easily verify that P is the codimension k skeleton of the Voronoi diagram of A, and critical points of p coincide with those of the distance function F contained in P.

Many assertions in this section are straightforward generalizations to the case of polynomial weights of what is known about conventional tropical varieties with constant weights. Since the proof of such assertions repeats the case of constant weights word by word, we omit the proof and refer the reader to canonical papers like [FS], [Kaz] or [Mi] for details. The only sources of new information are the assertions about the corner locus differential δ.

Lemma **2.5.** *1) We have $\partial(P, p) = 0$ for a weighted fan of codimension 0 in M, if and only if the function $p : P \to \mathbb{R}$ extends to a continuous function on M.*

2) The kernel and the image of $\delta : \mathcal{F}_0^1(M) \cap \ker \partial \to \mathcal{F}_1^0(M)$ are equal to $\{(M, l) \mid l$ is a linear function on $M\}$ and $\mathcal{F}_1^0(M) \cap \ker \partial$ respectively.

Part 2 is a new formulation of Lemma 1.6.

Proof of Part 1. Continuity of p at points of ∂P is equivalent to the equality $\partial(P, p) = 0$ by definition of $\partial(P, p)$. Continuity at other points follows from a toy version of the Riemann removable singularity theorem: if a real piecewise-polynomial function is continuous outside of a set of codimension 2, then it is continuous everywhere. □

Corner loci are tropical varieties.

Theorem **2.6.** *If (P, p) is a polynomially weighted tropical variety, then so is its corner locus $\delta(P, p)$.*

Proof. The statement can be reduced to the case of tropical (2-dimensional) surfaces with linear weights in the following three steps.

1) We consider a tropical variety $(P, p) \in \mathcal{F}_k^d(M)$ and wish to prove that the weight of $\partial\delta(P, p)$ vanishes at an arbitrary point $x \in \partial\partial P$. In a neighborhood of such point, P coincides with a preimage of a smooth 2-dimensional cone under a surjection of vector spaces $M \to M'$. Thus, without loss in generality, we assume that P equals such a preimage, and $x = 0$.

2) A linear section S of the projection $M \to M'$ of Step (1) contains the tropical (2-dimensional) surface $(P', p') = (P \cap S, p|_S)$, and if we prove the equality $\partial\delta(P', p') = 0$, then vanishing of the weight of $\partial\delta(P, p)$ at 0 will follow. Thus, without loss in generality, we assume that (P, p) is a tropical (2-dimensional) surface.

3) If all the partial derivatives of the weight $p(y)$ tend to 0, as $y \in P$ tends to 0, then so does $r(y)$ as $y \in \partial P$ tends to 0 (in the notation of Definition 2.4). Thus, we can delete all monomials of p except for those of degree 1, i.e. assume without loss of generality that the tropical surface (P, p) has linear weights, $\partial\delta(P, p) = (\{0\}, c)$ is a zero-dimensional tropical variety, and we wish to prove that $c = 0$.

In the latter assumptions we introduce a metric in M, identifying vectors with covectors, pseudomultivectors of maximal degree with constants, and pseudomultivectors of second maximal degree with vectors (see Remark in Subsection "Pseudovectors"). In particular, the weight p becomes a real-valued function, the weight of $\partial(P, p)$ becomes a vector-valued function, and we rewrite the desired statement in this "down to earth" setting as follows.

Let ∂P consist of rays generated by unit vectors v_i, and P consist of relative interiors of closed 2-dimensional convex polyhedral cones C_α. For every cone C_α, generated by vectors v_i and v_j, introduce vectors $v_{i,\alpha}, v_{j,\alpha}$ and p_α in its vector span, such that:
$v_{i,\alpha}$ is defined by the conditions $|v_i| = 1$, $v_{i,\alpha} \cdot v_i = 0$ and $v_{i,\alpha} \cdot v_j > 0$,
$v_{j,\alpha}$ is defined in the same way, with i and j interchanged,
p_α is defined by the equality $p_\alpha \cdot x = p(x)$ for every $x \in C_\alpha$.

In this notation, the equality $\partial(P, p) = 0$ at a point $v_i \in \partial P$ is written as

$$\sum_{\alpha :\, v_i \in C_\alpha} (v_i \cdot p_\alpha) v_{i,\alpha} = 0, \qquad (*_i)$$

and the desired equality $\partial\delta(P, p) = 0$ is written as

$$\sum_{\alpha, i :\, v_i \in C_\alpha} (v_{i,\alpha} \cdot p_\alpha) v_i = 0.$$

Summing up the equalities $(*_i)$ over all i, and collecting terms with the same α, we have

$$\sum_{\substack{\alpha, i, j :\, i \neq j, \\ v_i \in C_\alpha,\, v_j \in C_\alpha}} (v_i \cdot p_\alpha) v_{i,\alpha} + (v_j \cdot p_\alpha) v_{j,\alpha} = 0.$$

This equality coincides with the desired

$$\sum_{\substack{\alpha, i, j :\, i \neq j, \\ v_i \in C_\alpha,\, v_j \in C_\alpha}} (v_{i,\alpha} \cdot p_\alpha) v_i + (v_{j,\alpha} \cdot p_\alpha) v_j = 0$$

because of the following identity. $\square$

LEMMA **2.7.** *If (u, u') and (v, v') are two orthonormal bases of opposite orientation in $\mathbb{R}^2$, then, for any vector p, we have*

$$(u \cdot p)u' + (v \cdot p)v' = (u' \cdot p)u + (v' \cdot p)v.$$

Products and restrictions.

The cartesian product of weighted fans (P, p) in M and (Q, q) in N is the weighted fan $(P \times Q, p \wedge q) \in M \oplus N$. It is denoted by $(P, p) \times (Q, q)$.

