IDEALS GENERATED BY ADJACENT 2-MINORS

JÜRGEN HERZOG AND TAKAYUKI HIBI

ABSTRACT. Ideals generated by adjacent 2-minors are studied. First, the problem when such an ideal is a prime ideal as well as the problem when such an ideal possesses a quadratic Gröbner basis is solved. Second, we describe explicitly a primary decomposition of the radical ideal of an ideal generated by adjacent 2-minors, and challenge the question of classifying all ideals generated by adjacent 2-minors which are radical ideals. Finally, we discuss connectedness of contingency tables in algebraic statistics.

INTRODUCTION

Let $X = (x_{ij})_{\substack{i=1,\ldots,m \\ j=1,\ldots,m}}$ be an $m \times n$-matrix of indeterminates, and let K be an arbitrary field. The ideals of t-minors $I_t(X)$ in $K[X] = K[(x_{ij})_{\substack{i=1,\ldots,m \\ j=1,\ldots,m}}]$ are well understood. A standard reference for determinantal ideals are the lecture notes [2] by Bruns and Vetter. See also [1] for a short introduction to this subject. Determinantal ideals and the natural extensions of this class of ideals, including ladder determinantal ideals arise naturally in geometric contexts which partially explains the interest in them. One nice property of these ideals is that they are all Cohen–Macaulay prime ideals.

Motivated by applications to algebraic statistics one is lead to study ideals generated by an *arbitrary* set of 2-minors of X. We refer the interested reader to the article [3] of Diaconis, Eisenbud and Sturmfels where the encoding of the statistical problem to commutative algebra is nicely described. Here we just outline briefly some ideas presented in that paper. Let $\mathcal{B}$ be a subset of vectors of $\mathbb{Z}^n$. One defines the graph $G_{\mathcal{B}}$ whose vertex set is the set $\mathbb{N}^n$ of non-negative integer vectors. Two vectors $\mathbf{a}$ and $\mathbf{b}$ are connected by an edge of $G_{\mathcal{B}}$ if $\mathbf{a} - \mathbf{b} \in \pm\mathcal{B}$. One of the central problems is to describe the connected components of this graph. Here commutative algebra comes into play. One defines the ideal

$$I_{\mathcal{B}} = (\mathbf{x}^{\mathbf{b}_+} - \mathbf{x}^{\mathbf{b}_-} : \ \mathbf{b} \in \mathcal{B}),$$

where for a vector $\mathbf{a} \in \mathbb{Z}^n$, the vectors $\mathbf{a}_+, \mathbf{a}_- \in \mathbb{N}^n$ are the unique vectors with $\mathbf{a} = \mathbf{a}_+ - \mathbf{a}_-$. The crucial observation is the following [3, Theorem 1.1.]: two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{N}^n$ belong to the same component of $G_{\mathcal{B}}$ if and only if $\mathbf{x}^{\mathbf{a}} - \mathbf{x}^{\mathbf{b}} \in I_{\mathcal{B}}$. Hence the question arises how one can decide whether a specific binomial $f = \mathbf{x}^{\mathbf{a}} - \mathbf{x}^{\mathbf{b}}$ belongs to a given binomial ideal I. This is indeed a difficult problem, but often the following strategy yields necessary and sometimes necessary and sufficient conditions for f to

belong to I, which can be expressed in terms of feasible numerical conditions on the vectors $\mathbf{a}$ and $\mathbf{b}$. The idea is to write the given binomial ideal as an intersection $I = \bigcap_{k=1}^{r} J_k$ of ideals J_k. Then $f \in I$ if and only if $f \in J_k$ for all k. This strategy is useful only if each of the ideals J_k has a simple structure, so that it is possible to describe the conditions that guarantee that f belongs to J_k. A natural choice for such an intersection is a primary decomposition of I. By Eisenbud and Sturmfels [4] it is known that the primary components of I are again binomial ideals (in the more general sense that a binomial ideal may also contain monomials). In the case that I is a radical ideal the natural choice for the ideals J_k are the minimal prime ideals of I.

To be more specific we consider contingency tables with support in a subset $\mathcal{S} \subset [m] \times [n]$, where $\mathcal{S}$ is the set of certain unit boxes

$$\{(i,j), (i, j+1), (i+1, j), (i+1, j+1)\}$$

with $1 \leq i < m$ and $1 \leq j < n$. We call a set $T = \{a_{ij}: (i,j) \in \mathcal{S}\}$ of non-negative integers a contingency table with support in $\mathcal{S}$. From the statistical point of view it is of interest to generate random contingency tables with fixed row and column sum. The row sums of the table T are the sums $\sum_{j,\,(i,j)\in\mathcal{S}} a_{ij}$ and the column sums of T are the sums $\sum_{i,\,(i,j)\in\mathcal{S}} a_{ij}$.

Choosing $i_1 < i_2$ and $j_1 < j_2$ such that

$$Q = \{(i_1, j_1), (i_1, j_2), (i_2, j_1), (i_1, j_2)\} \subset \mathcal{S},$$

we may produce a new contingency table T' with same row and column sums by adding or subtracting the table $D(Q) = \{d_{ij}: (i,j) \in \mathcal{S}\}$, where $d_{ij} = 0$ if $(i,j) \notin Q$, $d_{i_1,j_1} = d_{i_2,j_2} = 1$ and $d_{i_1,j_2} = d_{i_2,j_1} = -1$, provided all entries of T' are again non-negative.

Given a set $\mathcal{D} = \{D_1, \ldots, D_r\}$ of tables as described in the preceding paragraph, we may start a random walk beginning with T by adding or subtracting in each step a randomly chosen table D_i. If in a step like this, which we call an adjacent move, the new table has non-negative entries we continue our walk, otherwise we go back one step and choose another D_i. The contingency tables which we can reach by this procedure from the given table T are called *connected* to T with respect to $\mathcal{D}$.

In this paper we are interested in walks where $\mathcal{D}$ consists of the tables $D(Q)$, where Q runs though all unit boxes in $\mathcal{S}$. By the above mentioned [3, Theorem 1.1] we are lead to study ideals generated by adjacent 2-minors, that is, minors of the form $x_{i,j}x_{i+1,j+1} - x_{i+1,j}x_{i,j+1}$. Following the general strategy described above it is desirable to understand the primary decomposition of such ideals, or at least their minimal prime ideals. Hoşten and Sullivant [8] describe in a very explicit way all the minimal prime ideals for the ideal generated by all adjacent 2-minors of an $m \times n$-matrix. In Theorem 3.3 we succeed in describing the minimal prime ideals of the ideal of a configuration of adjacent 2-minors under the mild assumption that this configuration is special, a concept which has first been introduced by Qureshi [9]. Special configurations include the case considered by Hoşten and Sullivant, but are much more general. Our description is not quite as explicit as that of Hoşten and Sullivant, but explicit enough to determine in each particular case all the minimal

prime ideals. Part of the result given in Theorem 3.3 can also be derived from [7, Corollary2.1] of Hoşten and Shapiro, since ideals generated by 2-adjacent minors are lattice basis ideals. Though the minimal prime ideals are known, the knowledge about the embedded prime ideals of an ideal generated by 2-adjacent minors is very little, let alone the knowledge on its primary decomposition. In [3] the primary decomposition of the ideal of all adjacent 2-minors of a 4×4-matrix is given, and in [7] that of a 3×5-matrix. It is hard to see a general pattern from these results.

Ideals generated by adjacent 2-minors tend to have a non-trivial radical, and are rarely prime ideals. In the first section of this paper we classify all ideals generated by adjacent 2-minors which are prime ideals. The result is described in Theorem 1.1. They are the ideals of adjacent 2-minors attached to a chessboard configuration with no 4-cycles.

One method to show that an ideal is a radical ideal, is to compute its initial ideal with respect to some monomial order. If the initial ideal is squarefree, then the given ideal is a radical ideal. In Section 2 we classify all ideals generated by adjacent 2-minors which have a quadratic Gröbner basis (Theorem 2.3). It turns out that these are the ideals of adjacent 2-minors corresponding to configurations whose components are monotone paths meeting is a suitable way. In particular, those ideals of adjacent 2-minors are radical ideals. In general the radical of an ideal of adjacent 2-minors attached to a special configuration can be naturally written as an intersections of prime ideals of relatively simple nature, see Theorem 3.2. These prime ideals are indexed by the so-called admissible sets. These are subsets of the set $\mathcal{S}$ which defines the configuration, and can be described in a purely combinatorial way.

In Section 4 we aim at classifying configurations whose ideal of adjacent 2-minors is a radical ideal. It is not so hard to see (cf. Proposition 4.2) that a connected special configuration whose ideal of adjacent 2-minors is a radical ideal should be a path or a cycle. Computations show that the cycles should have length at least 12. We expect that the ideal of adjacent 2-minors attached to any path is a radical ideal, and prove this in Theorem 4.3 under the additional assumption that the ideal has no embedded prime ideals.

In the last section of this paper we describe in detail what it means that special contingency tables are connected via adjacent moves.

