COLOURED QUIVERS FOR RIGID OBJECTS AND PARTIAL TRIANGULATIONS: THE UNPUNCTURED CASE

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Abstract. We associate a coloured quiver to a rigid object in a Hom-finite 2-Calabi–Yau triangulated category and to a partial triangulation on a marked (unpunctured) Riemann surface. We show that, in the case where the category is the generalised cluster category associated to a surface, the coloured quivers coincide. We also show that compatible notions of mutation can be defined and give an explicit description in the case of a disk. We show further that Iyama-Yoshino reduction can be interpreted as cutting along an arc in the surface.

Introduction

Let (S, M) be a pair consisting of a compact Riemann surface S with non-empty boundary and a set M of marked points on the boundary of S, with at least one marked point on each component of the boundary. We further assume that (S, M) has no component homeomorphic to a monogon, digon, or triangle. A *partial triangulation* $\mathscr{R}$ of (S, M) is a set of noncrossing simple arcs between the points in M. We define a mutation of such triangulations, involving replacing an arc α of $\mathscr{R}$ with a new arc depending on the surface and the rest of the partial triangulation. This allows us to associate a coloured quiver to each partial triangulation of M in a natural way. The coloured quiver is a directed graph in which each edge has an associated colour which, in general, can be any integer.

Let $\mathcal{C}$ be a Hom-finite, 2-Calabi–Yau, Krull-Schmidt triangulated category over a field k. A *rigid* object in $\mathcal{C}$ is an object R with no self-extensions, i.e. satisfying $\mathrm{Ext}^1_{\mathcal{C}}(R, R) = 0$. Rigid objects in $\mathcal{C}$ can also be mutated. In this case the mutation involves replacing an indecomposable summand X of R with a new summand depending on the relationship between X and the rest of the summands of R. As above, this allows us to associate a coloured quiver to each rigid object of $\mathcal{C}$ in a natural way.

In [BZ] the authors study the generalised cluster category $\mathcal{C}_{(S,M)}$ in the sense of Amiot [Ami09] associated to a surface (S, M) as above. Such a category is triangulated and satisfies the above requirements. It is shown in [BZ] that, given a choice of (complete) triangulation of (S, M), there is a bijection between the simple arcs in (S, M) (joining two points in M), up to homotopy, and the isomorphism classes of rigid indecomposable objects in $\mathcal{C}_{(S,M)}$. If X_α denotes the object corresponding to an arc α then $\mathrm{Ext}^1_{\mathcal{C}_{(S,M)}}(X_\alpha, X_\beta) = 0$ if and only if α and β do not cross. It follows that there is a bijection between partial triangulations of (S, M) and rigid objects in $\mathcal{C}_{(S,M)}$. Our main result is that the coloured quivers defined above coincide in this situation and that the two notions of mutation are compatible.

Suppose that α is a simple arc in (S, M) as above. Let X_α be the indecomposable rigid object corresponding to α. Iyama-Yoshino [IY08] have associated (in a more

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general context) a subquotient category $(\mathcal{C}_{(S,M)})_{X_\alpha}$ to X_α which we refer to as the Iyama-Yoshino reduction of $\mathcal{C}_{(S,M)}$ at X_α. The Iyama-Yoshino reduction is again triangulated. We show that $(\mathcal{C}_{(S,M)})_{X_\alpha}$ is equivalent to $\mathcal{C}_{(S,M)/\alpha}$ where $(S,M)/\alpha$ denotes the new marked surface obtained from (S,M) by cutting along α.

By studying the combinatorics, we are able to give an explicit description of the effect of mutation on coloured quivers associated to a disk with n marked points. The corresponding cluster category in this case was introduced independently in [CCS06] (in geometric terms) and in [BMR$^+$06] as the cluster category associated to a Dynkin quiver of type A_{n-3}. We also give a partial explicit description of coloured quiver mutation in the general case, together with a categorical proof. In general, there are quite interesting phenomena: we give an example to show that infinitely many colours can occur in one quiver, and also show that zero-coloured 2-cycles can occur (in contrast to the situation in [BT09]).

We remark that in the case of a cluster tilting object T in an acyclic cluster category the categorical mutation we define coincides with that considered in [BMR$^+$06]; also with that in the 2-Calabi-Yau case considered in [BIRS09, Pal09]. It also coincides in the maximal rigid case considered in [GLS06, BIRS09, IY08]. In this case, the coloured quiver we consider here encodes the same information as the matrix associated to T in [BMV10] provided there are no zero-coloured two-cycles. With this restriction, the mutation of this matrix coincides [BMV10, 1.1] with the mutation [FZ02] arising in the theory of cluster algebras. We note also that this fact for the cluster tilting case was shown in [BIRS09] under the assumption that there are no two-cycles or loops in the quiver of the endomorphism algebra; the cluster category case was considered in [BMR08] and the stable module category over a preprojective algebra was considered in [GLS06]. See also [BIRS] and [KY11], where mutation of quivers with potential has been studied in a categorical context. There has been a lot of work on this subject: see the survey [Kel10] for more details.

The geometric mutation of partial triangulations mentioned above specialises to the usual flip of an arc in the triangulation case (see [FST08, Defn. 3.5]). Coloured quivers similar to those considered here have been associated to m-cluster tilting objects in an $(m+1)$-Calabi-Yau category in [BT09] (in this case, the number of colours is fixed at $m+1$). The geometric mutation we define here should also be compared with the geometric mutation for m-allowable arcs in a disk [BT09, Sect. 11]; see also the geometric model of the m-cluster category of type A in [BM08].

We also note that the 2-Calabi-Yau tilting theorem of Keller-Reiten [KR07, Prop. 2.1] (see also Koenig-Zhu [KZ08, Cor. 4.4] and Iyama-Yoshino [IY08, Prop. 6.2]) was recently generalised [BM] to the general rigid object case, using Gabriel-Zisman localisation. This result (together with discussions with Aslak Bakke Buan) suggests that mutation of general rigid objects should be considered.

The paper is organised as follows. In Section 1 we set up notation and recall the results we need. In Section 2 we define the mutation and the coloured quiver of a rigid object in a triangulated category. In Section 3 we define mutation and the coloured quiver of a partial triangulation in a marked surface. In Section 4 we show that cutting along an arc corresponds categorically to Iyama-Yoshino reduction. In particular, the coloured quiver after cutting along an arc in a partial triangulation can be obtained from the coloured quiver of the partial triangulation by deleting a vertex. In Section 5 we show that, for a partial triangulation of a surface and the corresponding rigid object in the cluster category of the surface, the two notions of coloured quiver coincide. In Section 6 we show that mutation in the type A case can be described purely in terms of the coloured quiver and give an explicit description. We also give the example mentioned above in which the associated coloured quiver contains infinitely many colours. Finally, in Section 7, we give a partial explicit description and categorical interpretation of coloured quiver mutation.

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1. Preliminaries

1.1. **Riemann surfaces.**

In this section, we recall some definitions and results from [FST08] and [LF09].

We consider a pair (S, M) consisting of a compact Riemann surface with boundary S and a finite set M of marked points on the boundary of S, with at least one marked point on each boundary component. We refer to such a pair as a *marked surface.* We fix, once and for all, an orientation of S, inducing the clockwise orientation on each boundary component.

Note that:

- We do not assume the surface to be connected.
- We only consider unpunctured marked surfaces.

We will always assume that (S, M) does not have any component homeomorphic to a monogon, a digon or a triangle.

Up to homeomorphism, each component of (S, M) is determined by the following data:

- the genus g,
- the number of boundary components b and
- the number of marked points on each boundary component $\{n_1, \dots, n_b\}$.

An *arc* γ in (S, M) is (the isotopy class relative to endpoints of) a curve in S whose endpoints belong to M, which does not intersect itself (except possibly at endpoints) and which is not contractible to a point. The marked points on a boundary component divide it into segments, and we say that an arc isotopic to an arc along one of these segments is a *boundary arc.* The term *arc* will usually refer to a non-boundary arc.

The set of all (non-boundary) arcs in (S, M) is denoted by $A^0(S, M)$. Two arcs are said to be *non-crossing* if their isotopy classes contain representatives which do not cross, i.e. their crossing number is zero. If $\mathscr{R}$ is a collection of non-crossing arcs in (S, M), we will denote by $A^0_{\mathscr{R}}(S, M)$ the set of arcs in (S, M) which do not cross any arc in $\mathscr{R}$ and which do not belong to $\mathscr{R}$.

A partial triangulation of (S, M) is a collection of non-crossing arcs. A maximal collection of non-crossing arcs is called a *triangulation.* The number n of arcs in any triangulation of a connected marked surface is given by the formula:

$$n = 6g + 3b + c - 6,$$

where c is the number of marked points in M (see e.g. [FST08, Prop. 2.10]).

Let $\mathscr{T}$ be a triangulation. By [FST08, Sect. 4] and [LF09, Sect. 3] a quiver $Q = Q_{\mathscr{T}}$, together with a potential (a linear combination of cycles in $Q_{\mathscr{T}}$ up to cyclic permutation) $W_{\mathscr{T}}$ can be associated to $\mathscr{T}$ as follows. The vertices of Q are the arcs of the triangulation. There is an arrow from γ to γ' for each triangle in which γ' follows γ with respect to the orientation of S, and the potential $W_{\mathscr{T}}$ is the sum of all the 3-cycles; see Figure 1, where part of a triangulated surface is shown.

For an arrow $a \in Q_1$, the cyclic derivative ∂_a sends a cycle $a_1 \cdots a_d$ to the sum $\sum_{k=1}^{d} \delta_{a_k a}\, a_{k+1} \cdots a_d a_1 \cdots a_{k-1}$. It is extended to potentials by linearity. The *Jacobian algebra* of the quiver with potential $(Q_{\mathscr{T}}, W_{\mathscr{T}})$ is the quotient of the path algebra $kQ_{\mathscr{T}}$ by the ideal generated by the cyclic derivatives $\partial_a W_{\mathscr{T}}$, for all $a \in Q_1$.

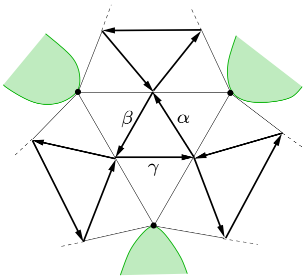

FIGURE 1. The quiver with potential associated to a triangulated surface. The potential $W = \alpha\beta\gamma + \cdots$.

Theorem: *Let $\mathscr{T}$ be a triangulation of a marked surface (S, M), and let $\mathscr{T}'$ be the triangulation obtained by flipping $\mathscr{T}$ at an arc γ. Then:*

(a) [LF09, Thm. 36] *The Jacobian algebra $J(Q_{\mathscr{T}}, W_{\mathscr{T}})$ is finite dimensional.*
(b) [ABCJP10, Thm. 2.7] *The Jacobian algebra $J(Q_{\mathscr{T}}, W_{\mathscr{T}})$ is gentle and Gorenstein of Gorenstein dimension 1.*
(c) [FST08, Prop. 4.8] *The quiver $Q_{\mathscr{T}'}$ is given by the Fomin–Zelevinsky mutation of $Q_{\mathscr{T}}$ at the vertex corresponding to γ.*
(d) [LF09, Thm. 30] *The quiver with potential $(Q_{\mathscr{T}'}, W_{\mathscr{T}'})$ is given by the QP mutation (see* [DWZ08, Sect. 5]*) of $(Q_{\mathscr{T}}, W_{\mathscr{T}})$ at the vertex corresponding to γ.*

1.2. **Cluster categories associated with Riemann surfaces.** Let k be a field. If $\mathcal{C}$ is a triangulated category, we will usually denote its shift functor by Σ. All the triangulated k-categories under consideration in this paper are assumed to be Krull–Schmidt, Hom-finite (all morphism spaces are finite-dimensional k-vector spaces) and admit non-zero *rigid objects* (objects R such that $\mathcal{C}(R, \Sigma R) = 0$). All rigid objects will be assumed basic (their summands are pairwise non-isomorphic). We will assume moreover that the triangulated categories are 2-Calabi–Yau, so that there are bifunctorial isomorphisms $\mathcal{C}(X, \Sigma Y) \simeq D\mathcal{C}(Y, \Sigma X)$ for all objects X, Y, where D is the vector space duality $D = \mathrm{Hom}_k(-, k)$. A rigid object T is called a *cluster tilting object* if, in addition, for all objects X in $\mathcal{C}$, $\mathcal{C}(X, \Sigma T) = 0 = \mathcal{C}(T, \Sigma X)$ implies that X belongs to $\operatorname{add} T$.