Lemma **2.8.**
1) If F and G are polynomially weighted tropical varieties, then so is $F\times G$.
2) In this case, we have the Leibnitz rule $\delta(F\times G)=(\delta F)\times G+F\times(\delta G)$.

We omit the proof, because both statements follow by definition.

A pair of smooth cones in M is said to be *bookwise*, if they are preimages of smooth cones of complementary dimension in a vector space N under a projection $M\to N$, and their union is not contained in a hypersurface. A point $x\in\overline{P}\cap\overline{Q}$ is said to be in the *stable intersection* $P\cap_s Q$ of smooth cones P and Q in M, if, in a small neighborhood of x, the pair (P,Q) coincides with a bookwise pair of cones. In this neighborhood, the smooth cones P and Q split into the union of their connected components $\sqcup_i P_i$ and $\sqcup_j Q_j$ respectively. Pick a small (relatively to the radius of the neighborhood) vector $\varepsilon\in M$ in general position with respect to P and Q, and define $\varepsilon_{i,j}=\begin{cases}1 & \text{if } P_i+\varepsilon \text{ intersects } Q_j\\ 0 & \text{otherwise}\end{cases}$ (the assumption of general position is that the intersections $(P_i+\varepsilon)\cap Q_j$ are transversal, and $P\cap\partial Q=\partial P\cap Q=\varnothing$ in the neighborhood of x).

If P and Q are the support sets of weighted fans (P,p) and (Q,q), then denote the limits of $p(y)$ and $q(z)$, as $y\in P_i$ and $z\in Q_j$ tend to x, by p_i and q_j respectively. Denote the sum $\sum_{i,j}\varepsilon_{i,j}\cdot p_i\wedge q_j$ by $s(x)$ for every $x\in P\cap_s Q$, observe that $P\cap_s Q$ is a smooth cone, and $s(x)$ is a pseudomultivector in $N_x(P\cap_s Q)$.

Definition **2.9.** The weighted fan $(P\cap_s Q,s)$ is called the *intersection product* of the weighted fans (P,p) and (Q,q), and is denoted by $(P,p)\cdot(Q,q)$.

Lemma **2.10.** *1) If F and G are polynomially weighted tropical varieties, then so is $F\cdot G$, and its definition does not depend on the choice of ε.*
2) Intersection product is associative.

We omit the proof as it repeats the one for tropical varieties with constant weights.

We are particularly interested in the following special case of the intersection product.

Definition **2.11.** Let F be a polynomially weighted tropical variety in M, and $L\subset M$ be a subspace of codimension l. Choose an arbitrary l-pseudovector $w\neq 0$ in N_0L, and denote the intersection product of the tropical varieties F and (L,w) by (P,p). Then the pair $(P,p/w)$ can be regarded as a polynomially weighted tropical variety in L, does not depend on the choice of ω, is said to be the *restriction of* F to the plane L, and is denoted $F|_L$.

Lemma 2.10.2 specializes to this case as follows:

Lemma **2.12.** *For any vector subspaces $K\subset L\subset M$, we have $(F|_L)|_K=F|_K$.*

Restrictions of corner loci.

Theorem **2.13.** *We have $\delta(F|_L)=(\delta F)|_L$.*

Proof. The statement can be reduced to the case of tropical (2-dimensional) surfaces with linear weights in the following four steps.

1) If the statement is proved for L being a hypersurface, then, in general case, we can choose a complete flag $L = L_l \subset L_{l-1} \subset \ldots \subset L_0 = M$ and observe that

$$\delta(F|_L) = \Big(\delta(F|_{L_l})\Big)\Big|_{L_l} = \Big(\delta(F|_{L_{l-1}})\Big)\Big|_{L_l} = \ldots = \Big(\delta(F|_{L_0})\Big)\Big|_{L_l} = (\delta F)|_L$$

by Lemma 2.12. Thus, without loss in generality, we assume that L is a hypersurface.

2) We consider a tropical variety $F = (P, p) \in \mathcal{F}_k^d(M)$ and wish to prove that the weights of $\delta(F|_L)$ and $(\delta F)|_L$ are equal at an arbitrary point $x \in \partial P \cap_s L$. In a neighborhood of such point, P coincides with a preimage of a smooth 2-dimensional cone under a surjection of vector spaces $M \to M'$, whose kernel is contained in L. Thus, without loss in generality, we assume that P equals such a preimage, and $x = 0$.

3) A linear section S of the projection $M \to M'$ contains the tropical (2-dimensional) surface $F' = (P \cap S, p|_S)$ and the hypersurface $L' = L \cap S$. If we prove the equality $\delta(F'|_{L'}) = (\delta F')|_{L'}$, then equality of the weights of $\delta(F|_L)$ and $(\delta F)|_L$ at 0 will follow. Thus, without loss in generality, we assume that $F = (P, p)$ is a tropical surface.

4) If all the partial derivatives of the weight $p(y)$ tend to 0, as $y \in P$ tends to 0, then so does $r(y)$ as $y \in \partial P$ tends to 0 (in the notation of Definition 2.4). Thus, we can delete all monomials of p except for those of degree 1, i.e. assume without loss of generality that the tropical surface $F = (P, p)$ has linear weights, L is a hypersurface, $\delta(F|_L) - (\delta F)|_L = (\{0\}, c)$ is a zero-dimensional tropical variety, and we wish to prove that $c = 0$.