1. Prime ideals generated by adjacent 2-minors

Let $X = (x_{ij})_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$ be a matrix of indeterminates, and let S be the polynomial ring over a field K in the variables x_{ij}. Let $\delta = [a_1, a_2 | b_1, b_2]$ be a 2-minor. The variables x_{a_i,b_j} are called the *vertices* and the sets $\{x_{a_1,b_1}, x_{a_1,b_2}\}]$, $\{x_{a_1,b_1}, x_{a_2,b_1}\}$, $\{x_{a_1,b_2}, x_{a_2,b_2}\}$ and $\{x_{a_2,b_1}, x_{a_2,b_2}\}$ the *edges* of the minor $[a_1, a_2 | b_1, b_2]$. The set of vertices of δ will be denoted by $V(\delta)$. The 2-minor $\delta = [a_1, a_2 | b_1, b_2]$ is called *adjacent* if $a_2 = a_1 + 1$ and $b_2 = b_1 + 1$.

Let $\mathcal{C}$ be any set of adjacent 2-minors. We call such a set also a configuration of adjacent 2-minors. We denote by $I(\mathcal{C})$ the ideal generated by the elements of $\mathcal{C}$. The set of vertices of $\mathcal{C}$, denoted $V(\mathcal{C})$, is the union of the vertices of its adjacent 2-minors.

Two distinct minors in $\delta, \gamma \in \mathcal{C}$ are called *connected* if there exist $\delta_1 \ldots, \delta_r \in \mathcal{C}$ such that $\delta = \delta_1$, $\gamma = \delta_r$, and δ_i and δ_{i+1} have a common edge.

Following Quereshi [9] we call a configuration of adjacent 2-minors *special*, if the following condition is satisfied: for any two adjacent 2-minors $\delta_1, \delta_2 \in \mathcal{C}$ which have exactly one vertex in common, there exists a $\delta \in \mathcal{C}$ which has a common edge with δ_1 and a common edge with δ_2. Any special configuration of adjacent 2-minors is a disjoint union of connected special configuration of adjacent 2-minors.

Any maximal subset D of $\mathcal{C}$ with the property that any two minors of D are connected, is called a *connected component of $\mathcal{C}$*. To $\mathcal{C}$ we attach a graph $G_{\mathcal{C}}$ as follows: the vertices of $G_{\mathcal{C}}$ are the connected components of $\mathcal{C}$. Let A and B be two connected components of C. Then there is an edge between A and B if there exists a minor $\delta \in A$ and a minor $\gamma \in B$ which have exactly one vertex in common. Note that $G_{\mathcal{C}}$ may have multiple edges.

A set of adjacent 2-minors is called a *chessboard configuration*, if any two minors of this set meet in at most one vertex. An example of a chessboard configuration is given in Figure 1. An ideal $I \subset S$ is called a *chessboard ideal* if it is generated by a chessboard configuration. Note that the graph $G_{\mathcal{C}}$ of a chessboard configuration is a simple bipartite graph.
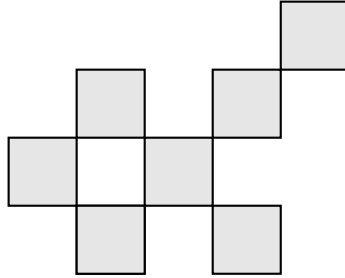

FIGURE 1.

Theorem 1.1. *Let I be an ideal generated by adjacent 2-minors. Then the following conditions are equivalent:*

(a) *I is a prime ideal.*
(b) *I is a chessboard ideal and $G_{\mathcal{C}}$ has no cycle of length 4.*

For the proof of this result we shall need some concepts related to lattice ideals.

Let $\mathcal{L} \subset \mathbb{Z}^n$ be a lattice. Let K be a field. The *lattice ideal* attached to $\mathcal{L}$ is the binomial ideal $I_{\mathcal{L}} \subset K[x_1, \ldots, x_n]$ generated by all binomials

$$\mathbf{x^a} - \mathbf{x^b} \quad \text{with} \quad \mathbf{a} - \mathbf{b} \in \mathcal{L}.$$

$\mathcal{L}$ is called saturated if for all $\mathbf{a} \in \mathbb{Z}^n$ and $c \in \mathbb{Z}$ such that $c\mathbf{a} \in \mathcal{L}$ it follows that $\mathbf{a} \in \mathcal{L}$. The lattice ideal $I_{\mathcal{L}}$ is a prime ideal if and only if $\mathcal{L}$ is saturated.

Let $\mathbf{v}_1, \ldots, \mathbf{v}_m$ be a basis of $\mathcal{L}$. Hoşten and Shapiro [7] call the ideal generated by the binomials $\mathbf{x}^{\mathbf{v}_i^+} - \mathbf{x}^{\mathbf{v}_i^-}$, $i = 1, \ldots, m$, a *lattice basis ideal* of $\mathcal{L}$. Here $\mathbf{v}^+$ denotes

the vector obtained from $\mathbf{v}$ by replacing all negative components of $\mathbf{v}$ by zero, and $\mathbf{v}^- = -(\mathbf{v} - \mathbf{v}^+)$.

Fischer and Shapiro [5], and Eisenbud and Sturmfels [4] showed

Proposition 1.2. *Let J be a lattice basis ideal of the saturated lattice $\mathcal{L} \subset \mathbb{Z}^n$. Then $J : (\prod_{i=1}^n x_i)^\infty = I_{\mathcal{L}}$.*

As a consequence of this proposition we obtain:

Lemma 1.3. *Let I be an ideal generated by adjacent 2-minors. Then I is a prime ideal if and only if all variables x_{ij} are nonzero divisors of S/I.*

Proof. It is known from [4] that the ideal generated by all adjacent 2-minors of X is a lattice basis ideal, and that the corresponding lattice ideal is just the ideal of all 2-minors of X. It follows that an ideal, such as I, which is generated by any set of adjacent 2-minors of X is again a lattice basis ideal and that its corresponding lattice $\mathcal{L}$ is saturated. Therefore its lattice ideal $I_{\mathcal{L}}$ is a prime ideal. Assume now that all variables x_{ij} are nonzero divisors of S/I. Then Proposition 1.2 implies that $I = I_{\mathcal{L}}$, so that I is a prime ideal. The converse implication is trivial. $\square$

For the proof of Theorem 1.1 we need the following two lemmata.

Lemma 1.4. *Let I be an ideal generated by adjacent 2-minors. For each of the minors we mark one of the monomials in the support as a potential initial monomial. Then there exists an ordering of the variables such that the marked monomials are indeed the initial monomials with respect to the lexicographic order induced by the given ordering of the variables.*

Proof. In general, suppose that, in the set $[N] = \{1, 2, \ldots, N\}$, for each pair $(i, i+1)$ an ordering either $i < i+1$ or $i > i+1$ is given. We claim that there is a total order $<$ on $[N]$ which preserves the given ordering. Working by induction on N we may assume that there is a total order $i_1 < \ldots < i_{N-1}$ on $[N-1]$ which preserve the given ordering for the pairs $(1, 2), \ldots, (N-2, N-1)$. If $N-1 < N$, then $i_1 < \ldots < i_{N-1} < N$ is a required total order $<$ on $[N]$. If $N-1 > N$, then $N < i_1 < \ldots < i_{N-1}$ is a required total order $<$ on $[N]$.

The above fact guarantees the existence of an ordering of the variables such that the marked monomials are indeed the initial monomials with respect to the lexicographic order induced by the given ordering of the variables, as can be seen in Example 1.5. $\square$

Example 1.5. In Figure 2 each of the squares represents an adjacent 2-minor, and the diagonal in each of the squares indicates the marked monomial of the corresponding 2-minor. For any lexicographic order for which the marked monomials in Figure 2 are the initial monomials the numbering of the variables in the top row must satisfy the following inequalities

$$1 < 2 > 3 < 4 > 5 > 6.$$

By using the general strategy given in the proof of Lemma 1.4 we relabel the top row of the vertices by the numbers 1 up to 6, and proceed in the same way in the next rows. The final result can be seen in Figure 3
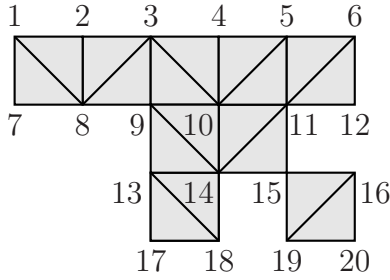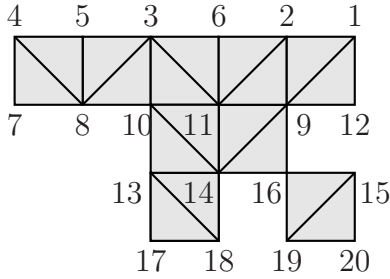
FIGURE 2.

FIGURE 3.

We call a vertex of a 2-minor in $\mathcal{C}$ *free*, if it does not belong to any other 2-minor of $\mathcal{C}$, and we call the 2-minor $\delta = ad - bc$ *free*, if either (i) a and d are free, or (ii) b and c are free.