The main examples of such categories that we consider are the (generalised) cluster categories associated with marked surfaces, the definition of which is recalled in the following sections.

1.2.1. *Ginzburg dg algebras.* Let (Q, W) be a quiver with potential (i.e. a QP). In this paper, we are mostly interested in QPs arising from triangulations of marked surfaces.

The Ginzburg dg-algebra $\Gamma(Q, W)$ is defined as follows: First define a graded quiver $\overline{Q}$. The vertices of $\overline{Q}$ are the vertices of Q, and the arrows are given as follows:

- the arrows of Q, of degree 0;
- for each arrow α in Q from i to j, an arrow α^* from j to i, of degree -1;
- for each vertex i in Q, a loop e_i^* at i, of degree -2.

The underlying graded algebra of $\Gamma(Q, W)$ is the path algebra of the graded quiver $\overline{Q}$. It is equipped with the unique differential d sending

- the arrows of degree 0 and each e_i to 0;
- the arrow α^* to $\partial_\alpha W$, for each $\alpha \in Q_1$, and
- the loop e_i^* to $e_i \left(\sum_\alpha [\alpha^*, \alpha]\right) e_i$, for $i \in Q_0$.

The cohomology of $\Gamma(Q, W)$ in degree zero is the Jacobian algebra $J(Q, W)$.

1.2.2. *Generalised cluster categories.* The cluster categories associated with acyclic quivers were introduced in [CCS06] in the A_n case and in [BMR$^+$06] in the acyclic case. Amiot defined, in [Ami09], the generalised cluster categories, associated with quivers with potentials whose Jacobian algebra is finite dimensional.

Let (Q, W) be a quiver with potential such that the Jacobian algebra $J(Q, W)$ is finite dimensional, and let $\Gamma = \Gamma(Q, W)$ be the associated Ginzburg dg algebra.

Let $\mathcal{D}\Gamma$ be the derived category of Γ, and let $\mathcal{D}^b\Gamma$ be the bounded derived category. The perfect derived category $\operatorname{per}\Gamma$ is the smallest triangulated subcategory of $\mathcal{D}\Gamma$ containing Γ and stable under taking direct summands.

Theorem [Kel09, Sect. 6]: *The Ginzburg dg algebra Γ is homologically smooth and 3-Calabi–Yau as a bimodule. In particular, there is an inclusion $\mathcal{D}^b\Gamma \subset \operatorname{per}\Gamma$.*

Definition [Ami09, Sect. 3]: *The (generalised) cluster category $\mathcal{C}_{(Q,W)}$ associated with the quiver with potential (Q, W) is the Verdier localisation* $\operatorname{per}\Gamma/\mathcal{D}^b\Gamma$.

This definition is motivated by the following:

Theorem [Ami09, Sect. 3]: *The cluster category $\mathcal{C}_{(Q,W)}$ is Hom-finite and 2-Calabi–Yau. Moreover, the image of Γ in $\mathcal{C}_{(Q,W)}$ is a cluster tilting object whose endomorphism algebra is isomorphic to the Jacobian algebra $J(Q, W)$. If Q is acyclic, then $W = 0$ and the triangulated categories $\mathcal{C}_{(Q,0)}$ and $\mathcal{C}_Q$ are equivalent.*

We also recall the 2-Calabi-Yau tilting theorem which applies in this context:

Theorem [KR07, Prop. 2.1] Let $\mathcal{C}$ be a triangulated Hom-finite Krull-Schmidt 2-Calabi-Yau category over a field k. If T is a cluster tilting object in $\mathcal{C}$, then the functor $\mathcal{C}(T, \Sigma -)$ induces an equivalence between the category $\mathcal{C}/T$ and the category of finite dimensional $\operatorname{End}_{\mathcal{C}}(T)$-modules.

Note that the assumption in the paper that k be algebraically closed is not required for this result. We also note that this result has been generalised (see [IY08, Prop. 6.2], [KZ08, Sect. 5.1]).

1.2.3. *Cluster categories from surfaces.* Let (S, M) be a marked surface, and let $\mathscr{T}$ be a triangulation of (S, M). Let (Q, W) be the quiver with potential associated with $\mathscr{T}$. The following particular case of a theorem of Keller–Yang shows that the cluster category $\mathcal{C}_{(Q,W)}$ does not depend on the choice of a triangulation. Let $\mathscr{T}'$ be a triangulation of (S, M) obtained from $\mathscr{T}$ by a flip. Denote by (Q', W') the associated quiver with potential.

Theorem [KY11]: *There is a triangle equivalence $\mathcal{C}_{(Q',W')} \simeq \mathcal{C}_{(Q,W)}$.*

Since any two triangulations of (S, M) are related by a sequence of flips, the theorem above shows that the cluster category $\mathcal{C}_{(Q,W)}$ is independent of the choice of the triangulation $\mathscr{T}$. The resulting category is denoted $\mathcal{C}_{(S,M)}$ and is called the cluster category associated with the marked surface (S, M). (We refer also to [BIRS, Theorem 5.1]).

These categories have been studied by Brüstle–Zhang in [BZ]. We now recall those of their main results which will be used in the article.

Fix a triangulation $\mathscr{T} = \{\gamma_1, \dots, \gamma_m\}$ of (S, M) with associated quiver with potential (Q, W). Let $T = T_1 \oplus \cdots \oplus T_m$ be the image of $\Gamma(Q, W)$ under $\operatorname{per}\Gamma(Q, W) \to \mathcal{C}_{(Q,W)} \simeq \mathcal{C}_{(S,M)}$. Note that T is a cluster tilting object.

With each arc γ not in $\mathscr{T}$ is associated [ABCJP10, Proposition 4.2] an indecomposable $J(Q,W)$-module $I(\gamma)$. Let X_γ be the unique (up to isomorphism) indecomposable object in $\mathcal{C}_{(S,M)}$ such that $\mathcal{C}_{(S,M)}(T,\Sigma X_\gamma) \simeq I(\gamma)$. Define $X_{\gamma_k} = T_k$, for $k = 1,\dots,m$.

Theorem [BZ]:

- *The map $\gamma \mapsto X_\gamma$ is a bijection between the arcs of (S,M) and the (isomorphism classes of) exceptional (i.e. indecomposable rigid) objects of $\mathcal{C}_{(S,M)}$.*
- *For any two exceptional objects X_α and X_β, we have $\operatorname{Ext}^1_{\mathcal{C}_{(S,M)}}(X_\alpha, X_\beta) = 0$ if and only if the arcs α and β do not cross.*
- *The shift functor of $\mathcal{C}_{(S,M)}$ acts on the arcs of (S,M) by moving both endpoints* clockwise *along the boundary to the next marked points.*

Note that a bijection with these properties is not unique in general.

We note that our choice of an orientation of the Riemann surface differs from that of [BZ], but coincides with that of [BT09, Section 11].

We extend the bijection in the first part of the previous Theorem to a bijection between partial triangulations of (S,M) and rigid objects in $\mathcal{C}_{(S,M)}$ in the obvious way.

1.3. **Iyama–Yoshino reduction.** For an object X in a triangulated category $\mathcal{C}$, we write ${}^\perp X$ for the full subcategory of $\mathcal{C}$ whose objects are those objects Y of $\mathcal{C}$ such that $\operatorname{Hom}_{\mathcal{C}}(Y,X) = 0$. The subcategory $X^\perp$ is similarly defined. For an additive subcategory $\mathcal{D}$ of $\mathcal{C}$, we write $\mathcal{C}/[\mathcal{D}]$ for the quotient category whose objects are the same as the objects of $\mathcal{C}$ with morphisms given by the morphisms of $\mathcal{C}$ modulo those morphisms factoring through $\mathcal{D}$. If $\mathcal{D}$ is the additive closure of an object X in $\mathcal{C}$ then we just write $\mathcal{C}/X$ for $\mathcal{C}/[\mathcal{D}]$.

Theorem: [IY08, 4.2, 4.7] *Let $\mathcal{C}$ be a 2-Calabi-Yau triangulated category and R a rigid object in $\mathcal{C}$. Then the subfactor category ${}^\perp(\Sigma R)/R$ of $\mathcal{C}$ is again a 2-Calabi-Yau triangulated category.*

We refer to the subfactor category ${}^\perp(\Sigma R)/R$ as the *Iyama-Yoshino reduction* of $\mathcal{C}$ at R and denote it $\mathcal{C}_R$. We denote its shift by Σ_R and the quotient functor ${}^\perp(\Sigma R) \longrightarrow \mathcal{C}_R$ by π_R. See also [BIRS09, II.2.1].

We recall a result of Keller:

Theorem: [Kel09, 7.4] *Let Q, W be a quiver with potential whose Jacobian algebra is finite dimensional. Let i be a vertex of Q and let $\overline{P_i}$ be the image of the indecomposable projective module over $\Gamma(Q,W)$ corresponding to i, under the quotient functor $\Pi : \operatorname{per}(\Gamma(Q,W)) \to \mathcal{C}_{Q,W}$. Then the Iyama-Yoshino reduction of $\mathcal{C}_{(Q,W)}$ at $\overline{P_i}$ is triangle equivalent to $\mathcal{C}_{(Q',W')}$, where Q' is Q with vertex i (and all incident arrows) removed, and W' is W with all cycles passing through i deleted.*

2. Coloured quivers for rigid objects

2.1. **Mutation and coloured quivers of rigid objects.** Let k be a field. Let $\mathcal{C}$ be a k-linear Hom-finite, Krull–Schmidt, 2-Calabi–Yau triangulated k-category. Let $R = R_1 \oplus \cdots \oplus R_m$ be a basic rigid object in $\mathcal{C}$ and let X be an indecomposable rigid object in $\mathcal{C}$ Ext-orthogonal to R, i.e. such that $\mathcal{C}(X,\Sigma R) = 0 = \mathcal{C}(R,\Sigma X)$.

For $k = 1,\dots,m$ and $c \in \mathbb{Z}$, consider triangles

$$X^{(c)} \xrightarrow{f^c} B^{(c)} \xrightarrow{g^c} X^{(c+1)} \longrightarrow \Sigma X^{(c)}$$

where f^c is a minimal left $\operatorname{add} R$-approximation and g^c is a minimal right $\operatorname{add} R$-approximation and where $X^{(0)} = X$. These will be called the *exchange triangles* for X with respect to R. They can be constructed using induction on c. We will often write $\kappa_R^{(c)}X$ for $X^{(c)}$, and κ for $\kappa^{(1)}$; $\kappa_R X$ will be referred to as the *twist* of X with respect to R. Note that $\kappa\kappa^{(c)} = \kappa^{(c+1)} = \kappa^{(c)}\kappa$ for all c.

These exchange triangles lift the triangles $X^{(c)} \longrightarrow 0 \longrightarrow \Sigma_R X^{(c)} \longrightarrow \Sigma_R X^{(c)}$ in the Iyama–Yoshino reduction ${}^{\perp}(\Sigma R)/R$ canonically to $\mathcal{C}$. Therefore, $X^{(c)}$ is indecomposable, rigid and Ext-orthogonal to $\operatorname{add} R$ for all c. This justifies the following definition:

Definition: *The* mutation *of R at R_k is the rigid object*

$$\mu_{R_k} R = R/R_k \oplus \kappa_{R/R_k} R_k.$$

We will often write μ_k for μ_{R_k} and call it the mutation at k.

We note that our use of Iyama–Yoshino to define the mutation above is similar to that of [BØO, Sect. 3] where cluster tilting objects are mutated at a non-indecomposable summand.

In [BT09], the authors associate coloured quivers to d-cluster–tilting objects in $(d+1)$-Calabi–Yau categories. Here we use the same definition to associate a coloured quiver to R.

Definition: *The coloured quiver $Q = Q_R$ associated with the rigid object R is defined as follows: The set of vertices is $Q_0 = \{1, \ldots, m\}$. The set $Q_1^{(c)}(i,j)$ of c-coloured arrows from i to j has cardinality given by the multiplicity of R_j in $B_i^{(c)}$, where*

$$R_i^{(c)} \xrightarrow{f_i^c} B_i^{(c)} \xrightarrow{g_i^c} R_i^{(c+1)} \longrightarrow \Sigma R_i^{(c)}$$

are the exchange triangles as above for R_i with respect to R/R_i.

