YOSHIDA LIFTS AND SELMER GROUPS

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ABSTRACT. Let f and g, of weights $k' > k \geq 2$, be normalised newforms for $\Gamma_0(N)$, for square-free $N > 1$, such that, for each Atkin-Lehner involution, the eigenvalues of f and g are equal. Let $\lambda \mid \ell$ be a large prime divisor of the algebraic part of the near-central critical value $L(f\otimes g, \frac{k+k'-2}{2})$. Under certain hypotheses, we prove that λ is the modulus of a congruence between the Hecke eigenvalues of a genus-two Yoshida lift of (Jacquet-Langlands correspondents of) f and g (vector-valued in general), and a non-endoscopic genus-two cusp form. In pursuit of this we also give a precise pullback formula for a genus-four Eisenstein series, and a general formula for the Petersson norm of a Yoshida lift.

Given such a congruence, using the 4-dimensional λ-adic Galois representation attached to a genus-two cusp form, we produce, in an appropriate Selmer group, an element of order λ, as required by the Bloch-Kato conjecture on values of L-functions. (Here we must assume that the Galois representation takes values in GSp_4.)

1. Introduction

This paper is about congruences between modular forms, modulo large prime divisors of normalised critical values of L-functions. The first instance of this might be considered to be Ramanujan's congruence modulo 691 between the Hecke eigenvalues of the cusp form Δ and an Eisenstein series of weight 12 for $\mathrm{SL}_2(\mathbb{Z})$, the prime 691 occurring in the critical value $\zeta(12)$. Congruences modulo p between Eisenstein series and cusp forms (now of weight 2 and level p) were used by Ribet [R1] to prove his converse to Herbrand's theorem. Interpreting the congruence as a reducibility modulo p of the 2-dimensional Galois representation attached to the cusp form, he used the non-trivial extension of 1-dimensional factors to construct elements of order p in the class group of $\mathbb{Q}(\zeta_p)$. Mazur and Wiles [MW] developed this idea further in their proof of Iwasawa's main conjecture. When Bloch and Kato [BK] proved most of their conjecture in the case of the Riemann zeta function, the Mazur-Wiles theorem was the main ingredient.

Let f and g, of weights $k' > k \geq 2$ be normalised newforms for $\Gamma_0(N)$, for square-free $N > 1$, such that, for each Atkin-Lehner involution, the eigenvalues of f and g are equal. Let $\lambda \mid \ell$ be a large prime divisor of the algebraic part of the near-central critical value $L(f\otimes g, \frac{k+k'-2}{2})$ (or equivalently of its partner $L(f\otimes g, \frac{k+k'}{2})$). In this paper, we seek a congruence modulo λ between the Hecke eigenvalues of a Yoshida lift $F = F_{f,g}$, and some other genus-2 Hecke eigenform G, of the same weight $\mathrm{Sym}^j \otimes \det^\kappa$, where $j = k-2$ and $\kappa = 2 + \frac{k'-k}{2}$, and level $\Gamma_0^{(2)}(N)$. (See §1.1 and later sections for definitions and notation.) Proposition 9.1 (and Corollary 9.2)

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is what we are able to prove. If p is any prime $p \nmid \ell N$ (where $\lambda \mid \ell$) and $\mu_G(p)$ is the eigenvalue of the Hecke operator $T(p)$ acting on G, then the congruence is

$$\mu_G(p) \equiv a_p(f) + p^{(k'-k)/2} a_p(g) \pmod{\lambda}.$$

Our proof is modelled on Katsurada's approach to proving congruences between Saito-Kurokawa lifts and non-lifts [Ka], modulo divisors of the near-central critical values of Hecke L-functions of genus-1 cuspidal eigenforms of level 1. Thus we consider a "pullback formula" for the restriction to $\mathfrak{H}_2 \times \mathfrak{H}_2$ of a genus-4 Eisenstein series to which a certain differential operator has been applied. The coefficient of $F \otimes F$ is some constant times a value of the standard L-function of F, divided by the Petersson norm of F.

Section 6 contains a proof of the required pullback formula, using differential operators from [B1] and [BSY], and taking care to determine the precise constants occurring. Section 8 contains the proof of a formula for the Petersson norm of the Yoshida lift F, generalising [BS1], which dealt with the analogous case where $k' = k = 2$ and F is scalar-valued of weight $\kappa = 2$. The value $L(f \otimes g, \frac{k+k'}{2})$ appears as a factor in this formula, thus introducing λ into a denominator in the pullback formula. The congruence is then proved by some application of Hecke operators to both sides. For this we need to know the integrality at λ of the left-hand-side (dealt with in Section 7), and, more problematically, that some Fourier coefficient of a canonical scaling of the Yoshida lift F is not divisible by λ. (At this point Katsurada was able to use an explicit formula for the Fourier coefficients of a Saito-Kurokawa lift.) What we need on Fourier coefficients of Yoshida lifts can be reduced to a weak condition on non-divisibility by λ of certain normalised L-values, in the case that N is prime, $w_N = -1$ and $k/2, k'/2$ are odd, using an averaging formula from [BS5].

Brown [Br] used the Galois interpretation of congruences (of Hecke eigenvalues) between Saito-Kurokawa lifts and non-lifts, to confirm a prediction of the Bloch-Kato conjecture. Likewise, in the earlier sections of this paper we use congruences between Yoshida lifts and non-lifts to produce non-zero elements of λ-torsion in the appropriate Bloch-Kato Selmer group. (See Proposition 5.1.) The required cohomology classes come from non-trivial extensions inside the mod λ reduction of Weissauer's 4-dimensional Galois representation attached to G. This mod λ representation is reducible thanks to the congruence.

The work of Brown is easily extended to other (not necessarily near-central) critical values of $L_f(s)$ if one assumes a conjecture of Harder [Ha, vdG] on the existence of congruences involving vector-valued genus-2 cusp forms. It is not possible likewise to extend the present work to other critical values of the tensor-product L-function using genus-2 Siegel modular forms. The problem is that we have two fixed parameters k' and k, not allowing any freedom to vary j and κ. This is explained in more detail at the end of [Du2].

M. Agarwal and K. Klosin, independently of us, had the idea of using congruences between Yoshida lifts and non-lifts to construct elements in Selmer groups, to support the Bloch-Kato conjecture for tensor product L-functions at the near central point [AK]. Their approach to proving such congruences is different, resulting in different conditions, and covers the scalar-valued case ($k = 2$). They use a Siegel-Eisenstein series with a character, as in [Br], and take pains to avoid our assumption (in Lemma 4.1 and Proposition 5.1) that λ is not a congruence prime for f or g.

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1.1. **Definitions and notation.** Let $\mathfrak{H}_n$ be the Siegel upper half plane of n by n complex symmetric matrices with positive-definite imaginary part. Let $\Gamma^{(n)} := \mathrm{Sp}(n,\mathbb{Z}) = \mathrm{Sp}_{2n}(\mathbb{Z}) = \{M \in \mathrm{GL}_{2n}(\mathbb{Z}) : {}^tMJM = J\}$, where $J = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$. For $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma^{(n)}$ and $Z \in \mathfrak{H}_n$, let $M(Z) := (AZ+B)(CZ+D)^{-1}$ and $J(M,Z) := CZ+D$. Let $\Gamma_0^{(n)}(N)$ be the subgroup of $\Gamma^{(n)}$ defined by the condition $N \mid C$. Let V be the space of a finite-dimensional representation ρ of $\mathrm{GL}(n,\mathbb{C})$. A holomorphic function $f : \mathfrak{H}_n \to V$ is said to belong to the space $M_\rho(\Gamma_0^{(n)}(N))$ of Siegel modular forms of genus n and weight ρ, for $\Gamma_0^{(n)}(N)$, if

$$f(M(Z)) = \rho(J(M,Z))f(Z) \quad \forall M \in \Gamma_0^{(n)}(N),\ Z \in \mathfrak{H}_n.$$

Such an f has a Fourier expansion

$$f(Z) = \sum_{S \geq 0} a(S)\mathbf{e}(\mathrm{Tr}(SZ)) = \sum_{S \geq 0} a(S,f)\mathbf{e}(\mathrm{Tr}(SZ)),$$

where the sum is over all positive semi-definite half-integral matrices, and $\mathbf{e}(z) := e^{2\pi i z}$.

Denote by $S_\rho(\Gamma_0^{(n)}(N))$, the subspace of cusp forms, those that vanish at the boundary. They are also characterised by $a(S,f) = 0$ unless S is positive-definite. When ρ is of the special form $\det^k \otimes \mathrm{Sym}^j(\mathbb{C}^n)$ (where $\mathbb{C}^n$ is the standard representation of $\mathrm{GL}_n(\mathbb{C})$), the Petersson inner product will be as in §2 of [Koz], and when also $n = 2$, the Hecke operators $T(m)$, for $(m,N) = 1$, will be defined as in §2 of [Ar], replacing $\mathrm{Sp}_4(\mathbb{Z})$ by $\Gamma_0^{(2)}(N)$. For a Hecke eigenform F, the incomplete spinor and standard L-functions $L^{(N)}(F,s,\mathrm{spin})$ and $L^{(N)}(F,s,\mathrm{St})$ may be defined in terms of Satake parameters as in [An], see also §20 of [vdG].

2. Critical values of the tensor product L-function

Let $f \in S_{k'}(\Gamma_0(N)), g \in S_k(\Gamma_0(N))$ be normalised newforms (with $k' > k \geq 2$), K some number field containing all the Hecke eigenvalues of f and g. Attached to f is a "premotivic structure" M_f over $\mathbb{Q}$ with coefficients in K. Thus there are 2-dimensional K-vector spaces $M_{f,B}$ and $M_{f,\mathrm{dR}}$ (the Betti and de Rham realisations) and, for each finite prime λ of O_K, a 2-dimensional K_λ-vector space $M_{f,\lambda}$, the λ-adic realisation. These come with various structures and comparison isomorphisms, such as $M_{f,B} \otimes_K K_\lambda \simeq M_{f,\lambda}$. See 1.1.1 of [DFG] for the precise definition of a premotivic structure, and 1.6.2 of [DFG] for the construction of M_f, which uses the cohomology, with, in general, non-constant coefficients, of modular curves, and pieces cut out using Hecke correspondences.

On $M_{f,B}$ there is an action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$, and the eigenspaces $M_{f,B}^{\pm}$ are 1-dimensional. On $M_{f,\mathrm{dR}}$ there is a decreasing filtration, with F^j a 1-dimensional space precisely for $1 \leq j \leq k'-1$. The de Rham isomorphism $M_{f,B} \otimes_K \mathbb{C} \simeq M_{f,\mathrm{dR}} \otimes_K \mathbb{C}$ induces isomorphisms between $M_{f,B}^{\pm} \otimes \mathbb{C}$ and $(M_{f,\mathrm{dR}}/F) \otimes \mathbb{C}$, where $F := F^1 = \ldots = F^{k'-1}$. Define $\omega^{\pm}$ to be the determinants of these isomorphisms.

These depend on the choice of K-bases for $M^{\pm}_{f,B}$ and $M_{f,\mathrm{dR}}/F$, so should be viewed as elements of $\mathbb{C}^{\times}/K^{\times}$. In exactly the same way there is also a premotivic structure M_g, but since $k' > k$, it turns out that it is the periods of f that will show up in the formula for the periods of the rank-4 premotivic structure $M_{f\otimes g} := M_f \otimes M_g$.

The eigenspaces $M^{\pm}_{f\otimes g,B}$ are 2-dimensional. On $M_{f\otimes g,\mathrm{dR}}$ there is a decreasing filtration, with F^t a 2-dimensional space precisely for $k \leq t \leq k'-1$. The de Rham isomorphism $M_{f\otimes g,B} \otimes_K \mathbb{C} \simeq M_{f\otimes g,\mathrm{dR}} \otimes_K \mathbb{C}$ induces an isomorphism between $M^{\pm}_{f,B} \otimes \mathbb{C}$ and $(M_{f\otimes g,\mathrm{dR}}/F') \otimes \mathbb{C}$, where $F' := F^k = \ldots = F^{k'-1}$. Define $\Omega^{\pm} \in \mathbb{C}^{\times}/K^{\times}$ to be the determinants of these isomorphisms.

For use in the next section, we shall choose an O_K-submodule $\mathfrak{M}_{f,B}$, generating $M_{f,B}$ over K, but not necessarily free, and likewise an $O_K[1/S]$-submodule $\mathfrak{M}_{f,\mathrm{dR}}$, generating $M_{f,\mathrm{dR}}$ over K, where S is the set of primes dividing $N(k'!)$. We take these as in 1.6.2 of [DFG]. They are part of the "S-integral premotivic structure" associated to f, and are defined using integral models and integral coefficients. Actually, it will be convenient to enlarge S so that $O_K[1/S]$ is a principal ideal domain, then replace $\mathfrak{M}_{f,B}$ and $\mathfrak{M}_{f,\mathrm{dR}}$ by their tensor products with the new $O_K[1/S]$. These will now be free, as will be any submodules, and the quotients we consider. Choosing bases, and using these to calculate the above determinants, we pin down the values of $\omega^{\pm}$ (up to S-units). Setting $\mathfrak{M}_{f\otimes g,B} := \mathfrak{M}_{f,B} \otimes \mathfrak{M}_{g,B}$ and $\mathfrak{M}_{f\otimes g,\mathrm{dR}} := \mathfrak{M}_{f,\mathrm{dR}} \otimes \mathfrak{M}_{g,\mathrm{dR}}$, similarly we pin down $\Omega^{\pm}$ (up to S-units). We just have to imagine not including in S any prime we care about.

For each prime λ of O_K (say $\lambda \mid \ell$), the λ-adic realisation $M_{f,\lambda}$ comes with a continuous linear action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. For each prime number $p \neq \ell$, the restriction to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ may be used to define a local L-factor $[\det(I - \mathrm{Frob}_p^{-1} p^{-s} | M^{I_p}_{f,\lambda})]^{-1}$ (which turns out to be independent of λ), and the Euler product is precisely $L_f(s)$. (Here I_p is an inertia subgroup at p, and Frob_p is a Frobenius element reducing to the generating p^{th}-power automorphism in $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$.) In exactly the same way we may use the Galois representation $M_{f\otimes g,\lambda} = M_{f,\lambda} \otimes M_{g,\lambda}$ to define the tensor product L-function $L_{f\otimes g}(s)$. According to Deligne's conjecture [De], for each integer t in the critical range $k \leq t \leq k'-1$,

$$L_{f\otimes g}(t)/\Omega(t) \in K,$$

where $\Omega(t) = (2\pi i)^t \Omega^{(-1)^t}$ is the Deligne period for the Tate twist $M_{f\otimes g}(t)$.

It is more convenient to use $\langle f, f\rangle$ than $\Omega^{\pm}$, so we consider the relation between the two. Calculating as in (5.18) of [Hi], using Lemma 5.1.6 of [De] and the latter part of 1.5.1 of [DFG], one recovers the well-known fact that, up to S-units,

$$\langle f, f\rangle = i^{k'-1}\omega^+\omega^- c(f), \tag{1}$$

where $c(f)$, the "cohomology congruence ideal", is, as the cup-product of basis elements for $\mathfrak{M}_{f,B}$, an integral ideal. Moreover, calculating as in Lemma 5.1 of [Du1], we find that

$$\Omega^+ = \Omega^- = 2(2\pi i)^{1-k}\omega^+\omega^-.$$

Hence Deligne's conjecture is equivalent to

$$\frac{L_{f\otimes g}(t)}{\pi^{2t-(k-1)}\langle f, f\rangle} \in K$$

(for each integer $k \leq t \leq k' - 1$). This is known to be true, using Shimura's Rankin-Selberg integral for $L_{f\otimes g}(s)$ [Sh4]. In the next section we consider the integral refinement of Deligne's conjecture.

3. The Bloch-Kato conjecture

We shall need the elements $\mathfrak{M}_{f,\lambda}$ of the S-integral premotivic structure, for each prime λ of O_K. These are as in 1.6.2 of [DFG]. For each λ, $\mathfrak{M}_{f,\lambda}$ is a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable O_λ-lattice in $M_{f,\lambda}$. Similarly we have $\mathfrak{M}_{g,\lambda}$, and $\mathfrak{M}_{f\otimes g,\lambda} := \mathfrak{M}_{f,\lambda} \otimes \mathfrak{M}_{g,\lambda}$.

Let $A_\lambda := M_{f\otimes g,\lambda}/\mathfrak{M}_{f\otimes g,\lambda}$, and $A[\lambda] := A_\lambda[\lambda]$ the λ-torsion subgroup. Let $\check{A}_\lambda := \check{M}_{f\otimes g,\lambda}/\check{\mathfrak{M}}_{f\otimes g,\lambda}$, where $\check{M}_{f\otimes g,\lambda}$ and $\check{\mathfrak{M}}_{f\otimes g,\lambda}$ are the vector space and O_λ-lattice dual to $M_{f\otimes g,\lambda}$ and $\mathfrak{M}_{f\otimes g,\lambda}$ respectively, with the natural $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action. Let $A := \oplus_\lambda A_\lambda$, etc.

Following [BK] (Section 3), for $p \neq \ell$ (where $\lambda \mid \ell$, including $p = \infty$) let

$$H^1_f(\mathbb{Q}_p, M_{f\otimes g,\lambda}(t)) = \ker\big(H^1(D_p, M_{f\otimes g,\lambda}(t)) \to H^1(I_p, M_{f\otimes g,\lambda}(t))\big).$$

Here D_p is a decomposition subgroup at a prime above p, I_p is the inertia subgroup, and $M_{f\otimes g,\lambda}(t)$ is a Tate twist of $M_{f\otimes g,\lambda}$, etc. The cohomology is for continuous cocycles and coboundaries. For $p = \ell$ let

$$H^1_f(\mathbb{Q}_\ell, M_{f\otimes g,\lambda}(t)) = \ker\big(H^1(D_\ell, M_{f\otimes g,\lambda}(t)) \to H^1(D_\ell, M_{f\otimes g,\lambda}(t) \otimes_{\mathbb{Q}_\ell} B_{\mathrm{crys}})\big).$$

(See Section 1 of [BK] or §2 of [Fo1] for the definition of Fontaine's ring B_{crys}.) Let $H^1_f(\mathbb{Q}, M_{f\otimes g,\lambda}(t))$ be the subspace of those elements of $H^1(\mathbb{Q}, M_{f\otimes g,\lambda}(t))$ that, for all primes p, have local restriction lying in $H^1_f(\mathbb{Q}_p, M_{f\otimes g,\lambda}(t))$. There is a natural exact sequence

$$0 \longrightarrow \mathfrak{M}_{f\otimes g,\lambda}(t) \longrightarrow M_{f\otimes g,\lambda}(t) \xrightarrow{\pi} A_\lambda(t) \longrightarrow 0.$$

Let $H^1_f(\mathbb{Q}_p, A_\lambda(t)) = \pi_* H^1_f(\mathbb{Q}_p, M_{f\otimes g,\lambda}(t))$. Define the λ-Selmer group $H^1_f(\mathbb{Q}, A_\lambda(t))$ to be the subgroup of elements of $H^1(\mathbb{Q}, A_\lambda(t))$ whose local restrictions lie in $H^1_f(\mathbb{Q}_p, A_\lambda(t))$ for all primes p. Note that the condition at $p = \infty$ is superfluous unless $\ell = 2$. Define the Shafarevich-Tate group

$$Ш(t) = \bigoplus_\lambda \frac{H^1_f(\mathbb{Q}, A_\lambda(t))}{\pi_* H^1_f(\mathbb{Q}, M_{f\otimes g,\lambda}(t))}.$$

Tamagawa factors $c_p(t)$ may be defined as in 11.3 of [Fo2] (where the notation is $\mathrm{Tam}^0 \ldots$). The λ part (for $\ell \neq p$) is trivial if $A_\lambda^{I_p}$ is divisible (for example if $p \nmid N$). The following is equivalent to the relevant cases of the Fontaine-Perrin-Riou extension of the Bloch-Kato conjecture to arbitrary weights (i.e. not just points right of the centre) and not-necessarily-rational coefficients. (This follows from 11.4 of [Fo2].)

Conjecture 3.1. *Suppose that* $k \leq t \leq k' - 1$. *Then we have the following equality of fractional ideals of* $O_K[1/S]$*:*

$$\frac{L_{f\otimes g}(t)}{\Omega(t)} = \frac{\prod_{p\leq\infty} c_p(t)\ \#Ш(t)}{\#H^0(\mathbb{Q}, A(t))\#H^0(\mathbb{Q}, \check{A}(1-t))}. \tag{2}$$

In other words,

$$\frac{L_{f\otimes g}(t)}{\pi^{2t-(k-1)}\langle f, f\rangle} = \frac{\prod_{p\leq\infty} c_p(t)\ \#Ш(t)}{\#H^0(\mathbb{Q}, A(t))\#H^0(\mathbb{Q}, \check{A}(1-t))c(f)}. \tag{3}$$

Let $f = \sum a_n(f)q^n$ etc. Let $\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}(M_{f,\lambda})$ be the 2-dimensional λ-adic Galois representation attached to f. Let $\overline{\rho}_f$ be its reduction $\pmod{\lambda}$, which is unambiguously defined if it is irreducible. Likewise ρ_g and $\overline{\rho}_g$.