In the latter assumptions we represent L as the zero set of a linear function $l : M \to \mathbb{R}$, introduce a metric $\mathcal{G}_0$ in L, and extend it to a (non-constant) metric $\mathcal{G}$ on $\{l > 0\}$ uniquely defined by the following conditions:

– every ray from the origin is orthogonal to the affine plane $\{l = c\}$ for every $c > 0$;

– the restriction of $\mathcal{G}$ to the affine plane $\{l = c\}$ coincides with the metric $\mathcal{G}_0$;

– the restriction of $\mathcal{G}$ to every ray from the origin coincides with dl^2.

The metric $\mathcal{G}$ identifies vectors with covectors, pseudomultivectors of maximal degree with constants, and pseudomultivectors of second maximal degree with vectors (see Subsection "Pseudovectors" for details). In particular, the weight p on the smooth cone $P \cap \{l > 0\}$ becomes a real-valued function, and we rewrite the desired statement in this "down to earth" (i.e. non-invariant) setting as follows.

Let $\partial P \cap \{l > 0\}$ consist of rays generated by vectors $v_i \in \{l = 1\}$, let $P \cap \{l > 0\}$ consist of relative interiors of closed 2-dimensional convex polyhedral cones C_α, and let L_α be the line of intersection of L with the vector span of C_α. The restriction of p to C_α is a linear real valued function on the vector span of C_α, and we denote its restriction to L_α by p_α. For each of the generators v_i of the cone C_α, we pick the unit vector $v_{i,\alpha} \in L_\alpha$, such that $v_{i,\alpha} + tv_i \in C_\alpha$ for $t \to +\infty$.

In this notation, the desired equality $\delta(F|_L) = (\delta F)|_L$ is written as

$$\sum_i \sum_{\alpha\,:\, v_i \in C_\alpha} p_\alpha(v_{i,\alpha}) = \sum_{\substack{i,\,\alpha\,:\, v_i \in C_\alpha \\ \dim(C_\alpha \cap L) = 1}} p_\alpha(v_{i,\alpha}).$$

The left hand side of this equality can be reduced to the right hand side by cancelling the pair of terms

$$p_\alpha(v_{i,\alpha}) + p_\alpha(v_{j,\alpha}) = 0$$

for every cone C_α, whose generators v_i and v_j are not contained in L. $\square$

Differential ring of polynomially weighted tropical varieties.

The operation of intersection product can be expressed in terms of cartesian product and restriction as usual:

LEMMA **2.14.** *Identifying the diagonal D of the sum $M \oplus M$ with the space M itself, we have $(F \times G)|_D = F \cdot G$ for every pair of polynomially weighted tropical varieties F and G in M.*

We omit the proof, because it follows by definition.

THEOREM **2.15.** *If F and G are polynomially weighted tropical varieties in M, then $\delta(F \cdot G) = \delta F \cdot G + F \cdot \delta G$.*

PROOF. By Lemma 2.14, the general case can be reduced to the case of $G = (L, c)$, where $L \subset M$ is a vector subspace and the weight c is a constant. This special case constitutes the statement of Theorem 2.13. $\square$

Let $\mathcal{K}_k^d$ be the space of all polynomially weighted tropical varieties (P, p) in the vector space M, such that $\operatorname{codim} P = k$, and p is locally a homogeneous polynomial of degree d. The direct sum of the spaces $\mathcal{K}_k^d$ over all $d \geqslant 0$ and $k = 0, \ldots, m$ is denoted by $\mathcal{K}$ and is called the *ring of tropical varieties with polynomial weights*. We summarize the results of this section as follows.

COROLLARY **2.16.** *$\mathcal{K} = \bigoplus \mathcal{K}_k^d$ is a bigraded differential ring with the multiplication*

$$\cdot \, : \, \mathcal{K}_k^c \oplus \mathcal{K}_l^d \to \mathcal{K}_{k+l}^{c+d}$$

of Definition 2.9 and the corner locus derivation

$$\delta \, : \, \mathcal{K}_k^d \to \mathcal{K}_{k+1}^{d-1}.$$

3 The isomorphisms.

Denote the subring $\bigoplus_d \mathcal{K}_0^d$ of $\mathcal{K}$ by $\mathcal{P}$, and the subring $\bigoplus_k \mathcal{K}_k^0$ by $\mathcal{C}$. Recall that all elements of P have the form $(M \setminus \Sigma, f)$, where M is the ambient vector space, $f : M \to \mathbb{R}$ is a continuous conewise-polynomial function, and Σ is the set of points where f is not smooth. Thus, we will always identify $\mathcal{P}$ with the ring of continuous conewise-polynomial functions on M. In $\mathcal{P}$, consider the ideal $\mathcal{L}$, generated by all linear functions on M. If the vector space M is endowed with an m-dimensional integer lattice, then, restricting our consideration to weighted cones, whose support sets are unions of rational polyhedral cones, we obtain subrings $\mathcal{K}_\mathbb{Q}, \mathcal{P}_\mathbb{Q}, \mathcal{C}_\mathbb{Q}, \mathcal{L}_\mathbb{Q}$ of the rings $\mathcal{K}, \mathcal{P}, \mathcal{C}, \mathcal{L}$. Since, in the presence of the lattice, pseudomultivectors of the maximal degree are identified with constants (see Subsection "Pseudovectors" of Section 2 for details), this definition of the rings $\mathcal{C}_\mathbb{Q}, \mathcal{P}_\mathbb{Q}$ and $\mathcal{L}_\mathbb{Q}$ agrees with the one given in Section 1.