When the category $\mathcal{C}$ is a (generalised) cluster category, it is often the case that a sequence of exchange triangles is periodic. With each vertex k of Q is thus associated an integer d_k (possibly infinite): the periodicity of the sequence of exchange triangles for R_k. In order to avoid keeping infinitely many arrows starting at each vertex when not necessary, the colours of the arrows starting at a vertex k are considered as elements in $\mathbb{Z}/d_k$. Note that the periodicity depends on the starting vertex.

Remark:

- Analogous definitions would apply to a functorially finite, strictly full rigid subcategory $\mathcal{R}$ of $\mathcal{C}$ closed under direct sums and direct summands, such that, for each indecomposable $R \in \mathcal{R}$, the subcategory $\mathcal{R} \setminus R$ is again functorially finite.
- Analogous definitions would also apply to rigid objects in a stably 2-Calabi–Yau Frobenius category. The use of Iyama–Yoshino reduction would be replaced by [BIRS09, Theorem I.2.6] (see also [GLS06, Lemma 5.1] and [AO, Sect. 4]).

2.2. **Mutation of rigid objects and Iyama–Yoshino reductions.** The following lemma shows that the mutation of rigid objects is well-behaved with respect to Iyama–Yoshino reductions. This will turn out to be helpful in simplifying the proof of Theorem 15 in Section 7.

Let R be a rigid object in $\mathcal{C}$. Let $\mathcal{C}_R = {}^{\perp}(\Sigma R)/(R)$ be the Iyama–Yoshino reduction of $\mathcal{C}$ with respect to R, with shift Σ_R.

Let T be a rigid object in $\mathcal{C}$, containing R as a direct summand. Assume that T_k is a summand of T but not of R, and consider the exchange triangle with respect to T/T_k:

$$(*) \qquad T_k \xrightarrow{f} B_k^{(0)} \xrightarrow{g} T_k^{(1)} \xrightarrow{\varepsilon} \Sigma T_k.$$

Here $B_k^{(0)}$ belongs to $\operatorname{add} \overline{T}$, where $T = T_k \oplus \overline{T}$.

Lemma 1. *The induced morphism* $\underline{f}$ *is a minimal left* $\pi_R(\overline{T})$*-approximation in* $\mathcal{C}_R$*.*

Proof. The triangle $(*)$ in $\mathcal{C}$ induces a triangle

$$T_k \xrightarrow{f} B_k^{(0)} \xrightarrow{g} T_k^{(1)} \longrightarrow \Sigma_R T_k,$$

in $\mathcal{C}_R$. We have:

$$\mathcal{C}_R(\Sigma_R^{-1}\pi_R(T_k^{(1)}), \pi_R(\overline{T})) \simeq \operatorname{Ext}^1_{\mathcal{C}_R}(\pi_R(T_k^{(1)}), \pi_R(\overline{T})) \simeq \operatorname{Ext}^1_{\mathcal{C}}(T_k^{(1)}, \overline{T}) = 0,$$

using [IY08, Lemma 4.8]. Hence, the morphism $\underline{f}$ is a left $\pi_R(\overline{T})$-approximation. It is left minimal since $T_k^{(1)}$ is indecomposable in $\mathcal{C}_R$. $\surd$

Remark: Write $B_k^{(0)} = R_k^{(0)} \oplus C_k^{(0)}$, with $C_k^{(0)}$ having no summands in common with R. Then the morphism $T_k \xrightarrow{f'} C_k^{(0)}$ is not a left $\overline{T}/R$-approximation in $\mathcal{C}$ in general.

Let Q be the coloured quiver of T in $\mathcal{C}$, and let $\underline{Q}$ be the coloured quiver of $\pi_R(T)$ in $\mathcal{C}_R$. Lemma 1 has the following immediate corollary:

Corollary 2. *The coloured quiver* $\underline{Q}$ *is the full subquiver of* Q *with vertices corresponding to the indecomposable summands of* T/R*.*

Moreover, computing the minimal $\overline{T}$-approximation $f \in \mathcal{C}$ in the triangle $(*)$ amounts to computing the minimal add T_j-approximation of T_k in the Iyama–Yoshino reduction $\mathcal{C}_{\overline{T}/T_j}$ for all $j \neq k$. More precisely:

Lemma 3. *Let* $R = R_1 \oplus \cdots \oplus R_m$ *be a rigid object in* $\mathcal{C}$ *and let* $1 \leq k \leq m$*. For each* $j = 1, \ldots, m$*, let* $\mathcal{C}_j$ *denote the Iyama–Yoshino reduction of* $\mathcal{C}$ *with respect to* $R/(R_k \oplus R_j)$*. For* $j \neq k$*, let* $f_j : R_k \longrightarrow R_j^{n_j}$ *be a map in* $\mathcal{C}$ *be such that* $R_k \xrightarrow{\underline{f_j}} R_j^{n_j}$ *is a minimal left* $\operatorname{add} R_j$*-approximation in* $\mathcal{C}_j$*. Then the morphism:*

$$R_k \xrightarrow{[f_j]} \bigoplus_{j \neq k} R_j^{n_j}$$

is a minimal left $\operatorname{add} R/R_k$*-approximation in* $\mathcal{C}$*.*

Proof. Let $i \neq k$, and let $R_k \xrightarrow{f} R_i$ be an arbitrary morphism in $\mathcal{C}$. Since $\underline{f_i}$ is an $\operatorname{add} R_i$-approximation in $\mathcal{C}_i$, there are morphisms $\bigoplus_{j \neq k} R_j^{n_j} \xrightarrow{g_1} R_i$, $R_k \xrightarrow{\beta_1} \bigoplus_{j \neq i,k} R_j^{a_j^{(1)}}$ and $\bigoplus_{j \neq i,k} R_j^{a_j^{(1)}} \xrightarrow{\alpha_1} R_i$ in $\mathcal{C}$ (for some $a_j^{(1)}$) such that $f = g_1[f_j] + \alpha_1\beta_1$. Note that α_1 must be a radical map, as no summand of its domain is isomorphic to R_i.

Reducing to $\mathcal{C}_j$ for some $j \neq i,k$, we see that the component $\beta_{1,j}$ of β_1 mapping to $R_j^{a_j^{(1)}}$ factors through $f_j : R_k \to R_j^{n_j}$ up to a map factoring through $\operatorname{add} \oplus_{l \neq j,k} R_l$. That is, we can write $\beta_{1,j}$ as $u_j f_j + w_j v_j$ for some $u_j : R_j^{n_j} \to R_j^{a_j^{(1)}}$, $v_j : R_k \to X_j$, and $w_j : X_j \to R_k$, where $X_j \in \operatorname{add} \oplus_{l \neq j,k} R_l$. Note that w_j is a radical map, since none of the summands in X_j are isomorphic to R_j.

Adding over all j for $j \neq i,k$, we obtain maps $\alpha_2 : \oplus_{l \neq k} R_l^{a_l^{(2)}} \to \oplus_{j \neq i,k} R_j^{a_j^{(1)}}$, $\beta_2 : R_k \to \oplus_{l \neq k} R_l^{a_l^{(2)}}$ and $\gamma_2 : \oplus_{j \neq k} R_j^{n_j} \to \oplus_{j \neq i,k} R_j^{a_j^{(1)}}$ (for some $a_l^{(2)}$) such that $\beta_1 = \alpha_2\beta_2 + \gamma_2[f_j]$. Setting $g_2 = \alpha_1\gamma_2$ we obtain $\alpha_1\beta_1 = \alpha_1\alpha_2\beta_2 + g_2[f_j]$, so

$$f = g_1[f_j] + \alpha_1\beta_1 = \alpha_1\alpha_2\beta_2 + (g_1 + g_2)[f_j].$$

Here, α_2 is a radical map, since all of its summands, the w_j, are radical. See Figure 2.

$$\begin{tikzcd}
\oplus_{j\neq k} R_j^{a_j^{(r)}} \arrow[r, "\alpha_r"'] & \cdots \arrow[r] & \oplus_{j\neq k} R_j^{a_j^{(2)}} \arrow[r, "\alpha_2"'] & \oplus_{j\neq i,k} R_j^{a_j^{(1)}} \arrow[r, "\alpha_1"'] & R_i \\
& & \cdots & & \\
& & & R_k \arrow[uulll, "\beta_r"] \arrow[uul, "\beta_2"] \arrow[uu, "\beta_1"] \arrow[uur, "f"'] \arrow[r, "{[f_j]}"'] & \oplus_{j\neq k} R_j^{n_j} \arrow[uu, "{g_1,g_2,\ldots,g_r}"']
\end{tikzcd}$$

FIGURE 2. Proof of Lemma 3

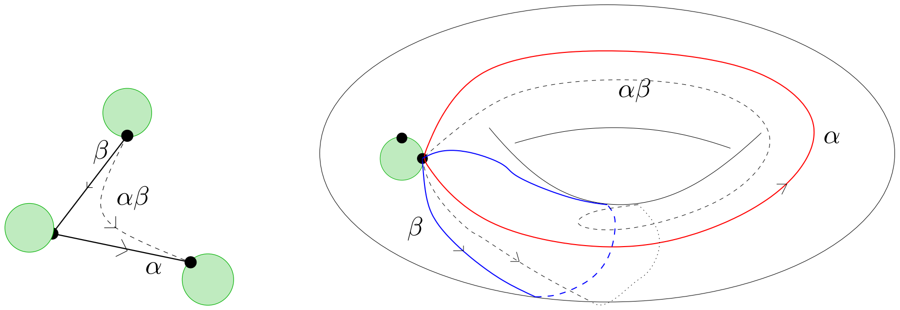

FIGURE 3. Composition of arcs

Iterating this step we construct, for all $r \geq 3$, morphisms $g_r : \bigoplus_{j\neq k} R_j^{n_j} \longrightarrow R_i$, $\alpha_r : \bigoplus_{j\neq k} R_j^{a_j^{(r)}} \longrightarrow \bigoplus_{j\neq k} R_j^{a_j^{(r-1)}}$, and $\beta_r : R_k \longrightarrow \bigoplus_{j\neq k} R_j^{a_j^{(r)}}$ for $r = 3, \ldots n$ (and some $a_l^{(r)}$), such that $f = \beta_n\alpha_n \cdots \alpha_1 + (g_1 + \cdots + g_n)[f_j]$ and each of the α_i is a radical map. Since $\mathcal{C}$ is Hom-finite, the radical of $\operatorname{End}(R)$ is nilpotent and the composition $\beta_n\alpha_n \cdots \alpha_1$ vanishes for n big enough. Therefore f factors through $[f_j]$ and $[f_j]$ is a left add R/R_k-approximation in $\mathcal{C}$. The left minimality of $[f_j]$ follows from the left minimality of each $\underline{f_j}$. $\surd$

3. COLOURED QUIVERS FOR PARTIAL TRIANGULATIONS

Let (S, M) be an unpunctured compact Riemann surface with boundary and marked points. We will always assume that each boundary component contains at least one marked point and that no component of (S, M) is a monogon, a digon or a triangle.

3.1. Composition of arcs.

3.1. Composition of arcs. Let α and β be two oriented arcs in (S, M) with $\beta(1) = \alpha(0)$. The composition $\alpha\beta$ is the arc given by

$$t \longmapsto \begin{cases} \beta(2t) & \text{if } 0 \leq t \leq 1/2 \\ \alpha(2t-1) & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

See Figure 3.

Note that the composition only makes sense for oriented arcs.

3.2. Twisting an arc with respect to a partial triangulation. In this section, our aim is to generalise the flip of triangulations to the twist of an arc with respect to a partial triangulation.