Lemma 1.6. *Let $\mathcal{C}$ be a chessboard configuration with $|\mathcal{C}| \geq 2$. Suppose $G_{\mathcal{C}}$ does not contain a cycle of length 4. Then the $G_{\mathcal{C}}$ contains at least two free 2-minors.*

Proof. We may assume there is at least one non-free 2-minor in $\mathcal{C}$, say $\delta = ad - bc$. Since we do not have a cycle of length 4, there exists a sequence of 2-minors in $\mathcal{C}$ as indicated in Figure 4. Then the left-most and the right-most 2-minor of this sequence is free. $\square$
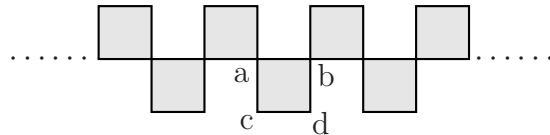

FIGURE 4.

Proof of Theorem 1.1. (a)$\Rightarrow$ (b): Let $\delta, \gamma \in I$ be two adjacent 2-minors which have an edge in common. Say, $\delta = ae - bd$ and $\gamma = bf - ce$. Then $b(af - cd) \in I$, but neither b nor $af - cd$ belongs to I. Therefore I must be a chessboard ideal.

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Suppose $G_{\mathcal{C}}$ contains a cycle of length 4. Then there exist in I adjacent two minors $\delta_1 = ae - bd$, $\delta_2 = ej - fi$, $\delta_3 = hl - ik$ and $\delta_4 = ch - dg$. Then $h(bcjk - afgl) \in I$, but neither h nor $bcjk - afgl$ belongs to I.

(b)$\Rightarrow$ (a): By virtue of Lemma 1.3 what we must prove is that all variables x_{ij} are nonzero divisors of S/I. Let $\mathcal{G}$ be the set of generating adjacent 2-minors of I. We may assume that $|\mathcal{G}| \geq 2$. Fix an arbitrary vertex x_{ij}. We claim that for each of the minors in $\mathcal{G}$ we may mark one of the monomials in the support as a potential initial monomial such that the variable x_{ij} appears in none of the potential initial monomials and that any two potential initial monomials are relatively prime.

Lemma 1.6 says that there exist at least two a free adjacent 2-minors in $\mathcal{G}$. Let $\delta = ad - bc$ be one of them and assume that a and d are free vertices of δ. We may assume that $x_{ij} \neq a$ and $x_{ij} \neq d$. Let $\mathcal{G}' = \mathcal{G} \setminus \{\delta\}$. By assumption of induction, for each of the minors of $\mathcal{G}'$ we may mark one of the monomials in the support as a potential initial monomial such that the variable x_{ij} appears in none of the potential initial monomials and that any two potential initial monomials are relatively prime. Then these markings together with the marking ad are the desired markings of the elements of $\mathcal{G}$.

According to Lemma 1.4 there exists an ordering of the variables such that with respect to the lexicographic order induced by this ordering the potential initial monomials become the initial monomials. Since initial monomials are relatively prime, it follows that $\mathcal{G}$ is a Gröbner basis of I, and since x_{ij} does not divide any initial monomial of an element in $\mathcal{G}$ it follows that x_{ij} that x_{ij} is a nonzero divisor of $S/\operatorname{in}(I)$. But then x_{ij} is a nonzero divisor of S/I as well. $\qquad\square$

2. Ideals generated by adjacent 2-minors with quadratic Gröbner basis

A configuration $\mathcal{P}$ of adjacent 2-minors is called a *path*, if there exists ordering $\delta_1, \ldots, \delta_r$ of the elements of $\mathcal{P}$ such that for all

$$\delta_j \cap \delta_i \subset \delta_{i-1} \cap \delta_i \quad \text{for all} \quad j < i, \quad \text{and} \quad \delta_{i-1} \cap \delta_i \quad \text{is an edge of } \delta_i.$$

A path $\mathcal{P}$ with path ordering $\delta_1, \ldots, \delta_r$ where $\delta_i = [a_i, a_i+1 | b_i, b_i+1]$ for $i = 1, \ldots, r$ is called *monotone*, if the sequences of integers $a_1, \cdots, a_r$ and $b_1, \ldots, b_r$ are monotone sequences. The monotone path $\mathcal{P}$ is called *monotone increasing (decreasing)* if the sequence $b_1, \ldots, b_r$ is increasing (decreasing). We define the *end points* of $\mathcal{P}$ to be (a_1, b_1) and $(a_r + 1, b_r + 1)$ if $\mathcal{P}$ is monotone increasing, and to be $(a_1, b_1 + 1)$ and $(a_r + 1, b_r)$ if $\mathcal{P}$ is monotone decreasing. If for $\mathcal{P}$ we have $a_1 = a_2 = \cdots = a_r$, or $b_1 = b_2 = \cdots = b_r$, then we call $\mathcal{P}$ a *line path*. Notice that a line graph is both monotone increasing and monotone decreasing.

Let $\delta = ad - bc$ be an adjacent 2-minor as shown in Figure 5. Then we call the monomial ad the *diagonal* of δ and the monomial bc the *anti-diagonal* of δ.

Lemma 2.1. *Let $\mathcal{P}$ be a monotone increasing (decreasing) path of 2-minors. Then for any monomial order $<$ for which $I(\mathcal{P})$ has a quadratic Gröbner basis, the initial monomials of the generators are all diagonals (anti-diagonals).*
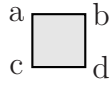
FIGURE 5.

Proof. Suppose first that $\mathcal{P}$ is a line path. If $I(\mathcal{P})$ has a quadratic Gröbner basis, then initial monomials of the 2-minors of $\mathcal{P}$ are all diagonals or all anti-diagonals, because otherwise there would be two 2-minors δ_1 and δ_2 in $\mathcal{P}$ connected by an edge such that $\text{in}(\delta_1)$ is a diagonal and $\text{in}(\delta_2)$ is an anti-diagonal. The S-polynomial of δ_1 and δ_2 is a binomial of degree 3 which belongs to the reduced Gröbner basis of I, a contradiction. If all initial monomials of the 2-minors in $\mathcal{P}$ are diagonals, we interpret $\mathcal{P}$ as a monotone increasing path, and if all initial monomials of the 2-minors in $\mathcal{P}$ are anti-diagonals, we interpret $\mathcal{P}$ as a monotone decreasing path.

Now assume that $\mathcal{P}$ is not a line path. We may assume that $\mathcal{P}$ is monotone increasing. (The argument for a monotone decreasing path is similar). Then, since $\mathcal{P}$ is not a line path it contains one of the following sub-paths displayed in Figure 6.

FIGURE 6.

For both sub-paths the initial monomials must be diagonals, otherwise $I(\mathcal{P})$ would not have a quadratic Gröbner basis. Then as in the case of line paths one sees that all the other initial monomials of $\mathcal{P}$ must be diagonals. $\square$

A configuration of adjacent 2-minors which are of the form shown in Figure 7, or are obtained by rotation from them, are called *square*, *pin* and *saddle*, respectively.
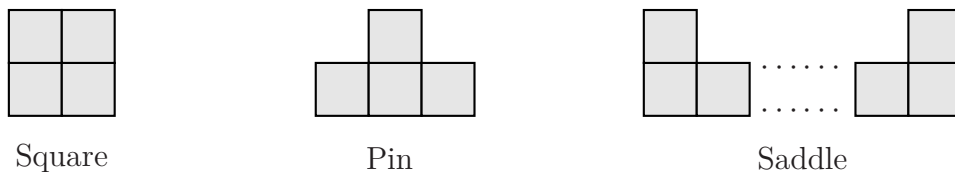

Square   Pin   Saddle

FIGURE 7.

Lemma 2.2. *Let $\mathcal{A}$ be a connected configuration of adjacent 2-minors. Then $\mathcal{A}$ is a monotone path if and only if $\mathcal{A}$ contains neither a square nor a pin nor a saddle.*

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Proof. Assume that $\mathcal{A} = \delta_1, \delta_2, \ldots, \delta_r$ with $\delta_i = [a_i, a_i + 1|b_i, b_i + 1]$ for $i = 1, \ldots, r$ is a monotone path. Without loss of generality we may assume the both sequences $a_1, \cdots, a_r$ and $b_1, \ldots, b_r$ are monotone increasing. We will show by induction on r that it contains no square, no pin and no saddle. For $r = 1$ the statement is obvious. Now let us assume that the assertion is true for $r - 1$. Since $\mathcal{A}' = \delta_1, \delta_2, \cdots, \delta_{r-1}$ is monotone increasing it follows that the coordinates of the minors δ_i for $i = 1, \ldots, r - 1$ sit inside the rectangle R with corners $(a_1, b_1), (a_{r-1} + 1, b_1), (a_{r-1} + 1, b_{r-1} + 1), (a_1, b_{r-1} + 1)$, and $\mathcal{A}'$ has no square, no pin and no saddle. Since $\mathcal{A}$ is monotone increasing, $\delta_r = [a_{r-1}, a_{r-1}+1|b_{r-1}+1, b_{r-1}+2]$ or $\delta_r = [a_{r-1}+1, a_{r-1}+2|b_{r-1}, b_{r-1}+1]$. It follows that if $\mathcal{A}$ would contain a square, a pin or a saddle, then the coordinates of one the minors δ_i, $i = 1, \ldots, r - 1$ would not be inside the rectangle R.