Lemma 3.2. (1) *Suppose that $\overline{\rho}_f$ and $\overline{\rho}_g$ are irreducible, that $\ell > k'$ and $\ell \nmid N$. Suppose (for some $p \parallel N$) that there is no normalised newform h of level dividing N/p and trivial character, of weight k' with $a_q(h) \equiv a_q(f) \pmod{\lambda}$ for all primes $q \nmid \ell N$, or of weight k with $a_q(h) \equiv a_q(g) \pmod{\lambda}$ for all primes $q \nmid \ell N$. Then the λ part of $c_p(t)$ is trivial (for any t).*

(2) *If $\lambda \mid \ell$ with $\ell \nmid N$ and $\ell > k' + k - 1$ then the λ part of $c_\ell(t)$ is trivial (for any t).*

Proof. (1) Applying a level-lowering theorem (Theorem 1.1 of [Di], see also [R2, R3]), $\overline{\rho}_f$ and $\overline{\rho}_g$ are both ramified at p. However, since $p \parallel N$, the action of I_p on each of $M_{f,\lambda}$ and $M_{g,\lambda}$ is unipotent, by Theorem 7.5 of [L]. It follows that both $\overline{\rho}_f \otimes \overline{\rho}_g$ and $\rho_f \otimes \rho_g$ have I_p-fixed subspace of dimension precisely 2, hence that $A_\lambda^{I_p}$ is divisible. As noted above, this implies that the λ-part of $c_p(t)$ is trivial.

(2) It follows from Lemma 5.7 of [DFG] (whose proof relies on an application, at the end of Section 2.2, of the results of [Fa]) that $\mathfrak{M}_{f\otimes g,\lambda}$ is the $O_\lambda[\mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)]$-module associated to the filtered ϕ-module $\mathfrak{M}_{f\otimes g,\mathrm{dR}} \otimes O_\lambda$ (identified with the crystalline realisation) by the functor they call $\mathbb{V}$. (This property is part of the definition of an S-integral premotivic structure given in Section 1.2 of [DFG].) Given this, the lemma follows from Theorem 4.1(iii) of [BK]. (That $\mathbb{V}$ is the same as the functor used in Theorem 4.1 of [BK] follows from the first paragraph of 2(h) of [Fa].)

□

Corollary 3.3. *Assume the conditions of Lemma 3.2, and also that (for some $k \leq t \leq k' - 1$)*

$$\mathrm{ord}_\lambda\left(\frac{L_{f\otimes g}(t)}{\pi^{2t-(k-1)}\langle f, f\rangle}\right) > 0.$$

Then the Bloch-Kato conjecture predicts that $\mathrm{ord}_\lambda(\#\text{Ш}(t)) > 0$, so predicts that the Selmer group $H^1_f(\mathbb{Q}, A_\lambda(t))$ is non-trivial.

The goal of this paper is to construct (under further hypotheses) a non-zero element of $H^1_f(\mathbb{Q}, A_\lambda(t))$, in the case that t is the near-central point $t = \frac{k'+k-2}{2}$.

Lemma 3.4. *If $\ell \nmid N$, $\ell > k'-1$ and $k < t < k'-1$ then the λ-parts of $\#H^0(\mathbb{Q}, A(t))$ and $\#H^0(\mathbb{Q}, \check{A}(1-t))$ are trivial.*

Proof. If not, then either $A[\lambda](t)$ or $\check{A}[\lambda](1-t)$ would have a trivial composition factor. The composition factors of $\overline{\rho}_f|_{I_\ell}$ are either χ^0, χ^{1-k} (in the ordinary case, with χ the cyclotomic character) or $\psi^{1-k}, \psi^{\ell(1-k)}$ (in the non-ordinary case, with ψ a fundamental character of level 2). This follows from theorems of Deligne and Fontaine, which are Theorems 2.5 and 2.6 of [Ed]. Noting that ψ has order $\ell^2 - 1$, with $\psi^{\ell+1} = \chi$, the composition factors of $(\overline{\rho}_f \otimes \overline{\rho}_g)|_{I_\ell}$ are of the form $\psi^a, \psi^b, \psi^c, \psi^d$, with $1 - \ell^2 < a, b, c, d \leq 0$ and each of a, b, c, d congruent to either $0, 1-k, 1-k'$ or $2-k-k' \pmod{\ell}$. Twisting by t is the same as multiplying by $\psi^{(\ell+1)t}$. This exponent is congruent to $t \pmod{\ell}$, and $k < t < k'-1$. Adding to this the possible values for $a, b, c, d \pmod{\ell}$ can never produce 0 or 1. Hence neither $A[\lambda](t)$ nor $\hat{A}[\lambda](1-t)$ can have a trivial composition factor (even when restricted to I_ℓ). □

4. A 4-dimensional Galois representation

Let f, g be as in §§2,3, both of exact level $N > 1$. Let $\lambda \mid \ell$ be a divisor of $\frac{L_{f\otimes g}(t)}{\pi^{2t-(k-1)}\langle f,f\rangle}$, with $\ell \nmid N(k')!$ and $t = (k'+k-2)/2$. Now suppose that f and g have the same Atkin-Lehner eigenvalues for each $p \mid N$, and let $F_{f,g}$ be some genus-2 Yoshida lift associated with a factorisation $N = N_1N_2$, as in §8 below. (It is of type $\mathrm{Sym}^j \otimes \det^\kappa$, with $j = k-2, \kappa = 2 + \frac{k'-k}{2}$. Note that $j + 2\kappa - 3 = k' - 1$.)

Suppose that there is a cusp form G for $\Gamma_0^{(2)}(N)$, an eigenvector for all the local Hecke algebras at $p \nmid N$, not itself a Yoshida lift of the same f and g, such that there is a congruence $\pmod{\lambda}$ of all Hecke eigenvalues (for $p \nmid N$) between G and $F_{f,g}$. In particular, if $\mu_G(p)$ is the eigenvalue for $T(p)$ on G (defined as in §2.1 of [Ar], replacing $\mathrm{Sp}_4(\mathbb{Z})$ by $\Gamma_0^{(2)}(N)$), then

$$\mu_G(p) \equiv a_p(f) + p^{(k'-k)/2} a_p(g) \pmod{\lambda}, \quad \text{for all } p \nmid N. \tag{4}$$

Under certain additional hypotheses, we prove in §9 below, the existence of such a G. (We enlarge K if necessary, to contain the Hecke eigenvalues of G.)

Let Π_G be an automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$ associated to G as in 3.2 of [Sc] and 3.5 of [AS]. (This Π_G is not necessarily uniquely determined by G, but its local components at $p \nmid N$ are.) By Theorem I of [We2], there is an associated continuous, linear representation

$$\rho_G : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_4(\overline{\mathbb{Q}}_\ell).$$

By enlarging K if necessary, we may assume that it takes values in $\mathrm{GL}_4(K_\lambda)$.

Lemma 4.1. *Suppose that there exists a G as above. Suppose also that λ is not a congruence prime for f in $S_{k'}(\Gamma_0(N))$ or g in $S_k(\Gamma_0(N))$, that $\ell > k'$, and that $\overline{\rho}_f$ and $\overline{\rho}_g$ are irreducible representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.*

(1) Π_G *is not a weak endoscopic lift.*
(2) Π_G *is not CAP.*

By λ not being a congruence prime for f in $S_{k'}(\Gamma_0(N))$, we mean that there does not exist a different Hecke eigenform $h \in S_{k'}(\Gamma_0(N))$, and a prime λ' dividing λ in a sufficiently large extension, such that $a_p(h) \equiv a_p(f) \pmod{\lambda'}$ for all primes $p \nmid \ell N$.

Proof. (1) If Π_G were a weak endoscopic lift then there would have to exist newforms $f' \in S_{k'}(\Gamma_0(N)), h \in S_k(\Gamma_0(N))$ such that $\mu_G(p) = a_p(f') + p^{(k'-k)/2}a_p(h)$ for almost all primes p. (See the introduction of [We2] for a precise definition of weak endoscopic lift, and (3) of Hypothesis A of [We2] for this consequence.) We have then

$$a_p(f') + p^{(k'-k)/2} a_p(h) \equiv a_p(f) + p^{(k'-k)/2} a_p(g) \pmod{\lambda},$$

for almost all primes p. Consequently

$$\overline{\rho}_f \oplus \overline{\rho}_g\,((k-k')/2) \simeq \overline{\rho}_{f'} \oplus \overline{\rho}_h((k-k')/2).$$

Now $\overline{\rho}_{f'}$ could not be isomorphic to $\overline{\rho}_g\,((k-k')/2)$, since the restrictions to I_ℓ give different characters (using $\ell > k'$). The only way to reconcile the two sides of the above isomorphism is for $\overline{\rho}_f \simeq \overline{\rho}_{f'}$. Given that λ is not a congruence prime for f in $S_{k'}(\Gamma_0(N))$, we must have $f' = f$, and similarly $h = g$. It follows from (4) and (6) of Hypothesis A of [We2] that Π_G must be associated to some Yoshida lift $F'_{f,g}$ of f and g. (Those $p \mid N$ for which the local component is Π_v^+ rather than Π_v^- are the divisors of N_1.) By (6) of Hypothesis A of [We2], the multiplicity of Π_G in the discrete spectrum is one. By Lemmes 1.2.8 and 1.2.10 of [SU], the local representation Π_p of $\mathrm{GSp}(4, \mathbb{Q}_p)$, for $p \mid N$, is that labelled VIa in [Sc]. By Table 3 of [Sc], the spaces of $\Gamma_0^{(2)}(\mathbb{Z}_p)$-fixed vectors in Π_p are 1-dimensional. It follows that (up to scaling), $G = F'_{f,g}$, contrary to hypothesis.

(2) By Corollary 4.5 of [PS], Π_G could only be CAP for a Siegel parabolic subgroup, but then, as on p.74 of [We2], we would have $k = 2$ and

$$\mu_G(p) = a_p(f') + \chi(p)p^{k'/2} + \chi(p)p^{(k'/2)-1},$$

for some newform $f' \in S_{k'}(\Gamma_0(N))$ and χ a quadratic or trivial character. This is incompatible with $\mu_G(p) \equiv a_p(f) + p^{(k'-k)/2}a_p(g) \pmod{\lambda}$ and the irreducibility of $\overline{\rho}_f$ and $\overline{\rho}_g$.

□

Note that the proof of Hypothesis A (on which Theorem I also depends) is not in [We2], but has now appeared in [We3].

Lemma 4.2. *Let G be as in Lemma 4.1. Then the representation ρ_G is irreducible.*

Proof. Suppose that ρ_G is reducible. It cannot have any 1-dimensional composition factor, since $\overline{\rho}_G$ has 2-dimensional irreducible composition factors $\overline{\rho}_f$ and $\overline{\rho}_g((k-k')/2)$. (The factors are well-defined, even though $\overline{\rho}_G$ isn't.) Looking at the list, in 3.2.6 of [SU], of possibilities for the composition factors of ρ_G, we must be in Cas B, (iv) or (v). But as in 3.2.6 of [SU], Π_G would be CAP in one case, a weak endoscopic lift in the other, and both of these are ruled out by Lemma 4.1. □

Let V, a 4-dimensional vector space over K_λ, be the space of the representation ρ_G. Choose a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant O_λ-lattice T in V, and let $W := V/T$. Let $\overline{\rho}_G$ be the representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $W[\lambda] \simeq T/\lambda T$. This depends on the choice of T, but we may choose T in such a way that $\overline{\rho}_G$ has $\overline{\rho}_g((k-k')/2)$ as a submodule and $\overline{\rho}_f$ as a quotient. Assume that this has been done.

Lemma 4.3. T *may be chosen in such a way that furthermore $\overline{\rho}_f$ is not a submodule of $\overline{\rho}_G$, i.e. so that the extension of $\overline{\rho}_f$ by $\overline{\rho}_g((k-k')/2)$ is not split.*

Proof. We argue as in the proof of Proposition 2.1 of [R1]. Choose an O_λ-basis for T, so that $\rho_G(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})) \subset \mathrm{GL}_4(O_\lambda)$. Assuming the lemma is false, we prove by induction that for all $i \geq 1$ there exists $M_i = \begin{pmatrix} I_2 & S_i \\ 0_2 & I_2 \end{pmatrix} \in \mathrm{GL}_4(O_\lambda)$ such that $M_i\,\rho_G(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}))M_i^{-1}$ consists of matrices of the form $\begin{pmatrix} A & \lambda^i B \\ \lambda C & D \end{pmatrix}$, with $A, B, C, D \in M_2(O_\lambda)$. Then letting $S = \lim S_i$ and $M = \begin{pmatrix} I_2 & S \\ 0_2 & I_2 \end{pmatrix}$, $M\,\rho_G(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}))M^{-1}$ consists of matrices of the form $\begin{pmatrix} A & 0_2 \\ \lambda C & D \end{pmatrix}$, contradicting the irreducibility of ρ_G.

By assumption, $\overline{\rho}_f$ is a submodule of $\overline{\rho}_G$ (i.e. $\overline{\rho}_G$ is semi-simple), so we have M_1. This is the base step. Now suppose that we have M_i. We must try to produce M_{i+1}. Let $P = \begin{pmatrix} I_2 & 0_2 \\ 0_2 & \lambda I_2 \end{pmatrix}$. Then $P^i M_i\,\rho_G(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}))M_i^{-1}P^{-i}$ consists of matrices of the form $\begin{pmatrix} A & B \\ \lambda^{i+1}C & D \end{pmatrix}$. Now let U be a matrix of the form $\begin{pmatrix} I_2 & B' \\ 0_2 & I_2 \end{pmatrix}$ such that $UP^iM_i\,\rho_G(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}))M_i^{-1}P^{-i}U^{-1}$ consists of matrices of the form $\begin{pmatrix} \tilde{A} & \lambda\tilde{B} \\ \lambda^{i+1}\tilde{C} & \tilde{D} \end{pmatrix}$. This exists because we are assuming that not only $\overline{\rho}_G$, but any other reduction with submodule $\overline{\rho}_g((k-k')/2)$, is semi-simple. Now just let $M_{i+1} = P^{-i}UP^iM_i$. Note that since $P^{-i}UP^i = \begin{pmatrix} I_2 & \lambda^iB' \\ 0_2 & I_2 \end{pmatrix}$, it is clear that M_{i+1} is of the form $\begin{pmatrix} I_2 & S_{i+1} \\ 0_2 & I_2 \end{pmatrix}$, with $S_{i+1} \equiv S_i \pmod{\lambda^i}$. □

We remark that, though the first T chosen may give semi-simple $\overline{\rho}_G$, the lemma shows there will be another choice that gives a non-trivial extension. Compare with the situation for 5-torsion on elliptic curves in the isogeny class of conductor 11.

5. A non-zero element in a Bloch-Kato Selmer group

Let G be as in the previous section. Then by Lemma 4.3, $\overline{\rho}_G$ is a non-trivial extension of $\overline{\rho}_f$ by $\overline{\rho}_g((k-k')/2)$:

$$0 \longrightarrow \overline{\rho}_g((k-k')/2) \longrightarrow \overline{\rho}_G \longrightarrow \overline{\rho}_f \longrightarrow 0.$$

Applying $\mathrm{Hom}_{\mathbb{F}_\lambda}(\overline{\rho}_f, \_\_)$ to the exact sequence, and pulling back the inclusion of the trivial module in $\mathrm{Hom}_{\mathbb{F}_\lambda}(\overline{\rho}_f, \overline{\rho}_f)$, we get a non-trivial extension of the trivial module by $\mathrm{Hom}(\overline{\rho}_f, \overline{\rho}_g((k-k')/2)$. Thus we get a non-zero class in $H^1(\mathbb{Q}, \mathrm{Hom}_{\mathbb{F}_\lambda}(\overline{\rho}_f, \overline{\rho}_g((k-k')/2)))$, in the standard way. (Lifting the identity to a section $s \in \mathrm{Hom}_{\mathbb{F}_\lambda}(\overline{\rho}_f, \overline{\rho}_G)$, a representing cocycle is $g \mapsto g.s - s$, where $(g.s)(x) = g(s(g^{-1}(x)))$.)

Now the dual of $\overline{\rho}_f$ is $\overline{\rho}_f(k'-1)$, so

$$\mathrm{Hom}_{\mathbb{F}_\lambda}(\overline{\rho}_f, \overline{\rho}_g((k-k')/2)) \simeq \overline{\rho}_f(k'-1) \otimes \overline{\rho}_g((k-k')/2) \simeq \overline{\rho}_f \otimes \overline{\rho}_g((k'+k-2)/2).$$

In the notation of §3, this is $A[\lambda]((k'+k-2)/2)$. So we have a non-zero class $c \in H^1(\mathbb{Q}, A[\lambda]((k'+k-2)/2))$. By Lemma 3.4, $H^0(\mathbb{Q}, A_\lambda((k'+k-2)/2))$ is trivial, so we get a non-zero class $d \in H^1(\mathbb{Q}, A_\lambda((k'+k-2)/2))$, the image of c under the map induced by inclusion.

Proposition 5.1. *Let $f \in S_{k'}(\Gamma_0(N)), g \in S_k(\Gamma_0(N))$ be normalised newforms of square-free level $N > 1$, with $k' > k \geq 2$. Suppose that at each prime $p \mid N$, f and g share the eigenvalue of the Atkin-Lehner involution. Let $\lambda \mid \ell$ be a divisor of $\frac{L_{f\otimes g}((k'+k-2)/2)}{\pi^{k'-1}\langle f,f\rangle}$, with $\ell \nmid N$ and $\ell > \frac{3k'+k-2}{2}$. Suppose also that λ is not a congruence prime for f in $S_{k'}(\Gamma_0(N))$ or g in $S_k(\Gamma_0(N))$, and that $\overline{\rho}_f$ and $\overline{\rho}_g$ are irreducible representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. With $G \in S_\rho(\Gamma_0^{(2)}(N))$ as above, suppose that the representation $\rho_G : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_4(K_\lambda)$ takes values in $\mathrm{GSp}_4(K_\lambda)$. Then the Bloch-Kato Selmer group $H^1_f(\mathbb{Q}, A_\lambda((k'+k-2)/2))$ is non-zero.*

Proof. We will show that the non-zero element $d \in H^1(\mathbb{Q}, A_\lambda((k'+k-2)/2))$ satisfies $\mathrm{res}_p(d) \in H^1_f(\mathbb{Q}_p, A_\lambda((k'+k-2)/2))$ for each prime p.

(1) If $p \nmid \ell N$ then $\rho_G|_{I_p}$ is trivial, so certainly

$$0 \longrightarrow \overline{\rho}_g((k-k')/2)|_{I_p} \longrightarrow \overline{\rho}_G|_{I_p} \longrightarrow \overline{\rho}_f|_{I_p} \longrightarrow 0$$

splits, showing that $\mathrm{res}_p(c) \in \ker(H^1(\mathbb{Q}_p, A[\lambda]((k'+k-2)/2)) \to H^1(I_p, A[\lambda]((k'+k-2)/2)))$, hence that $\mathrm{res}_p(d) \in \ker(H^1(\mathbb{Q}_p, A_\lambda((k'+k-2)/2)) \to H^1(I_p, A_\lambda((k'+k-2)/2)))$. Since $A_\lambda^{I_p}$ is divisible (in this case the whole of A_λ), this shows that $\mathrm{res}_p(d) \in H^1_f(\mathbb{Q}_p, A_\lambda((k'+k-2)/2))$, as in Lemma 7.4 of [Br].

(2) If $p = \ell$ then we may prove $\mathrm{res}_p(d) \in H^1_f(\mathbb{Q}_p, A_\lambda((k'+k-2)/2))$ just as in Lemma 7.2 of [Du1]. Since $\ell \nmid N$, $\rho_G|_{D_\ell}$ is crystalline; see Theorem 3.2(ii) of [U], which refers to [Fa] and [CF]. It is for this case that we need the condition $\ell > \frac{3k'+k-2}{2}$. This $\frac{3k'+k-2}{2}$ arises as the span of the "weights" $\{1-k', 0\}$ of $\overline{\rho}_f^*$ and $\{(k'-k)/2, (k'+k-2)/2\}$ of $\overline{\rho}_g((k-k')/2)$. See the proof of Lemma 7.2 of [Du1] for comparison.