We give a combinatorial (i.e. not involving geometry and topology of toric varieties) description of the isomorphism $I : \mathcal{P}/\mathcal{L} \to \mathcal{C}$ and its specialization $I_\mathbb{Q} : \mathcal{P}_\mathbb{Q}/\mathcal{L}_\mathbb{Q} \to \mathcal{C}_\mathbb{Q}$, which in particular gives a new explicit formula for the mixed volume of polytopes in terms of the product of their support functions. For the sake of completeness, we also recall the construction of the isomorphisms $\mathcal{H} \to \mathcal{P}_{\mathbb{Q}}/\mathcal{L}_{\mathbb{Q}}$ and $\mathcal{H} \to \mathcal{C}_{\mathbb{Q}}$ (where $\mathcal{H}$ is the direct limit of the cohomology rings of m-dimensional toric varieties, as explained in Section 1). In the next section, we discuss what happens to the isomorphism $I : \mathcal{P}/\mathcal{L} \to \mathcal{C}$, as we replace the ambient vector space M with a tropical variety.

Isomorphism $\mathcal{P}/\mathcal{L} \to \mathcal{C}$.

Define the map $I : \mathcal{P} \to \mathcal{C}$ on $\mathcal{K}_0^d$ as $\delta^d/d!$.

THEOREM **3.1.** *We have $I(\mathcal{L}) = 0$, and $I : \mathcal{P}/\mathcal{L} \to \mathcal{C}$ is a ring isomorphism.*

REMARK. If we pick a simple fan Δ, and restrict our consideration to polynomially weighted tropical varieties, whose support sets are unions of cones from Δ, then the statement remains valid, and the proof is the same.

REMARK. Although the linear map $\delta^d : \mathcal{K}_0^k \to \mathcal{K}_d^{k-d}$ is surjective for $d = k$, and the kernel of $\delta^d : \mathcal{K}_{k-d}^d \to \mathcal{K}_k^0$ is generated by linear functions for $d = k$, none of this remains true for other values of d. For instance, introducing the standard metric $dx^2 + dy^2$ in the coordinate plane, and thus representing weights of plane tropical curves as real-valued functions, the restriction of the function $|x| - |y|$ to the set $\{xy = 0\}$ can be regarded as a tropical curve $F \in \mathcal{K}_1^1$, and we have $\delta F = 0$. However, F cannot be represented as the corner locus of a conewise quadratic function, and cannot be represented as $\sum_i l_i F_i$ for linear functions $l_i : \mathbb{R}^2 \to \mathbb{R}$ and tropical curves F_i with constant weights. (The first statement can be verified by definition, and the second one is true because otherwise $F = \delta(\sum_i l_i \delta^{(-1)} F_i)$, contradicting the first statement.) It would be interesting to explicitly describe the kernel of $\delta^d : \mathcal{K}_{k-d}^d \to \mathcal{K}_k^0$ and the image of $\delta^d : \mathcal{K}_0^k \to \mathcal{K}_d^{k-d}$.

PROOF. Since $\delta^{d+1}(\mathcal{K}_0^d) = 0$, we have

$$\delta^{k+l} F \cdot G \;=\; \sum_j C_{k+l}^j \cdot \delta^j F \cdot \delta^{k+l-j} G \;=\; C_{k+l}^k \cdot \delta^k F \cdot \delta^l G$$

for every pair of tropical varieties $F \in \mathcal{K}_0^k$ and $G \in \mathcal{K}_0^l$, hence I is indeed a ring homomorphism. Since $\delta(M, l) = 0$ for every linear function l, then $I(\mathcal{L}) = 0$. Since the restriction of I to the degree 1 is an isomorphism $\mathcal{K}_0^1 \to \mathcal{K}_1^0$ by Lemma 2.5.2, and the ring $\mathcal{C}$ is generated by $\mathcal{K}_1^0$ (see e.g. [Kaz]), then the homomorphism I is surjective.

The image $\mathcal{M}$ of the component $\mathcal{K}_0^m$ in the quotient $\mathcal{P}/\mathcal{L}$ is 1-dimensional and generated by the weighted fan $L = \big(\{l_1 > 0, \ldots, l_m > 0\}, l_1 \cdot \ldots \cdot l_m\big)$ for a collection of linearly independent linear functions $l_1, \ldots, l_m$ on M. The pairing $F, G \mapsto \frac{F \cdot G}{L} \in \mathbb{R}$ on $\mathcal{P}_{\mathbb{Q}}/\mathcal{L}_{\mathbb{Q}}$ is perfect (see e.g. [Br2]), i.e. every non-zero element $F \in \mathcal{P}/\mathcal{L}$ admits an element $G \in \mathcal{P}/\mathcal{L}$ of complementary dimension, such that $F \cdot G = c \cdot L \mod \mathcal{L}$ for a non-zero number c. Since $I(L)$ is non-zero in $\mathcal{C}$ (one can readily compute $I(L)$ explicitly by definition), then $I(F) \cdot I(G) = c \cdot I(L) \neq 0$, which implies that $I(F)$ is non-zero. Thus I is injective. □

Proof of Proposition 1.3.

Introducing a metric in $\mathbb{R}^n$ and writing δ^n explicitly by definition, we note that the weight of the zero-dimensional tropical variety $\delta^n(f)$ for a continuous conewise-polynomial function $f : \mathbb{R}^n \to \mathbb{R}$ is exactly the sum in the statement of Proposition 1.3 (note that $\delta^n(f)$ is even easier to compute, because some similar terms are collected). We can thus formulate Proposition 1.3 as follows.

Theorem 3.2. *We have*

$$\frac{\delta^n}{n!}\Big(A_1(\cdot)\cdot\ldots\cdot A_n(\cdot)\Big)=(\{0\},A_1\cdot\ldots\cdot A_n)$$

for every collection of polytopes $A_1,\ldots,A_n$ in $\mathbb{R}^n$.