Conversely suppose that $\mathcal{A}$ contains no square, no pin and no saddle. Then $\mathcal{A}'$ contains no square, no pin and no saddle as well. Thus arguing by induction on r, we may assume that $\mathcal{A}'$ is a monotone path. Without loss of generality we may even assume that $a_1 \leq a_2 \leq \cdots \leq a_{r-1}$ and $b_1 \leq b_2 \leq \cdots \leq b_{r-1}$. Now let δ_r be connected to δ_i (via an edge). If $i \in \{2, \ldots, r - 2\}$, then $\mathcal{A}$ contains a square, a pin or a saddle which involves δ_r, a contradiction. If $i = 1$ or $i = r - 1$, and $\mathcal{A}$ is not monotone, then contains $\mathcal{A}$ contains a square or a saddle involving δ_r. $\quad\square$

With the notation introduced we have

Theorem 2.3. *Let $\mathcal{C}$ be a configuration of adjacent 2-minors. Then the following conditions are equivalent:*

(a) *$I(\mathcal{C})$ has a quadratic Gröbner basis with respect to the lexicographic order induced by a suitable order of the variables.*

(b) (i) *Each connected component of $\mathcal{C}$ is a monotone path.*

    (ii) *If $\mathcal{A}$ and $\mathcal{B}$ are components of $\mathcal{C}$ which meet in a vertex which is not an end point of $\mathcal{A}$ or not and end point of $\mathcal{B}$, and if $\mathcal{A}$ is monotone increasing, then $\mathcal{B}$ must be monotone decreasing, and vice versa.*

(c) *The initial ideal of $I(\mathcal{C})$ with respect to the lexicographic order induced by a suitable order of the variables is a complete intersection.*

Proof. (a) $\Rightarrow$ (b): (i) Suppose there is component $\mathcal{A}$ of $\mathcal{C}$ which is not a monotone path. Then, according to Lemma 2.2, $\mathcal{A}$ contains a square, a pin or a saddle. In all three cases, no matter how we label the vertices of the component $\mathcal{A}$, it will contain, up to a rotation or reflection, two adjacent 2-minors with leading terms as indicated in Figure 8.
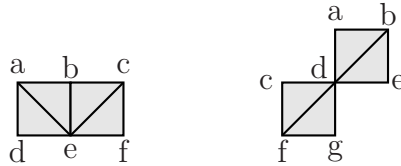

FIGURE 8.

In the first case the S-polynomial of the two minors is $abf - bcd$ and in the second case it is $aef - bcg$. We claim that in both cases these binomials belong to the reduced Gröbner basis of $I(\mathcal{C})$, which contradicts our assumption (a).

Indeed, first observe that the adjacent 2-minors generating the ideal $I(\mathcal{C})$ is the unique minimal set of binomials generating $I(\mathcal{C})$. Therefore, the initial monomials of degree 2 are exactly the initial monomials of these binomials. Suppose now that $abf - bcd$ does not belong to the reduced Gröbner basis of I, then one of the monomials ab, af or bf must be the leading of an adjacent 2-minor, which is impossible. In the same way one argues in the second case.

(ii) Assume $\mathcal{A}$ and $\mathcal{B}$ have a vertex c in common. Then c must be a corner of $\mathcal{A}$ and $\mathcal{B}$, that is, a vertex which belongs to exactly one 2-minor of $\mathcal{A}$ and exactly one 2-minor of $\mathcal{B}$, see Figure 9.
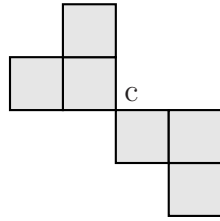

FIGURE 9.

If for both components of the initial monomials are the diagonals (anti-diagonals), then the S-polynomial of the 2-minor in $\mathcal{A}$ with vertex c and the 2-minor of $\mathcal{B}$ with vertex c is a binomial of degree three whose initial monomial is not divisible by any initial monomial of the generators of $\mathcal{C}$, unless c is an end point of both $\mathcal{A}$ and $\mathcal{B}$. Thus the desired conclusion follows from Lemma 2.1.

(b) $\Rightarrow$ (c): The condition (b) implies that any pair of initial monomials of two distinct binomial generators of $I(\mathcal{C})$ are relatively prime. Hence the initial ideal is a complete intersection.

(c) $\Rightarrow$ (d): Since the initial monomial of the 2-minors generating $I(\mathcal{C})$ belong to any reduced Gröbner basis of $I(\mathcal{C})$, they must form a regular sequence. This implies that S-polynomials of any two generating 2-minors of $I(\mathcal{C})$ reduce to 0. Therefore $I(\mathcal{C})$ has a quadratic Gröbner basis. $\qquad\square$

Corollary 2.4. *Let $\mathcal{C}$ be a configuration satisfying the conditions of Theorem 2.3(b). Then $I(\mathcal{C})$ is a radical ideal generated by a regular sequence.*

Proof. Let $\mathcal{C} = \delta_1, \ldots, \delta_r$. By Theorem 2.3 there exist a monomial order $<$ such that $\mathrm{in}_<(\delta_1), \ldots, \mathrm{in}_<(\delta_r)$ is a regular sequence. It follows that $\delta_1, \ldots, \delta_r$ is a regular sequence. Since the initial monomials are squarefree and form a Gröbner basis of $I(\mathcal{C})$ it follows that $I(\mathcal{C})$ is a radical ideal, see for example [6, Proof of Corollary 2.2]. $\qquad\square$

To demonstrate Theorem 2.3 we consider the following two examples displayed in Figure 10
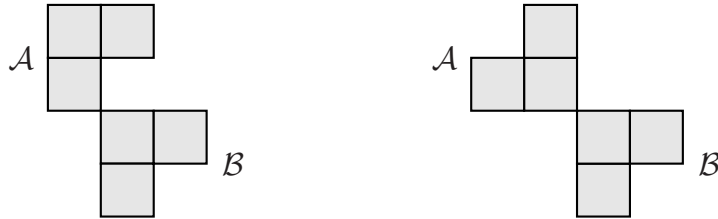
FIGURE 10.

In both examples the component $\mathcal{A}$ and the component $\mathcal{B}$ are monotone increasing paths. In the first example $\mathcal{A}$ and $\mathcal{B}$ meet in a vertex which is an end point of $\mathcal{A}$, therefore condition (b)(ii) of Theorem 2.3 is satisfied, and the ideal $I(\mathcal{A} \cup \mathcal{B})$ has a quadratic Gröbner basis. However in the second example $\mathcal{A}$ and $\mathcal{B}$ meet in a vertex which is not and end point of $\mathcal{A}$ and not and end point of $\mathcal{B}$. Therefore condition (b)(ii) of Theorem 2.3 is not satisfied, and the ideal $I(\mathcal{A} \cup \mathcal{B})$ does not have a quadratic Gröbner basis for the lexicographic order induced by any order of the variables.

3. Minimal prime ideals of special configurations of adjacent 2-minors

Let $\mathcal{C}$ be a connected configuration of adjacent 2-minors. In this section we want to describe a primary decomposition of $\sqrt{I(\mathcal{C})}$. For this purpose we have to introduce some terminology: let $\mathcal{C} = \delta_1, \delta_2, \ldots, \delta_r$ be an arbitrary configuration of adjacent 2-minors. A subset W of the vertex set of $\mathcal{C}$ is called *admissible*, if for each index i either $W \cap V(\delta_i) = \emptyset$ or $W \cap V(\delta_i)$ contains an edge of δ_i. For example, the admissible sets of the configuration shown in Figure 11
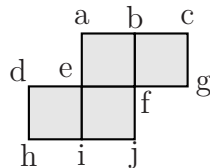

FIGURE 11.

are the following

$$\emptyset, \{c, g\}, \{d, h\}, \{a, e, i\}, \{b, f, j\}, \{a, b, c\}, \ldots, \{a, b, c, d, e, f, g, h, i, j\}.$$

Let $W \subset V(\mathcal{C})$ be an admissible set. We define an ideal $P_W(\mathcal{C})$ containing $I(\mathcal{C})$ as follows: the generators of $P_W(\mathcal{C})$ are the variables belonging to W and all 2-minors $\delta = [a_1, a_2 | b_1, b_2]$ (not necessarily adjacent) such that all vertices (i, j) with $a_1 \leq i \leq a_2$ and $b_1 \leq j \leq b_2$ belong to $V(\mathcal{C}) \setminus W$.

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Note that $P_W(\mathcal{C}) = (W, P_\emptyset(\mathcal{C}'))$ where $\mathcal{C}' = \{\delta \in \mathcal{C}: V(\delta) \cap W = \emptyset\}$. We denote by $\mathcal{G}(\mathcal{C}')$ the set of 2-minors of $P_\emptyset(\mathcal{C}')$ and call it the set of *inner* 2-*minors* of $\mathcal{C}'$.

For example, if we take the configuration displayed in Figure 11, then

$$\begin{aligned}
P_\emptyset(\mathcal{C}) &= (af - be, aj - bi, ej - fi, ag - ce, bg - cf, di - eh, dj - fh), \\
P_{\{d,h\}}(\mathcal{C}) &= (d, h, af - be, aj - bi, ej - fi, ag - ce, bg - cf).
\end{aligned}$$

Lemma 3.1. *Let $\mathcal{C}$ be a special configuration of adjacent 2-minors. Then for any admissible set $W \subset V(\mathcal{P})$ the ideal $P_W(\mathcal{C})$ is a prime ideal.*

Proof. Let $P_W(\mathcal{C}) = P_W(\mathcal{C}) = (W, P_\emptyset(\mathcal{C}'))$, where $\mathcal{C}' = \{\delta \in \mathcal{C}: V(\delta) \cap W = \emptyset\}$.