Let $\mathscr{R}$ be a partial triangulation of (S, M), i.e. a collection of non-crossing arcs $\gamma_1, \ldots, \gamma_m$. Let α be an arc in (S, M) which does not cross $\mathscr{R}$ and does not belong

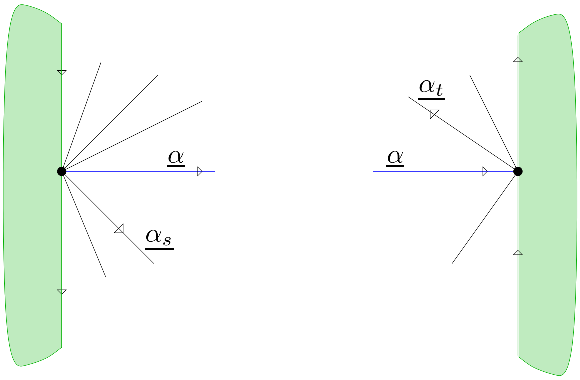

FIGURE 4. Orientation of α_s and α_t

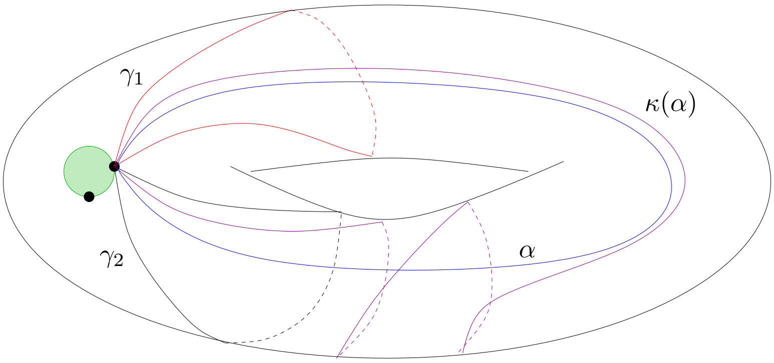

FIGURE 5. An example of a twist

to $\mathscr{R}$, i.e. $\alpha \in A^0_{\mathscr{R}}(S,M)$. We define the *twist* of α with respect to $\mathscr{R}$ as follows: Choose an orientation $\underline{\alpha}$ of α. Consider the arcs of the partial triangulation $\mathscr{R}$ which admit $\underline{\alpha}(0)$ as an endpoint. Restrict to a neighbourhood of $\underline{\alpha}(0)$ small enough not to contain any loop. The orientation of the boundary containing $\underline{\alpha}(0)$ induces an ordering on the parts of the arcs included in the neighbourhood (see Figure 4). Let α_s be the arc, in $\mathscr{R}$ or boundary, which follows α in this ordering (note that it is not allowed to be α itself). Similarly, define α_t with respect to the endpoint $\alpha(1)$. These will be called the *arcs following* α *in* $\mathscr{R}$.

We give α_s and α_t the orientations $\underline{\alpha_s}$ and $\underline{\alpha_t}$ described in the local pictures of Figure 4. Note that this orientation coincides with the orientation of the boundary if α_s or α_t is a boundary arc.

For an oriented arc $\underline{\beta}$, let $[\underline{\beta}]$ denote the underlying unoriented arc. Define the *twist* of the arc α with respect to $\mathscr{R}$ to be the underlying unoriented arc of the composition:

$$\kappa_{\mathscr{R}}(\alpha) = [\underline{\alpha_t}\,\underline{\alpha}\,\underline{\alpha_s}^{-1}].$$

See Figure 5 for an example of a twist. Note that the definition of the arc $\kappa_{\mathscr{R}}(\alpha)$ does not depend on the choice of an orientation for α. It is easy to check, using a case-by-case analysis depending on whether or not α, α_s, α_t are loops and the order in which they appear at their end-points, that $\kappa_{\mathscr{R}}(\alpha)$ does not cross any arc in $\mathscr{R}$, i.e. that $\kappa_{\mathscr{R}}(\alpha) \in A^0_{\mathscr{R}}(S,M)$.

The twist with respect to $\mathscr{R}$ is invertible with inverse $\kappa^{-1}_{\mathscr{R}}$, which can also be defined similarly.

3.3. **Mutation of partial triangulations.** Let $\mathscr{R}$ be a partial triangulation of (S, M), and let $\gamma \in \mathscr{R}$. Write $\mathscr{R} = \overline{\mathscr{R}} \sqcup \{\gamma\}$. The *mutation* of $\mathscr{R}$ at γ is the partial triangulation

$$\mu_\gamma \mathscr{R} = \overline{\mathscr{R}} \sqcup \left\{\kappa_{\overline{\mathscr{R}}}(\gamma)\right\}.$$

If $\mathscr{R}$ is a triangulation, then, for $\gamma \in \mathscr{R}$, $\mu_\gamma \mathscr{R}$ is the usual flip of $\mathscr{R}$ at the arc $\gamma \in \mathscr{R}$; see [FST08, Defn. 3.5].

3.4. **Coloured quivers.** Let α and β be two arcs in (S, M), and let $\mathscr{R}$ be a partial triangulation not containing α. For all $c \in \mathbb{Z}$, define the numbers $q^{(c)}_{\mathscr{R}}(\alpha, \beta)$ by:

$$\begin{aligned} q_{\mathscr{R}}(\alpha, \beta) &= \begin{cases} 2 & \text{if } \beta = \alpha_s = \alpha_t \\ 1 & \text{if } \beta \in \{\alpha_s, \alpha_t\} \text{ and } \alpha_s \neq \alpha_t \\ 0 & \text{otherwise,} \end{cases} \\ q^{(c)}_{\mathscr{R}}(\tau, \tau') &= q_{\mathscr{R}}(\kappa^c_{\mathscr{R}}(\tau), \tau'). \end{aligned}$$

With a partial triangulation $\mathscr{R} = \{\gamma_1, \ldots, \gamma_m\}$, we associate a coloured quiver $Q_{\mathscr{R}}$:

Definition *The coloured quiver $Q_{\mathscr{R}}$ associated with the partial triangulation $\mathscr{R}$ is defined as follows: The set of vertices is $Q_0 = \{1, \ldots, m\}$. The set $Q_1^{(c)}(i, j)$ of c-coloured arrows from vertex i to vertex j has cardinality $q^{(c)}_{\mathscr{R}_i}(\gamma_i, \gamma_j)$, where $\mathscr{R}_i = \mathscr{R} \setminus \{\gamma_i\}$.*

Here also, the colours can be thought of as elements of $\mathbb{Z}/d_k$ where d_k is the periodicity of twisting the arc corresponding to the starting vertex.

For an example of a coloured quiver associated to a partial triangulation of a torus and the effect of mutation on the quiver, see Section 6.2.

4. Cutting along an arc and CY reduction

Let (S, M) be as in Section 3.

4.1. **Cutting along an arc.** Let α be an arc on (S, M) not homotopic to a point or a boundary arc. Fix a representative of α, also denoted by α, whose intersection with the boundary of S consists only of its endpoints. Then the marked surface obtained from (S, M) by *cutting along the arc* α is the Riemann surface with boundary obtained by cutting along the arc α together with the image of the marked points M after cutting. Up to homeomorphism, it does not depend on the choice of representative of α. We will denote it by $(S, M)/\alpha$. Note that if α is not a loop, then each endpoint of α gives rise to two distinct marked points in $(S, M)/\alpha$. If α is a loop, its endpoint gives rise to three distinct marked points in $(S, M)/\alpha$.

The resulting marked surface cannot contain a monogon as a connected component, since α is not homotopic to a point. No connected component can be a bigon, since α is not a boundary arc. If a component homeomorphic to a triangle has been created, we remove it.

There is a natural bijection between the arcs on (S', M') and the arcs of (S, M) which do not cross the arc α. Moreover, the (partial) triangulations of (S', M') correspond, through this bijection, to the (partial) triangulations of (S, M) containing the arc α.

Remark 4. *The surface $(S, M)/\alpha$ can also be constructed as follows. Let $\mathscr{T}$ be a triangulation of (S, M) containing α. The surface (S, M) is then obtained from the triangles of the triangulation by gluing matching sides of triangles in a prescribed orientation. The surface $(S, M)/\alpha$ is obtained from the same triangles by respecting the same gluings except for the sides which correspond to α, which are not glued together anymore.*

Given a collection $\mathscr{R}$ of non-crossing arcs, one can cut successively along each arc. Whatever order is chosen yields the same new surface, by Remark 4. The corresponding surface will be called the *reduction* of (S,M) with respect to $\mathscr{R}$, and will be denoted by $(S,M)/\mathscr{R}$. We will denote the natural bijection between $A^0_{\mathscr{R}}(S,M)$ and $A^0((S,M)/\mathscr{R})$ by $\pi_{\mathscr{R}}$.

4.2. **Compatibility with CY reduction.** Let R be a basic rigid object in $\mathcal{C}_{(S,M)}$, and let $\mathscr{R}$ be the associated partial triangulation. We denote by $\mathcal{C}_R = {}^{\perp}(\Sigma R)/(R)$ the Calabi–Yau reduction of $\mathcal{C}_{(S,M)}$ with respect to R, and by $(S,M)/\mathscr{R}$ the marked surface obtained from (S,M) by cutting along the arcs of $\mathscr{R}$.

Proposition 5. *The triangulated categories $\mathcal{C}_{(S,M)/\mathscr{R}}$ and $\mathcal{C}_R$ are equivalent.*

Proof. Complete the collection of arcs $\mathscr{R}$ to a triangulation $\mathscr{T}$. Let (Q,W) be the QP associated with $\mathscr{T}$. By definition, there is an equivalence of triangulated categories $\mathcal{C}_{(S,M)} \simeq \mathcal{C}_{(Q,W)}$. By [Kel09, Theorem 7.4] (see section 1.3), the category $\mathcal{C}_R$ is triangle equivalent to the cluster category $\mathcal{C}_{(Q',W')}$, where (Q',W') is obtained from (Q,W) by deleting the vertices of Q which correspond to arcs in $\mathscr{R}$, and all adjacent arrows. On the other hand, the arcs in $\mathscr{T}$ not in $\mathscr{R}$ induce a triangulation of the surface $(S,M)/\mathscr{R}$. It follows from Remark 4 that (Q',W') is the QP associated with this triangulation. Thus $\mathcal{C}_{(S,M)/\mathscr{R}}$ is equivalent to $\mathcal{C}_{(Q',W')}$. $\surd$

Remark: Lemma 7 shows that the equivalence above is well-behaved with respect to well-chosen bijections between arcs and exceptional objects.

Figure 6 shows the effect of cutting along an arc in a triangulation of a torus with a single boundary component containing two marked points. We cut along the red arc (numbered 3) and obtain a cylinder with four marked points as shown, with triangulation given by the remaining arcs. In the last step, the cylinder has been rotated around to get a simpler picture. The effect on the corresponding quiver with potential is shown in Figure 7.

Proposition 6. *Let (S,M) be a marked surface and $\mathscr{R}$ a partial triangulation of (S,M). Let $\mathscr{R}'$ be a collection of arcs containing $\mathscr{R}$. Then the coloured quiver associated to $\pi_{\mathscr{R}}(\mathscr{R}'\setminus\mathscr{R})$ in $(S,M)/\mathscr{R}$ coincides with the coloured quiver associated to $\mathscr{R}'$ in (S,M) with the vertices corresponding to $\mathscr{R}$ and all arrows incident with them removed.*

Proof. It is clear that the vertices of each coloured quiver correspond to the arcs in $\mathscr{R}'\setminus\mathscr{R}$. In the definition of the twist $\kappa_{\mathscr{R}}$ (see Section 3.2), no distinction is made between arcs in $\mathscr{R}$ and boundary arcs. Then, looking at the definition of the coloured quiver of a partial triangulation (see Section 3.4) we see that the arrows between arcs in $\mathscr{R}'\setminus\mathscr{R}$ are the same when considered in either coloured quiver. The result follows. $\surd$

We now give an example. In Figure 8, we start with a partial triangulation of a torus with a single boundary component with two marked points. This has been obtained by removing arcs 4 and 5 from the triangulation considered in Figure 6. As before, we cut along the red arc (numbered 3) and obtain a cylinder with four marked points as shown, with a partial triangulation given by the remaining arcs. Figure 9 gives the corresponding coloured quiver associated to the partial triangulation in Figure 8, together with the new quiver obtained after cutting along the red arc (numbered 3), i.e. with vertex 3 and all arrows incident with it removed.