(3) Now consider the case that $p \mid N$. As in the proof of Lemma 3.2(1), the action of I_p on $\mathfrak{M}_{f,\lambda}/\lambda\mathfrak{M}_{f,\lambda}$ and $\mathfrak{M}_{g,\lambda}/\lambda\mathfrak{M}_{g,\lambda}$ is non-trivial and unipotent. Hence we may choose a basis for $W[\lambda]$ (notation as in the previous section) such that for any $\sigma \in I_p$, $\overline{\rho}_G(\sigma)$ is represented by $\exp(t_\ell(\sigma)\tilde{N})$, with $t_\ell : I_p \to \mathbb{Z}_\ell(1)$ the standard tamely ramified character and $\tilde{N}$ of the form $\tilde{N} = \begin{pmatrix} A & B \\ 0_2 & A \end{pmatrix}$, with $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. By Theorem 2.2.5(1) of [GT], $\tilde{N}^2 = 0$. To see that the conditions of that theorem are satisfied here, firstly ρ_G is irreducible by Lemma 4.2, secondly ρ_G is symplectic by hypothesis. Lastly, given that the local component Π_p of Π_G has a non-zero vector fixed by $\Gamma_0^{(2)}(\mathbb{Z}_p)$ but none fixed by $\mathrm{GSp}_4(\mathbb{Z}_p)$, an inspection of Table 3 in [Sc] reveals that it is always the case that either the subspace of Π_p fixed by the Siegel parahoric $\Gamma_0^{(2)}(\mathbb{Z}_p)$, or that fixed by a Klingen parahoric, is 1-dimensional. (Note that if Π_p had a non-zero vector fixed by $\mathrm{GSp}_4(\mathbb{Z}_p)$ then, by Theorem I of [We2], ρ_G would be unramified at p, contrary to $\overline{\rho}_G$ having $\overline{\rho}_f$ as a subfactor.)

Since $\tilde{N}^2 = 0$, B must be of the form $B = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$. Writing elements of $\mathrm{Hom}_{\mathbb{F}_\lambda}(\overline{\rho}_f, \overline{\rho}_g((k-k')/2))$ as 2-by-2 matrices in the obvious way, a short calculation shows that $c|_{I_p}$ is represented by the cocycle $\sigma \mapsto \begin{pmatrix} 0 & t_\ell(\sigma)b \\ 0 & 0 \end{pmatrix}$, which is the coboundary $\sigma \mapsto \sigma\left(\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}\right) - \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$. Since $c|_{I_p} = 0$, $d|_{I_p} = 0$. As already noted in the proof of Lemma 3.2, $A_\lambda^{I_p}$ is divisible, so we may deduce as in (1) that $\mathrm{res}_p(d) \in H^1_f(\mathbb{Q}_p, A_\lambda((k'+k-2)/2))$.

□

Remark 5.2. *If Π_G has multiplicity one in the discrete spectrum, then the condition about ρ_G being symplectic is satisfied, by Theorem IV of* [We2]. *(The symplectic form comes from Poincaré duality.) It is expected always to hold. See the discussion following 6.4 in* [Du1] *for more on this.*

6. The doubling method with differential operators

We mainly recall some properties of the doubling method in the setting of holomorphic Siegel modular forms (with invariant differential operators). As long as one does not insist on explict constants and explict Γ-factors, everything works more generally for arbitrary polynomial representations as automorphy factors, see [BS3, I1].

6.1. **Construction of holomorphic differential operators.** We construct holomorphic differential operators on $\mathfrak{H}_{2n}$ with certain equivariance properties. We combine the constructions from [B1] and [BSY]; a similar strategy was also used by [Koz].
We decompose $Z \in \mathfrak{H}_{2n}$ as

$$Z = (z_{ij}) = \begin{pmatrix} z_1 & z_2 \\ z_2^t & z_4 \end{pmatrix} \qquad (z_1, z_4 \in \mathfrak{H}_n).$$

We also use the natural embedding $Sp(n) \times Sp(n) \hookrightarrow Sp(2n)$, defined by

$$(M_1, M_2) \mapsto M_1^{\uparrow} \cdot M_2^{\downarrow} := \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}, \qquad M_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} \in Sp(n).$$

The differential operator matrix $\partial = (\partial_{ij})$ with $\partial_{ij} = \frac{1+\delta_{ij}}{2} \frac{\partial}{\partial z_{ij}}$ will then be decomposed in block matrices of size n, denoted by

$$\partial = \begin{pmatrix} \partial_1 & \partial_2 \\ \partial_2^t & \partial_4 \end{pmatrix}.$$

We realize the symmetric tensor representation $\sigma_\nu := Sym^\nu$ of $GL(n, \mathbb{C})$ in the usual way on the space $V_\nu := \mathbb{C}[X_1, \dots X_n]_\nu$ (of homogeneous polynomials of degree ν). For V_ν -valued functions f on $\mathfrak{H}_n$, $\alpha, \beta \in \mathbb{C}$ and $M \in Sp(n, \mathbb{R})$ we define the slash-operator by

$$(f \mid_{\alpha,\beta,\sigma_\nu} M)(z) := \det(cz+d)^{-\alpha} \det(c\bar{z}+d)^{-\beta} \sigma_\nu(cz+d)^{-1} f(M\langle z \rangle).$$

We may ignore the ambiguity of the powers $\alpha, \beta \in \mathbb{C}$ most of the time. If $\beta = 0$ or $\nu = 0$ we just omit them from the slash operator.

Proposition 6.1. *For nonnegative integers μ, ν there is a (nonzero) holomorphic differential operator $\mathbb{D}_\alpha(\mu,\nu)$ mapping scalar-valued C^∞ functions F on $\mathfrak{H}_{2n}$ to $V_\nu \otimes V_\nu$-valued functions on $\mathfrak{H}_n \times \mathfrak{H}_n$, satisfying*

$$(5) \qquad \mathbb{D}_\alpha(\mu,\nu)(F \mid_{\alpha,\beta} (M_1^{\uparrow} M_2^{\downarrow}) = (\mathbb{D}_\alpha(\mu,\nu)(F)) \mid^{z_1}_{\alpha+\mu,\beta,\sigma_\nu} M_1 \mid^{z_4}_{\alpha+\mu,\beta,\sigma_\nu} M_2$$

for all $M_1, M_2 \in Sp(n, \mathbb{R})$; the upper index at the slash operator indicates, for which variables M_i is applied.
More precisely, there is a $V_\nu \otimes V_\nu$ -valued nonzero polynomial $Q(\alpha, \mathbf{T}) = Q_\alpha^{(\mu,\nu)}(\mathbf{T})$ in the variables α and $\mathbf{T}$ (where $\mathbf{T}$ is a symmetric $2n \times 2n$ matrix of variables), with rational coefficients, such that

$$\mathbb{D}_\alpha(\mu,\nu) = Q_\alpha^{(\mu,\nu)}(\partial_{ij}) \mid_{z_2=0}.$$

The differential operator $\mathbb{D}_\alpha(\mu,\nu)$ has the additional symmetry property

$$\mathbb{D}_\alpha(\mu,\nu)(F \mid V) = \mathbb{D}_\alpha(\mu,\nu)(F)^\star,$$

where V *is the operator defined on functions on* $\mathfrak{H}_{2n}$ *by*

$$F \longmapsto (F \mid V)\left(\begin{pmatrix} z_1 & z_2 \\ z_2^t & z_4 \end{pmatrix}\right) = F\left(\begin{pmatrix} z_4 & z_2^t \\ z_2 & z_1 \end{pmatrix}\right)$$

and for a function g *on* $\mathfrak{H}_n \times \mathfrak{H}_n$ *we put* $g^\star(z,w) := g(w,z)$.

Remark 6.2. *We allow arbitrary "complex weights"* α *here; note that there is no ambiguity in this as long as we use the same branch of* $\log\det(CZ+D)$ *to define the* $\det(CZ+D)^s$ *on both sides of (5).*
Note also that the differential operators do not depend at all on β.

Proof. We recall from [B1] the existence of an explicitly given differential operator

$$\mathcal{D}_\alpha = (-1)^n C_n\left(\alpha - n + \frac{1}{2}\right)\det(\partial_2) + \ldots + \det(z_2)\cdot\det(\partial_{ij})$$

with

$$C_n(s) := s\left(s+\frac{1}{2}\right)\ldots\left(s+\frac{n-1}{2}\right) = \frac{\Gamma_n(s+\frac{n+1}{2})}{\Gamma_n(s+\frac{n-1}{2})} \qquad \left(\Gamma_n(s) = \pi^{\frac{n(n-1)}{4}}\prod_{j=0}^{n-1}\Gamma\left(s-\frac{j}{2}\right)\right).$$

This operator is compatible with the action of $Sp(n,\mathbb{R})\times Sp(n,\mathbb{R}) \hookrightarrow Sp(2n,\mathbb{R})$, increasing the weight α by one (without restriction!), i.e.

$$\mathcal{D}_\alpha(F\mid_{\alpha,\beta} M_1^\uparrow\cdot M_2^\downarrow) = (\mathcal{D}_\alpha F)\mid_{\alpha+1,\beta} M_1^\uparrow\cdot M_2^\downarrow, \qquad (M_i\in Sp(n,\mathbb{R}))$$

We put

$$\mathcal{D}_\alpha^\mu := \mathcal{D}_{\alpha+\mu-1}\circ\cdots\circ\mathcal{D}_\alpha.$$

Remark 6.3. *The combinatorics of this operator is not known explictly for general* μ.

The second type of differential operators maps scalar-valued functions on $\mathfrak{H}_{2n}$ to $\mathbb{C}[X_1,\ldots,X_n]_\nu\otimes\mathbb{C}[Y_1,\ldots,Y_n]_\nu$-valued functions on $\mathfrak{H}_n\times\mathfrak{H}_n$, changing the automorphy factor from $\det^\alpha$ on $GL(2n,\mathbb{C})$ to $(\det^\alpha\otimes Sym^\nu)\boxtimes(\det^\alpha\otimes Sym^\nu)$ on $GL(n,\mathbb{C})\times GL(n,\mathbb{C})$. This operator was introduced in [BSY]; it is a special feature that we know the combinatorics in this case quite explictly:

(6)

$$L_\alpha^\nu := \frac{1}{(2\pi i)^\nu\alpha^{[\nu]}}\left(\sum_{0\le 2j\le\nu}\frac{1}{j!(\nu-2j)!(2-\alpha-\nu)^{[j]}}(D_\uparrow D_\downarrow)^j(D-D_\uparrow-D_\downarrow)^{\nu-2j}\right)_{z_2=0};$$

here we use the same notation as in [BSY]:

$$\begin{aligned}
\alpha^{[j]} &= \alpha(\alpha+1)\ldots(\alpha+j-1) = \frac{\Gamma(\alpha+j)}{\Gamma(\alpha)}\\
D &= \partial[(X_1,\ldots,X_n,Y_1,\ldots,Y_n)^t]\\
D_\uparrow &= \partial[(X_1,\ldots,X_n,0,\ldots,0)^t]\\
D_\downarrow &= \partial[(0,\ldots,0;Y_1,\ldots,Y_n)^t],
\end{aligned}$$

where $A[x] := x^t Ax$; we remark that

$$D-D_\uparrow-D_\downarrow = (X_1,\ldots X_n;0,\ldots 0)\cdot\partial_2\cdot(0,\ldots,0;Y_1,\ldots Y_n)^t.$$

In [BSY] the weight was a natural number k, but everything works also for arbitrary complex α instead. (Due to the normalization of [BSY], we have to omit certain finitely many α.)
We put

$$\mathbb{D}_\alpha(\mu,\nu) := L^\nu_{\alpha+\mu} \circ \mathcal{D}^\mu_\alpha.$$

This operator has all the requested properties, except for the fact that the coefficients are not polynomials in α but rational functions. □

6.2. **Some combinatorics.** Then we consider the function $h_{\alpha,\beta}$ defined on $\mathbb{H}_{2n}$ by

$$h_{\alpha,\beta}(Z) := \det(z_1+z_2+z_2^t+z_4)^{-\alpha}\overline{\det(z_1+z_2+z_2^t+z_4)}^{-\beta}$$

and we note that (following [BCG])

$$\mathcal{D}^\mu_\alpha h_{\alpha,\beta} = A_{\alpha,\mu}\cdot h_{\alpha+\mu,\beta}$$

with

$$A_{\alpha,\mu} = \frac{\Gamma_n(\alpha+\mu)}{\Gamma_n(\alpha)}\frac{\Gamma_n(\alpha+\mu-\frac{n}{2})}{\Gamma_n(\alpha-\frac{n}{2})}$$

and also

$$L^\nu_\alpha h_{\alpha,\beta} = B_{\alpha,\nu}\sigma_\nu(z_1+z_4)^{-1}\left(\sum X_iY_i\right)^\nu \det(z_1+z_4)^{-\alpha}\overline{\det(z_1+z_4)}^{-\beta}$$

with

$$B_{\alpha,\nu} = \frac{1}{(-2\pi i)^\nu \nu!}\frac{\Gamma(2\alpha-2+\nu)}{\Gamma(2\alpha-2)}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha+\nu-1)},$$

following [BSY, Lemma 4.2].

For later purposes we summarize here some additional properties of these differential operators:
First we note that $\mathbb{D}_\alpha(\mu,\nu)$ is a homogeneous polynomial (of degree $n\mu+\nu$) in the partial derivatives; we decompose it as

$$\mathbb{D}_\alpha(\mu,\nu) = \mathcal{M}+\mathcal{R},$$

where the "main term " $\mathcal{M}$ denotes the part free of derivatives w.r.t. z_1 or z_4.

Lemma 6.4. *a) All the monomials occuring in the "remainder term" $\mathcal{R}$ have positive degree in the partial derivatives w.r.t. z_1 and z_4.*
b) The "main term" $\mathcal{M}$ is of the form

$$\mathcal{M} = C_\alpha(\mu,\nu)\left(D-D^\uparrow-D^\downarrow\right)^\nu \det(\partial_2)^\mu$$

with

$$C_\alpha(\mu,\nu) = \frac{1}{(\alpha+\mu)^{[\nu]}\nu!}\prod_{j=0}^{\mu-1} C_n\left(\alpha-n+\frac{\mu+\nu'+j}{2}\right) \qquad \left(\nu' := \frac{\nu}{n}\right).$$

c) For the polynomial $Q^{\mu,\nu}_\alpha(\mathbf{T})$ with the symmetric matrix $\mathbf{T} = \begin{pmatrix} \mathbf{T}_1 & \mathbf{T}_2 \\ \mathbf{T}_2^t & \mathbf{T}_4 \end{pmatrix}$ of size $2n$ this means

$$(7) \qquad Q^{\mu,\nu}_\alpha(\mathbf{T}) = C_\alpha(\mu,\nu)\left(2(X_1,\dots,X_n)\mathbf{T}_2(Y_1,\dots,Y_n)^t\right)^\nu \det(\mathbf{T}_2)^\mu + (*),$$

where (*) contains only contributions with positive degree in $\mathbf{T}_1$ and $\mathbf{T}_4$.

Proof. a) The formula (12) in [B1] shows that in $\mathcal{D}_\alpha$ an entry of ∂_1 always appears together with an entry of ∂_4. The same is then true for $\mathcal{D}^\mu$. Furthermore, the explict formula (6) for $L^\nu_{\alpha+\mu}$ shows that only the contribution of $j=0$ is free of partial derivatives w.r.t. z_1; it is at the same time the only contribution free of derivatives w.r.t. z_4.
b) We define an element $M = M(X_1,\dots,X_n;Y_1,\dots,Y_n)$ of $V_\nu \otimes V_\nu$ by

$$M := \mathbb{D}_\alpha(\mu,\nu)(\operatorname{exptr}(z_2)) = \mathcal{M}(\operatorname{exptr}(z_2)).$$

The transformation properties of $\mathbb{D}_\alpha(\mu,\nu)$, applied for

$$\begin{pmatrix} A^t & 0 \\ 0 & A^{-1} \end{pmatrix}^\uparrow, \quad \begin{pmatrix} A & 0 \\ 0 & A^{-t} \end{pmatrix}^\downarrow \qquad (A \in GL(n,\mathbb{R}))$$

yield

$$M((X_1,\dots,X_n)\cdot A;Y_1,\dots,Y_n) = M(X_1,\dots,X_n;(Y_1,\dots,Y_n)A^t) \quad (A \in GL(n,\mathbb{C})).$$

Such a vector in $V_\nu \otimes V_\nu$ is unique up to constants and is therefore a scalar multiple of $(\sum X_i Y_i)^\nu$, i.e. $M = c\cdot(2\sum_i X_iY_i)^\nu$ for an appropriate constant $c = C_\alpha(\mu,\nu)$. To understand $\mathcal{M}$ we study its action on those functions on $\mathbb{H}_{2n}$, which depend only on z_2; it is enough to look at functions of type $f_T(z_2) := \operatorname{exptr}(Tz_2)$ with $T \in \mathbb{R}^{(n,n)}$, $\det(T) \neq 0$. Then

$$\begin{aligned}
\mathbb{D}_\alpha(\mu,\nu)f_T &= \det(T)^{-\alpha}\mathbb{D}_\alpha(\mu,\nu)(f_{1_n} \mid_\alpha \begin{pmatrix} T & 0 \\ 0 & T^{-t} \end{pmatrix}) \\
&= \det(T)^{-\alpha}\,(\mathbb{D}_\alpha(\mu,\nu)f_{1_n}) \mid^{z_1}_{\alpha+\mu,\nu} \begin{pmatrix} T & 0 \\ 0 & T^{-t} \end{pmatrix} \\
&= \det(T)^\mu c\cdot(2\sum_i X_i T^t Y_i)^\nu \\
&= c(D - D^\uparrow - D^\downarrow)^\nu \det(\partial_2)^\mu f_T.
\end{aligned}$$

It remains to determine the coefficient $C_\alpha(\mu,\nu)$; we compute $\mathbb{D}_s(\mu,\nu)\det(z_2)^s$ in two ways, using the standard formulas (see e.g. [BCG, Section 1])

$$\det(\partial_2)\det(z_2)^s = C_n\left(\frac{s}{2}\right)\det(z_2)^{s-1}$$

$$\mathcal{D}_\alpha \det(z_2)^s = (-1)^n C_n\left(\frac{s}{2}\right) C_n(\alpha - n + \frac{s}{2})\det(z_2)^{s-1}.$$

Then

$$\mathbb{D}_\alpha(\mu,\nu)\det(z_2)^s = C_\alpha(\mu,\nu)\left(\prod_{j=0}^{\mu-1} C_n\left(\frac{s-j}{2}\right)\right)\{\left(D - D^\uparrow - D^\downarrow\right)^\nu \det(z_2)^{s-\mu}\}\mid_{z_2=0}$$

and on the other hand

$$\mathbb{D}_\alpha(\mu,\nu)\det(z_2)^s = L^\nu_{\alpha+\mu}(\mathcal{D}^\mu_\alpha \det(z_2)^s)$$

$$= \prod_{j=0}^{\mu-1} C_n\left(\frac{s-j}{2}\right) C_n\left(\alpha - n + \frac{s+j}{2}\right)\{L^\nu_{\alpha+\mu}\det(z_2)^{s-\mu}\}\mid_{z_2=0}$$

$$= \prod_{j=0}^{\mu-1} C_n\left(\frac{s-j}{2}\right) C_n\left(\alpha - n + \frac{s+j}{2}\right)\frac{1}{(\alpha+\mu)^{[\nu]}\nu!}\{\left(D - D^\uparrow - D^\downarrow\right)^\nu \det(z_2)^{s-\mu}\}\mid_{z_2=0}$$

If $\nu = n\nu'$ is a multiple of n, then $s := \mu + \nu'$ gives nonzero contributions and we get

$$C_\alpha(\mu,\nu) = \frac{1}{(\alpha+\mu)^{[\nu]}\nu!}\prod_{j=0}^{\mu-1} C_n\left(\alpha - n + \frac{\mu+\nu'+j}{2}\right).$$

Actually, this formula makes sense (and is also valid) for arbitrary ν. □

6.3. **Doubling method with the differential operators $\mathbb{D}_\alpha(\mu,\nu)$.** The inner product $(\sum a_i X_i, \sum b_i X_i) = \sum a_i \overline{b_i}$ on $V_1 := \mathbb{C}[X_1,\dots X_n]_1$ induces a "produit scalaire adapté" (see [Go]) on the ν-fold symmetric tensor product $V_\nu = \mathrm{Sym}^\nu(V_1) = \mathbb{C}[X_1,\dots X_n]_\nu$ by

$$\{\alpha_1\cdot\dots\cdot\alpha_\nu, \beta_1\cdot\dots\cdot\beta_\nu\} = \frac{1}{\nu!}\sum_\tau \prod_{j=1}^{\nu}(\alpha_{\tau(j)},\beta_j) \qquad (\alpha_i,\beta_j\in V_1),$$

where τ runs over the symmetric group of order ν. This inner product is invariant under the action of unitary matrices via Sym^ν.
Note that for all $\mathbf{v}\in\mathbb{C}[X_1,\dots,X_n]_\nu$ we have

$$\left\{\mathbf{v}, \left(\sum X_i Y_i\right)^\nu\right\} = \tilde{\mathbf{v}},$$

where $\tilde{\mathbf{v}}$ denotes the same polynomial as $\mathbf{v}$, but with the variables Y_i instead of the X_i.
We describe here the general pullback formula for level N Eisenstein series (N squarefree).
We put

$$G_k^{(2n)}(Z,s) = \sum_{M\in\Gamma_0^{(2n)}(N)_\infty\backslash\Gamma_0^{(2n)}(N)} \det(CZ+D)^{-k-s}\det(C\overline{Z}+D)^{-s}.$$

For a cusp form $F\in S_\rho(\Gamma_0^{(n)}(N))$ with $\rho = \det^{k+\mu}\otimes\sigma_\nu$ and $z = x+iy, w = u+i\mathbf{v}\in\mathfrak{H}_n$ we get

$$\int_{\Gamma_0^{(n)}(N)\backslash\mathbb{H}_n}\left\{\rho(\sqrt{y})F(z), \rho(\sqrt{y})\mathbb{D}_{s+k}(\mu,\nu)G_k^{(2n)}\left(\left(\begin{array}{cc} z & 0\\ 0 & -\bar{w}\end{array}\right),\bar{s}\right)\det(y)^s\det(\mathbf{v})^s\right\}d\omega_n$$

$$= \gamma_n(k,\mu,\nu,s)\sum_M F(w)\mid T_N(M)\det(M)^{-k-2s}. \tag{8}$$

Here $d\omega_n = \det(y)^{-n-1}dxdy$, M runs over all (integral) elementary divisor matrices of size n with $M\equiv 0 \bmod N$, and $T_N(M)$ denotes the Hecke operator associated to the double coset $\Gamma_0^{(n)}(N)\left(\begin{array}{cc} 0 & -M^{-1}\\ M & 0\end{array}\right)\Gamma_0^{(n)}(N)$.