Note that $\mathcal{C}'$ is again a special configuration of 2-adjacent minors. Indeed, let $\delta_1, \delta_2 \in \mathcal{C}'$ be two adjacent 2-minors with exactly one common vertex. Since $\mathcal{C}$ is special, there exists $\delta \in \mathcal{C}$ which has a common edge with δ_1 and a common edge with δ_2. Since $\delta \notin \mathcal{C}'$, the set W contains an edge of δ. This implies that $W \cap V(\delta_1) \neq \emptyset$ or $W \cap V(\delta_2) \neq \emptyset$, contradicting the fact that $V(\mathcal{C}') \cap W = \emptyset$.

By a result of Qureshi [9], $P_\emptyset(\mathcal{C}')$ is a prime ideal. Therefore $P_W(\mathcal{C})$ is a prime ideal. $\qquad\square$

Theorem 3.2. *Let $\mathcal{C}$ be a special configuration of adjacent 2-minors. Then*

$$\sqrt{I(\mathcal{C})} = \bigcap_W P_W(\mathcal{C}),$$

where the intersection is taken over all admissible sets $W \subset V(\mathcal{C})$.

Proof. We show that if P is a minimal prime ideal of $I(\mathcal{C})$, then there exists an admissible set $W \subset V(\mathcal{C})$ such that $P = P_W(\mathcal{C})$.

So now let P be any minimal prime ideal of $I(\mathcal{C})$, and let W be the set of variables among the generators of P. We claim that W is admissible. Indeed, suppose that $W \cap V(\delta) \neq \emptyset$ for some adjacent 2-minor of $\mathcal{C}$. Say, $\delta = ad - bc$ and $a \in W$. Then $bc \in P$. Hence, since P is a prime ideal, it follows that $b \in P$ or $c \in P$. Thus W contains the edge $\{a, c\}$ or the edge $\{a, b\}$ of δ.

Since $I(\mathcal{C}) \subset P$ it follows that $(W, I(\mathcal{C})) \subset P$. Observe that $(W, I(\mathcal{C})) = (W, I(\mathcal{C}'))$, where $W \cap V(\mathcal{C}') = \emptyset$ and $\mathcal{C}'$ is again a special configuration, see the proof of Lemma 3.1. Modulo W we obtain a minimal prime ideal $\bar{P}$, which contains no variables, of the ideal $I(\mathcal{C}')$.

By the result of Qureshi [9] the ideal $P_\emptyset(\mathcal{C}')$ is a prime ideal containing $I(\mathcal{C}')$. Thus the assertion of the theorem follows once we have shown that $P_\emptyset(\mathcal{C}') \subset \bar{P}$.

Since $P_\emptyset(\mathcal{C}')$ is generated by the union of the set of 2-minors of certain $r \times s$-matrices, it suffices to show that if P is a prime ideal having no variables among its generators and containing all adjacent 2-minors of the $r \times s$-matrix X, then it contains all 2-minors of X. In order to prove this, let $\delta = [a_1, a_2 | b_1, b_2]$ be an arbitrary 2-minor of X. We prove that $\delta \in P$ by induction on $(a_2 - a_1) + (b_2 - b_1)$. For $(a_2 - a_1) + (b_2 - b_1) = 2$, this is the case by assumption. Now let $(a_2 - a_1) + (b_2 - b_1) > 2$. We may assume that $a_2 - a_1 > 1$. Let $\delta_1 = [a_1, a_2 - 1 | b_1, b_2]$ and $\delta_2 = [a_2 - 1, a_2 | b_1, b_2]$. Then $x_{a_2 - 1, b_1} \delta = x_{a_2, b_1} \delta_1 + x_{a_1, b_1} \delta_2$. Therefore, by induction hypothesis $x_{a_2 - 1, b_1} \delta \in P$. Since P is a prime ideal, and $x_{a+k-1,1} \notin P$ it follows that $\delta \in P$, as desired. $\qquad\square$

In general it seems to be pretty hard to find the primary decomposition for ideals generated by adjacent 2-minors. This seems to be even difficult for ideals described in Theorem 2.3. For example, the primary decomposition (computed with the help of Singular) of the ideal $I(\mathcal{C})$ of adjacent 2-minors shown in Figure 12 is the following:
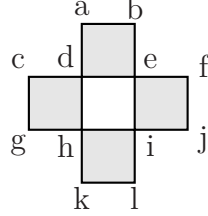

FIGURE 12.

$$
\begin{aligned}
I(\mathcal{C}) &= (ae - bd, ch - dg, ej - fi, hl - ik) \\
&= (ik - hl, fi - ej, dg - ch, bd - ae, bcjk - afgl) \cap (d, e, h, i).
\end{aligned}
$$

It turns out that $I(\mathcal{C})$ is a radical ideal. On the other hand, if we add the minor $di - eh$ we get a connected configuration $\mathcal{C}'$ of adjacent 2-minors. The ideal $I(\mathcal{C}')$ is not radical, because it contains a pin, see Proposition 4.2. Indeed, one has

$$
\begin{aligned}
\sqrt{I(\mathcal{C}')} &= (ae - bd, ch - dg, ej - fi, hl - ik, di - eh, fghl - chjl, \\
&\quad\ bfhl - aejl, bchk - achl, bcfh - acej)
\end{aligned}
$$

Applying Proposition 3.3, we get

$$
\begin{aligned}
\sqrt{I(\mathcal{C}')} &= (ae - bd, ch - dg, ej - fi, hl - ik, di - eh, fghl - chjl, \\
&\quad\ bfhl - aejl, bchk - achl, bcfh - acej) \\
&= (-ik + hl, -fi + ej, -ek + dl, -fh + dj, -eh + di, -fg + cj, -eg + ci, \\
&\quad\ -dg + ch, -bk + al, -bh + ai, -bd + ae) \\
&\cap\ (d, e, h, i) \cap (a, d, h, i, j) \cap (d, e, f, h, k) \cap (c, d, e, i, l) \cap (b, e, g, h, i) \\
&\cap\ (a, d, h, k, ej - fi) \cap (c, d, e, f, hl - ik) \cap (b, e, i, l, ch - dg) \\
&\cap\ (g, h, i, j, ae - bd).
\end{aligned}
$$

The presentation of $\sqrt{I(\mathcal{C})}$ as an intersection of prime ideals as given in Theorem 3.2 is usually not irredundant. In order to obtain an irredundant intersection, we have to identify the minimal prime ideals of $I(\mathcal{C})$ among the prime ideals $P_W(\mathcal{C})$.

Theorem 3.3. *Let $\mathcal{C}$ be a special configuration of adjacent 2-minors, and let $V, W \subset V(\mathcal{C})$ be admissible sets of $\mathcal{C}$, and let $P_V(\mathcal{C}) = (V, \mathcal{G}(\mathcal{C}'))$ and $P_W(\mathcal{C}) = (W, \mathcal{G}(\mathcal{C}''))$, as given in Lemma 3.1. Then*

(a) *$P_V(\mathcal{C}) \subset P_W(\mathcal{C})$ if and only if $V \subset W$, and for all elements*

$$
\delta \in \mathcal{G}(\mathcal{C}') \setminus \mathcal{G}(\mathcal{C}'')
$$

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one has that $W \cap V(\delta)$ contains an edge of δ.

(b) *$P_W(\mathcal{C}) = (W, \mathcal{G}(\mathcal{C}''))$ is a minimal prime ideal of $I(\mathcal{C})$ if and only if for all admissible subsets $V \subset W$ with $P_V(\mathcal{C}) = (V, \mathcal{G}(\mathcal{C}'))$ there exists*

$$\delta \in \mathcal{G}(\mathcal{C}') \setminus \mathcal{G}(\mathcal{C}'')$$

such that the set $W \cap V(\delta)$ does not contain and edge of δ.

Proof. (a) Suppose that $P_V(\mathcal{C}) \subset P_W(\mathcal{C})$. The only variables in $P_W(\mathcal{C})$ are those belonging to W. This shows that $V \subset W$. The inclusion $P_V(\mathcal{C}) \subset P_W(\mathcal{C})$ implies that $\delta \in (W, \mathcal{G}(\mathcal{C}''))$ for all $\delta \in \mathcal{G}(\mathcal{C}')$. Suppose $W \cap V(\delta) = \emptyset$. Then δ belongs to $P_\emptyset(\mathcal{C}'') = (\mathcal{G}(\mathcal{C}''))$. Let $f = u - v \in \mathcal{G}(\mathcal{C}'')$. Neither u now v appears in another element of $\mathcal{G}(\mathcal{C}'')$. Therefore any binomial of degree 2 in $P_\emptyset(\mathcal{C}'')$ belongs to $\mathcal{G}(\mathcal{C}'')$. In particular, $\delta \in \mathcal{G}(\mathcal{C}'')$, a contradiction. Therefore, $W \cap V(\delta) \neq \emptyset$. Suppose that $W \cap V(\delta)$ does not contain an edge of $\delta = ad - bc$. We may assume that $a \in W \cap V(\delta)$. Then, since $\delta \in P_W(\mathcal{C})$, it follows that $bc \in P_W(\mathcal{C})$. Since $P_W(\mathcal{C})$ is a prime ideal, we conclude that $b \in P_W(\mathcal{C})$ or $c \in P_W(\mathcal{C})$. Then $b \in W$ or $c \in W$ and hence either the edge $\{a, b\}$ or the edge $\{a, c\}$ belongs to $W \cap V(\delta)$.