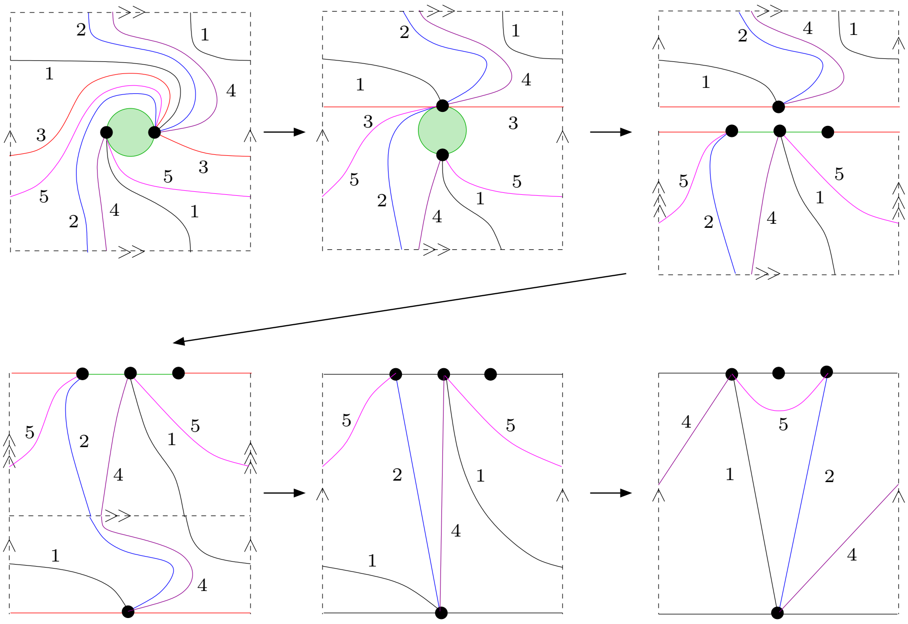

Figure 6. Cutting along an arc, numbered 3, in a torus to get a cylinder: triangulation case.

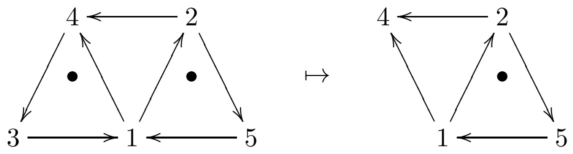

Figure 7. The change in the quiver with potential from the cut in Figure 6. The potential in each case is given by the sum of the 3-cycles containing black dots.

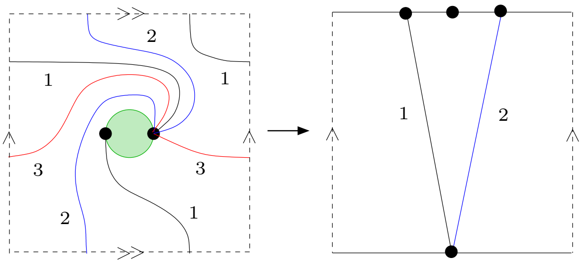

Figure 8. Cutting along an arc in a torus to get a cylinder: partial triangulation case.

5. Compatibility

5.1. **Compatibility of the mutations.** Let $\mathscr{R}$ be a partial triangulation of (S, M). Complete $\mathscr{R}$ to a triangulation $\mathscr{T}$ of (S, M), and let T be the associated cluster tilting object in $\mathcal{C} = \mathcal{C}_{(S,M)}$. Let R be the direct summand of T corresponding to

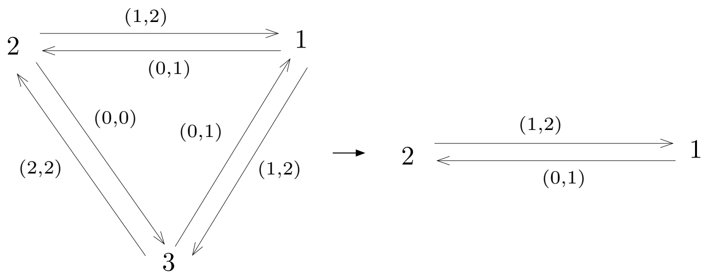

FIGURE 9. The effect of cutting along arc 3 on the coloured quiver of the partial triangulation in Figure 8.

$\mathscr{R}$. We thus obtain a map $\alpha \mapsto X_\alpha$ between the arcs of (S,M) and the isomorphism classes of exceptional objects in $\mathcal{C}_{(S,M)}$ (see section 1.2.3). We denote by $\pi_{\mathscr{R}}$ the bijection $A^0_{\mathscr{R}}(S,M) \to A^0\big((S,M)/\mathscr{R}\big)$; recall also that π_R denotes the functor ${}^{\perp}(\Sigma R) \to \mathcal{C}_R$. Consider the partial triangulation $\pi_{\mathscr{R}}(\mathscr{T} \setminus \mathscr{R})$ of $(S,M)/\mathscr{R}$. Note that $T' := \pi_R(T)$ is cluster tilting in $\mathcal{C}_{(S,M)/\mathscr{R}} \simeq \mathcal{C}_R$ by [IY08, Theorem 4.9]. This cluster tilting object induces a bijection $\beta \mapsto Y_\beta$ between the arcs in $(S,M)/\mathscr{R}$ and the exceptional objects in $\mathcal{C}_R$.

Lemma 7. *Let α be an arc in $A^0_{\mathscr{R}}(S,M)$. Then the image of X_α under π_R is isomorphic to $Y_{\pi_{\mathscr{R}}\alpha}$.*

Proof. Using [JP, Proposition 3.5], the modules associated with $\pi_R X_\alpha$ and $Y_{\pi_{\mathscr{R}}\alpha}$ are seen to be isomorphic. $\surd$

Let α be an arc in (S,M) which is not in $\mathscr{R}$ and which does not cross $\mathscr{R}$, i.e. $\alpha \in A^0_{\mathscr{R}}(S,M)$. Fix an orientation $\underline{\alpha}$ of α and let α_s and α_t be the two (possibly boundary) arcs following α in $\mathscr{R}$ (as defined in section 3.2). Recall that $\kappa_{\mathscr{R}}(\alpha)$ is defined to be $[\underline{\alpha_t}\underline{\alpha}\underline{\alpha_s}^{-1}]$.

If γ is any arc in (S,M) then recall we have (from [BZ]; see Section 1.2.3):

$$\Sigma X_\gamma = X_{\kappa_\phi(\gamma)}, \tag{1}$$

where ϕ denotes the empty set of arcs in (S,M). Thus $\kappa_\phi(\alpha)$ is obtained from the arc α by composition with the two boundary arcs which follow α (see Section 3.2).

The following corollary describes the twist of an arc in terms of the action of the shift functor of an Iyama-Yoshino reduction.

Corollary 8. *Under the bijection $A^0_{\mathscr{R}}(S,M) \leftrightarrow A^0\big((S,M)/\mathscr{R}\big)$, the induced action of the shift functor of $\mathcal{C}_{(S,M)/\mathscr{R}}$ on $A^0_{\mathscr{R}}(S,M)$ coincides with that of the twist $\kappa_{\mathscr{R}}$. In other words, we have a commutative diagram:*

$$\begin{tikzcd}
A^0\big((S,M)/\mathscr{R}\big) \arrow[r, "shift"] & A^0\big((S,M)/\mathscr{R}\big) \\
A^0_{\mathscr{R}}(S,M) \arrow[u, "\pi_{\mathscr{R}}"'] \arrow[r, "\kappa_{\mathscr{R}}"] & A^0_{\mathscr{R}}(S,M). \arrow[u, "\pi_{\mathscr{R}}"']
\end{tikzcd}$$

Proof. Let $\alpha \in A^0_{\mathscr{R}}(S,M)$. By (1), noting that $\mathscr{R}$ becomes part of the boundary of $(S,M)/\mathscr{R}$, we have $\Sigma_R Y_{\pi_{\mathscr{R}}(\alpha)} \simeq Y_{\pi_{\mathscr{R}}\kappa_{\mathscr{R}}(\alpha)}$. The result follows. $\surd$

We now have the ingredients we need in order to show that the two mutations (of partial triangulations and rigid objects) are compatible.

Proposition 9. *Let R be the rigid object in $\mathcal{C}_{(S,M)}$ associated with the partial triangulation $\mathscr{R}$ as above. Let α be an arc in $A^0_{\mathscr{R}}(S,M)$. Then we have:*

$$\kappa_R X_\alpha \simeq X_{\kappa_{\mathscr{R}}(\alpha)} \text{ and } \mu_k R \simeq X_{\mu_k \mathscr{R}}.$$

Proof. Since α does not cross $\mathscr{R}$, it follows from [BZ] (see Section 1.2.3) that $X_\alpha \in {}^\perp(\Sigma R)$. Similarly, $X_{\kappa_{\mathscr{R}}(\alpha)} \in {}^\perp(\Sigma R)$, since $\kappa_{\mathscr{R}}(\alpha)$ does not cross $\mathscr{R}$. By the description of the shift Σ_R of $\mathcal{C}_R$ in [IY08, 4.1], $\pi_R(\kappa_R(X_\alpha)) \simeq \Sigma_R(\pi_R X_\alpha)$ in $\mathcal{C}_R$. By Lemma 7, $\Sigma_R(\pi_R X_\alpha) \simeq \Sigma_R Y_{\pi_{\mathscr{R}}(\alpha)}$. By Corollary 8, we have $\Sigma_R Y_{\pi_{\mathscr{R}}(\alpha)} \simeq Y_{\pi_{\mathscr{R}}\kappa_{\mathscr{R}}(\alpha)}$. By Lemma 7, $Y_{\pi_{\mathscr{R}}\kappa_{\mathscr{R}}(\alpha)} \simeq \pi_R X_{\kappa_{\mathscr{R}}}(\alpha)$. Hence $\pi_R(\kappa_R X_\alpha) \simeq \pi_R(X_{\kappa_{\mathscr{R}}(\alpha)})$.

Note that $\kappa_R X_\alpha$ is an indecomposable object in ${}^\perp(\Sigma R)$ which is not in $\operatorname{add} R$ (see Section 2.1). Since $\kappa_{\mathscr{R}}\alpha$ does not cross $\mathscr{R}$ and does not lie in $\mathscr{R}$, the same is true of $X_{\kappa_{\mathscr{R}}(\alpha)}$. It follows that $\kappa_R X_\alpha \simeq X_{\kappa_{\mathscr{R}}(\alpha)}$, proving the first part of the Proposition. The second part follows. $\surd$

5.2. **Compatibility of the coloured quivers.** As in the previous section, let $\alpha \in A^0_{\mathscr{R}}(S,M)$; we fix an orientation of α and let α_s and α_t be the two (possibly boundary) arcs following α in $\mathscr{R}$ (as defined in section 3.2). Note that it is possible that $\alpha_s = \alpha_t$. We choose a triangulation $\mathscr{T}$ of (S,M) containing $\mathscr{R}$ and α and let T be the corresponding cluster tilting object, containing R and as a direct summand and X_α as an indecomposable direct summand. Recall that $X_\gamma = 0$ if γ is a boundary arc.

Lemma 10. *There is a minimal left* $\operatorname{add} R$*-approximation of X_α in $\mathcal{C}_{(S,M)}$ of the form*

$$X_\alpha \longrightarrow X_{\alpha_s} \oplus X_{\alpha_t}.$$

Proof. By the 2-Calabi-Yau tilting theorem (see Section 1.2.2), the functor $H = \mathcal{C}(T, \Sigma -)$ induces an equivalence between $\mathcal{C}/T$ and $\operatorname{mod} J(Q,W)$. Hence H induces an equivalence between $\Sigma^{-1} \operatorname{add} T$ and the category $\mathcal{P}$ of projective modules over $J(Q,W)$. Let $P_\alpha = H(\Sigma^{-1} X_\alpha)$ for each arc α in $\mathscr{T}$ and let $\mathcal{P}_R = H(\Sigma^{-1} \operatorname{add} R)$. Then it is enough to show that there is a minimal left $\mathcal{P}_R$-approximation of P_α in $\operatorname{mod} J(Q,W)$ of the form

$$P_\alpha \longrightarrow P_{\alpha_s} \oplus P_{\alpha_t}.$$

We recall that $J(Q,W)$ is gentle (see Section 1.1). In particular, the defining relations are all zero-relations. Let $\gamma_1, \gamma_2, \ldots, \gamma_j$ be the arcs in $\mathscr{T}$ incident with $\alpha(0)$ which are after α in the order induced by the orientation of the boundary at $\alpha(0)$ (and listed in that order); see Section 3.2. Similarly, let $\delta_1, \delta_2, \ldots, \delta_k$ be the arcs in $\mathscr{T}$ around $\alpha(1)$ which are after α in the order induced by the orientation of the boundary at $\alpha(1)$.