Proof. We have

$$\frac{\delta^n}{n!}\Big(A_1(\cdot)\cdot\ldots\cdot A_n(\cdot)\Big)=I\Big(A_1(\cdot)\cdot\ldots\cdot A_n(\cdot)\Big)=I\Big(A_1(\cdot)\Big)\cdot\ldots\cdot I\Big(A_n(\cdot)\Big),$$

for any collection of polytopes $A_1,\ldots,A_n$, because I is a ring isomorphism (see Theorem 3.1), and is defined as $\delta^n/n!$ for a homogeneous conewise polynomial of degree n. For conewise-linear functions it is defined as δ, so we have

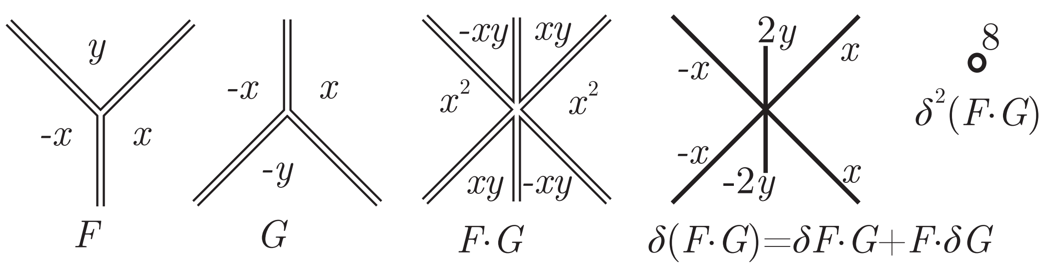
$$I\Big(A_i(\cdot)\Big)=\delta A_i(\cdot).$$

The tropical Bernstein formula is valid for arbitrary tropical varieties with constant weights, not only for rational ones (see e.g. [Kaz]):

$$\delta A_1(\cdot)\cdot\ldots\cdot\delta A_n(\cdot)=(\{0\},A_1\cdot\ldots\cdot A_n).$$

These three equalities imply the desired one. □

For instance, the mixed area of the pair of triangles on Picture 1 can be counted as follows (their support functions are denoted by F and G):



Picture 3.

The count of the mixed area of the right pair of polygons on Picture 1 proceeds in the same way, because the product of their support functions is the same as for the left pair.

Remark. The notion of corner loci of polynomially weighted tropical varieties simplifies the proof of many known useful formulas for mixed volumes. To give an example, denote the maximal face of a polytope $A\subset\mathbb{R}^n$, on which a non-zero covector $\gamma\in(\mathbb{R}^n)^*$ attains its maximal value $A(\gamma)$, by A^γ, note that the $(n-1)$-dimensional mixed volume $A_2^\gamma\cdot\ldots\cdot A_n^\gamma$ makes sense for any polytopes $A_2,\ldots,A_n$ in the euclidean space $\mathbb{R}^n$, and let $\langle\gamma\rangle$ be the ray generated by γ. Applying the tropical Kouchnirenko-Bernstein formula to both parts of the equality

$$\delta A_1(\cdot)\cdot\ldots\cdot\delta A_n(\cdot)=\delta\Big(A_1(\cdot)\delta A_2(\cdot)\cdot\ldots\cdot\delta A_n(\cdot)\Big),\qquad(*)$$

we have $\delta A_1(\cdot)\cdot\ldots\cdot\delta A_n(\cdot)=(\{0\},A_1\cdot\ldots\cdot A_n)$ and $\delta A_2(\cdot)\cdot\ldots\cdot\delta A_n(\cdot)$ is the union of all external normal rays to the facets of $A_2+\ldots+A_n$, with the constant weight $A_2^\gamma\cdot\ldots\cdot A_n^\gamma$ associated to every ray $\langle\gamma\rangle$. As a result, the equality $(*)$ turns into the well known

$$A_1\cdot\ldots\cdot A_n=\sum_{|\gamma|=1}A_1(\gamma)\Bigl(A_2^\gamma\cdot\ldots\cdot A_n^\gamma\Bigr).$$

Isomorphisms $\mathcal{H}\to\mathcal{P}_{\mathbb{Q}}/\mathcal{L}_{\mathbb{Q}}$ and $\mathcal{H}\to\mathcal{C}_{\mathbb{Q}}$.

The models $\mathcal{P}_{\mathbb{Q}}/\mathcal{L}_{\mathbb{Q}}$ and $\mathcal{C}_{\mathbb{Q}}$ for the cohomology ring $\mathcal{H}$ are Poincare dual to each other in the following sense. Pick a simple fan Γ in M, and consider a k-dimensional cohomological cycle γ in the corresponding toric variety $\mathbb{T}^\Gamma$ as an element of $\mathcal{H}$. We have the following two ways to describe γ explicitly. Let $\mathbb{T}^C$ be the closure of the orbit of $\mathbb{T}^\Gamma$, corresponding to the cone $C\in\Gamma$. The fundamental cycles of the subvarieties $\mathbb{T}^C$ over all cones C generate the homology group of $\mathbb{T}^\Gamma$, and their Poincare duals generate the cohomology. Represent γ as $\sum_C\gamma_C\cdot\mathbb{T}^C,\ \gamma_C\in\mathbb{R}$, and denote the intersection number $\gamma\cdot\mathbb{T}^C\in\mathbb{R}$ by γ^C for every cone C of codimension k. Denote the collection of all such cones by Γ^k. Then the cycle γ is uniquely determined by each of these two Poincare dual collections of numbers

$$(\gamma_C,\ C\in\Gamma^{m-k})\text{ and }(\gamma^C,\ C\in\Gamma^k).$$

The image of γ under the isomorphisms

$$I_P:\mathcal{H}\to\mathcal{P}_{\mathbb{Q}}/\mathcal{L}_{\mathbb{Q}}\quad\text{ and }\quad I_C:\mathcal{H}\to\mathcal{C}_{\mathbb{Q}}$$

can be described in terms of these two collections as follows.