The 'if' part of statement (a) is obvious.

(b) is a simple consequence of Theorem 3.2 and statement (a). □

In Figure 13 we display all the minimal prime ideals $I(\mathcal{P})$ for the path $\mathcal{P}$ shown in Figure 11. The fat dots mark the admissible sets and the dark shadowed areas, the regions where the inner 2-minors have to be taken.
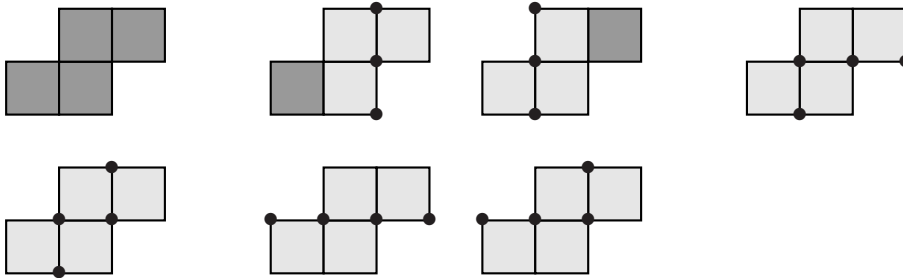

FIGURE 13.

4. SPECIAL CONFIGURATIONS WHICH ARE RADICAL

This section is devoted to study special configuration of adjacent 2-minors $\mathcal{C}$ for which $I(\mathcal{C})$ is a radical ideal. Any special configuration $\mathcal{C}$ is a disjoint union $\bigcup_{i=1}^{k} \mathcal{C}^i$ of connected special configurations of adjacent 2-minors. It follows that $I(\mathcal{C})$ is radical if and only if each $I(\mathcal{C}^i)$ is radical. Thus when we discuss the radical property we may always assume that $\mathcal{C}$ is connected.

We call a configuration $\mathcal{C}$ of adjacent 2-minors a *cycle*, if each for each $\delta \in \mathcal{C}$ there exist exactly two $\delta_1, \delta_2 \in \mathcal{C}$ such that δ and δ_1 have a common edge and δ and δ_2 have a common edge.

Lemma 4.1. *Let $\mathcal{C}$ be a connected special configuration which does not contain a pin. Then $\mathcal{C}$ is a path or a cycle.*

Proof. If $\mathcal{C}$ does not contain a pin, then for each adjacent 2-minor $\delta \in \mathcal{C}$ there exists at most two adjacent 2-minors in $\mathcal{C}$ which have a common edge which have a common edge with δ. Thus, if $\mathcal{C}$ is not a cycle but connected, there exists $\delta_1, \delta_2 \in \mathcal{C}$ such that δ_1 has a common edge only with δ_1. Now in the configuration $\mathcal{C}' = \mathcal{C} \setminus \{\delta_1\}$ the element δ_2 has at most one edge in common with another element of $\mathcal{C}'$. If δ_2 has no edge in common with another element of $\mathcal{C}'$, then $\mathcal{C} = \{\delta_1, \delta_2\}$. Otherwise, continuing this argument, a simple induction argument yields the desired conclusion. $\square$

Proposition 4.2. *Let $\mathcal{C}$ be a connected special configuration of adjacent 2-minors. If $I(\mathcal{C})$ is a radical ideal, then $\mathcal{C}$ is a path or a cycle.*

Proof. By Lemma 4.1 it is enough to prove that $\mathcal{C}$ does not contain a pin. Suppose $\mathcal{C}$ contains the pin $\mathcal{C}'$ as in Figure 14. Then $q = acej - bcfh \notin I(\mathcal{C}')$ but $q^2 \in I(\mathcal{C}') \subset$
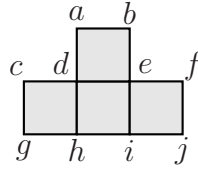

FIGURE 14.

$I(\mathcal{C})$. We consider two cases. In the first case suppose that the adjacent 2-minors $kd - ac$ and $bf - le$ do not belong to $\mathcal{C}$, see Figure 15.
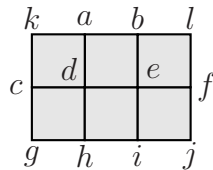

FIGURE 15.

Then $q \notin (I(\mathcal{C}), W)$ where W is the set of vertices which do not belong to $\mathcal{C}'$. It follows that $q \notin I(\mathcal{C})$. In the second case we may assume that $ac - kd \in \mathcal{C}$. Let $\mathcal{C}''$ be the configuration with the adjacent 2-minors $kd - ac, ae - bd, ch - dg, di - eh$. Then $r = kdi - aeg \notin I(\mathcal{C}'')$ but $r^2 \in I(\mathcal{C}'') \subset I(\mathcal{C})$. Then $r \notin (I(\mathcal{C}), V)$ where V is

the set of vertices in $\mathcal{C}$ which do not belong to $\mathcal{C}'$. It follows that $r \notin I(\mathcal{C})$. Thus in both cases we see that $I(\mathcal{C})$ is not a radical ideal. $\qquad\square$

For the cycle $\mathcal{C}$ displayed in Figure 16 the ideal $I(\mathcal{C})$ is not radical. Indeed we have $f = b^2 hino - abhjno \notin I(\mathcal{C})$, but $f^2 \in I(\mathcal{C})$. By computational evidence, we expect that the ideal of adjacent 2-minors of a cycle is a radical ideal if and only if the length of the cycle is ≥ 12. On the other hand, if $\mathcal{P}$ is a monotone path, we know
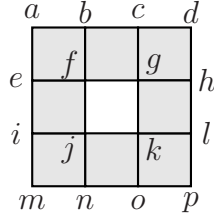

FIGURE 16.

from Theorem 2.3 that $I(\mathcal{P})$ has a squarefree initial ideal. This implies that $I(\mathcal{P})$ is a radical ideal. More generally, we expect that ideal of adjacent 2-minors of a path $\mathcal{P}$ is always a radical ideal, and prove this under the assumption that the ideal $I(\mathcal{P})$ has no embedded prime ideals. In [3, Theorem 4.2] the primary decomposition of the ideal of adjacent 2-minors of a 4×4-matrix is given, from which it can be seen that in general the ideal of adjacent 2-minors of a special configuration may have embedded prime ideals.

Theorem 4.3. *Let $\mathcal{P}$ be path, and suppose that $I(\mathcal{P})$ has no embedded prime ideals. Then $I(\mathcal{P})$ is a radical ideal.*

The proof will require several steps.

Lemma 4.4. *Let $I \subset S$ be an ideal, and let $a, b \in S$ such that a is a nonzero divisor modulo (b, I). Then*
$$(ab, I) = (a, I) \cap (b, I).$$

Proof. Obviously one has $(ab, I) \subset (a, I) \cap (b, I)$. Conversely, let $f \in (a, I) \cap (b, I)$. Then
$$f = ag_1 + h_1 = bg_2 + h_2 \quad \text{with} \quad g_1, g_2 \in S \quad \text{and} \quad h_1, h_2 \in I.$$
Therefore, $ag_1 \in (b, I)$. Since a is a nonzero divisor modulo (b, I), it follows that $g_1 = cb + h$ for some $c \in S$ and $h \in I$. Hence we get that $f = ag_1 + h_1 = a(cb + h) + h_1 = abc + (ah + h_1)$. Thus $f \in (ab, I)$ $\qquad\square$

Lemma 4.5. *Let $\mathcal{P} = \delta_1, \delta_2, \ldots, \delta_r$ be a path which is not a line, and let $i > 1$ be the smallest index for which δ_i has a free vertex c (which we call a corner of the path). Let a be the free vertex of δ_1 whose first or second coordinate coincides with that of c, see Figure 17. Then a does not belong to any minimal prime ideal of $I(\mathcal{P})$.*

In particular, if $I(\mathcal{P})$ has no embedded prime ideals, the element a is a nonzero divisor of $S/I(\mathcal{P})$.

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Proof. We may assume that, like in Figure 17, the first 2-adjacent minors up to the first corner form a horizontal path. Let W be an admissible set with $a \in W$. In our discussion we refer to the notation given in Figure 17. Then, since W is admissible, we have $b \in W$ or $c \in W$.