Because of the zero-relations in $J(Q,W)$, the only non-zero paths in Q starting at α are paths:

$$\alpha \longrightarrow \gamma_1 \longrightarrow \gamma_2 \longrightarrow \cdots \longrightarrow \gamma_j$$

and

$$\alpha \longrightarrow \delta_1 \longrightarrow \delta_2 \longrightarrow \cdots \longrightarrow \delta_k.$$

Thus the only non-zero morphisms from P_α to some indecomposable projective module lie in the composition chains:

$$P_\alpha \longrightarrow P_{\gamma_1} \longrightarrow P_{\gamma_2} \longrightarrow \cdots \longrightarrow P_{\gamma_j}$$

and

$$P_\alpha \longrightarrow P_{\delta_1} \longrightarrow P_{\delta_2} \longrightarrow \cdots \longrightarrow P_{\delta_k},$$

or are linear combinations of these (noting that the chains may overlap).

If α_t is a boundary arc, but α_s is not, then α_s occurs in the first chain above. It is easy to see that the non-zero map $P_\alpha \longrightarrow P_{\alpha_s}$ coming from the chain of compositions is a left minimal $\mathcal{P}_R$-approximation and we are done. The argument is similar if α_s is a boundary arc but α_t is not. If both α_s and α_t are boundary arcs then the zero map is a left minimal $\mathcal{P}_R$-approximation.

We are left with the case where neither α_s nor α_t is a boundary arc. Thus $\alpha_s = \gamma_i$ for some i while $\alpha_t = \delta_{i'}$ for some i'. Let f_s and f_t be the non-zero morphisms arising from the above chains of compositions and let $f : P_\alpha \longrightarrow P_{\alpha_s} \oplus P_{\alpha_t}$ be the map with components f_s, f_t. It follows from the above that f is a left $\mathcal{P}_R$-approximation of P_α. It remains to check that f is left minimal.

We note that if we had $f_s = kh$ for some $h : P_\alpha \longrightarrow P_\beta$ and $k : P_\beta \longrightarrow P_{\alpha_s}$ for some $\beta \in \mathscr{R}$ then k would have to be an isomorphism since the path in Q from α to α_s is not equal to any other path in Q from α to α_s, and α_s is the first arc in $\mathscr{R}$ appearing along this path. A similar statement holds for f_t.

If f were not left minimal, a summand of form $0 \longrightarrow P_{\alpha_s}$ (respectively, $0 \longrightarrow P_{\alpha_t}$) would split off and we would have a left $\mathcal{P}_R$-approximation of the form $g_s : P_\alpha \longrightarrow P_{\alpha_s}$ (respectively, $g_t : P_\alpha \longrightarrow P_{\alpha_t}$). We consider only the first case (the second case requires a similar argument). In this case, f_t factors through g_s, i.e. $f_t = vg_s$ for some map $v : P_{\alpha_s} \to P_{\alpha_t}$. By the above, v is an isomorphism and $g_s = v^{-1} f_t$.

Again, since g_s is a left $\mathcal{P}_R$-approximation, we also have that f_s factors through g_s, i.e. $f_s = wg_s$ for some $w : P_{\alpha_s} \longrightarrow P_{\alpha_t}$. By the above, w is an isomorphism. Hence we have $f_s = wv^{-1} f_t$ where wv^{-1} is an isomorphism. This is a contradiction since f_s and f_t arise from two different paths starting at α. The result is proved. $\surd$

Theorem 11. *Let R be the rigid object in $\mathcal{C}_{(S,M)}$ associated with the partial triangulation $\mathscr{R}$. Then the coloured quivers $Q_{\mathscr{R}}$ and Q_R coincide.*

Proof. By Proposition 9, it is enough to prove that the sets of 0-coloured arrows coincide. This follows from Lemma 10. $\surd$

6. Some examples

6.1. **The A_n case.** In this section, we assume that the category $\mathcal{C}$ is the cluster category of type A_n.

Suppose that R is a basic rigid object in $\mathcal{C}$. In Section 2.1 we have associated a coloured quiver Q with R. If R_k is an indecomposable direct summand of R then the rigid object $\mu_k R$ also has a coloured quiver, $\widetilde{Q}$, associated with it, and we can ask if $\widetilde{Q}$ can be computed from Q. This is known in the d-cluster-tilting object case of a $d{+}1$-Calabi-Yau category [BT09, Thm. 2.1] but is not known for a general rigid object. In Section 7 we will indicate some results in this direction with a categorical proof, but here we give a complete answer for the cluster category of type A using a combinatorial (geometric) proof. In this case, the corresponding surface is a disk with $n+3$ marked points (see [CCS06]), which we shall denote (S,M); as usual, we denote by $\mathscr{R}$ the set of noncrossing arcs in (S,M) corresponding to the indecomposable direct summands of R.

The following is easy to check (d_i was defined in section 2.1):

Lemma 12. *Let $i \in I$. Then*

(a) $d_i = \max\{c \geq 0 \,:\, q_{ij}^{(c)} \neq 0\} + \min\{c \geq 0 \,:\, q_{ji}^{(c)} \neq 0\} + 1.$

(b) *For all* $c \in \mathbb{Z}$, $q_{ii}^{(c)} = 0$.

Let R_i be a summand of R and let α_i denote the corresponding arc in (S,M). If the arc corresponding to R_i in $(S,M)/(\mathscr{R} \setminus \alpha_i)$ is a diameter of a $2d$-sided polygon for some d then twisting α_i with respect to $\mathscr{R} \setminus \mathscr{R}_i$ has order d, while in all other cases the order of this twisting coincides with the number of sides of the polygon. See Figure 10 for an example.

It is easy to see that this situation occurs if and only if for any j such that i and j are ends of a common arrow in Q, there is a unique colour c such that $q_{ij}^{(c)} \neq 0$,

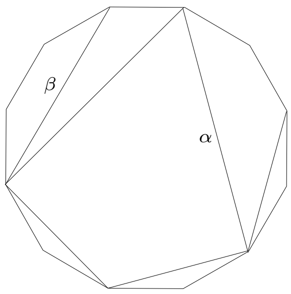

FIGURE 10. The arc α has order 3 under twisting and lies in a hexagon, while the arc β has order 5 and lies in a pentagon.

and thus this can be detected in Q. This irregularity makes it difficult to compute the new coloured quiver after mutation of $\mathscr{R}$ at an arc. But, since it is detectable in Q, we can make a first pass and add second arrows between i and j to Q in such cases. We do this when computing the coloured quiver Q of $\mathscr{R}$ by continuing to mutate $\mathscr{R}$ at α_i a further d times and adding the new arrows arising to Q; see step (ii) below. At the end, this procedure must be reversed.

Thus, given a coloured quiver Q of a rigid object and vertex k as above, we consider the following algorithm to define a new quiver $\widetilde{Q}$. The letters i, j always refer to distinct vertices.

Type A algorithm:

(i) Compute d_i for all $i \in I$ using the formula in Lemma 12.

(ii) For any pair of vertices i, j for which there is a unique c with $q_{ij}^{(c)} \neq 0$, set $q_{ij}^{(c+d_i)} = 1$. For all such i arising in this way, replace d_i with $2d_i$.

We note that after this step, for any two vertices i, j, there will either be no arrows between i and j or exactly two arrows in each direction. In the latter case, we indicate the two colours in a given direction by $l \leq l'$ for some letter l and we will write the colours as a pair labelling a single arrow.

(iii) Suppose we have the following arrows:

$$\begin{tikzcd}
i \arrow[r, shift left, "{(a,a')}"] & k \arrow[l, shift left, "{(0,b')}"] & & j \arrow[r, shift left, "{(c,c')}"] & k \arrow[l, shift left, "{(d,d')}"]
\end{tikzcd}$$

where $d \neq 0$. Add the following arrows:

$$\begin{tikzcd}
i \arrow[r, shift left, "{(d,d+a'-a)}"] & j \arrow[l, shift left, "{(c,c')}"]
\end{tikzcd}$$

and cancel any pairs of arrows between i and j in the same direction whose colours differ by 1.

(iv) Suppose we have the following full subquiver of Q where $j \neq k$:

$$\begin{tikzcd}
k \arrow[r, shift left, "{(0,b')}"] & i \arrow[l, shift left, "{(a,a')}"] \arrow[r, shift left, "{(d,d')}"] & j \arrow[l, shift left, "{(c,c')}"]
\end{tikzcd} .$$

Then change the arrows between i and j to:

$$i \underset{(c,c')}{\overset{(d,d+b')}{\rightleftarrows}} j\ .$$

(v) Apply the following rule to all vertices i with an arrow to or from k:

$$i \underset{(b,b')}{\overset{(a,a')}{\rightleftarrows}} k \ \mapsto \begin{cases} i \underset{(b-1,b'-1)}{\overset{(a+1,a'+1)}{\rightleftarrows}} k & \text{if } b \neq 0; \\ i \underset{(b'-1,a+b')}{\overset{(0,a'-a)}{\rightleftarrows}} k & \text{if } b = 0. \end{cases}$$

If $b = 0$ then d_i is replaced with $d_i + b' - a = b' + a' - a$.

(vi) Whenever we have arrows:

$$i \underset{(d,d')}{\overset{(c,c+\frac{1}{2}d_i)}{\rightleftarrows}} j\ ,$$

remove the arrow with colour $c + \frac{1}{2}d_i$ and replace d_i with $\frac{1}{2}d_i$.

Proposition 13. *Let $R = \bigoplus_{i\in I} R_i$ be a rigid object in the cluster category of type A_n, with associated quiver Q. Let R_k be a summand of R and let $\widetilde{Q}$ denote the quiver of $\mu_k(R)$. Then $\widetilde{Q}$ can be computed using the above algorithm.*

Proof. Let Γ be the quiver obtained from Q after Step (ii) of the algorithm has been applied and let $\widetilde{\Gamma}$ be the quiver so obtained from $\widetilde{Q}$. Since Step (vi) takes $\widetilde{\Gamma}$ to $\widetilde{Q}$ it is enough to show that Steps (iii) to (v) take Γ to $\widetilde{\Gamma}$.

We consider each possible configuration of the arcs corresponding to the summands of R in the disc and check that in each case, the above rule gives the correct answer. It is easy to check that the algorithm works in the case where there are no arrows of colour zero starting at k. The other possible configurations are given in Figures 11,12,13 and 14, together with the corresponding quivers Γ and $\widetilde{\Gamma}$. In each case, the label on part of the boundary indicates the number of boundary segments between the two nearest arc ends on the boundary. (The black dot indicates the end of arc k to indicate that this has changed after the mutation). Case I can be regarded as an instance of Step (iii) with $a = r$, $a' = r + s$, $b = 0$, $b' = q + t + 1$, $c = t$, $c' = p + t$, $d = q$ and $d' = q + r + 1$, followed by Step (v). Case II can be regarded as an instance of Step (iii) with $a = q + t + 1$, $a' = q + r + t + 1$, $b = 0$, $b' = s$, $c = t$, $c' = p + t$, $d = q + 1$ and $d' = q + s + 1$, followed by Step (v). Case III can be regarded as instance of Step (iv) with $a = s$, $a' = q + s + t + 1$, $b = 0$, $b' = r$, $c = t$, $c' = p + t$, $d = q$ and $d' = q + s + 1$, followed by Step (v). In Case IV, only Step (v) is applied. The result follows. $\surd$

6.2. **An example with infinitely many colours.** We consider again the example from Figure 8, i.e. a torus with a single boundary component with two marked points. We show again the partial triangulation of this surface in Figure 15. Several copies are drawn to make it easier to see mutations at each of the arcs. The corresponding coloured quiver is given below the surface: note that removing any of the three arcs leaves a hexagon; it follows that mutation at any of the arcs has order 3, and we get finitely many colours: 0, 1 and 2, appearing as labels on the arrows.