To compute the archimedean factor γ one should keep in mind that the unfolding of the integral leads to an integration over $\mathfrak{H}_n$ involving $\mathbb{D}_{k+s}(\mu,\nu)h_{k+s,s}$. Then γ is naturally a product of (essentially) three factors

$$\gamma_n(k,\mu,\nu,s) = i^{nk+n\mu+\nu}2^{n(n-k-\mu-2s-\nu+1)}A_{k+s,\mu}B_{k+\mu+s,\nu}I(s+k+\mu-n-1,\nu)$$

with a Hua type integral

$$I(\alpha,\nu)=\frac{\pi^{\frac{n(n+1)}{2}}}{\alpha+n+\nu}\prod_{j=1}^{n-1}\frac{(2\alpha+2j+1)(n+j+2\alpha)^{[\nu]}}{(\alpha+j)\Gamma(\nu+n+j+2\alpha+1)}.$$

We refer to [BSY, Sect.3], see also [B3, 2.2] for details.

6.4. **Doubling method with the differential operators $\mathbb{D}_k(\mu,\nu)$.** The differential operator $\mathbb{D}_{k+s}(\mu,\nu)$ was applied directly to the Eisenstein series of "weight" $k+s$. If we use the Hecke summation not in $s=0$ but in $s_1:=\frac{2n+1}{2}-k$ for an Eisenstein series of degree $2n$, we should better use a differential operator acting on the weight k Eisenstein series $E_k^{(2n)}:=G_k^{(2n)}\cdot(\det \mathrm{Im}Z)^s$ to get holomorphic modular forms (in particular theta series) after evaluating in $s=s_1$. One might try to use the calculations of Takayanagi [Tak]. Note however that the results of [Tak] are applicable only for the case $\mu=0$; to incorporate the differential operator $\mathcal{D}_k^{\mu}$ there is quite complicated, see also [Koz]. We avoid this difficulty by observing that the two types of differential operators are actually not that different:
By

$$F\longmapsto \mathcal{D}_{k,s}(\mu,\nu)(F):=\det(y)^s\det(\mathbf{v})^s\mathbb{D}_{k+s}(\mu,\nu)(\det(Y)^{-s}\times F)$$

we can define a new (nonholomorphic) differential operator mapping functions F on $\mathfrak{H}_{2n}$ to $\mathbb{C}[X_1,\dots,X_n]_\nu\otimes\mathbb{C}[Y_1,\dots,Y_n]_\nu$ valued functions on $\mathfrak{H}_n\times\mathfrak{H}_n$; this operator has exactly the same transformation properties as $\mathbb{D}_k(\mu,\nu)$.
Starting from the observation that $\mathcal{D}_{k,s}(\mu,\nu)$ maps holomorphic functions on $\mathfrak{H}_{2n}$ to nearly holomorphic functions on $\mathfrak{H}_n\times\mathfrak{H}_n$ we get from the theory of Shimura [Sh2, Sh3] in the same way as in [BCG, section 1] an operator identity

$$\mathcal{D}_{k,s}(\mu,\nu)=\sum_{\rho_i,\rho_j}\delta_{\rho_j}^{(z_1)}\otimes\delta_{\rho_j}^{(z_4)}\circ\mathcal{D}_s(\rho_i,\rho_j). \tag{9}$$

Here the ρ_i,ρ_j run over finitely many polynomial representations of $GL(n,\mathbb{C})$ and $\mathcal{D}_s(\rho_i,\rho_j)$ denotes a $V_{\rho_1}\otimes V_{\rho_j}$-valued holomorphic differential operator (a polynomial in the $\partial_{i,j}$, evaluated at $z_2=0$; it changes the automphy factor $\det^k$ on $GL(2n,\mathbb{C})$ to $(\det^k\otimes\rho_1)\boxtimes(\det^k\otimes\rho_2)$ on $GL(n,\mathbb{C})\times GL(n,\mathbb{C})$). As is usual in the theory of nearly holomorphic functions, we have to avoid finitely many weights k here. Furthermore the δ_{ρ_i}, δ_{ρ_j} are non-holomorphic differential operators on $\mathfrak{H}_n$, changing automorphy factors from $\det^k\otimes\rho$ to $\det^{k+\mu}\otimes Sym^\nu$. Actually, by invariant theory, holomorphic differential operators $\mathcal{D}_s(\rho_i,\rho_j)$ with the transformation properties described above only exist in the case $\rho_i=\rho_j$ see [I1].
If $\delta_\rho^{(z_1)}\otimes\delta_\rho^{(z_4)}$ is the identity, then $\rho=\det^{k+\mu}\otimes Sym^\nu$ and (at least for $k\geq n$) $\mathcal{D}_s(\rho,\rho)$ is a scalar multiple of $\mathbb{D}_k(\mu,\nu)$, because the space of such differential operators is one-dimensional. The decomposition (9) can then be rewritten as

$$p_s(k)\mathcal{D}_{k,s}(\mu,\nu)=d_s(k)\mathbb{D}_k(\mu,\nu)+\mathcal{K} \tag{10}$$

where $p_s(k)$ and $d_s(k)$ are polynomials in k and $\mathcal{K}$ is a nonholomorphic differential operator with the same transformation properties as $\mathbb{D}_k(\mu,\nu)$ and with the additional property that $\mathcal{K}(F)$ is orthogonal to all holomorphic cusp forms in the variables z_1 or z_4 (for any C^∞ automorphic form on $\mathbb{H}_{2n}$ with suitable growth properties). Note that (10) holds now for *all* weights k, if we request the finitely many exceptions from (9) to be among the zeroes of $p_s(k)$. We also observe that $\mathcal{D}_{k,s}(\mu,\nu)$ is a homogeneous polynomial of degree $n\mu+\nu$ in the variables $(\partial_{ij})_{|z_2=0}$

and the entries of y_1^{-1} and y_4^{-1} and $\mathcal{K}$ consists only of monomials whose joint degree in ∂_1 and y_1^{-1} as well as in ∂_4 and y_4^{-1} are both positive, in particular, $\mathcal{K}$ cannot contribute monomials that only involve entries of ∂_2.
Therefore (as in [BCG]) we may compare the coefficients of $\det(\partial_2)^\mu \left(\sum_{i,j} \frac{\partial}{\partial_{i,n+j}} X_i Y_j\right)^\nu$ on both sides: We get

$$p_s(k) C_{k+s}(\mu,\nu) = d_s(k) C_k(\mu,\nu).$$

From this we obtain a version of the pullback formula (8)

$$\begin{aligned} &\int_{\Gamma_0^{(n)}(N)\backslash\mathbb{H}_n} \left\{\rho(\sqrt{y})F(z), \rho(\sqrt{y})\mathbb{D}_k(\mu,\nu)E_k^{(2n)}\left(\begin{pmatrix} z & 0 \\ 0 & -\bar{w}\end{pmatrix}, \bar{s}\right)\right\} d\omega_n \\ (11)\qquad &= \frac{p_s(k)}{d_s(k)} \cdot \gamma_n(k,\mu,\nu,s) \sum_M F \mid T_N(M) \det(M)^{-k-2s}. \end{aligned}$$

We need the result above for the pullback formula applied for a degree 4, weight 2 Eisenstein series at $s_1 = \frac{1}{2}$: we consider the holomorphic modular form

$$\mathcal{E}_2^{(4)} := \mathrm{Res}_{s=s_1} E_2^{(4)}(Z,s)$$

Then we get for a cusp form $F \in S_\rho(\Gamma_0^{(2)}(N))$, with $\rho = \det^{2+\mu} \otimes \mathrm{Sym}^\nu$,

$$\begin{aligned} \langle F, \mathbb{D}_2(\mu,\nu)\mathcal{E}_2^{(4)}(*,-\bar{w})\rangle &= \mathrm{Res}_{s=s_1} \langle F, \mathbb{D}_2(\mu,\nu) E_2^{(4)}(*,-\bar{w})\rangle \\ &= \mathrm{Res}_{s=s_1} \frac{p_s(2)}{d_s(2)} \langle F, \mathbb{D}_{2+s}(\mu,\nu) G_2^{(4)} \det(y)^s \det(v)^s\rangle \\ (12)\qquad &= c \cdot \mathrm{Res}_{s=s_1}\left(\sum_M F(w) \mid T_N(M)\det(M)^{-2-2s}\right). \end{aligned}$$

The relevant constant is then

$$c = \frac{C_2(\mu,\nu)}{C_{2+\frac{1}{2}}(\mu,\nu)} \gamma_2\left(2,\mu,\nu,\frac{1}{2}\right). \tag{13}$$

6.5. **Standard-L-functions at $s = 1$ and $s = 2$, in particular for Yoshida lifts of degree 2.**

6.5.1. *An Euler product.* If $F \in S_\rho(\Gamma_0^{(n)}(N))$ is an eigenform of all the Hecke operators $T_N(M)$ with eigenvalues $\lambda_N(M)$, then the Dirichlet series of these eigenvalues can be written in terms of the (good part of) the standard L-function $D_F^{(N)}(s)$:

$$\begin{aligned} &\sum \lambda_N(M)\det(M)^{-s} = \\ &\quad \left(\sum_{\det(M)\mid N^\infty} \det(M)^{-s}\right) \times \frac{1}{\zeta^{(N)}(s)\prod_{i=1}^n \zeta^{(N)}(2s-2i)} D_F^{(N)}(s-n). \end{aligned}$$

The integral representations studied above allow us to investigate (for degree 2) the behaviour of such a standard L-function at $s = 1$ and $s = 2$; we remark that $s = 1$ is not a critical value for the standard L-function! Note that with $n = 2$ in

the formula just above, $s-n=1$ when $s=3$, which matches $s=1/2$ (so $2s+2=3$) in (12).
If F is actually a Yoshida lift of level N associated to two elliptic cuspidal newforms $f \in S_{k'}(\Gamma_0(N)), g \in S_k(\Gamma_0(N))$, with $k' \geq k$, then $F \in S_\rho(\Gamma_0^{(2)}(N))$ with $\rho = \det^{2+\frac{k'-k}{2}} \otimes \sigma_{k-2}$ is indeed an eigenform of all the Hecke operators $T_N(M)$:
(14)

$$\sum_M F \mid T_N(M) \det(M)^{-s} = \frac{\lambda}{N^{ns}} \zeta^{(N)}(s-2) L^{(N)}\left(f \otimes g, s+\frac{k'+k}{2}-3\right) \Lambda_N(s-2) \cdot F$$

where $\lambda = \pm N^{n(n-1)/2} = \pm N$ (with the sign depending only on N),

$$\Lambda_N(s) = \prod_{p|N} \prod_{j=1}^{2} (1-p^{-s-2+j})^{-1}$$

and

$$L^{(N)}(f_1 \otimes f_2, s) := \prod_{p \nmid N} (1-\alpha_p\beta_p p^{-s})(1-\alpha_p\beta_p' p^{-s})(1-\alpha_p'\beta_p p^{-s})(1-\alpha_p'\beta_p' p^{-s}).$$

Moreover $F \mid_\rho \begin{pmatrix} 0_2 & -1_2 \\ N \cdot 1_2 & 0_2 \end{pmatrix}$ is also an eigenfunction of all the $T_N(M)$ with the same eigenvalues as F; for details on the facts mentioned above we refer to [BS1, BS3].

6.5.2. *A version of the pullback formula for the Eisenstein series attached to the cusp zero.* We can consider the same doubling method using the Eisenstein series

$$\begin{aligned} \mathfrak{F}_k^{(2n)}(Z,s) &:= \sum_{C,D} \det(CZ+D)^{-k-s} \det(C\bar{Z}+D)^{-s}, \\ \mathbb{F}_k^{(2n)}(Z,s) &:= \mathfrak{F}_k^{(2n)}(Z,s) \times \det(Y)^s, \end{aligned}$$

where (C,D) runs over non-associated coprime symmetric pairs with the additional condition "$\det(C)$ coprime to N" (this is the Eisenstein series "attached to the cusp zero"). The reason for using both versions is that in our previous papers [BS1, BS3] we mainly worked with $E_k^{(2n)}$, whereas the Fourier expansion is more easily accessible for the Eisenstein series $\mathbb{F}_k^{(2n)}$.
The two doubling integrals are linked to each other by the elementary relation

$$E_k^{(2n)}(Z,s) \mid_k \begin{pmatrix} 0_{2n} & -1_{2n} \\ N \cdot 1_{2n} & 0_{2n} \end{pmatrix} = N^{-kn-2ns} \mathbb{F}_k^{(2n)}(Z,s).$$

Due to this relation, substituting $\mathfrak{F}$ for E in the doubling method just means (for Yoshida-lifts) a modication by a power of N (the factor N^{-ns} in (14) goes away). For the case of arbitrary cusp forms we refer to [BCG, BKS].
We write down the relevant cases explicitly for the Yoshida lift F from above:
The residue of the standard L-function at $s=1$ corresponds to a near center value for $L(f_1 \otimes f_2, s)$:
The equation (12) then becomes (with $\mathcal{F}_2^{(4)} := \mathrm{Res}_{s=\frac{1}{2}} \mathbb{F}_2^{(4)}$)

$$\left\langle F, \mathbb{D}_2\left(\frac{k'-k}{2}, k-2\right) \mathcal{F}_2^{(4)}(*, -\bar{w}) \right\rangle \tag{15}$$

$$= c\lambda \prod_{p|N}(1-p^{-1})\Lambda_N(1)\frac{1}{\zeta^{(N)}(3)\zeta^{(N)}(4)\zeta^{(N)}(2)}L^{(N)}\left(f\otimes g, \frac{k'+k}{2}\right)\cdot F(w)$$

with

$$c = \frac{C_2(\frac{k'-k}{2}, k-2)}{C_{2+\frac{1}{2}}(\frac{k'-k}{2}, k-2)}\cdot \gamma_2\left(\frac{k'-k}{2}, k-2, \frac{1}{2}\right).$$

To treat the critical value of the standard L-function at $s=2$, we can directly use the formula (8), taking tacitly into account that $\mathcal{F}_4^{(4)}(Z) := \mathbb{F}_4^{(4)}(Z,s)\,|_{s=0}$ defines a holomorphic modular form (see [Sh1, Prop.10.1]) by Hecke summation.
This yields

$$\text{(16)}\ \left\langle F, \mathbb{D}_4\left(\frac{k'-k}{2}-2, k-2\right)\mathcal{F}_4^{(4)}(*, -\overline{w})\right\rangle$$
$$= \gamma_2\left(4, \frac{k'-k}{2}-2, k-2, 0\right)\times$$
$$(\pm N)\ \Lambda_N(2)\frac{\zeta^{(N)}(2)}{\zeta^{(N)}(4)\zeta^{(N)}(6)\zeta^{(N)}(4)}L^{(N)}\left(f\otimes g, \frac{k'+k}{2}+1\right)\cdot F(w).$$

In the case of a general cusp form $F \in S_\rho(\Gamma_0^{(2)}(N))$, which we assume to be an eigenfunction of the Hecke operators “away from N”, we can write

$$\left\langle F, \mathbb{D}_4\left(\frac{k'-k}{2}-2\right)\mathcal{F}_4^{(4)}(*, -\overline{w})\right\rangle$$
$$= \gamma_2\left(4, \frac{k'-k}{2}-2, k-2, 0\right)\times \frac{D_f^{(N)}(2)}{\zeta^{(N)}(4)\zeta^{(N)}(6)\zeta^{(N)}(4)}\mathcal{T}(F)(w)$$

where $\mathcal{T}$ is an (infinite) sum of Hecke operators at the bad places.

7. Integrality properties

The known results about integrality of Fourier coefficients of Eisenstein series are not sufficent for our purposes because they deal only with level one and large weights. We do not aim at the most general case, but just describe how to adapt the reasoning in [B4] to the cases necessary for our purposes.

7.1. **The Eisenstein series.** We collect some facts about the Fourier coefficients of Eisenstein series

$$F_k^m(Z) := \mathbb{F}_k^m(Z,s)_{|s=0}$$

for even $m = 2n$ with $k \geq \frac{m+4}{2}$
This function is known to define a holomorphic modular form with Fourier expansion

$$F_m^k(Z) = \sum_{T\geq 0} a_m^k(T,N)exp(2\pi i tr(TZ)).$$

We first treat T of maximal rank. We denote by $d(T) := (-1)^n \det(2T)$ the discriminant of T and by χ_T the corresponding quadratic character, defined by $\chi_T(.) := \left(\frac{d(T)}{.}\right)$.

Then $a_m^k(T,N)=0$ unless $T>0$, see e.g. [BCG, prop.5.2].
If $T>0$ then the Fourier coefficient is of type

$$a_m^k(T)=A_m^k\det(T)^{k-\frac{m+1}{2}}\prod_{p\nmid N}\alpha_p(T,k)$$

where $\alpha_p(T,k)$ denotes the usual local singular series and

$$A_m^k=(-1)^{\frac{mk}{2}}\frac{2^m}{\Gamma_m(k)}\pi^{mk}.$$

We can express the nonarchimedian part by a normalizing factor and polynomials in p^{-k}:

$$\begin{aligned}\prod_{p\nmid N}\alpha_p(T,k) &= \frac{1}{\zeta^{(N)}(k)\prod_{j=1}^n\zeta^{(N)}(2k-2j)}\times\\ &\quad\sum_G \det(G)^{-2k+m-1}L^{(N)}(k-n,\chi_{T[G^{-1}]})\prod_{p\nmid N}\beta_p(T[G^{-1}],k).\end{aligned}$$

Here G runs over

$$GL(n,\mathbb{Z})\backslash\{M\in\mathbb{Z}^{(n,n)}\mid\det(M)\text{ coprime to }N\}$$

and the $\beta_p(T)$ denote the "normalized primitive local densities". In general they are polynomials in p^{-k} with integer coefficients and they are equal to one for all p coprime to $d(T)$, see e.g. [B4].
Let f_T be the conductor of the quadratic character χ_T and η_T the corresponding primitive character. Then

$$\begin{aligned}L^{(N)}(k-n,\chi_T) &= \prod_{p\mid N}(1-\chi_T(p)p^{-k+n})L(k-n,\chi_T)\\ &= \prod_{p\mid N}(1-\chi_T(p)p^{-k+n})\prod_{p\mid d(T)}(1-\eta(p)p^{n-k})L(k-n,\eta_T)\end{aligned}$$

We quote from [B4] that

$$\left(\frac{d(T)}{f_T}\right)^{k-\frac{m}{2}}\prod_{p\nmid N}(1-\eta_T(p)p^{n-k})\beta_p(T,k)\in\mathbb{Z}.$$

We may therefore just ignore this factor. Then as in [B4] we use the functional equation of the Riemann zeta function and the Dirichlet L-functions attached to quadratic characters.
We get (for $4\mid k$) that

$$a_m^k(T,N)\in\prod_{p\mid N}\left((1-p^{-k})\prod_{j=1}^n(1-p^{-2k+2j})\right)2^n\frac{k}{B_k}\frac{1}{\mathcal{N}_{2k-m}^*}\prod_{j=1}^n\frac{k-j}{B_{2k-2j}}\cdot\frac{1}{N^{k-n}}\cdot\mathbb{Z}$$

Here the factor N^{k-n} takes care of the possible denominator arising from $\prod_{p\mid N}(1-\chi_T(p)p^{-k+n})$ and

$$\mathcal{N}_{2k-m}^*:=\prod_{p\mid\mathcal{N}_{2k-m}}p^{1+\nu_p(k-n)},$$

where $\mathcal{N}_{2k-m}$ is the denominator of the Bernoulli number B_{2k-m}.
If $k \equiv 2 \bmod 4$ there is a similar formula, see [B3].