First suppose that $c \in W$. If $W = \{a, c\}$, then $P_\emptyset(\mathcal{P})$ is a proper subset of $P_W(\mathcal{P})$, and so $P_W(\mathcal{P})$ is not a minimal prime ideal of $I(\mathcal{C})$. Hence, we may assume that $\{a, c\}$ is a proper subset of W. In case of $d \in W$, it follows that $V = W \setminus \{a\}$ is an admissible set with $\mathcal{G}(V) = \mathcal{G}(W)$. In case of $b \in W$ it follows that $V = W \setminus \{c\}$ is an admissible set with $\mathcal{G}(V) = \mathcal{G}(W)$. Hence in both cases it follows from Theorem 3.3 that $P_W(\mathcal{P})$ is not a minimal prime ideal of $I(\mathcal{C})$. On the other hand, in case of $d \notin W$ and $b \notin W$, it follows that $V = W \setminus \{a, b\}$ is an admissible set with either $\mathcal{G}(V) = \mathcal{G}(W)$ or $\mathcal{G}(W) = \mathcal{G}(V) \cup \{ad - bc\}$. Hence by Theorem 3.3, $P_W(\mathcal{P})$ is not a minimal prime ideal of $I(\mathcal{C})$.

In the second case, suppose that $c \notin W$. Then $b \in W$. Let $a = (i, j)$ and $p = (k, j)$ with $k > i + 1$, and let $[a, p] = \{(l, j) \colon i \le l \le k\}$. Then $b \in [a, c]$. If $W = [a, p]$, then $P_W(\mathcal{P})$ is not a minimal prime ideal of $I(\mathcal{C})$, because in that case $P_\emptyset(\mathcal{P})$ is a proper subset of $P_W(\mathcal{P})$. On the other hand, if W is a proper subset of $[a, p]$, then W is not admissible. Hence there exists $e \in [a, p]$ such that $[a, e] \subset W$ and moreover f, as indicated in Figure 17, belongs to W. We may assume that $[b, f'] \cap W = \emptyset$. Let $V = W \setminus [a, e']$. Then V is admissible. Since $\mathcal{G}(V) \setminus \mathcal{G}(W)$ consists of those adjacent 2-minors which are indicated in Figure 17 as the dark shadowed area, it follows from Theorem 3.3 that $P_W(\mathcal{P})$ is not a minimal prime ideal of $I(\mathcal{C})$. $\square$
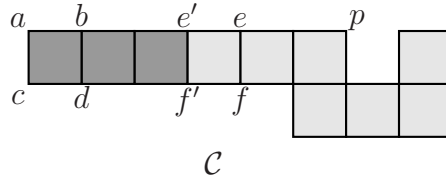

FIGURE 17.

The vertex c in Figure 17 is the first corner of the path $\mathcal{C}$. Therefore, according to Lemma 4.5, the element a is not contained in any minimal prime ideal of $I(\mathcal{C})$.

Proof of Theorem 4.3. Let $\mathcal{P} = \delta_1, \delta_2, \ldots, \delta_r$ be a path and choose the vertex $a \in \delta_1$ as described in Proposition 4.5. Then our hypothesis implies that a is a nonzero divisor modulo $I(\mathcal{P})$. The graded version of Lemma 4.4.9 in [1] implies then that $I(\mathcal{P})$ is a radical ideal if and only if $(a, I(\mathcal{P}))$ is a radical ideal. Thus it suffices to show that $(a, I(\mathcal{P}))$ is a radical ideal.

Let $\delta_1 = ad - bc$ and $\mathcal{P}' = \delta_2, \ldots, \delta_r$ be the path which is obtained from $\mathcal{P}$ by removing δ_1. Then $(a, I(\mathcal{P})) = (a, bc, I(\mathcal{P}'))$. Thus $(a, I(\mathcal{P}))$ is a radical ideal if $(bc, I(\mathcal{P}'))$ is a radical ideal, because a is a variable which does not appear in $(bc, I(\mathcal{P}'))$. Since c is regular modulo $(b, I(\mathcal{P}'))$, we may apply Lemma 4.4 and get

that $(bc, I(\mathcal{P}')) = (b, I(\mathcal{P}')) \cap (c, I(\mathcal{P}'))$. By using induction of the length of the path we may assume that $I(\mathcal{P}')$ is a radical ideal. Since c does not appear $I(\mathcal{P}')$ it follows that $(c, I(\mathcal{P}))$ is a radical ideal. Thus it remains to be shown that $(b, I(\mathcal{P}'))$ is a radical ideal. Observe that b is one of the vertices of δ_2. If it is a free vertex we can argue as before. So we may assume that b is not free. The following Figure 18(i) and Figure 18(ii) indicate (up to rotation and reflection) the possible positions of b in $\mathcal{P}'$.

(i)   (ii)

FIGURE 18.

In the case of Figure 18(i) we have $(b, I(\mathcal{P}')) = (b, df, gf, I(\mathcal{P}''))$ where $\mathcal{P}'' = \delta_4, \ldots, \delta_r$. Since the variable b does not appear in $(df, gf, I(\mathcal{P}''))$ it follows that $(b, I(\mathcal{P}'))$ is a radical ideal if and only if $(df, gf, I(\mathcal{P}''))$ is a radical ideal. Applying Lemma 4.4 we see that $(df, gf, I(\mathcal{P}'')) = (d, gf, I(\mathcal{P}'')) \cap (f, I(\mathcal{P}''))$. By induction hypothesis we may assume that $(f, I(\mathcal{P}''))$ is a radical ideal. Thus it remains to be shown that $(d, gf, I(\mathcal{P}''))$ is a radical ideal which is the case if $(gf, I(\mathcal{P}''))$ is a radical ideal. Once again we apply Lemma 4.4 and get $(gf, I(\mathcal{P}'')) = (g, I(\mathcal{P}'')) \cap (f, I(\mathcal{P}''))$. By assumption of induction we deduce as before that both ideals $(g, I(\mathcal{P}''))$ and $(f, I(\mathcal{P}''))$ are radical ideals. Therefore, $(gf, I(\mathcal{P}''))$ is a radical ideal. $\square$

In the case of Figure 18(ii) a similar argument works.

5. SPECIAL CONTINGENCY TABLES

We call a contingency table $T = (a_{ij})$ with support in $\mathcal{S}$ *special*, if $\mathcal{S} = \{(i, j): x_{ij} \in V(\mathcal{C})\}$ where $\mathcal{C}$ is a special configuration. If $\mathcal{S}' \subset \mathcal{S}$ we denote by $T'_{\mathcal{S}}$ the restriction of T to $\mathcal{S}'$. In other words, $T_{\mathcal{S}'} = (a_{ij})_{(i,j) \in \mathcal{S}'}$.

Figure 19 shows the contingency table corresponding to the configuration $\mathcal{C}$ of 2-adjacent minors as shown in Figure 11. One has to observe that in the corresponding contingency table the entries are displayed in cells whose center coordinates correspond to the vertices of $\mathcal{C}$.

In this section we want to discuss connectedness of special contingency tables with respect to adjacent moves. For that we shall need the following result.

Proposition 5.1. *Let $\mathcal{C}$ be a special configuration of 2-adjacent minors. We write $\mathcal{C} = \bigcup_{k=1}^{r} \mathcal{C}_k$ as a disjoint union of connected special configurations of 2-adjacent minors, and set $\mathcal{S} = \{(i, j): x_{ij} \in V(\mathcal{C})\}$ and $\mathcal{S}_k = \{(i, j): x_{ij} \in V(\mathcal{C}_k)\}$ for $k = 1, \ldots, r$. Then $I(\mathcal{C})$ is a lattice basis ideal for the saturated lattice $\mathcal{L}$ consisting of all tables $T = \{a_{ij}: (i, j) \in \mathcal{S}\}$ such that for $k = 1, \ldots, r$ the tables $T_{\mathcal{S}_k}$ have row*

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and column sums equal to zero. Moreover, the lattice ideal $I_{\mathcal{L}}$ of $\mathcal{L}$ is generated by all inner 2-minors of $\mathcal{C}$, that is, $I_{\mathcal{L}} = P_{\emptyset}(\mathcal{C}) = (P_{\emptyset}(\mathcal{C}_1), \ldots, P_{\emptyset}(\mathcal{C}_r))$.

Proof. We observed already in the proof of Lemma 1.3 that for any configuration $\mathcal{C}$ of adjacent 2-minors the ideal $I(\mathcal{C})$ is a lattice basis ideal of a saturated lattice $\mathcal{L}$. The basis elements of the lattice which correspond to the adjacent 2-minors in $\mathcal{C}$ are tables of the form $D = (d_{ij})$ such that there exists an integer $k \in \{1, \ldots, r\}$ and integers i' and j' such that $(i', j'), (i', j'+1), (i'+1, j'), (i'+1, j'+1) \in \mathcal{S}_k$, $d_{i',j'} = d_{i'+1,j'+1} = 1$, $d_{i',j'+1} = d_{i'+1,j'} = -1$, and all for all other pairs (i, j) one has $d_{i,j} = 0$. Since any other element in $\mathcal{L}$ is $\mathbb{Z}$-linear combination of such tables, it follows $\mathcal{L}$ consists of all tables $T = \{a_{ij} \colon (i, j) \in \mathcal{S}\}$ as described in the proposition. As noted in Section 1, we have that $I_{\mathcal{L}} = I(\mathcal{C}) \colon \prod_{(i,j) \in \mathcal{S}} x_{ij}$. Thus Theorem 3.2 implies that $I_{\mathcal{L}} = P_{\emptyset}(\mathcal{C})$ which is exactly the ideal of inner 2-minors of $\mathcal{C}$. $\square$

	a	b	c
d	e	f	g
h	i	j	

FIGURE 19.