Now suppose we mutate at arc 1. We obtain the partial triangulation in Figure 16; the corresponding quiver is given below the picture of the surface. Here, an

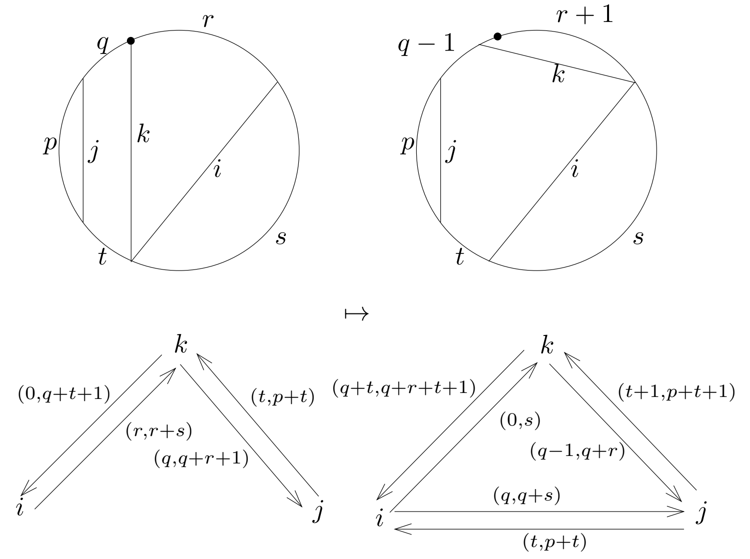

Figure 11. Case I. Here we have $p \geq 2$, $q \geq 1$, $r \geq 1$, $s \geq 2$, $t \geq 0$. Note that d_i changes from $r+s+1$ to $q+t+s+1$.

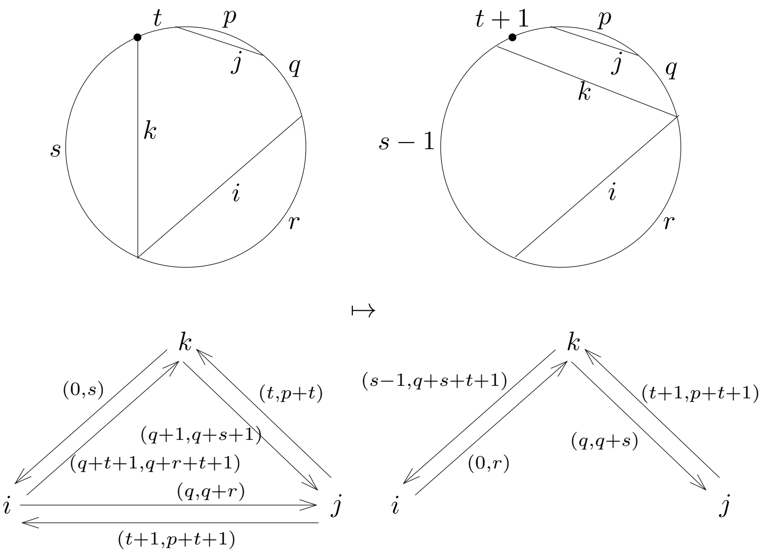

Figure 12. Case II. Here we have $p \geq 2$, $q \geq 0$, $r \geq 2$, $s \geq 2$, $t \geq 0$. Note that d_i changes from $q+r+t+2$ to $r+s$.

arrow is labelled with $\mathbb{Z}$ to represent an infinite number of arrows, one coloured n for each integer n. This infinity of arrows comes from mutating at arc number 2. If we cut along the remaining arcs in the partial triangulation, we obtain a cylinder. Then, after each mutation a small neighbourhood of the triangulation is the same at each end (which explains the regularity), but as more and more mutations are made the arc wraps itself more and more around the cylinder. Thus we see that, even if the quiver is locally finite to start with, after a mutation it might not be.

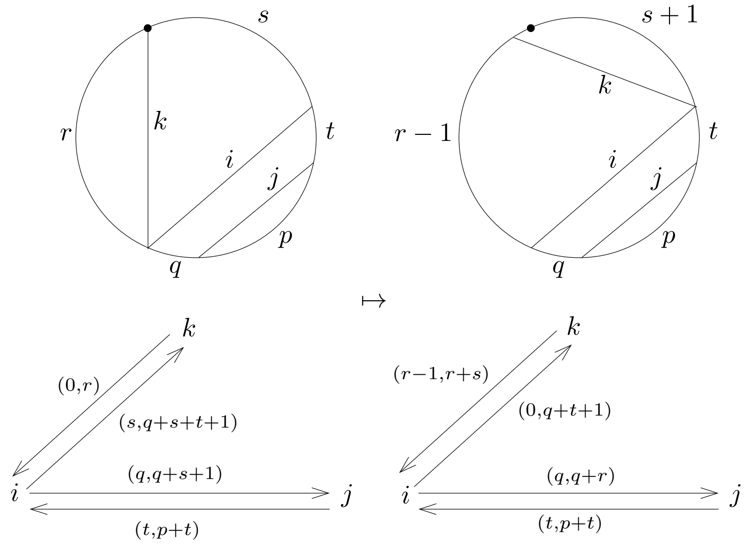

FIGURE 13. Case III. Here we have $p \geq 2$, $q \geq 0$, $r \geq 2$, $s \geq 0$, $t \geq 0$, $q + t \geq 1$. Note that d_i changes from $q + s + t + 2$ to $q + r + t + 1$.

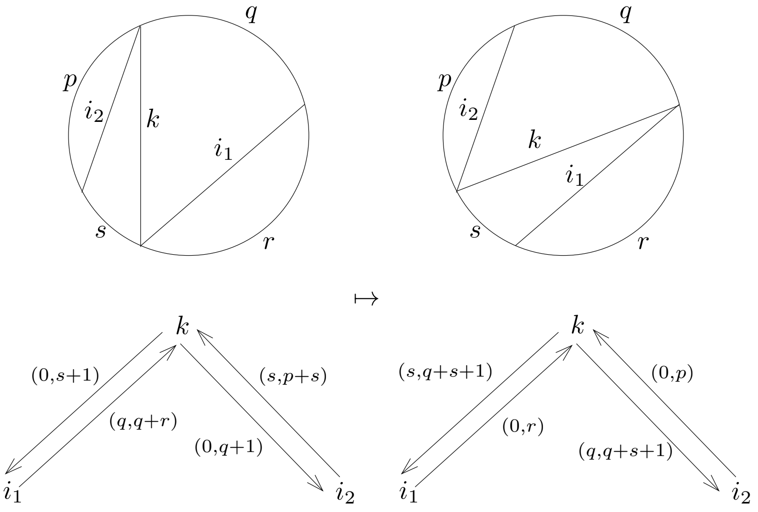

FIGURE 14. Case IV. Here we have $p \geq 2$, $q \geq 1$, $r \geq 2$, $s \geq 1$. Note that d_{i_1} changes from $q + r + 1$ to $r + s + 1$ and d_{i_2} changes from $p + s + 1$ to $p + q + 1$.

Remark 14. *We note that in the example in Figure 16, the coloured quiver contains a two-cycle of arrows both coloured zero, a situation that does not arise in the coloured quivers arising in m-cluster categories* [BT09, Sect. 2].

7. PARTIAL CATEGORICAL INTERPRETATION

Let $\mathcal{C}$ be any Hom-finite, Krull–Schmidt, 2-Calabi–Yau triangulated k-category. Let Q be the coloured quiver associated with a rigid object $R \in \mathcal{C}$. Let $\widetilde{Q}$ be the coloured quiver associated with $\mu_k R$, for some k. The periodicity associated with the vertex i of Q (resp. $\widetilde{Q}$) is denoted by d_i (resp. d'_i).

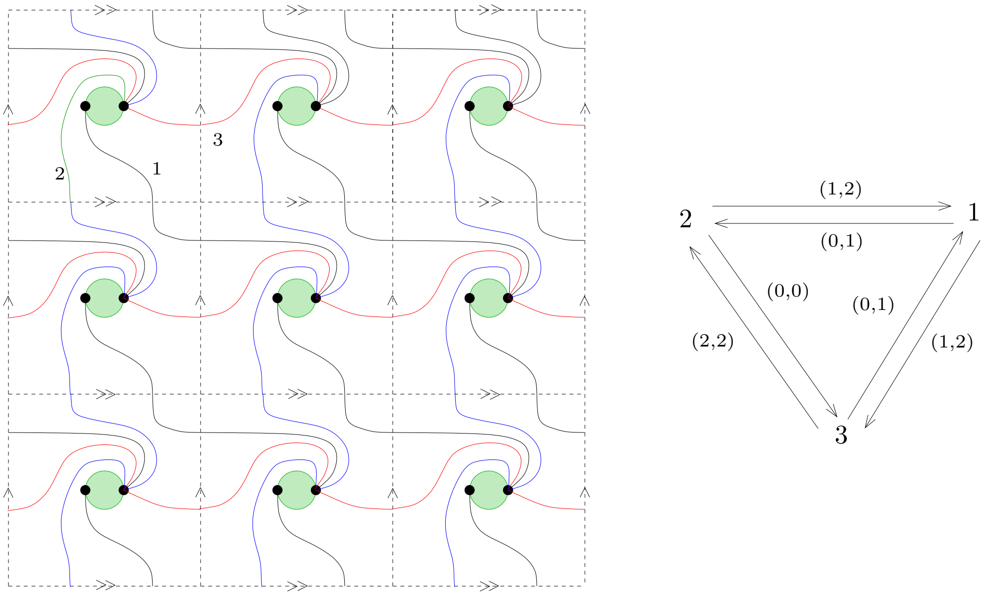

FIGURE 15. A partial triangulation of a torus with a single boundary component and two marked points and the corresponding coloured quiver.

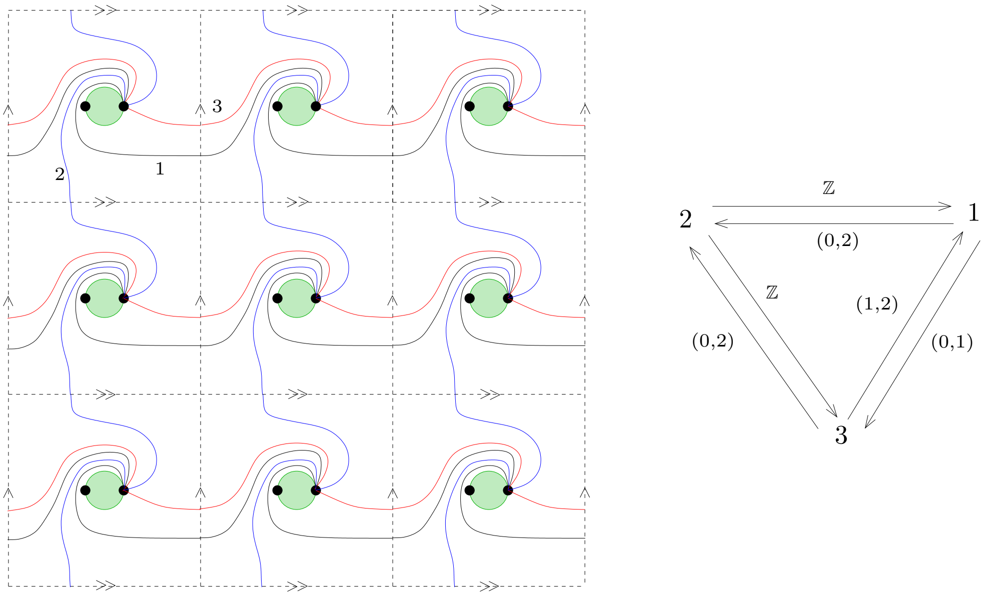

FIGURE 16. The result of mutating the partial triangulation in Figure 15 at arc 1 and the corresponding coloured quiver.

Theorem 15. (i) *We have:*

- $d_k = d'_k$ *and*
- *for any* $j \in Q_0$ *and any* $c \in \mathbb{Z}/d_k$, $\widetilde{q}^{(c)}_{k,j} = q^{(c+1)}_{k,j}$.

(ii) *Let* $i, j \in Q_0$ *be such that* $q^{(0)}_{k,j} = 0 = q^{(0)}_{k,i}$. *Then we have:*

- $d'_i = d_i$, $d'_j = d_j$;

- *for any* $c \in \mathbb{Z}/d_j$, $\widetilde{q}_{j,k}^{(c)} = q_{j,k}^{(c-1)}$*;*
- *for any* $c \in \mathbb{Z}/d_i$, $\widetilde{q}_{i,j}^{(c)} = q_{i,j}^{(c)}$*.*

We note that, for the cases covered by the theorem, the new coloured quiver depends only on the old coloured quiver and not on the particular choice of rigid object or category $\mathcal{C}$.