In case of lower weights (i.e. $\frac{m+4}{2} \leq k \leq m$) we have to assure that Fourier coefficients of lower rank do not occur. For this we have to study the Fourier expansion of $F^k_m(Z, s)$ near the point $s = 0$ (after analytic continuation). The analysis of the order of singular series, Γ-factors and confluent hypergeometric functions done mainly by Shimura [Sh1] (see also the very explicit counting of orders in Haruki's work [Har]) shows that indeed nontrivial Fourier coefficients occur only for T of maximal rank.

Remark 7.1. *The Fourier coefficients of* $F^4_4(N)$ *are in*

$$\prod_{p|N} \left((1-p^{-4})^2(1-p^{-6})\right) \frac{9}{2N^2} \cdot \mathbb{Z} \subseteq \frac{9}{N^{16}} \cdot \mathbb{Z}\left[\frac{1}{2}\right].$$

7.2. **The differential operators.** By definition, the coefficients of the differential operator $\mathcal{D}^\mu_k$ are in $\mathbb{Z}[1/2]$; here we view $\mathcal{D}^\mu_k$ as a polynomial in the variables z_2 and ∂_{ij}.
Concerning the integrality properties of L^ν_k, we just remark that because of

$$(2-k-\nu)^{[j]} = (-1)^j(k+\nu-j-1)^{[j]} = (-1)^j \frac{(k+\nu-2)!}{(k+\nu-j-1)!}$$

it is sufficient to look at

$$\frac{(k+\nu-j-1)!}{k^{[\nu]} j!(\nu-2j)!(k+\nu-2)!} \qquad (0 \leq j \leq \left[\frac{\nu}{2}\right]).$$

Taking into account that $\frac{\nu!}{j!(\nu-2j)!} \in \mathbb{Z}$ and

$$\frac{(k+\nu-j-1)!}{(k+\nu-2)!} \in \frac{1}{(k+\nu-[\frac{\nu}{2}])\ldots(k+\nu-2)} \cdot \mathbb{Z}$$

we see that the coefficients of L^ν_k are in

$$\frac{1}{k^{[\nu]}\nu!(k+\nu-2)\ldots(k+\nu-[\frac{\nu}{2}])} \cdot \mathbb{Z}.$$

Putting things together, we see that $\mathbb{D}_k(\mu, \nu)$ has coefficients in

$$\frac{1}{(k+\mu)^{[\nu]}\nu!(k+\mu+\nu-2)\ldots(k+\mu+\nu-[\frac{\nu}{2}])} \cdot \mathbb{Z}\left[1/2\right].$$

Remark 7.2. *The Fourier coefficients of* $\mathbb{D}_4(\mu, \nu)\mathbf{F}^4_4$ *are in*

$$\frac{1}{(4+\mu)^{[\nu]}\nu!(4+\mu+\nu-2)\ldots(4+\mu+\nu-[\frac{\nu}{2}])} \times \frac{9}{N^{16}}\mathbb{Z}\left[1/2\right].$$

This remark does not claim, that the denominator given there is the best possible one, there may be additional cancelations of denominators coming from the restriction.

8. The Petersson norm of the Yoshida lift

Take $f = \sum a_n q^n, g = \sum b_n q^n$ as in the introduction, of weights k' and k respectively and assume that for all primes p dividing the common (square-free) level N of f, g both functions have the same Atkin-Lehner eigenvalue ϵ_p. Let $k' = 2\nu_1 + 2, k = 2\nu_2 + 2$. Choose a factorization $N = N_1 N_2$, where N_1 is the product of an odd number of prime factors, and let $D = D(N_1, N_2)$ be the definite quaternion algebra over $\mathbb{Q}$, ramified at ∞ and the primes dividing N_1. Let $R = R(N_1, N_2)$ be an Eichler order of level $N = N_1 N_2$ in $D(N_1, N_2)$ with (left) ideal class number h.

We recall (and slightly modify) some notation from §1 of [BS3]: For $\nu \in \mathbb{N}$ let $U_\nu^{(0)}$ be the space of homogeneous harmonic polynomials of degree ν on $\mathbb{R}^3$ and view $P \in U_\nu^{(0)}$ as a polynomial on $D_\infty^{(0)} = \{x \in D_\infty | \mathrm{tr}(x) = 0\}$ by putting $P(\sum_{i=1}^3 x_i e_i) = P(x_1, x_2, x_3)$ for an orthonormal basis $\{e_i\}$ of $D_\infty^{(0)}$ with respect to the norm form n on D. The representations τ_ν of $D_\infty^\times/\mathbb{R}^\times$ of highest weight (ν) on $U_\nu^{(0)}$ given by $(\tau_\nu(y))(P)(x) = P(y^{-1}xy)$ for $\nu \in \mathbb{N}$ give all the isomorphism classes of irreducible rational representations of $D_\infty^\times/\mathbb{R}^\times$.

For an irreducible rational representation (V_τ, τ) (with $\tau = \tau_\nu$ as above) of $D_\infty^\times/\mathbb{R}^\times$ we denote by $\mathcal{A}(D_\mathbb{A}^\times, R_\mathbb{A}^\times, \tau)$ the space of functions $\phi : D_\mathbb{A}^\times \to V_\tau$ satisfying $\phi(\gamma x u) = \tau(u_\infty^{-1})\phi(x)$ for $\gamma \in D_\mathbb{Q}^\times$ and $u = u_\infty u_f \in R_\mathbb{A}^\times$, where $R_\mathbb{A}^\times = D_\infty^\times \times \prod_p R_p^\times$ is the adelic group of units of R. Let $D_\mathbb{A}^\times = \cup_{i=1}^r D^\times y_i R_\mathbb{A}^\times$ be a double coset decomposition with $y_{i,\infty} = 1$ and $n(y_i) = 1$. A function in $\mathcal{A}(D_\mathbb{A}^\times, R_\mathbb{A}^\times, \tau)$ is then determined by its values at the y_i. We put $I_{ij} = y_i R y_j^{-1}$, $R_i = I_{ii}$ and let e_i be the number of units of the order R_i. On the space $\mathcal{A}(D_\mathbb{A}^\times, R_\mathbb{A}^\times, \tau)$ we have for $p \nmid N$ Hecke operators $\tilde{T}(p)$ defined by $\tilde{T}(p)\phi(x) = \int_{D_p^\times} \phi(xy^{-1})\chi_p(y)dy$ where χ_p is the characteristic function of $\{y \in R_p | n(y) \in p\mathbb{Z}_p^\times\}$. They commute with the involutions $\tilde{w}_p$ and are given explicitly by $\tilde{T}(p)\phi(y_i) = \sum_{j=1}^r B_{ij}^\nu(p)\phi(y_j)$, where the Brandt matrix entry $B_{ij}^\nu(p)$ is given as

$$B_{ij}(p) = B_{ij}^{(\nu)}(p) = \frac{1}{e_j} \sum_{\substack{x \in y_j R y_i^{-1} \\ n(x) = p}} \tau(x) \ ,$$

hence is itself an endomorphism of the representation space $U_\nu^{(0)}$ of τ.

From [Ei, H-S, Shz, J-L] we know then that the essential part $\mathcal{A}_{\mathrm{ess}}(D_\mathbb{A}^\times, R_\mathbb{A}^\times, \tau)$ consisting of functions ϕ that are orthogonal (under the natural inner product) to all $\psi \in \mathcal{A}(D_\mathbb{A}^\times, (R'_\mathbb{A})^\times, \tau)$ for orders R' strictly containing R is invariant under the $\tilde{T}(p)$ for $p \nmid N$ and the $\tilde{w}_p$ for $p \nmid N$ and hence has a basis of common eigenfunctions of all the $\tilde{T}(p)$ for $p \nmid N$. Moreover in $\mathcal{A}_{\mathrm{ess}}(D_\mathbb{A}^\times, R_\mathbb{A}^\times, \tau)$ strong multiplicity one holds, i.e., each system of eigenvalues of the $\tilde{T}(p)$ for $p \nmid N$ occurs at most once, and the eigenfunctions are in one to one correspondence with the newforms in the space $S^{2+2\nu}(N)$ of elliptic cusp forms of weight $2 + 2\nu$ for the group $\Gamma_0(N)$ that are eigenfunctions of all Hecke operators (if τ is the trivial representation and R is a maximal order one has to restrict here to functions orthogonal to the constant function 1 on the quaternion side in order to obtain cusp forms on the modular forms side).

Let $\phi_1 = \phi_1^{(N_1,N_2)} : D_{\mathbb{A}}^\times \to U_{\nu_1}^{(0)}$ and $\phi_2 = \phi_2^{(N_1,N_2)} : D_{\mathbb{A}}^\times \to U_{\nu_2}^{(0)}$ correspond to f and g respectively with respect to the choice of N_1, N_2 and hence of $D = D(N_1, N_2)$. Let $F = F_{f,g} = F_{\phi_1,\phi_2}$ (which of course also depends on the choice of N_1, N_2) be the Yoshida lift; it takes values in the space W_ρ of the symmetric tensor representation $\rho = \det^\kappa \otimes \mathrm{Sym}^j(\mathbb{C}^2)$, $j = k-2, \kappa = 2 + \frac{k'-k}{2}$ and is a Siegel cusp form $F \in S_\rho(\Gamma_0^{(2)}(N))$. To describe it explicitly we notice that the group of proper similitudes of the quadratic form $q(x) = n(x)$ on D (with associated symmetric bilinear form $B(x,y) = \mathrm{tr}(x\bar{y})$, where tr denotes the reduced trace on D) is isomorphic to $(D^\times \times D^\times)/Z(D^\times)$ (as algebraic group) via $(y, y') \mapsto \sigma_{y,y'}$ with $\sigma_{y,y'}(x) = yx(y')^{-1}$, the special orthogonal group is then the image of $\{(y, y') \in D^\times \times D^\times \mid n(y) = n(y')\}$.

We denote by H the orthogonal group of (D, n), by H^+ the special orthogonal group and by K (resp. K^+) the group of isometries (resp. isometries of determinant 1) of the lattice R in D. It is well known that the $H^+(\mathbb{R})$-space $U_{\nu_1}^{(0)} \otimes U_{\nu_2}^{(0)}$ is isomorphic to the $H^+(\mathbb{R})$-space U_{ν_1,ν_2} of $\mathbb{C}[X_1, X_2]$-valued harmonic forms on D_∞^2 transforming according to the representation of $GL_2(\mathbb{R})$ of highest weight $(\nu_1 + \nu_2, \nu_1 - \nu_2)$; an intertwining map Ψ has been given in [BS5, Section 3]. It is also well known [KV] that the representation λ_{ν_1,ν_2} of $H^+(\mathbb{R})$ on U_{ν_1,ν_2} is irreducible of highest weight $(\nu_1 + \nu_2, \nu_1 - \nu_2)$. If $\nu_1 > \nu_2$ it can be extended in a unique way to an irreducible representation of $H(\mathbb{R})$ on the space $U_{\nu_1,\nu_2,s} := (U_{\nu_1}^{(0)} \otimes U_{\nu_2}^{(0)}) \oplus (U_{\nu_2}^{(0)} \otimes U_{\nu_1}^{(0)}) =: U_\lambda$ which we denote by $(\tau_1 \otimes \tau_2) =: \lambda$ for simplicity, on this space $\sigma_{y,y'} \in H^+(\mathbb{R})$ acts via $\tau_1(y) \otimes \tau_2(y')$ on the summand $U_{\nu_1}^{(0)} \otimes U_{\nu_2}^{(0)}$ and via $\tau_2(y) \otimes \tau_1(y')$ on the summand $U_{\nu_2}^{(0)} \otimes U_{\nu_1}^{(0)}$. For $\nu_1 = \nu_2$ there are two possible extensions to representations $(\tau_1 \otimes \tau_2)_\pm$ on U_{ν_1,ν_2}; we denote this space with the representation $(\tau_1 \otimes \tau_2)_+ =: \lambda$ on it by U_λ again (and don't consider the minus variant in the sequel).

We recall then from [KV, We1, BS3] that the space $\mathcal{H}_q(\rho)$ consisting of all q-pluriharmonic polynomials $P : M_{4,2}(\mathbb{C}) \to W_\rho$ such that $P(xg) = (\rho(g^t))P(x)$ for all $g \in GL_2(\mathbb{C})$ is isomorphic to (U_λ, λ) as a representation space of $H(\mathbb{R})$. The space $\mathcal{H}_q(\rho)$ carries an essentially unique $H(\mathbb{R})$-invariant scalar product $\langle \ , \ \rangle_{\mathcal{H}_q(\rho)}$, and in the usual way we can find a reproducing $H(\mathbb{R})$ invariant kernel $P_{\mathrm{Geg}} \in \mathcal{H}_q(\rho) \otimes \mathcal{H}_q(\rho)$ (generalized Gegenbauer polynomial) , i. e., P_{Geg} is a polynomial on $D_\infty^2 \oplus D_\infty^2$ taking values in $W_\rho \otimes W_\rho$ which as function of each of the variables

i) is a q-pluriharmonic polynomial in $\mathcal{H}_q(\rho)$,
ii) is symmetric in both variables
iii) satisfies $P_{\mathrm{Geg}}(h\mathbf{x}, h\tilde{\mathbf{x}}) = P_{\mathrm{Geg}}(\mathbf{x}, \tilde{\mathbf{x}})$ for $h \in H(\mathbb{R})$
iv) satisfies $\langle P_{\mathrm{Geg}}(\mathbf{x}, \cdot), P(\cdot)\rangle_{\mathcal{H}_q(\rho)} = P(\mathbf{x})$ for all $P \in \mathcal{H}_q(\rho)$.

In fact, since such a polynomial is characterized by the first three properties up to scalar multiples we can construct it (in a more general situtation) with the help of the differential operator $\mathbb{D}_\alpha(\mu, \nu)$ and the polynomial $Q_\alpha^{\mu,\nu}$ from 6.1:

For $k \in \mathbb{N}$ and nonnegative integers μ, ν we define a polynomial map

$$\widetilde{P_{\mathrm{Geg}}}^{(k,\mu,\nu)} : \mathbb{C}^{2k,n} \times \mathbb{C}^{2k,n} \longrightarrow V_\nu \otimes V_\nu$$

by

$$\widetilde{P_{\mathrm{Geg}}}^{(k,\mu,\nu)}(\mathbf{Y_1}, \mathbf{Y_2}) := Q_k^{(\mu,\nu)}\left(\begin{pmatrix} \mathbf{Y}_1^t\mathbf{Y} & \mathbf{Y}_1^t\mathbf{Y}_2 \\ \mathbf{Y}_2^t\mathbf{Y}_1 & \mathbf{Y}_2^t\mathbf{Y}_2 \end{pmatrix}\right)$$

Then $\widetilde{P_{\mathrm{Geg}}}^{(k,\mu,\nu)}$ is symmetric and pluriharmonic in $\mathbf{Y_1}$ and $\mathbf{Y_2}$, see [I1] ; moreover, for $A, B \in GL(n,\mathbb{C})$ we have

$$\widetilde{P_{\mathrm{Geg}}}^{(k,\mu,\nu)}(\mathbf{Y_1}\cdot A, \mathbf{Y_2}\cdot B) = \det(A)^{\mu}\det(B)^{\mu}\sigma_\nu(A)\otimes\sigma_\nu(B)(\widetilde{P_{\mathrm{Geg}}}^{(k,\mu,\nu)}(\mathbf{Y}_1,\mathbf{Y}_2).$$

For $g \in O(2k,\mathbb{C})$ we get

$$\widetilde{P_{\mathrm{Geg}}}^{(k,\mu,\nu)}(g\mathbf{Y}_1,\mathbf{Y}_2) = \widetilde{P_{\mathrm{Geg}}}^{(k,\mu,\nu)}(\mathbf{Y}_1, g^{-1}\mathbf{Y}_2).$$

If we consider a $2k$-dimensional positive definite real quadratic space with positive definite quadratic form q and associated bilinear form B (so that $B(x,x) = 2q(x)$) we write $q(x_1,\dots,x_{2n}) = (B(x_i,x_j)/2)_{i,j}$ for (half) the $2n\times 2n$ Gram matrix associated to the $2n$-tuple of vectors $(x_1,\dots,x_{2n})$ and put in a similar way as above for $(\mathbf{y},\mathbf{y}')\in V^{2n}$

$$P_{\mathrm{Geg}}^{(k,\mu,\nu)}(\mathbf{y},\mathbf{y}') = Q_k^{\mu,\nu}(q(\mathbf{y},\mathbf{y}')),$$

this gives a nonzero polynomial with values in $V_\nu\otimes V_\nu$ which is symmetric in the variables $\mathbf{y},\mathbf{y}'$, is q-pluriharmonic in each of the variables with the proper transformation under the right action of GL_n and is invariant under the diagonal action of the orthogonal group of q; it is hence a scalar multiple of the $V_\nu\otimes V_\nu$-valued Gegenbauer polynomial on this space.

If we apply the differential operator $\mathbb{D}_k(\mu,\nu)$ to a degree $2n$ theta series $\Theta_S^{2n}(Z) := \sum_{R\in\mathbb{Z}^{2k,2n}}\exp 2\pi i\mathrm{tr}(R^tSRZ)$ written in matrix notation we get

$$(\mathbb{D}_k(\mu,\nu)\Theta_S^{2n})(z_1,z_4)$$
$$= \sum_{R_1,R_2\in\mathbb{Z}^{(2k,n)}}(2\pi i)^{n\mu}Q_k^{(\mu,\nu)}\left(\begin{pmatrix} S[R_1] & R_1^tSR_2\\ R_2^tSR_1 & S[R_2]\end{pmatrix}\right)\exp 2\pi i\mathrm{tr}(S[R_1]z_1 + S[R_2]z_4);$$

writing the theta series in lattice notation as the degree $2n$ theta series

$$\theta_\Lambda^{(2n)}(Z) = \sum_{\mathbf{x}\in\Lambda^{2n}}\exp(2\pi i\mathrm{tr}(q(\mathbf{x})Z))$$

of a lattice Λ on V we obtain in the same way

(17)
$$\begin{aligned}\mathbb{D}_k(\mu,\nu)\theta_\Lambda^{(2n)}(z_1,z_4)&\\ &= (2\pi i)^{n\mu}\sum_{(\mathbf{y},\mathbf{y}')\in\Lambda^{2n}}P_{\mathrm{Geg}}(\mathbf{y},\mathbf{y}')\exp(2\pi i\mathrm{tr}(q(\mathbf{y})z_1 + q(\mathbf{y}')z_4))\\ &= (2\pi i)^{n\mu}\sum_{(\mathbf{y})\in\Lambda^{n}}\theta_\Lambda^{(n,\nu)}(z_4)(\mathbf{y})\exp(2\pi i\mathrm{tr}(q(\mathbf{y})z_1)),\end{aligned}$$

where we have written

$$\theta_\Lambda^{(n,\nu)}(z_4)(\mathbf{y}) := \sum_{(\mathbf{y}')\in\Lambda^n}P_{\mathrm{Geg}}(\mathbf{y},\mathbf{y}')\exp(2\pi i\mathrm{tr}(q(\mathbf{y}')z_4)). \tag{18}$$

Going through the construction above in our quaternionic situation with $V_\nu = W_\rho$ we see that we can normalize the scalar product on $\mathcal{H}_q(\rho)$ in such a way that the polynomial P_{Geg} obtained in the way just described is indeed the reproducing kernel for this space. We choose this normalization in what follows and write

$$\theta_{ij,\rho}(Z)(\tilde{\mathbf{x}}) := \sum_{\mathbf{x}\in(y_iRy_j^{-1})^2}P_{\mathrm{Geg}}(\mathbf{x},\tilde{\mathbf{x}})\exp(2\pi i\mathrm{tr}(q(\tilde{\mathbf{x}})Z))\in W_\rho\otimes W_\rho$$

(so that $\theta_{ij,\rho}(Z)$ is (for each Z in the Siegel upper half space $\mathfrak{H}_2$) an element of $\mathcal{H}_q(\rho) \otimes W_\rho$.) For an arbitrary lattice Λ on D the theta series $\theta_{\Lambda,\rho}$ is defined analogously as given in equation (18).
We denote by $\mathcal{P}$ the (essentially unique) isomorphism from U_λ to $\mathcal{H}_q(\rho)$. With the help of the map Ψ from [BS5] mentioned above we can fix a normalization and write $\mathcal{P}(R_1 \otimes R_2)$ for $R_j \in U_{\nu_j}^{(0)}$ as

$$\begin{aligned} &\mathcal{P}(R_1 \otimes R_2)(d_1, d_2)(X_1, X_2) \\ &\quad = (\mathcal{D}(n(d_1X_1 + d_2X_2)^{\nu_2}\tau_2(d_1X_1 + d_2X_2)R_2)R_1)(\mathrm{Im}(d_1\overline{d_2})), \end{aligned} \tag{19}$$

where we associate as usual to a polynomial $R \in \mathbb{C}[t_1, t_2, t_3])$ the differential operator $\mathcal{D}(R) = R(\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}, \frac{\partial}{\partial t_3})$, set $\mathrm{Im}(d) = d - \bar{d}$ and write all vectors as coordinate vectors with respect to an orthonormal basis.