For a rational subspace $L\subset\mathbb{R}^m$, pick a basis $v_1,\ldots,v_l$ of the integer lattice $L\cap\mathbb{Z}^m$ and the corresponding orientation α on L, and define a pseudovector $e(L)$ on L by the equality $e(L)_\alpha=v_1\wedge\ldots\wedge v_l$ (this definition does not depend on the choice of $v_1,\ldots,v_l$). Defining $P=\cup_{C\in\Gamma^k}C$, and $p(x)=\gamma^C\cdot e(N_xP)$ for $x\in C$, we have

$$I_C(\gamma)=(P,p).$$

For a simple cone $C\subset\mathbb{R}^m$, generated by primitive linearly independent vectors $v_1,\ldots,v_l$, denote the polynomial function $v^1\cdot\ldots\cdot v^l:C\to\mathbb{R}$ by $e(C)$, where linear functions $v^i:C\to\mathbb{R}$ are dual to the vectors v_j in the sense that $v^i\cdot v_j=\delta^i_j$. Define $q(x)=\gamma_C\cdot e(C)$ for $s\in C,\ C\in\Gamma^{m-k}$, then the function q on the union $\cup_{C\in\Gamma^{m-k}}C$ admits a unique continuous polynomial extension of degree at most k onto every cone of the fan Γ. Gluing these extensions into a continuous conewise-polynomial function $q:M\to\mathbb{R}$ of degree at most k, and denoting $\cup_{C\in\Gamma^0}C$ by Q, we have

$$I_P(\gamma)=(Q,q).$$

4 Intersection theory on tropical varieties.

We first show that the intersection theory on a smooth tropical variety is locally induced from the ambient vector space, and then discuss the general case. We use notation, introduced in Section 2.

Intersection theory on smooth tropical varieties.

A tropical variety with conewise-constant weights is considered smooth, if its support set locally looks like a matroid fan (see e.g. [FR] for the definition). The first motivation for this terminology is to see that the tropicalization of $V \cap (\mathbb{C} \setminus \{0\})^n$ for an affine subspace $V \subset \mathbb{C}^n$ is a matroid fan.

Theorem **4.1.** *Let (P, p) and (Q, q) be tropical varieties with conewise-constant weights, such that $p \neq 0$, and P is a matroid fan, containig Q. Then (Q, q) can be represented as $(P, p) \cdot V$ for some tropical variety V with conewise-constant weights.*

Remark**.** If we assume that q is conewise-polynomial of degree d, and allow V to have conewise-polynomial weights of degree at most d, then both the statement and the proof of the theorem remain valid. In this text, we restrict our attention to tropical varieties, whose support sets consist of cones with vertices at the origin. One could also consider "affine" tropical varieties, whose support sets are unions of arbitrary polyhedra of the same dimension. If we assume that Q is "affine", then both the statement and the proof of the theorem remain valid. However, we cannot expect similar statement for "affine" P: if P is the union of two parallel lines, and Q is a point on one of them, then $(Q, q) = (P, p) \cdot V$ is impossible. Theorem 4.1 is also not valid for the simplest non-smooth tropical variety (see the last example in this section).

The intersection theory on smooth tropical varieties, developed in [FR], [Al], [Sh], etc., is locally induced from the ambient space in the following sense:

> The product of tropical varieties G_1 and G_2 in (P, p), as defined in [FR], [Al], [Sh], equals $\widetilde{G}_1 \cdot \widetilde{G}_1 \cdot (P, p)$ for tropical varieties $\widetilde{G}_i$ such that $G_i = \widetilde{G}_i \cdot (P, p)$. Such $\widetilde{G}_i$ always locally exist by Theorem 4.1.

In particular, the isomorphism of Theorem 3.1 implies the following:

Corollary **4.2.** *The ring of tropical varieties in a matroid fan P (as constructed in [FR], [Al] and [Sh]) is generated by the divisors of rational functions on P (in the terminology of these works).*

We recall that, for every linear map $l : M \to N$ of vector spaces, and for tropical varieties F in M and G in N, one defines the image and the inverse image l_*F and l^*G, such that $l_*(F \cdot l^*G) = l_*(F) \cdot G$, and l^* is a ring homomorphism (see e.g. [Mi] or [Kaz]). Let $i : M \to M \times M$ be the diagonal inclusion of the ambient vector space $M \supset P$ in the setting of Theorem 4.1. We first prove its following special case.

Lemma **4.3.** *1) If $(P', p') = (P, p) \times (P, p)$, then its diagonal $i_*(P, p)$ can be represented as $(P', p') \cdot \Sigma$ for some tropical variety Σ in $M \oplus M$.*
2) The product of tropical varieties G_1 and G_2 in (P, p), as defined in [FR], [Al], [Sh], equals $(G_1 \times G_2) \cdot \Sigma$.

Proof. In [FR], continuous conewise-linear functions $h_1, \ldots, h_k$ on the closure of P were constructed, such that $\delta^k(P', p'h_1 \ldots h_k) = i_*(P, p)$. Extending the product $h_1 \ldots h_k$ to a continuous conewise-polynomial function on $M \times M$, we can consider this function as a codimension 0 tropical variety G with weights of degree k. Then $(P', p'h_1 \dots h_k) = (P', p') \cdot G$, and $\delta^k(P', p'h_1 \dots h_k) = (P', p') \cdot \delta^k G$, so we can take $\Sigma = \delta^k G$. $\square$

Since the diagonal of the square of (P, p) is represented as we need, the following abstract nonsense argument extends this representation to all subvarieties of (P, p).