Let $\mathcal{C}$ be a special configuration. Recall from Section 3 that all the minimal prime ideals of $I(\mathcal{C})$ are of the form $P_W(\mathcal{C}) = (W, P_{\emptyset}(\mathcal{C}'))$ where $W \subset V(\mathcal{C})$ is an admissible set and $\mathcal{C}' = \{\delta \in \mathcal{C} \colon V(\delta) \cap W = \emptyset\}$.

Let $\mathcal{S} = \{(i, j) \colon x_{ij} \in V(\mathcal{C})$ and $T = \{c_{ij} \colon (i, j) \in \mathcal{S}\}$ be a table with $c_{ij} \in \mathbb{Z}$. We define the support of T to be the set $\operatorname{supp} T = \{x_{ij} \in \mathcal{S} \colon c_{ij} \neq 0\}$. Now we have

Corollary 5.2. *Let $\mathcal{C}$ be a special configuration of 2-adjacent minors with $\mathcal{S} = V(\mathcal{C})$, and let $W \subset \mathcal{S}$ be an admissible set of $\mathcal{C}$. Then the binomial $f = \prod_{(i,j) \in \mathcal{S}} x_{ij}^{a_{ij}} - \prod_{(i,j) \in \mathcal{S}} x_{ij}^{b_{ij}}$ with $T = \{a_{ij} \colon (i, j) \in \mathcal{S}\}$ and $T' = \{b_{ij} \colon (i, j) \in \mathcal{S}\}$ belongs to $P_W(\mathcal{C}) = (W, P_{\emptyset}(\mathcal{C}'))$ if and only if one of the following conditions (i) and (ii) is satisfied:*

(i) *the inequalities*

$$\sum_{x_{ij} \in W} a_{ij} \geq 1 \quad and \quad \sum_{x_{ij} \in W} b_{ij} \geq 1$$

hold;

(ii) *one has $\operatorname{supp}(T - T') \subset V(\mathcal{C}')$ and the tables $T_{\mathcal{S}_K}$ and $T'_{\mathcal{S}_k}$ have the same row and column sums for all k, where $\mathcal{C}_1, \ldots, \mathcal{C}_r$ are the connected special configurations such that $\mathcal{C}' = \bigcup_{k=1}^{r} \mathcal{C}_k$ is the (unique) disjoint union of the $\mathcal{C}_k$, and $\mathcal{S}_k = \{(i, j) \colon x_{ij} \in V(\mathcal{C}_k)\}$ for all k.*

Proof. Suppose that $f \in P_W(\mathcal{C})$ and that $\sum_{x_{ij} \in W} a_{ij} = 0$ or $\sum_{x_{ij} \in W} b_{ij} = 0$. Say, $\sum_{x_{ij} \in W} a_{ij} = 0$. Then $\sum_{x_{ij} \in W} b_{ij} = 0$, because otherwise it would follow that

$\prod_{(i,j)\in\mathcal{S}} x_{ij}^{a_{ij}} \in P_\emptyset(\mathcal{C}')S$. Since $P_\emptyset(\mathcal{C}')S$ is a prime ideal, this would imply that one of the variables $x_{ij} \in V(\mathcal{C})$ would belong to $P_\emptyset(\mathcal{C}')S$, a contradiction. Therefore, $f \in P_\emptyset(\mathcal{C}')S$. Let $S' \subset S$ be the polynomial ring in the variables $x_{ij} \in V(\mathcal{C}')$. Then $P_\emptyset(\mathcal{C}')$ is an ideal of S'. Let $f = ug$, where u is a monomial and g is a binomial whose monomial terms have greatest common divisor 1, say, $g = \prod_{(i,j)\in\mathcal{S}} x_{ij}^{c_{ij}} - \prod_{(i,j)\in\mathcal{S}} x_{ij}^{d_{ij}}$ with $T_0 = \{c_{ij}\colon (i,j) \in \mathcal{S}\}$ and $T_0' = \{d_{ij}\colon (i,j) \in \mathcal{S}\}$. Since $f \in P_\emptyset(\mathcal{C}')S$ it follows that $g \in P_\emptyset(\mathcal{C}')$. Therefore, since $T - T' = T_0 - T_0'$, we get that $\mathrm{supp}(T - T') = \mathrm{supp}(T_0 - T_0') \subset V(\mathcal{C}')$. Hence Proposition 5.1 implies that $(T_0)_{\mathcal{S}_k}$ and $(T_0')_{\mathcal{S}_k}$ have the same row and column sums. Since $T = T_0 + E$ and $T' = T_0' + E$ where E is the table given by the exponents of the monomial u, we conclude that $T_{\mathcal{S}_k}$ and $T_{\mathcal{S}_k}'$ have the same row and column sums, too.

Conversely suppose that $\sum_{(i,j)\in W} a_{ij} \geq 1$ and $\sum_{(i,j)\in W} b_{ij} \geq 1$. Then $f \in (W)$, and hence $f \in P_W(\mathcal{C})$. On the other hand, if $\mathrm{supp}(T - T') \subset V(\mathcal{C}')$ and the tables $T_{\mathcal{S}_k}$ and $T_{\mathcal{S}_k}'$ have the same row and column sums for all k, then, with the notation introduced above, we have that $\mathrm{supp}(T_0 - T_0') \subset V(\mathcal{C}')$ and the tables $(T_0)_{\mathcal{S}_k}$ and $(T_0')_{\mathcal{S}_k}$ have the same row and column sums for all k. Hence Proposition 5.1 yields that $g \in P_\emptyset(\mathcal{C}')$, and this implies that $f \in P_W(\mathcal{C})$. $\qquad\square$

Corollary 5.3. *Let $T = \{a_{ij}\colon (i,j) \in \mathcal{S}\}$ and $T' = \{b_{ij}\colon (i,j) \in \mathcal{S}\}$ be two special contingency tables supported in $\mathcal{S}$.*

(a) If T and T' are connected with respect to adjacent moves, then, for T and T' either (i) or (ii) of Corollary 5.2 is satisfied for all admissible sets $W \subset \mathcal{S}$.

(b) Let $\mathcal{C}$ be the configuration of 2-adjacent minors with $\mathcal{S} = \{\{(i,j)\colon x_{ij} \in V(\mathcal{C})\}$, and assume that $I(\mathcal{C})$ is a radical ideal. Then T and T' are connected with respect to adjacent moves, if and only if, for T and T' either (i) or (ii) of Corollary 5.2 is satisfied for all admissible sets $W \subset \mathcal{S}$.

In the above statements it is enough to consider admissible sets which correspond to minimal prime ideals of $I(\mathcal{C})$ as characterized in Theorem 3.3(b).

Proof. By [3, Theorem 1.1] T and T' are connected with respect to adjacent moves if and only if the monomial $f = \prod_{(i,j)\in\mathcal{S}} x_{ij}^{a_{ij}} - \prod_{(i,j)\in\mathcal{S}} x_{ij}^{b_{ij}}$ belongs to $I(\mathcal{C})$, where $\mathcal{C}$ is the configuration of adjacent 2-minors with $\mathcal{S} = \{(i,j)\colon x_{ij} \in V(\mathcal{C})\}$. Since $\mathcal{C}$ is special it follows from Theorem 3.2 that $f \in P_W(\mathcal{C})$ for all admissible sets $W \in V(\mathcal{C})$. Thus Corollary 5.2 implies statement (a). In the case that $I(\mathcal{C})$ is a radical ideal, $f \in I(\mathcal{C})$ if and only if $f \in P_W(\mathcal{C})$ for all admissible sets $W \in V(\mathcal{C})$. This proves (b). $\qquad\square$

It follows from Corollary 5.3 and Figure 13 that two contingency tables T and T' as in Figure 19 with the same row and column sums are connected via adjacent moves if and only the following conditions are satisfied: either both tables satisfies

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the following inequalities

$$
\begin{aligned}
a + e + i &\geq 1 \\
b + f + j &\geq 1 \\
e + f + g + i &\geq 1 \\
b + e + f + i &\geq 1 \\
d + e + f + g &\geq 1 \\
b + d + e + f &\geq 1.
\end{aligned}
$$

or else $a + e + i = 0$ or $b + f + j = 0$ for one of the tables and the remaining 4 inequalities hold for both tables. In this case, if $a + e + i = 0$ for one table, then one should have that $\mathrm{supp}(T - T') \subset \{b, c, f, g\}$, and if $b + f + j = 0$ for one table, then one should have that $\mathrm{supp}(T - T') \subset \{d, e, h, i\}$.

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Jürgen Herzog, Fachbereich Mathematik, Universität Duisburg–Essen, Campus Essen, 45117 Essen, Germany

E-mail address: `juergen.herzog@uni-essen.de`

Takayuki Hibi, Department of Pure and Applied Mathematics, Graduate School of Information Science and Technology, Osaka University, Toyonaka, Osaka 560-0043, Japan

E-mail address: `hibi@math.sci.osaka-u.ac.jp`