Definition 8.1. *With notation as above we define the Yoshida lift of* (ϕ_1, ϕ_2)*, or also of* (f, g) *with respect to* (N_1, N_2)*, to be given by*

$$F(Z) := Y^{(2)}(\phi_1, \phi_2)(Z) := \sum_{i,j=1}^{r} \frac{1}{e_i e_j} \langle \mathcal{P}(\phi_1(y_i) \otimes \phi_2(y_j)), \theta_{ij,\rho}(Z) \rangle_{\mathcal{H}_q(\rho)} \in W_\rho.$$

Lemma 8.2. (1) *One has* $\theta_{ij,\rho}(Z)(\mathbf{x}) = \theta_{ji,\rho}(Z)(\bar{\mathbf{x}})$ *(where* $\bar{\mathbf{x}} = (\bar{x_1}, \bar{x_2})$ *denotes the quaternionic conjugate of the pair* $\mathbf{x} = (x_1, x_2)$*).*

(2)

$$\begin{aligned} 2F(Z) &= 2Y^{(2)}(\phi_1, \phi_2, Z) \\ &= \sum_{i,j=1}^{r} \frac{1}{e_i e_j} \sum_{\mathbf{x} \in (y_i R y_j^{-1})^2} \mathcal{P}(\phi_1(y_i) \otimes \phi_2(y_j) + \phi_2(y_i) \otimes \phi_1(y_j)))(x_1, x_2) \times \\ &\qquad \times \exp(2\pi i \mathrm{tr}(q(\mathbf{x})Z)). \end{aligned}$$

(3) *Denote by* $\langle F, \theta_{ij,\rho} \rangle_{\mathrm{Pet}}$ *the Petersson product of the vector valued Siegel modular forms* F *and* $\theta_{ij,\rho}$*. Then the function* $\xi : (y_i, y_j) \mapsto \langle F, \theta_{ij,\rho} \rangle_{\mathrm{Pet}} \in \mathcal{H}_q(\rho)$ *has the symmetry property* $\xi(y_i, y_j)(\mathbf{x}) = \xi(y_j, y_i)(\bar{\mathbf{x}})$*. It induces a unique function, denoted by* $\tilde{\xi}$*, on* $H(\mathbb{A})$ *satisfying* $\tilde{\xi}(\sigma_{y_i,y_j}) = \xi(y_i, y_j)$ *and*

$$\tilde{\xi}(\gamma\sigma k) = \lambda(k_\infty^{-1})\tilde{\xi}(\sigma) \text{ for } \sigma \in H(\mathbb{A}), \gamma \in H(\mathbb{Q}), k = (k_\nu)_\nu \in H(R_{\mathbb{A}}),$$

where we denote by $H(R_{\mathbb{A}})$ *the group of adelic isometries of the lattice* R *on* D.

Proof. This is easily seen to be a consequence of the fact that the lattice $I_{ij} = y_i R y_j^{-1}$ is the quaternionic conjugate of the lattice $I_{ji} = y_j R y_i^{-1}$ and that quaternionic conjugation is an element of the (global) orthogonal, but not of the special orthogonal group of (D, n). □

As in [BS1] we need to show that ξ is proportional to the function ξ_{ϕ_1,ϕ_2} : $(y_i, y_j) \mapsto \phi_1(y_i) \otimes \phi_2(y_j) + \phi_2(y_i) \otimes \phi_1(y_j)$ that appears in our formula for the Yoshida lifting, and to determine the factor of proportionality occurring.

Lemma 8.3. *With notations as in the previous lemma one has*

$$\langle F, \theta_{ij,\rho} \rangle_{\mathrm{Pet}} = c_5 \mathcal{P}(\phi_1(y_i) \otimes \phi_2(y_j) + \phi_2(y_i) \otimes \phi_1(y_j))$$

and

$$\langle F, F\rangle_{\mathrm{Pet}} = c_5\langle \mathcal{P}(\phi_1\otimes\phi_2), \mathcal{P}(\phi_1\otimes\phi_2)\rangle,$$

with some constant $c_5 \neq 0$*, where the latter inner product is the natural inner product on* $\mathcal{H}_q(\rho)$*-valued functions on* $D_{\mathbb{A}}^\times \times D_{\mathbb{A}}^\times$ *satisfying the usual invariance properties under* $R_{\mathbb{A}}^\times$ *and* $D_{\mathbf{Q}}^\times$*, which is defined by*

$$\langle \mathcal{P}(\phi_1\otimes\phi_2), \mathcal{P}(\phi_1\otimes\phi_2)\rangle = \sum_{i,j=1}^{r} \frac{1}{e_i e_j}\langle \mathcal{P}(\phi_1(y_i)\otimes\phi_2(y_j)), \mathcal{P}(\phi_1(y_i)\otimes\phi_2(y_j))\rangle_{\mathcal{H}_q(\rho)}.$$

Proof. The proof proceeds in essentially the same way as in [BS1]: We notice first that the space of all ξ with the symmetry property mentioned (or equivalently the space of functions $\tilde{\xi}$ on $H(\mathbb{A})$ with the invariance property given) has a basis consisting of the $\xi_{\phi_1,\phi_2} = \xi_{\phi_2,\phi_1}$, where (ϕ_1, ϕ_2) runs through the pairs of eigenforms in $(\mathcal{A}(D_{\mathbb{A}}^\times, (R_{\mathbb{A}})^\times, \tau_1) \times (\mathcal{A}(D_{\mathbb{A}}^\times, (R_{\mathbb{A}})^\times, \tau_2)$ and where the pairs are unordered if $\nu_1 = \nu_2$.

The Hecke operators $T_i'(p)$ on the spaces $\mathcal{A}(D_{\mathbb{A}}^\times, R_{\mathbb{A}}^\times, \tau_i)$ (for $i = 1, 2$) via Brandt matrices described above induce Hecke operators $\hat{T}(p)$ on the space of ξ as above that are given by

$$\xi|\hat{T}(p)(y_i, y_j) = \sum_{k=1}^{r} \tilde{B}_{jk}^{(\mathrm{right})}(p)\xi(y_i, y_k) + \sum_{l=1}^{r} \tilde{B}_{il}^{(\mathrm{left})}(p)\xi(y_l, y_j),$$

where for $\nu_1 > \nu_2$ we let $\tilde{B}_{jk}^{(\mathrm{right})}(p)$ act on $U = U_{\nu_1}^{(0)} \otimes U_{\nu_2}^{(0)} \oplus U_{\nu_2}^{(0)} \otimes U_{\nu_1}^{(0)}$ via $\mathrm{id}\otimes B_{jk}^{\nu_2}(p) \oplus \mathrm{id}\otimes B_{jk}^{\nu_1}(p)$ and $\tilde{B}_{il}^{(\mathrm{left})}(p)$ as $B_{jk}^{\nu_1}(p)\otimes \mathrm{id} \oplus B_{jk}^{\nu_2}(p)\otimes \mathrm{id}$, and where for $\nu_1 = \nu_2$ the action of $\tilde{B}^{(\mathrm{left})}, \tilde{B}^{(\mathrm{right})}$ on $U = U_{\nu_1}^{(0)} \otimes U_{\nu_2}^{(0)}$ is simply the action of the Brandt matrix on the respective factor of the tensor product.

In the same way as sketched in [BS1, 10 b)] we obtain then (using the calculations of Hecke operators from [Y1, Y2]) first

$$\langle F, \theta_{ij,\rho}|T(p)\rangle_{\mathrm{Pet}} = \xi|\hat{T}(p)(y_i, y_j).$$

Since, again by Yoshida's computations of Hecke operators (see also [BS3]), we know that F is an eigenfunction of $T(p)$ with eigenvalue $\lambda_p(f) + \lambda_p(g)$ this implies that ξ is an eigenfunction with the same eigenvalue for $\hat{T}(p)$. A computation that uses the eigenfunction property of ϕ_1, ϕ_2 for the action of the Hecke operators on the spaces $\mathcal{A}(D_{\mathbb{A}}^\times, (R_{\mathbb{A}})^\times, \tau_1), \mathcal{A}(D_{\mathbb{A}}^\times, (R_{\mathbb{A}})^\times, \tau_2)$ shows that the same is true for the function ξ_{ϕ_1,ϕ_2}.

Since ϕ_1, ϕ_2 are in the essential parts of $\mathcal{A}(D_{\mathbb{A}}^\times, (R_{\mathbb{A}})^\times, \tau_1), \mathcal{A}(D_{\mathbb{A}}^\times, (R_{\mathbb{A}})^\times, \tau_2)$, their eigenvalue systems occur with strong multiplicity one in these spaces, and as in Section 10 of [BS1] we can conclude that that ξ and ξ_{ϕ_1,ϕ_2} are indeed proportional, i.e., we have

$$\langle F, \theta_{ij,\rho}\rangle_{\mathrm{Pet}} = c_5\mathcal{P}(\phi_1(y_i)\otimes\phi_2(y_j) + \phi_2(y_i)\otimes\phi_1(y_j))$$

with some constant $c_5 \neq 0$.

From this we see:

$$\begin{aligned}\langle F, F\rangle_{\mathrm{Pet}} &= \langle F, \sum_{i,j=1}^{r} \frac{1}{e_i e_j} \langle \mathcal{P}(\phi_1(y_i)\otimes\phi_2(y_j)), \theta_{ij,\rho}\rangle_{\mathcal{H}_q(\rho)}\rangle_{\mathrm{Pet}} \\ &= \sum_{i,j=1}^{r} \frac{c_5}{e_i e_j} \langle \mathcal{P}(\phi_1(y_i)\otimes\phi_2(y_j)), \mathcal{P}(\phi_1(y_i)\otimes\phi_2(y_j)+\phi_2(y_i)\otimes\phi_1(y_j))\rangle_{\mathcal{H}_q(\rho)} \\ &= c_5\langle \mathcal{P}(\phi_1\otimes\phi_2), \mathcal{P}(\phi_1\otimes\phi_2)\rangle.\end{aligned}$$

□

In order to compute the constant c_5 we will first need the generalization of Lemma 9.1 of [BS1] to the present situation:

Lemma 8.4. (1) *If Λ is a lattice on some quaternion algebra D' with $n(\Lambda) \subseteq \mathbb{Z}$, of level dividing N, and with $\mathrm{disc}(\Lambda) \neq N^2$ the theta series $\theta_{\Lambda,\rho}$ is orthogonal to all Yoshida lifts $Y^{(2)}(\phi_1, \phi_2)$ of level N.*

(2) *If If Λ is a lattice on some quaternion algebra $D' \neq D$ with $n(\Lambda) \in \mathbb{Z}$, of level N , and with $\mathrm{disc}(\Lambda) = N^2$ the theta series $\theta_{\Lambda,\rho}$ is orthogonal to all Yoshida lifts $Y^{(2)}(\phi_1, \phi_2)$ of level N associated to D.*

Proof. The proof of Lemma 9.1 of [BS1] unfortunately contains some misprints: In line 4 on p. 81 the minus sign in front of the whole factor should not be there and the exponent at p should be $n(n+1)/2$ (which is equal to 3 in our present situation), in line 5 the exponent at p should be $n(n-1)/2$ (hence 1 in our case), in line 9 the factor p in the right hand side of the equation should be omitted, and in line 14 the exponent at p should be 1 instead of 3.

Apart from these corrections the argument given there carries over to our situation unchanged. In particular, the results from Section 7 of [BS1] that were used in the proof of that lemma remain true and their proof carries over if one uses the reformulation of Evdokimov's result from [Ev] sketched in Section 4 of [BS3]. □

We recall from [BS1] that we have

$$\mathcal{E}_2^{(4)}(Z_1, Z_2) = \sum_{r=1}^{t} \alpha_r \sum_{\{K_r\}} \frac{1}{|O(K_r)|} \theta_{K_r}^{(2)}(Z_1)\theta_{K_r}^{(2)}(Z_2),$$

where we denote by $L_1, \ldots, L_t$ representatives of the genera of lattices of rank 4, square discriminant and level dividing $N = N_1N_2$, the summation over $\{K_r\}$ runs over a set of representatives of the isometry classes in the genus of L_r and α_r are some constants that are explicitly determined in [BS1].

Hence by (18) we obtain

$$\begin{aligned}&\left(\mathbb{D}_2\left(\frac{k'-k}{2}-2, k-2\right)(\mathcal{E}_2^{(4)})\right)(Z_1, Z_2) \\ &= c_3 \sum_{r=1}^{t} \alpha_r \sum_{\{K_r\}} \frac{1}{|O(K_r)|} \sum_{(\mathbf{x}_1,\mathbf{x}_2)\in K_r^2\times K_r^2} P_{\mathrm{Geg}}(\mathbf{x}_1, \mathbf{x}_2)\exp(2\pi i \mathrm{tr}(q(\mathbf{x}_1)Z_1 + q(\mathbf{x}_2)Z_2))\end{aligned}$$

with $c_3 = (2\pi i)^{k'-k}$, and similarly for the Eisenstein series $\mathcal{F}_2^{(4)}$ attached to the cusp zero, with the α_r replaced by β_r as in [BS1].

The reproducing property of P_{Geg} implies then

$$\sum_{(\mathbf{x}_1,\mathbf{x}_2)\in K_r^2\times K_r^2} P_{\mathrm{Geg}}(\mathbf{x}_1,\mathbf{x}_2)\exp(2\pi i \mathrm{tr}(q(\mathbf{x}_1)Z_1+q(\mathbf{x}_2)Z_2))$$
$$=\langle\langle\theta_{K,\rho}(Z_1)(\mathbf{u}_1)\otimes\theta_{K,\rho}(Z_2)(\mathbf{u}_2),P_{\mathrm{Geg}}(\mathbf{u}_1,\mathbf{u}_2)\rangle_{\mathcal{H}_q(\rho)}\rangle_{\mathcal{H}_q(\rho)}.$$

Using the fact that by Lemma 8.4 the Yoshida lifting F is orthogonal to all $\theta_{K,\rho}$ where K is not in the genus of the given Eichler order of level N_1N_2 we see that the part of the sum for $\mathbb{D}(\mathcal{F}_2^{(4)})(Z_1,Z_2)$ which contributes to the Petersson product with F can be written as

$$c_3\beta_1\sum_{i,j}\frac{1}{e_ie_j}\langle\langle\theta_{ij,\rho}(Z_1)(\mathbf{u}_1)\otimes\theta_{ij,\rho}(Z_2)(\mathbf{u}_2),P_{\mathrm{Geg}}(\mathbf{u}_1,\mathbf{u}_2)\rangle_{\mathcal{H}_q(\rho)}\rangle_{\mathcal{H}_q(\rho)}.$$

We further recall that by (15) we have

$$\langle F,\mathbb{D}(\mathcal{F}_2^{(4)})(*,-\bar{w})\rangle_{\mathrm{Pet}}=c_4L^{(N)}\left(f\otimes g,\frac{k+k'}{2}\right)F(w)$$

with

$$c_4=\lambda\prod_{p|N}(1-p^{-1})\Lambda_N(1)\frac{1}{\zeta^{(N)}(3)\zeta^{(N)}(4)\zeta^{(N)}(2)}\frac{C_2\left(\frac{k'-k}{2},k-2\right)}{C_{2+\frac{1}{2}}\left(\frac{k'-k}{2},k-2\right)}\cdot\gamma_2\left(\frac{k'-k}{2},k-2,\frac{1}{2}\right).$$

Proposition 8.5. *With notations as above we have*

$$\langle F,F\rangle_{\mathrm{Pet}}=\frac{c_4}{2c_3\beta_1}L^{(N)}\left(f\otimes g,\frac{k+k'}{2}\right)\langle\mathcal{P}(\phi_1\otimes\phi_2),\mathcal{P}(\phi_1\otimes\phi_2)\rangle.$$

Proof. From what we saw above and using Lemma 8.3 we get

$$\langle F,\mathbb{D}(\mathcal{F}_2^{(4)})(*,-\bar{Z})\rangle_{\mathrm{Pet}}$$
$$=c_3\beta_1\sum_{i,j}\frac{1}{e_ie_j}\langle\langle F(*),\theta_{ij,\rho}(-\bar{Z})(\mathbf{u}_1)\otimes\langle\theta_{ij,\rho}(*)(\mathbf{u}_2)\rangle_{\mathrm{Pet}},P_{\mathrm{Geg}}(\mathbf{u}_1,\mathbf{u}_2)\rangle_{\mathcal{H}_q(\rho)}\rangle_{\mathcal{H}_q(\rho)}$$
$$=c_5c_3\beta_1\sum_{i,j}\frac{1}{e_ie_j}\langle\langle\overline{\theta_{ij,\rho}(-\bar{Z})}\otimes\mathcal{P}(\phi_1(y_i)\otimes\phi_2(y_j)+\phi_2(y_i)\otimes\phi_1(y_j)),P_{\mathrm{Geg}}\rangle_{\mathcal{H}_q(\rho)}\rangle_{\mathcal{H}_q(\rho)}$$
$$=c_5c_3\alpha_1\sum_{i,j}\frac{1}{e_ie_j}\langle\theta_{ij,\rho}(Z),\mathcal{P}(\phi_1(y_i)\otimes\phi_2(y_j)+\phi_2(y_i)\otimes\phi_1(y_j))\rangle_{\mathcal{H}_q(\rho)}$$
$$=2c_3c_5\beta_1F.$$

Comparing with

$$\langle F,\mathbb{D}(\mathcal{F}_2^{(4)})(*,-\bar{Z}))\rangle_{\mathrm{Pet}}=c_4L^{(N)}\left(f\otimes g,\frac{k+k'}{2}\right)F(Z)$$

we obtain

$$c_5=\frac{c_4L^{(N)}(f\otimes g,\frac{k+k'}{2})}{2\beta_1c_3},$$

which together with Lemma 8.3 yields the assertion. □

In order to make use of the above proposition in the next section we will also need to compare $\langle\mathcal{P}(\phi_1\otimes\phi_2),\mathcal{P}(\phi_1\otimes\phi_2)\rangle$ with $\langle\phi_1,\phi_1\rangle\langle\phi_2,\phi_2\rangle$, where we have $\langle\phi_\mu,\phi_\mu\rangle=\sum_{i=1}^r\frac{\langle\phi_\mu(y_i),\phi_\mu(y_i)\rangle_\mu}{e_i}$ for $\mu=1,2$, with $\langle\ ,\ \rangle_\mu$ denoting the (suitably normalized, see below) scalar product on $U_{\nu_\mu}^{(0)}$.