Proof of Theorem 4.1. Denote the tropical variety (P, p) by F, (Q, q) by G, and $(M, 1)$ by H. By [FR], Theorem 4.5(4), we have

$$(F \times G) \cdot \Sigma = i_* G,$$

where Σ is constructed in Lemma 4.3. Let us now consider the diagonal inclusion $j : M \oplus M \to (M \oplus M) \oplus (M \oplus M)$, the projection $\pi : \oplus^3 M \to \oplus^2 M$ that sends (b, c, d) to $(c,\, d - b)$, and $\pi' : \oplus^4 M \to \oplus^3 M$ that sends (a, b, c, d) to $(a, c,\, d - b)$, so that $\pi' = (\mathrm{id}, \pi)$. In this notation, the latter equality becomes

$$(F \times G \times \Sigma) \cdot j_*(H \times H) = j_* i_* G$$

by definition of the product. Note that $j_*(H \times H) = \pi'^* i_* H$, thus we have

$$(F \times G \times \Sigma) \cdot \pi'^* i_* H = j_* i_* G.$$

Applying π'_* to both sides, we have

$$\Big(F \times \pi_*(G \times \Sigma)\Big) \cdot i_* H = i_* G.$$

Denoting the restriction of $\pi_*(G \times \Sigma)$ to $M \oplus \{0\} \subset M \oplus M$ by Σ_G, Lemma 2.12 implies that $(F \times \Sigma_G) \cdot i_* H = i_* G$, which means the desired $F \cdot \Sigma_G = G$. $\square$

We now formalize the properties of (P, p) that we need for Theorem 4.1.

Definition **4.4.** Let (P, p) be a k-dimensional tropical variety in a vector space M, with p conewise-constant and positive.
1) (P, p) is said to be *normal*, if P is the union of disjoint (relatively open) k-dimensional cones, such that, for every m, every m cones with a common facet generate at least $(m+k-2)$-dimensional space.
2) (P, p) is said to be *diagonalizable*, if $P \times P$ admits a conewise-polynomial (not necessary continuous) weight q, such that $(P \times P, q)$ is tropical, and $\delta^k(P \times P, q) = i_*(P, p)$.

Lemma **4.5.** *If (P, p) is a normal tropical variety, and (P, q) is a tropical variety, then $q = ph$ for some continuous conewise-polynomial $h : M \to \mathbb{R}$.*

Proof. If no m of vectors $v_0, \dots, v_m$ in $\mathbb{R}^m$ are linearly independent, and $\sum_i a_i v_i = \sum_i b_i v_i = 0$, then b_i / a_i does not depend on i. Applying this statement to the values of p and q at a point of the boundary ∂P, we conclude that q/p is a continuous function on the closure of P, and thus can be extended to the desired function $h : M \to \mathbb{R}$. $\square$

Corollary **4.6.** *If (P, p) is normal and diagonalizable, then every tropical variety (Q, q) with conewise-constant weights and $Q \subset P$ can be represented as $(P, p) \cdot V$ for another tropical variety V with conewise-constant weights.*

By the preceding lemma, the proof is the same as for Theorem 4.1.

Conjecture. Every normal tropical variety is diagonalizable.

Cohomology of tropical varieties.

Intersection theory on tropical varieties (see e.g. [Mi], [AR], [Katz]) can be formulated in our terms as follows. Let $F=(P,p)$ be a tropical variety with constant weights in a vector space M, and consider the map $m:\mathcal{K}\to\mathcal{K}$ of multiplication by F, so that $m(G)=F\cdot G$.

Definition **4.7.** The images $m(\mathcal{K}^0_k)$ and $m(\mathcal{K}^d_0)$ are called the *homology* and the *equivariant cohomology* of F, and are denoted by $H_k(F)$ and $HH^d(F)$ respectively. The map m brings the ring structure of the ring $\mathcal{K}$ to the direct sums $H_\bullet(F)=\bigoplus_k H_k(F)$ and $HH^\bullet(F)=\bigoplus_d HH^d(F)$, so that the product of $m(G_1)$ and $m(G_2)$ equals $m(G_1\cdot G_2)$. We always consider $H_\bullet(F)$ and $HH^\bullet(F)$ as rings with respect to this ring structure (not with respect to the one induced by the inclusions $H_\bullet(F)\subset\mathcal{K}$ and $HH^\bullet(F)\subset\mathcal{K}$). The *Poincare duality* $D_F: HH^d(F)\to H_d(F)$ is defined as $D_F(G)=\frac{1}{d!}\delta^d(G)$. The *cohomology ring* $H^\bullet(F)$ of the tropical variety F is the quotient of the equivariant cohomology $HH^\bullet(F)$ by the ideal $\ker D_F$.

This definition makes sense because of the following facts.

Lemma **4.8.** *1) If $D_F(g)=0$ then $D_F(g\cdot h)=0$ for every $h\in HH^\bullet(F)$.*
2) The induced map $D_F: H^\bullet(F)\to H_\bullet(F)$ is a ring isomorphism.

Proof. Part 1 for $\deg h=c$ and multiplicativity of D_F follow from the equality $\delta^{d+c}(gh)\cdot F=\delta^d(g)\cdot\delta^c(h)\cdot F$, which follows from the Leibnitz rule for δ and from $\delta^{d+1}g=\delta^{c+1}h=0$. Surjectivity of D_F follows from surjectivity in Theorem 3.1. □

Example. If $F=(M,1)$ is the vector space of dimension m, then $H^\bullet(F)$ and $HH^\bullet(F)$ are the direct limits of cohomology and equivariant cohomology of m-dimensional toric varieties (see Section 1 for details).