Lemma 8.6. *Write* $\widetilde{G_a^{(\nu)}}(x) = (B(a,x))^\nu$ *for* $a, x \in D_{\mathbb{C}}^{(0)} := D_\infty^{(0)} \otimes \mathbb{C}$ *with* $n(a) = 0$ *and let* $\nu_1 \geq \nu_2$. *Then*

(1)

$$\begin{aligned}\mathcal{P}(\widetilde{G_a^{(\nu_1)}} \otimes \widetilde{G_a^{(\nu_2)}})(d_1, d_2)(X_1, X_2) \\ = \frac{\nu_1!}{\nu_2!}(B(a, d_1)X_1 + B(a, d_2)X_2)^{2\nu_2} \widetilde{G_a^{(\nu_1-\nu_2)}}(\mathrm{Im}(d_1\overline{d_2})).\end{aligned} \tag{20}$$

(2) *For* $a \in D_{\mathbb{C}}^{(0)}$ *as above there is* $b \in D_{\mathbb{C}} := D \otimes \mathbb{C}$ *with* $ab = 0, a\bar{b} = a, n(b) = 0$, *and for such a* b *we have*

$$\begin{aligned}\lim_{\lambda\to 0} \frac{1}{\lambda^{\nu_1-\nu_2}} P_{\mathrm{Geg}}((a, a+\lambda b), (d_1, d_2))(Y_1, Y_2, X_1, X_2) \\ = c_6(B(a, d_1)X_1 + B(a, d_2)X_2)^{2\nu_2} \widetilde{G_a^{(\nu_1-\nu_2)}}(\mathrm{Im}(d_1\overline{d_2}).\end{aligned} \tag{21}$$

with $c_6 = C_2(\nu_1 - \nu_2, 2\nu_2)$ *as in Lemma 6.4.*

Proof. (1) From the formula for the map $\mathcal{P}$ in equation (19) we get

$$\begin{aligned}&\mathcal{P}(G_a^{(\nu_1)} \otimes G_a^{(\nu_2)})(d_1, d_2)(X_1, X_2) \\ &\quad = \frac{\nu_1!}{\nu_2!}(B(a, (d_1X_1 + d_2X_2)a(\overline{d_1}X_1 + \overline{d_2X_2})))^{\nu_2} G_a^{(\nu_1-\nu_2)}(\mathrm{Im}(d_1\overline{d_2})).\end{aligned}$$

Using $\bar{a} = -a, a^2 = 0$ and $xa = a\bar{x} - B(a, x)$ for $x \in D_{\mathbb{C}}$ we get $B(a, ya\bar{x}) = B(a, x)B(a, y)$ for $x, y \in D_{\mathbb{C}}$. We extend this identity to the polynomial ring, insert for x, y one of d_1X_1, d_2X_2 and obtain $B(a, (d_1X_1 + d_2X_2)a(\overline{d_1}X_1 + \overline{d_2X_2})) = (B(a, d_1)X_1 + B(a, d_2)X_2)^2$, which yields the assertion.

(2) For simplicity we identify $D_{\mathbb{C}}$ with the matrix ring $M_2(\mathbb{C})$ and fix $a = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), b = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ (we will need this Lemma only for one particular choice of a, b). Equation (7) in Lemma 6.4 gives us

$$\begin{aligned}&P_{\mathrm{Geg}}((a, a+\lambda b), (d_1, d_2))(Y_1, Y_2, X_1, X_2) \\ &\quad = c_6\left((Y_1, Y_2)\begin{pmatrix} B(a, d_1) & B(a, d_2) \\ B(a+\lambda b, d_1) & B(a+\lambda b, d_2)\end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}\right)^{2\nu_2} \\ &\quad \times \det\left(\begin{pmatrix} B(a, d_1) & B(a, d_2) \\ B(a+\lambda b, d_1) & B(a+\lambda b, d_2)\end{pmatrix}\right)^{\nu_1-\nu_2}.\end{aligned}$$

Dividing by $\lambda^{\nu_1-\nu_2}$ and taking the limit for $\lambda \to 0$ we get

$$c_6((Y_1 + Y_2)(B(a, d_1)X_1 + B(a, d_2)X_2))^{2\nu_2} \det\left(\begin{pmatrix} B(a, d_1) & B(a, d_2) \\ B(b, d_1) & B(b, d_2)\end{pmatrix}\right)^{\nu_1-\nu_2}.$$

Computing the determinant for our choice of a, b, writing d_1, d_2 as matrices $\left(\begin{smallmatrix} x_1 & x_2 \\ x_3 & x_4 \end{smallmatrix}\right), \left(\begin{smallmatrix} y_1 & y_2 \\ y_3 & y_4 \end{smallmatrix}\right)$ and using that quaternionic conjugation sends a matrix $\left(\begin{smallmatrix} x_1 & x_2 \\ x_3 & x_4 \end{smallmatrix}\right)$ to its classical adjoint $\left(\begin{smallmatrix} x_4 & -x_2 \\ -x_3 & x_1 \end{smallmatrix}\right)$ one checks that both $\det(\dots)^{\nu_1-\nu_2}$ and $\widetilde{G_a^{(\nu_1-\nu_2)}}(\mathrm{Im}(d_1\overline{d_2}))$ evaluate to $(x_3y_4 - x_4y_3)^{\nu_1-\nu_2}$, which proves the assertion.

∎

Proposition 8.7. *Let $R_1 \in U^{(0)}_{\nu_1}, R_2 \in U^{(0)}_{\nu_2}$ be given and let the scalar products $\langle \quad , \quad \rangle_\mu$ on $U^{(0)}_{\nu_\mu}$ for $\mu = 1, 2$ be normalized such that the Gegenbauer polynomial*

$$G^{(\nu_\mu)}(x,y) = \frac{2^{\nu_\mu}}{\Gamma(1/2)} \sum_{j=0}^{[\nu_\mu/2]} (-1)^j \frac{1}{j}!(\nu_\mu - 2j)!\Gamma(\nu_\mu - j + \frac{1}{2})(\mathrm{tr}(x\bar{y}))^{\nu_\mu - 2j}(n(x)n(y))^j$$

(see [BS5, p. 47]*) is the reproducing kernel for $U^{(0)}_{\nu_\mu}$. Then one has*

$$\langle \mathcal{P}(R_1 \otimes R_2), \mathcal{P}(R_1 \otimes R_2)\rangle_{\mathcal{H}_q(\rho)} = c_7 \langle R_1, R_1\rangle_1 \langle R_2, R_2\rangle_2$$

with $c_7 = c_6 \frac{\nu_1!}{\nu_2!}\binom{2\nu_1}{\nu_1}\binom{2\nu_2}{\nu_2}$.

Proof. Since $\mathcal{P}$ is an intertwining map between finite dimensional irreducible unitary representations of the compact orthogonal group it is clear that the right hand side and the left hand side of the asserted equality are proportional. It suffices therefore to evaluate both sides for a particular choice of R_1, R_2. We choose $R_1 = G^{(\nu_1)}_a, R_2 = G^{(\nu_2)}_a$ with $G^{(\nu_\mu)}_a(y) = G^{(\nu_\mu)}(a, y)$. The reproducing property of the Gegenbauer polynomial gives

$$\langle P_{\mathrm{Geg}}(a, a + \lambda b), \mathcal{P}(G^{(\nu_1)}_a \otimes G^{(\nu_2)}_a)\rangle_{\mathcal{H}_q(\rho)} = (\mathcal{P}(G^{(\nu_1)}_a \otimes G^{(\nu_2)}_a))^c(a, a + \lambda b),$$

where we denote by the exponent c at $(\mathcal{P}(G^{(\nu_1)}_a \otimes G^{(\nu_2)}_a))$ complex conjugation of the coefficients of this polynomial (in order to avoid confusion with quaternionic conjugation). With the particular choice of a, b form the previous lemma we obtain, using $a\bar{b} = a$ and $a^2 = 0$ and writing a^c for the vector obtained from a by complex conjugation of the coordinates with respect to an orthonormal basis of D_∞,

$$\begin{aligned}
\frac{1}{\lambda^{\nu_1 - \nu_2}} &(\mathcal{P}(G^{(\nu_1)}_a \otimes G^{(\nu_2)}_a))^c(a, a + \lambda b) \\
&= \binom{2\nu_1}{\nu_1}\binom{2\nu_2}{\nu_2}(B(a^c, a))^{2\nu_2}(B(a^c, \mathrm{Im}(a(\overline{a + \lambda b}))))^{\nu_1 - \nu_2} \\
&= \binom{2\nu_1}{\nu_1}\binom{2\nu_2}{\nu_2}(B(a^c, a))^{\nu_1 + \nu_2} \\
&= (G^{(\nu_1)}_a)(a^c)(G^{(\nu_2)}_a)(a^c) \\
&= \langle G^{(\nu_1)}_a, G^{(\nu_1)}_a\rangle_1 \langle G^{(\nu_2)}_a, G^{(\nu_2)}_a\rangle_2.
\end{aligned}$$

Inserting the formulas from Lemma 8.6 proves the assertion. ∎

Corollary 8.8. *With notations as in Proposition 8.5 and $\mathcal{P}$ normalized as above one has*

$$\langle F, F\rangle_{\mathrm{Pet}} = \frac{c_4 c_7}{2 c_3 \beta_1} L^{(N)}\left(f \otimes g, \frac{k + k'}{2}\right) \langle \phi_1, \phi_1\rangle_1 \langle \phi_2, \phi_2\rangle_2$$

Let F be a Yoshida lift of f and g as above and define $F_{\mathrm{can}} = \frac{F}{\sqrt{\langle \mathcal{P}(\phi_1 \otimes \phi_2), \mathcal{P}(\phi_1 \otimes \phi_2)\rangle}}$. Any rescaling of ϕ_1, ϕ_2 or $\mathcal{P}$ affects the numerator and denominator in the same way, so this may be viewed as a canonical choice of scaling of F. We can now express this canonical choice of F explicitly.

Proposition 8.9. *Let $\phi^{(0)}_1, \phi^{(0)}_2$ be normalized by $\langle \phi^{(0)}_1, \phi^{(0)}_1\rangle_1 = \langle \phi^{(0)}_2, \phi^{(0)}_2\rangle_2 = 1$ and let $\mathcal{P}$ be normalized as above. Then one has*

$$F_{\mathrm{can}} = \frac{1}{c_7} Y^2(\phi^{(0)}_1, \phi^{(0)}_2).$$

Note that the Fourier coefficients of F_{can} are algebraic. From the results of [BS5, BS4] it is clear that the square of the (scalar valued) average over matrices T of fixed fundamental discriminant $-d$ of the Fourier coefficients $A(F,T) \in W_\rho$ of the Yoshida lifting F_{can} is proportional to the product of the central critical values of the twists with the quadratic character χ_{-d} of the L-functions of the elliptic modular forms f and g; notice that the averaging procedure for the W_ρ-valued Fourier coefficients involves a scalar product of $A(F,T)$ with the vector $\rho(T^{-1/2})\mathbf{v}_0$, where $\mathbf{v}_0$ is an $O_n(\mathbb{R})$-invariant vector in W_ρ. We can now make this proportionality as explicit as the result of [BS2] for the scalar valued case.

Proposition 8.10. *Assume that ν_1, ν_2 are even and that both f, g have a $+$-sign in the functional equation. Choose N_1, N_2 such that the (common) Atkin-Lehner eigenvalue ϵ_p of f, g at p is -1 if and only if $p \mid N_1$. Let $-d < 0$ be a fundamental discriminant with $(\frac{-d}{p})\epsilon_p = 1$ for all primes p dividing $N_d = N/\gcd(N,d)$. We let $F = F_{\mathrm{can}}$ be the canonical Yoshida lifting of f, g with respect to N_1, N_2 and put*

$$a(F,d) = \frac{\sqrt{d}}{2} \sum_{\substack{\{T\} \\ \operatorname{disc} T = -d}} \frac{1}{\epsilon(T)} \int\limits_{T[\mathbf{x}] \le 1} A(F,T)(x_1,x_2)\,dx_1\,dx_2$$

where $A(F,T)$ is the Fourier coefficient at T of F, the summation is over integral equivalence classes of T, and $\epsilon(T)$ is the number of automorphs (units) of T, i.e., the number of $g \in GL_2(\mathbb{Z})$ with ${}^t gTg = T$.

Then one has

(22)
$$(a(F,d))^2 = c_8 \frac{L(1+\nu_1, f)L(,1+\nu_2, g)L(1+\nu_1, f \otimes \chi_{-d})L(1+\nu_2, g \otimes \chi_{-d})}{\langle f, f\rangle \langle g, g\rangle}$$

with $c_8^{-1} = 2^6(\nu_2+1)^2\pi^{2+2\nu_1+2\nu_2}$.

Proof. Corollary 4.3 of [BS5] gives

$$\left(\frac{d}{4}\right)^{\frac{\nu_1+\nu_2}{2}} \sigma_0(N_d)a(F,d) = \frac{c}{2} a(\mathcal{W}(\phi_1), d)a(\mathcal{W}(\phi_2), d),$$

where the $a(\mathcal{W}(\phi_\mu), d)$ are the Fourier coefficients of the Waldspurger liftings $\mathcal{W}(\phi_\mu) = \sum_{j=1}^{r} \frac{1}{e_j} \sum_{x \in L_j} \phi(y_j)(x) \exp(2\pi i n(x) z)$ associated to the lattices $L_j = D^{(0)} \cap (\mathbb{Z}1 + 2R_j)$ and where $c = \frac{(-1)^{\nu_2} 2\pi}{2\nu_2+2}$. Inserting the explicit version of Waldspurger's theorem from [Koh, BS4] gives the assertion. □

Remark 8.11. (1) *The restrictive conditions on f, g, N_1, d in the proposition are chosen in order to to prevent that $a(F,d)$ becomes zero for trivial reasons.*

(2) *Since $\frac{\sqrt{d}}{2} \int\limits_{T[\mathbf{x}] \le 1} x_1^i x_2^j\,dx_1\,dx_2$ is zero for i or j odd and equal to*

$$\int_0^{\frac{\pi}{2}} \cos^i(\alpha)\sin^j(\alpha)\,d\alpha = \frac{\Gamma(i_1+\frac{1}{2})\Gamma(j_1+\frac{1}{2})}{2\Gamma(i_1+j_1+1)}$$

for even $i = 2i_1, j = 2j_1$, we have:

If for a prime λ not dividing $2\nu_2!$ and some $j \in \mathbb{N}$ one has $\lambda^j \nmid a(F,d)/\pi$, then there is some T of discriminant $-d$ such that λ^j does not divide all coefficients of the polynomial $A(F,T)$.

9. A congruence of Hecke eigenvalues

As above, let f and g be cuspidal Hecke eigenforms for $\Gamma_0(N)$, of weights $k' > k \geq 2$. For critical $k \leq t < k'$, define $L_{\mathrm{alg}}(f \otimes g, t) := \frac{L(f\otimes g,t)}{\pi^{2t-(k-1)}\langle f,f\rangle}$. (Alternatively one could divide by a canonical Deligne period–it makes no difference to the proposition below.) Let K be a number field containing all the Hecke eigenvalues of f and g. Let F be a Yoshida lift of f and g, lying in $S_\rho(\Gamma_0^{(2)}(N))$ say, and define as in the previous section $F_{\mathrm{can}} = \frac{F}{\sqrt{\langle \mathcal{P}(\phi_1\otimes\phi_2),\mathcal{P}(\phi_1\otimes\phi_2)\rangle}}$. In fact we have such an F and F_{can} for each factorisation $N = N_1N_2$ with an odd number of prime factors in N_1, and we label these F_i and $F_{i,\mathrm{can}}$ for $1 \leq i \leq u$, say. Note that by Lemma 8.4, these different Yoshida lifts of the same f and g are mutually orthogonal with respect to the Petersson inner product. Let's say $F = F_1$ arbitrarily.

As in §2.1 of [Ar] the operators $T(m)$, for $(m,N) = 1$ (generated over $\mathbb{Z}$ by the $T(p)$ and $T(p^2)$, see (2.2) of [Ar]) are self-adjoint for the Petersson inner product, and commute amongst themselves, so $S_\rho(\Gamma_0^{(2)}(N))$ has a basis of simultaneous eigenvectors for such $T(m)$. Also, these $T(m)$, acting on elements of $S_\rho(\Gamma_0^{(2)}(N))$, preserve integrality (at any given prime) of Fourier coefficients, by (2.13) of [Sa]. If $G \in S_\rho(\Gamma_0^{(2)}(N))$ is an eigenform (for the $T(m)$, with $(m,N) = 1$), then the Hecke eigenvalues for G are algebraic integers. This follows from Theorem I of [We2], which says that the characteristic polynomial of $\rho_G(\mathrm{Frob}_p^{-1})$ (c.f. §4 above) is $1 - \mu_G(p)X + (\mu_G(p)^2 - \mu_G(p^2) - p^{k'-2})X^2 - p^{k'-1}\mu_G(p)X^3 + p^{2(k'-1)}X^4$ (c.f. (2.2) of [Ar]), and that the eigenvalues of $\rho_G(\mathrm{Frob}_p^{-1})$ are algebraic integers. Moreover, as p varies for fixed G, the $\mu_G(p)$ and $\mu_G(p^2)$ generate a finite extension of $\mathbb{Q}$.

Proposition 9.1. *Suppose that $k'-k \geq 6$. Suppose that λ is a prime of K such that $\mathrm{ord}_\lambda\left(L_{\mathrm{alg}}\left(f\otimes g, \frac{k'+k}{2}\right)\right) > 0$ but $\mathrm{ord}_\lambda\left(L_{\mathrm{alg}}\left(f\otimes g, \frac{k'+k}{2}+1\right)\right) = 0$, and let ℓ be the rational prime that λ divides. Suppose that $\ell \nmid N$ and $\ell > k'-2$. Assume that there exist a half-integral symmetric 2-by-2 matrix A, and an integer $0 \leq b \leq k-2$ such that, if for $1 \leq i \leq u$, a_i denotes the coefficient of the monomial $x^b y^{k-2-b}$ in the A-Fourier coefficient in $F_{i,\mathrm{can}}$, then $\mathrm{ord}_\lambda(\sum_{i=1}^u a_i^2) \leq 0$. Then there is a cusp form $G \in S_\rho(\Gamma_0^{(2)}(N))$, an eigenvector for all the $T(m)$, with $(m,N) = 1$, not itself a Yoshida lift of the same f and g, such that there is a congruence of Hecke eigenvalues between G and F:*

$$\mu_G(m) \equiv \mu_F(m) \pmod{\lambda},\ \textit{for all } (m,N) = 1.$$

(We make K sufficiently large to contain the Hecke eigenvalues of G.)

Proof. Since $k'-k \geq 6$, $\frac{k'-k}{2} - 2 > 0$, so $\mathbb{D}_4\left(\frac{k'-k}{2} - 2, k-2\right)\mathcal{F}_4^{(4)}(Z,W)$ is a cusp form. Let $\{F_1, F_2, \ldots, F_r\}$ be a basis of $S_\rho(\Gamma_0^{(2)}(N))$ consisting of eigenforms for all the local Hecke algebras at $p \nmid N$, with $F_1, \ldots, F_u$ the Yoshida lifts of f and g, as above.

It is easy to show that $\mathbb{D}_4\left(\frac{k'-k}{2}-2,k-2\right)\mathcal{F}_4^{(4)}(Z,W)=\sum_{i,j=1}^r c_{i,j}F_i(Z)F_j(W)$, for some $c_{i,j}$. By (16), $c_{1,1}$ is equal to the right hand side of (16), divided by $F(w)\langle F,F\rangle$, and $c_{1,j}=0$ for $j\neq 1$. Similarly for all the $c_{i,i}$ for $1\leq i\leq u$. Using Proposition 8.5, we find

$$c_{1,1}=c'\frac{L_{\mathrm{alg}}\left(f\otimes g,\frac{k'+k}{2}+1\right)}{L_{\mathrm{alg}}\left(f\otimes g,\frac{k'+k}{2}\right)\langle\mathcal{P}(\phi_1\otimes\phi_2),\mathcal{P}(\phi_1\otimes\phi_2)\rangle},\tag{23}$$

where
(24)

$$c'=\gamma_2\left(4,\frac{k'-k}{2}-2,k-2,0\right)(\pm N)\Lambda_N(2)\frac{\zeta^{(N)}(2)\,\pi^2}{\zeta^{(N)}(4)\zeta^{(N)}(6)\zeta^{(N)}(4)}\prod_{p\mid N}\frac{(1-p^{-3})}{(1-p^{-1})}.$$

(The last term takes into account the fact that we have passed from incomplete to complete L-functions.)

We now choose A and b as in the statement of the proposition. For $u+1\leq i\leq r$ let a_i' be the coefficient of x^by^{k-2-b} in the A-Fourier coefficient of F_i . Imitating §4 of [Ka], let $\mathcal{F}_{4,\rho,A}(Z)$ be the coefficient of $x_w^by_w^{k-2-b}$ in the coefficient of $\mathbf{e}(\mathrm{Tr}(AW))$ in
$\mathbb{D}_4\left(\frac{k'-k}{2}-2,k-2\right)\mathcal{F}_4^{(4)}(Z,W)$. Then

$$\mathcal{F}_{4,\rho,A}(Z)=\sum_{i=1}^{u}e_iF_{i,\mathrm{can}}(Z)+\sum_{i\geq u+1}e_i'F_i(Z),\tag{25}$$

where, for $1\leq i\leq u$, $e_i=c'\frac{L_{\mathrm{alg}}\left(f\otimes g,\frac{k'+k}{2}+1\right)}{L_{\mathrm{alg}}\left(f\otimes g,\frac{k'+k}{2}\right)}a_i$. Careful checking of all the things that go into c' shows that it is a rational number, and that it follows from $\ell>k'-2$ that $\mathrm{ord}_\ell(c')\leq 0$. The coefficients of $\mathcal{F}_{4,\rho,A}$ are integral at λ, by Remarks 7.1 and 7.2. Given all this, we can apply the method of Lemma 5.1 of [Ka], to deduce that there is a congruence $\pmod{\lambda}$ of Hecke eigenvalues (for all $T(m)$, with $(m,N)=1$) between F and some other $F_i=G$, say, with $i\geq u+1$.