Example. In general, the group $HH^1(F)$ is well known as the group of rational functions on F ([AR]) or the group of mixed Minkowski weights ([Katz]), the degree 1 component of D_F is the intersection map, and $H_1(F)$ is the group of Weil divisors. Note that $H^1(F)$ is a non-trivial (in general) quotient of the group of Cartier divisors, see the second remark after Theorem 3.1 for an example.

Example. In our notation, the self-intersection number of the classical line $L=\{x=y,\ z=0\}$ on the tropical plane $F=\delta\max(0,x,y,z)$ in $\mathbb{R}^3$ can be computed as follows. Recall that the support set P of F is the regular part of the singular locus of $\max(0,x,y,z)$, and that the standard metric $x^2+y^2+z^2$ on $\mathbb{R}^3$ allows us to consider weights of tropical varieties as numbers (rather than as pseudovectors). In [AR], the line $(L,1)$ is represented as $D_F(g\cdot F)$, where a continuous conewise linear function g on $\mathbb{R}^3$ is uniquely defined on P by the following two properties: its restriction to every connected component of $P\setminus L$ is linear, and, on the boundary of these connected components, we have $g(1,1,1)=g(0,-1,0)=g(0,0,-1)=g(-1,-1,0)=0,\ g(1,1,0)=-1,\ g(-1,0,0)=1$. One checks by definition that $\delta(g^2\cdot F)$ is the ray generated by $(1,1,0)$ with the linear weight $-\sqrt{2}x$ on it (this is the weight in the standard metric; the weight in the "integer metric" is $-2x$). Thus the desired self-intersection number $L \circ L = D_F(g^2 \cdot F) = \frac{1}{2}\delta^2(g^2 \cdot F) = \frac{1}{2}\frac{\partial(-\sqrt{2}x)}{\partial(x/\sqrt{2})}$ equals -1, which agrees with [AR].

Note that, in addition to $H_d(F)$ and $H^d(F)$, one can consider larger groups for the tropical variety $F = (P, p)$ (they will be non-trivial even for the trivial weight $p = 0$ on a non-empty P): the group $\overline{H}_d(F) \supset H_d(F)$ consists of all tropical varieties with constant weights that are contained in P and have codimension d in it (it is usually called the group of codimension d cycles on F), the group $\overline{HH}^d(F) \supset HH^d(F)$ consists of all polynomially weighted tropical varieties of the form (P, q) for a homogeneous (not necessarily continuous) conewise polynomial q of degree d on P, the Poincare dual $D_F : \overline{HH}^d(F) \to \overline{H}_d(F)$ is defined by $D_F(G) = \frac{1}{d!}\delta^d(G)$, and the group $\overline{H}^d(F) \supset H^d(F)$ is the quotient of $\overline{HH}^d(F)$ by $\ker D_F$. These larger groups do not have a natural ring structure, although we still have the following cap products.

Definition **4.9.** The *cap product* of $g \in \overline{H}^d(F)$ and $G \in \overline{H}_k(F)$ is
$g \cap G = \delta^d(g' \cdot G) \in \overline{H}_{k-d}(F)$ for $g = g' \cdot F \in H^d(F)$, $G \in \overline{H}_k(F)$, and
$g \cap G = \delta^d(g \cdot G') \in \overline{H}_{k-d}(F)$ for $g \in \overline{H}^d(F)$, $G = G' \cdot F \in H_k(F)$.

If F is smooth, then $H_\bullet(F) = \overline{H}_\bullet(F)$ and $H^\bullet(F) = \overline{H}^\bullet(F)$ (see Theorem 4.1), and the latter equality remains valid for normal F (see Lemma 4.5). However, it would be interesting to study pairwise difference between the groups $H_\bullet$, $\overline{H}^\bullet$ and $\overline{H}_\bullet$ for arbitrary F, because they may be different:

Example (B.Kazarnovskii). Let A be the union of two planes $xz = 0$ in $\mathbb{R}^3$, and let L be the x-coordinate line. Then $L \subset A$ cannot be represented as the product of the tropical surface $(A, 1)$ and another tropical surface with constant weights. However, this line $(L, 1) \in \overline{H}_1(A)$ is Poincare dual to $(A, p) \in \overline{HH}^1(A)$, where $p : A \to \mathbb{R}$ equals $|y|/2$ for $z = 0$ and equals z for $z \neq 0$.

This example implies that $H_1(A) \neq \overline{H}_1(A)$, although the Poincare duality $D_F : \overline{H}^\bullet(A) \to \overline{H}_\bullet(A)$ is still an isomorphism (by Theorem 3.1, applied to the planes $x = 0$ and $z = 0$). A stronger version of the preceding conjecture is as follows (this, in particular, motivates the second remark after Theorem 3.1):

Conjecture. The Poincare duality $D_F : \overline{H}^\bullet(F) \to \overline{H}_\bullet(F)$ is an isomorphism for every normal tropical variety F.

Acknowledgements. Theorems 1.2 and 3.1 were discussed and proved in the framework of the "Algebra, Geometry and Topology" seminar of the University of Toronto, lead by A. Khovanskii in 2006; I am very grateful to A. Khovanskii, who suggested Theorem 1.2, to K. Kaveh, M. Mazin, other participants of the seminar, G. Gusev, E. Katz and B. Kazarnovskii for helpful attention and fruitful discussions. I want to thank N. A'Campo, G.-M. Greuel, D. Siersma, O. Viro and other participants of Conference on Singularities, Geometry and Topology in honour of the 60th Anniversary of Sabir Gusein-Zade (2010, El Escorial) for many important remarks and suggestions.

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