In a little more detail, we suppose that no such G exists, so that for each $u+1\leq i\leq r$ there exists an m_i, with $(m_i,N)=1$, such that if $\mu_{F_i}(m_i)$ is the eigenvalue of $T(m_i)$ on F_i then $\mu_{F_i}(m_i)\not\equiv\mu_F(m_i)\pmod{\lambda}$. (We may enlarge K to contain all the Hecke eigenvalues for all the F_i.) Applying $\prod_{i=u+1}^r(T(m_i)-\mu_{F_i}(m_i))$ to both sides of (25), we get something on the left that is integral at λ. On the right all the F_i terms, for $i\geq u+1$, disappear, while the remaining terms get multiplied by $\prod_{i=u+1}^r(\mu_F(m_i)-\mu_{F_i}(m_i))$, which is not divisible by λ, so on the right-hand-side the coefficient of $x_z^by_z^{k-2-b}$ in the coefficient of $\mathbf{e}(\mathrm{Tr}(AZ))$, namely

$$c'\prod_{i=u+1}^{r}(\mu_F(m_i)-\mu_{F_i}(m_i))\frac{L_{\mathrm{alg}}\left(f\otimes g,\frac{k'+k}{2}+1\right)}{L_{\mathrm{alg}}\left(f\otimes g,\frac{k'+k}{2}\right)}\left(\sum_{i=1}^{u}a_i^2\right),$$

is non-integral at λ, which is a contradiction. $\square$

Using Proposition 8.10 and Remark 8.11(2), we obtain the following.

Corollary 9.2. *Suppose that* $k'-k\geq 6$, *with* $k/2$ *and* $k'/2$ *odd, that* N *is prime, and that the common eigenvalue* w_N *for* f *and* g *is* -1. *Suppose that* λ *is a prime of* K *such that* $\mathrm{ord}_\lambda\left(L_{\mathrm{alg}}\left(f\otimes g,\frac{k'+k}{2}\right)\right)>0$ *but* $\mathrm{ord}_\lambda\left(L_{\mathrm{alg}}\left(f\otimes g,\frac{k'+k}{2}+1\right)\right)=0$, *with* $\ell \nmid N$ *and* $\ell > k'-2$, *where* ℓ *is the rational prime that* λ *divides. Suppose that there is some fundamental discriminant* $-d < 0$ *that is a quadratic residue modulo* p *for all primes* p *dividing* $N_d = N/\gcd(N,d)$, *such that*

$$\mathrm{ord}_\lambda\left(\frac{L(k'/2,f)L(k'/2,f\otimes\chi_{-d})}{\pi^{k'}\langle f,f\rangle}\frac{L(k/2,g)L(k/2,g\otimes\chi_{-d})}{\pi^{k}\langle g,g\rangle}\right)\leq 0.$$

Then there is a cusp form $G \in S_\rho(\Gamma_0^{(2)}(N))$, *an eigenvector for all the* $T(m)$, *with* $(m,N)=1$, *not a multiple of* F, *such that there is a congruence of Hecke eigenvalues between* G *and* F*:*

$$\mu_G(m) \equiv \mu_F(m) \pmod{\lambda},\ \textit{for all } (m,N)=1.$$

(We make K *sufficiently large to contain the Hecke eigenvalues of* G*.)*

9.1. **Examples.**

(1) When $k=2$ and $k'=4$ (so $j=0$ and $\kappa=3$), one may check that, for $N=23,29,31,37$ or 43, the dimension of $S_3(\Gamma_0^{(2)}(N))$ $(2,4,4,9,14$ respectively, using Theorem 2.2 in [I2]) is the same as that of the subspace spanned by Yoshida lifts of $f\in S_4(\Gamma_0(N))$ and $g\in S_2(\Gamma_0(N))$. This appears to leave no room for G (recall Lemma 4.1). However, we calculated $L_{\mathrm{alg}}(f\otimes g,3)$ in the case $N=23$, using Theorem 2 of [Sh4] and Stein's tables [St]. (The two choices for g are conjugate over $\mathbb{Q}(\sqrt{5})$.) For the near-central value, this calculation involves an Eisenstein series of weight 2, to which a non-holomorphic adjustment must be made. The result was that $L_{\mathrm{alg}}(f\otimes g,3)=32/3$, so there is in fact no divisor λ, dividing a large prime ℓ, for which a congruence with some G is required.

(2) The previous paragraph leaves open the possibility that the condition $k'-k\geq 6$, in Proposition 9.1, is purely technical. However, the following example shows that it is essential. Let $k=2$ and $k'=6$ (so $j=0$ and $\kappa=4$) and $N=11$. As is well-known, $S_2(\Gamma_0(11))$ is 1-dimensional, spanned by $g=q-2q^2-q^3+\ldots$, for which $w_{11}=-1$. Using [St], $\dim S_6(\Gamma_0(11))=4$, with the $w_{11}=-1$ eigenspace 3-dimensional, spanned by the embeddings of a newform $f=q+\beta q^2+\ldots$, where $\beta^3-90\beta+188=0$. The discriminant of this polynomial is $2^4 3^3 19\cdot 239$. Using Theorem 2 of [Sh4] we find that $L_{\mathrm{alg}}(f\otimes g,4)=-\frac{4^5\alpha}{3}$, with $\mathrm{Norm}(\alpha)=-\frac{17\cdot 76157}{2^4 3^4 5^2 11^2 19\cdot 239}$. In fact α is divisible by the prime ideals $(17,\beta+1)$ and $(76157,\beta+74208)$.

The dimension of $S_4(\Gamma_0^{(2)}(11))$ is 7, from the table in §2.4 of [I2]. This fact was also obtained by Poor and Yuen, who gave an explicit basis for this space using theta series, [PY]. We are indebted to D. Yuen for calculating for us a Hecke eigenbasis, which included the three Yoshida lifts, a non-lift with rational eigenvalues, and three conjugate non-lifts with eigenvalues and Fourier coefficients in the same cubic field as f and the Yoshida lifts. He looked for congruences modulo primes dividing 17 or 76157 (or any other large primes), but found that there were none, though it appears that each Yoshida lift has *Fourier coefficients* (not just Hecke eigenvalues) congruent mod 5 to those of a corresponding non-lift (suitably normalised).

(3) We should expect that any example of f and g we look at, with prime level N, common $w_N=-1$, weights $k'>k\geq 2$ with $k'-k\geq 6$ and $k'/2,k/2$ odd, is very likely to satisfy the remaining condition of Corollary 9.2, for some λ. It seems though that finding an example where one can directly

observe the congruence guaranteed by Corollary 9.2 would be difficult. Already for $k = 2, k' = 10$ and $N = 11$ we have $\dim S_6(\Gamma_0^{(2)}(11)) = 31$ (from the table in 7-11 of [Has]). For us, f and g are of level $N > 1$, and Yoshida lifts do not exist at level 1. However, recently J. Bergström (in collaboration with C. Faber and G. van der Geer) has found experimentally what appear to be eleven examples of congruences of exactly the same shape, but for f and g of level 1. For example, it appears that there is a genus-2 cusp form of level 1 and weight $\mathrm{Sym}^{20} \otimes \det^5$ such that

$$\mu_G(p) \equiv a_p(f) + p^3 a_p(g) \pmod{\lambda},$$

with $\lambda \mid 227$, where f and g are cuspidal Hecke eigenforms of genus 1, level 1 and weights $k' = 28, k = 22$ respectively. Bergström et. al. have checked this for $p \leq 17$. Using Theorem 2 of [Sh4], we have checked that $L(f \otimes g, 25) = \frac{4^{27}\pi^{29}}{108(24!)} \cdot \alpha(f, f)$, with $\mathrm{Norm}(\alpha) = \frac{7.17.227}{2.3^6.5^4.131.139}$. In two more examples, with $(k', k, \ell) = (28, 18, 223)$ and $(28, 20, 2647)$, we have likewise checked that the prime occurring in the modulus of an apparent congruence also appears in the near-central tensor-product L-value, in accord with the Bloch-Kato conjecture. We are grateful to Bergström for permission to mention his unpublished data.

References

[AK] M. Agarwal, K. Klosin, Yoshida lifts and the Bloch-Kato conjecture for the convolution L-function, in preparation.

[An] A. N. Andrianov, *Quadratic forms and Hecke operators*, Grundl. d. Math. Wiss. 286, Springer, Berlin 1987.

[Ar] T. Arakawa, Vector valued Siegel's modular forms of degree two and the associated Andrianov L-functions, *Manuscripta Math.* **44** (1983), 155–185.

[AS] M. Asgari, R. Schmidt, Siegel modular forms and representations, *Manuscripta Math.* **104** (2001), 173–200.

[BK] S. Bloch, K. Kato, L-functions and Tamagawa numbers of motives, The Grothendieck Festschrift Volume I, 333–400, Progress in Mathematics, 86, Birkhäuser, Boston, 1990.

[B1] S. Böcherer, Über die Fourier-Jacobi-Entwicklung Siegelscher Eisensteinreihen II, *Math. Z.* **189** (1985), 81–110.

[B2] S. Böcherer, Siegel modular forms and theta series, *AMS Proc. Symp. Pure Math.,* Vol. 49 (1989), part 2, 3–17.

[B3] S. Böcherer, Über die Funktionalgleichung automorpher L-Funktionen zur Siegelschen Modulgruppe, *J. f. d. reine u. angew. Math.* **362** (1985), 146–168.

[B4] S. Böcherer, Über die Fourierkoeffizienten der Siegelschen Eisensteinreihen, *Manuscripta math.* **45** (1984), 273–288.

[BKS] S. Böcherer, H. Katsurada, R. Schulze-Pillot, On the basis problem for Siegel modular forms with level, pp.13–28 in: Modular forms on Schiermonnikoog. Cambridge University Press 2008.

[BSY] S. Böcherer, T. Satoh, T. Yamazaki, On the pullback of a differential operator and its application to vector valued Eisenstein series, *Comment. Math. Univ. St. Paul.* **42** (1992), 1–22.

[BCG] S. Böcherer, C. G. Schmidt, p-adic measures attached to Siegel modular forms, *Ann. Inst. Fourier* **50** (2000), 1375–1443.

[BS1] S. Böcherer, R. Schulze-Pillot, Siegel modular forms and theta series attached to quaternion algebras, *Nagoya Math. J.* **121** (1991), 35–96.

[BS2] S. Böcherer, R. Schulze-Pillot, The Dirichlet series of Koecher and Maass and modular forms of weight $\frac{3}{2}$. *Math. Z.* **209** (1992), no. 2, 273–287.

[BS3] S. Böcherer, R. Schulze-Pillot, Siegel modular forms and theta series attached to quaternion algebras, II, *Nagoya Math. J.* **147** (1997), 71–106.

[BS4] S. Böcherer, R. Schulze-Pillot, Vector valued theta series and Waldspurger's theorem. *Abh. Math. Sem. Univ. Hamburg* **64** (1994), 211–233.

[BS5] S. Böcherer, R. Schulze-Pillot, Mellin transforms of vector valued theta series attached to quaternion algebras, *Math. Nachr.* **169** (1994), 31–57.

[Br] J. Brown, Saito-Kurokawa lifts and applications to the Bloch-Kato conjecture, *Compositio Math.* **143** (2007), 290–322.

[CF] C.-L. Chai, G. Faltings, *Degeneration of Abelian Varieties,* Ergeb. Math. Grenzgeb. (3), Vol. 22, Springer, Berlin, 1990.

[De] P. Deligne, Valeurs de Fonctions L et Périodes d'Intégrales, *AMS Proc. Symp. Pure Math.,* Vol. 33 (1979), part 2, 313–346.

[Di] F. Diamond, The refined conjecture of Serre, *in* Elliptic Curves, Modular Forms and Fermat's Last Theorem, J. Coates, S.T. Yau, eds., 172–186, International Press, Cambridge MA, 1995.

[DFG] F. Diamond, M. Flach, L. Guo, The Tamagawa number conjecture of adjoint motives of modular forms, *Ann. Sci. École Norm. Sup. (4)* **37** (2004), 663–727.

[Du1] N. Dummigan, Symmetric square L-functions and Shafarevich-Tate groups, II, *Int. J. Number Theory* **5** (2009), 1321–1345.

[Du2] N. Dummigan, Selmer groups for tensor product L-functions, pp. 37–46 in *Automorphic representations, automorphic* L*-functions and arithmetic,* R.I.M.S. Kokyuroku 1659, July 2009.

[Ed] B. Edixhoven, The weight in Serre's conjectures on modular forms, *Invent. Math.* **109** (1992), 563–594.

[Ei] M. Eichler, The basis problem for modular forms and the traces of the Hecke operators, p. 76-151 in Modular functions of one variable I, Lecture Notes Math. 320, Berlin-Heidelberg-New York 1973.

[Ev] S.A. Evdokimov, Action of the irregular Hecke operator of index p on the theta series of a quadratic form, *J. Sov. Math.* **38** (1987), 2078–2081.

[Fa] G. Faltings, Crystalline cohomology and p-adic Galois representations, *in* Algebraic analysis, geometry and number theory (J. Igusa, ed.), 25–80, Johns Hopkins University Press, Baltimore, 1989.

[Fl] M. Flach, A generalisation of the Cassels-Tate pairing, *J. f. d. reine u. angew. Math.* **412** (1990), 113–127.

[Fo1] J.-M. Fontaine, Le corps de périodes p-adiques, *Astérisque* **223** (1994), 59–111.

[Fo2] J.-M. Fontaine, Valeurs spéciales des fonctions L des motifs, Séminaire Bourbaki, Vol. 1991/92. *Astérisque* **206** (1992), Exp. No. 751, 4, 205–249.

[FL] J.-M. Fontaine, G. Lafaille, Construction de représentations p-adiques, *Ann. Sci. E.N.S.* **15** (1982), 547–608.

[Fu] M. Furusawa, On L-functions for $\mathrm{GSp}(4) \times \mathrm{GL}(2)$ and their special values, *J. f. d. reine u. angew. Math.* **438** (1993), 187–218.

[GT] A. Genestier, J. Tilouine, Systèmes de Taylor-Wiles pour GSp(4), *Astérisque* **302** (2005), 177–290.

[Go] R. Godement, Seminaire Cartan 10 (1957/58) Exp.4-9

[Ha] G. Harder, A congruence between a Siegel and an elliptic modular form, manuscript, 2003, *reproduced in* The 1-2-3 of Modular Forms (J. H. Bruinier et. al.), 247–262, Springer-Verlag, Berlin Heidelberg, 2008.

[Har] A. Haruki, Explicit formulae of Siegel Eisenstein series, *Manuscripta math.* **92** (1997), 107-134.

[Has] K. Hashimoto, The dimension of the spaces of cusp forms on Siegel upper half-plane of degree two, I, *J. Fac. Sci. Univ. Tokyo, Sect. IA, Math.* **30** (1983), 403–488.

[Hi] H. Hida, *Modular forms and Galois cohomology.* Cambridge University Press, 2000.

[H-S] H. Hijikata, H. Saito, On the representability of modular forms by theta series, p. 13-21 in Number Theory, Algebraic Geometry and Commutative Algebra, in honor of Y. Akizuki, Tokyo 1973

[I1] T. Ibukiyama, On differential operators on automorphic forms and invariant pluri-harmonic polynomials, *Comment. Math. Univ. Sanct. Paul.* **48** (1999), 103–118.

[I2] T. Ibukiyama, Dimension formulas of Siegel modular forms of weight 3 and supersingular abelian varieties, *in* Siegel Modular Forms and Abelian Varieties (T. Ibukiyama, ed.), Proceedings of the Fourth Spring Conference on Modular Forms and Related Topics, Hamana Lake, Japan, 2007. Ryushido, 2007.

[J-L] H. Jacquet, R. Langlands, Automorphic forms on $\mathsf{GL}(2)$, Lect. Notes in Math. 114, Berlin-Heidelberg-New York 1970

[KV] M. Kashiwara, M. Vergne, On the Segal-Shale-Weil representations and harmonic polynomials, Invent. Math. 44 (1987), 1-44

[Ka] H. Katsurada, Congruence of Siegel modular forms and special values of their standard zeta functions, *Math. Z.* **259** (2008), 97–111.

[Koh] W. Kohnen, Fourier coefficients of modular forms of half-integral weight. Math. Ann. 271 (1985), no. 2, 237–268.

[Koz] N. Kozima, On special values of standard L-functions attached to vector valued Siegel modular forms, *Kodai Math. J.* **23** (2000), 255–265.

[L] R. P. Langlands, Modular forms and ℓ-adic representations, pp. 362–499 in *Modular functions of one variable II,* Lecture Notes in Math. Vol. 349, Springer, Berlin, 1973.

[MW] B. Mazur, A. Wiles, Class fields of abelian extensions of $\mathbb{Q}$, *Invent. Math.* **76** (1984), 179–330.

[PS] A. Pitale, R. Schmidt, Ramanujan-type results for Siegel cusp forms of degree 2, *J. Ramanujan Math. Soc.* **24** (2009), 87–111.

[PY] C. Poor, D.S. Yuen, Dimensions of cusp forms for $\Gamma_0(p)$ in degree two and small weights, *Abh. Math. Sem. Univ. Hamburg.* **77** (2007), 59–80.

[R1] K. Ribet, A modular construction of unramified p-extensions of $\mathbb{Q}(\mu_p)$, *Invent. Math.* **34** (1976), 151–162.

[R2] K. Ribet, On modular representations of Gal($\overline{\mathbb{Q}}/\mathbb{Q}$) arising from modular forms, *Invent. Math.* **100** (1990), 431–476.

[R3] K. Ribet, Report on (mod l) representations of Gal($\overline{\mathbb{Q}}/\mathbb{Q}$), *in* Motives, A.M.S. Proc. Symp. Pure Math. **55**:2 (1994), 639–676.

[Sa] T. Satoh, On certain vector valued Siegel modular forms of degree two, *Math. Ann.* **274** (1986), 335–352.

[Sc] R. Schmidt, Iwahori-spherical representations of GSp(4) and Siegel modular forms of degree 2 with square-free level, *J. Math. Soc. Japan* **57** (2005), 259–293.

[Shz] H. Shimizu, Theta series and automorphic forms on GL_2, *J. Math. Soc. Japan* **24** (1972), 638–683.

[Sh1] G. Shimura, On Eisenstein series, *Duke Math. J.* **50** (1983), 417–476.

[Sh2] G. Shimura, On a class of nearly holomorphic automorphic forms, *Annals of Math.* **123** (1986), 347–406.

[Sh3] G. Shimura, Nearly holomorphic functions on hermitian symmetric spaces, *Math. Ann.* **278** (1987), 1–28.

[Sh4] G. Shimura, The special values of the zeta functions associated with cusp forms, *Comm. Pure Applied Math.* **29** (1976), 783–804.

[SU] C. Skinner, E. Urban, Sur les déformations p-adiques de certaines représentations automorphes, *J. Inst. Math. Jussieu* **5** (2006), 629–698.

[St] W. Stein, The Modular Forms Database: Tables,
http://modular.fas.harvard.edu/Tables/tables.html

[Tak] H. Takayanagi, Vector valued Siegel modular forms and their L-functions; application of a differential operator, *Japan J. Math.* **19** (1994), 251–297.

[Tay] R. Taylor, On the ℓ-adic cohomology of Siegel threefolds, *Invent. Math.* **114** (1993), 289–310.

[U] E. Urban, Selmer groups and the Eisenstein-Klingen ideal, *Duke Math. J.* **106** (2001), 485–525.

[vdG] G. van der Geer, Siegel Modular Forms and Their Applications, *in* The 1-2-3 of Modular Forms (J. H. Bruinier et. al.), 181–245, Springer-Verlag, Berlin Heidelberg, 2008.

[We1] R. Weissauer, Vektorwertige Siegelsche Modulformen kleinen Gewichts, *J. f. d. reine u. angew. Math.* **343** (1983), 184-202

[We2] R. Weissauer, Four dimensional Galois representations, *Astérisque* **302** (2005), 67–150.

[We3] R. Weissauer, Endoscopy for GSp(4) and the Cohomology of Siegel Modular Threefolds, Lecture Notes in Math., Vol. 1968, 368 pp., Springer, Berlin, 2009.

[Y1] H. Yoshida, Siegel's modular forms and the arithmetic of quadratic forms, *Invent. Math.* **60** (1980), 193–248.

[Y2] H. Yoshida, The action of Hecke operators on theta series, p. 197-238 in: Algebraic and topological theories - to the memory of Dr. T. Miyata -, 1985.

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