7.1. **Proof of Theorem 15.** We break the proof of Theorem 15 down into smaller steps, which we present as individual lemmas.

Let $R \in \mathcal{C}$ be rigid and let Q be the associated coloured quiver.

Lemma 16. *We have* $d'_k = d_k$ *and, for any* $c \in \mathbb{Z}/d_k$ *and any* $j \in Q_0$*,*

$$\widetilde{q}_{k,j}^{(c)} = q_{k,j}^{(c+1)}.$$

Proof. The exchange triangles for $R_k^{(1)}$ can be deduced from those for R_k, so that we have $(R_k^{(1)})^{(c)} = R_k^{(c+1)}$. √

Lemma 17. *Let* $j \in Q_0$ *be such that* $q_{k,j}^{(0)} = 0$*. Then* $d'_j = d_j$ *and for any* $c \in \mathbb{Z}/d_j$ *we have:*

$$\widetilde{q}_{j,k}^{(c)} = q_{j,k}^{(c-1)}.$$

Proof. By Corollary 2, we may assume that $R = R_j \oplus R_k$. Since $q_{k,j}^{(0)} = 0$, the first exchange triangle for R_k with respect to R_j is:

$$R_k \longrightarrow 0 \longrightarrow R_k^{(1)} \xrightarrow{=} \Sigma R_k.$$

Let

$$\ldots,\ R_j^{(-1)} \longrightarrow R_k^{s_{-1}} \longrightarrow R_j \longrightarrow \Sigma R_j^{(-1)}\ ,\ R_j \longrightarrow R_k^{s_0} \longrightarrow R_j^{(1)} \longrightarrow \Sigma R_j\ ,\ldots$$

be the exchange triangles for R_j with respect to R_k. Since $q_{k,j}^{(0)} = 0$, we have $s_{-1} = 0$ and R_j is isomorphic to $\Sigma R_j^{(-1)}$. The exchange triangles for R_j with respect to $R_k^{(1)} = \Sigma R_k$ are thus obtained from those with respect to R_k by applying the shift functor:

$$\begin{tikzcd}
 & \vdots & & \\
\Sigma R_j^{(-3)} \arrow[r] & \Sigma R_k^{s_{-3}} \arrow[r] & \Sigma R_j^{(-2)} \arrow[r] & {} \\
\Sigma R_j^{(-2)} \arrow[r] & \Sigma R_k^{s_{-2}} \arrow[r] & R_j \arrow[r] & {} \\
R_j \arrow[r] & 0 \arrow[r] & \Sigma R_j \arrow[r] & {} \\
\Sigma R_j \arrow[r] & \Sigma R_k^{s_0} \arrow[r] & \Sigma R_j^{(1)} \arrow[r] & {} \\
\Sigma R_j^{(1)} \arrow[r] & \Sigma R_k^{s_1} \arrow[r] & \Sigma R_j^{(2)} \arrow[r] & {} \\
 & \vdots & &
\end{tikzcd}$$

√

Lemma 18. *Let* $i, j \in Q_0$ *be such that* $q_{k,j}^{(0)} = 0 = q_{k,i}^{(0)}$*. Then, for any* $c \in \mathbb{Z}/d_i$*, we have:*

$$\widetilde{q}_{i,j}^{(c)} = q_{i,j}^{(c)}.$$

Proof. By Corollary 2, we may assume that $R = R_i \oplus R_j \oplus R_k$. Let

$$\ldots,\ R_i^{(-1)} \to R_j^{t_{-1}} \to R_i \to \Sigma R_i^{(-1)},\ R_i \to R_k^{s_0} \oplus R_j^{t_0} \to R_i^{(1)} \to \Sigma R_i\ ,\ldots$$

be the exchange triangles in $\mathcal{C}$ for R_i with respect to $R_j \oplus R_k$. We denote by $R_i^{(c)*}$ the twists of R_i with respect to $\mu_k R / R_i$. Our assumptions have the following consequences:

(i) $R_k^{(1)} = \Sigma R_k$ and
(ii) the spaces $\mathcal{C}(R_k, R_j)$ and $\mathcal{C}(R_k, R_i)$ vanish.

Let $\underline{\mathcal{C}}$ be the Iyama–Yoshino reduction of $\mathcal{C}$ with respect to $\Sigma R_k = R_k^{(1)}$. The image in $\underline{\mathcal{C}}$ of a morphism $f \in \mathcal{C}$ is denoted $\underline{f}$.

By induction on $c \geq 0$, we are going to construct:

(a) A minimal left $\operatorname{add} R_j$-approximation $R_i^{(c)*} \longrightarrow R_j^{t_c}$ in $\underline{\mathcal{C}}$;
(b) a triangle $X_{c+1} \to R_i^{(c+1)} \to R_i^{(c+1)*} \to \Sigma X_{c+1}$ in $\mathcal{C}$, with X_{c+1} in $\operatorname{add} R_k$;
(a') a minimal right $\operatorname{add} R_j$-approximation $R_j^{t_{-c-1}} \longrightarrow R_i^{(-c)*}$ in $\underline{\mathcal{C}}$ and
(b') a triangle $X_{-c-1} \to R_i^{(-c-1)} \to R_i^{(-c-1)*} \to \Sigma X_{-c-1}$ in $\mathcal{C}$, with X_{-c-1} in $\operatorname{add} R_k$.

The result then follows from (a) and (a') by Corollary 2.

Let us first prove that (a) and (b) hold for $c = 0$. Note that, by (ii), both R_i and R_j belong to $(\Sigma^{-1} R_k^{(1)})^{\perp}$, so that (a) makes sense. Let us denote by $\begin{bmatrix} f' \\ f \end{bmatrix}$ the map $R_i \to R_k^{s_0} \oplus R_j^{t_0}$. Since $\mathcal{C}(R_k, R_j) = 0$, the map $R_i \xrightarrow{f} R_j^{t_0}$ is a left $\operatorname{add} R_j$-approximation in $\mathcal{C}$, thus so is $\underline{f}$ in $\underline{\mathcal{C}}$. Let $g \in \operatorname{End}_{\mathcal{C}}(R_j^{t_0})$ be such that $\underline{g}\,\underline{f} = \underline{f}$. Then $\left[\begin{smallmatrix} 1 & 0 \\ 0 & g \end{smallmatrix}\right] \begin{bmatrix} f' \\ f \end{bmatrix} - \begin{bmatrix} f' \\ f \end{bmatrix}$ factors through $\operatorname{add} \Sigma R_k$. Since R_i belongs to ${}^{\perp}(\Sigma R_k)$, we have in fact $\left[\begin{smallmatrix} 1 & 0 \\ 0 & g \end{smallmatrix}\right] \begin{bmatrix} f' \\ f \end{bmatrix} = \begin{bmatrix} f' \\ f \end{bmatrix}$. By minimality, g is an isomorphism. Thus $\underline{f}$ is left-minimal. By Lemma 3 and Lemma 17, the first exchange triangle with respect to $\mu_k R$ for R_i is

$$R_i \longrightarrow R_j^{t_0} \longrightarrow R_i^{(1)*} \longrightarrow \Sigma R_i.$$

The triangle (b) is easily constructed by applying the octahedral axiom to the composition $R_i \longrightarrow R_k^{s_0} \oplus R_j^{t_0} \xrightarrow{\text{proj}} R_j^{t_0}$ as follows:

$$\begin{tikzcd}
 & R_k^{s_0} \arrow[r, equal] \arrow[d] & R_k^{s_0} \arrow[d] & \\
R_i \arrow[r] \arrow[d, equal] & R_k^{s_0} \oplus R_j^{t_0} \arrow[r] \arrow[d] & R_i^{(1)} \arrow[r] \arrow[d] & \Sigma R_i \arrow[d, equal] \\
R_i \arrow[r] & R_j^{t_0} \arrow[r] \arrow[d, "0"'] & R_i^{(1)*} \arrow[r] \arrow[d] & \Sigma R_i \\
 & \Sigma R_k^{s_0} \arrow[r, equal] & \Sigma R_k^{s_0}. &
\end{tikzcd}$$

Assume that (a) and (b) hold for some c, and let us first prove that (a) holds for $c+1$. Note that, by construction, $R_i^{(c+1)*}$ belongs to $R_k^{\perp} = \Sigma^{-1}(R_k^{(1)})^{\perp}$ and so does R_j, by (ii), so (a) makes sense. Write X for X_{c+1}. Since X belongs to $\operatorname{add} R_k$, the space $\mathcal{C}(X, R_j)$ vanishes and the morphism $R_i^{(c+1)} \to R_k^{s_{c+1}} \oplus R_j^{t_{c+1}}$ induces a morphism of triangles:

$$\begin{tikzcd}
X \arrow[r] \arrow[d, dashed] & R_i^{(c+1)} \arrow[r] \arrow[d] & R_i^{(c+1)*} \arrow[r] \arrow[d, dashed, "m"] & \Sigma X \arrow[d, dashed] \\
R_k^{s_{c+1}} \arrow[r] & R_k^{s_{c+1}} \oplus R_j^{t_{c+1}} \arrow[r] & R_j^{t_{c+1}} \arrow[r, "0"] & \Sigma R_k^{s_{c+1}}.
\end{tikzcd}$$

We claim that $\underline{m}$ is a minimal left $\operatorname{add} R_j$-approximation in $\underline{\mathcal{C}}$. Let f belong to $\mathcal{C}(R_i^{(c+1)*}, R_j)$. The following diagram illustrates the proof:

$$\begin{tikzcd}
\Sigma^{-1}R_i^{(c+2)} \arrow[d, "v"'] & & \\
R_i^{(c+1)} \arrow[r, "q"] \arrow[d, "u"'] & R_i^{(c+1)*} \arrow[r, "p"] \arrow[d, "m"'] \arrow[ddr, "f"] & \Sigma X \\
R_k^{s_{c+1}} \oplus R_j^{t_{c+1}} \arrow[r, "\pi_2"] \arrow[d] \arrow[drr, dashed, "a"'] & R_j^{t_{c+1}} \arrow[dr, dashed, "b"] & \\
R_i^{(c+2)} & & R_j
\end{tikzcd}$$

where π_2 denotes the second projection. Since the space $\mathcal{C}(R_i^{(c+2)}, \Sigma R_j)$ vanishes, we have $fqv = 0$ and there exists a morphism a such that $fq = au$. By (ii), the morphism a factors through π_2. Let b be such that $a = b\pi_2$. We then have $fq = b\pi_2 u = bmq$, and the morphism $f - bm$ factors through p. Since the object X belongs to $\operatorname{add} R_k$, this implies that $f - bm$ lies in the ideal (ΣR_k). That is $\underline{m}$ is a left $\operatorname{add} R_j$-approximation in $\underline{\mathcal{C}}$. Let $g \in \operatorname{End}_{\mathcal{C}}(R_j^{t_{c+1}})$ be such that $\underline{g}\,\underline{m} = \underline{m}$, that is $gm - m$ belongs to the ideal (ΣR_k). This implies that the composition $(m - gm)q$ vanishes since $\mathcal{C}(R_i^{(c+1)}, \Sigma R_k) = 0$. Let $h \in \mathcal{C}(\Sigma X, R_j^{t_{c+1}})$ be such that $gm = m + hp$. We have: $gmq = mq + hpq = mq$. Since $mq = \pi_2 u$ is left minimal, the morphism g is an isomorphism in $\mathcal{C}$, thus so is $\underline{g}$ in $\underline{\mathcal{C}}$. Hence (a) holds for $c+1$.

Let us now prove that (b) holds for $c+1$. By Lemma 3, Lemma 17 and (a) for $c+1$, we have a minimal left $\operatorname{add} R_j \oplus \Sigma R_k$ approximation of $R_i^{(c+1)*}$ in $\mathcal{C}$ of the form $\left[\begin{smallmatrix} m \\ r \end{smallmatrix}\right]$ for some $r : R_i^{(c+1)*} \longrightarrow \Sigma R_k^{s_c}$, which we complete to an exchange triangle

$$R_i^{(c+1)*} \xrightarrow{\left[\begin{smallmatrix} m \\ r \end{smallmatrix}\right]} R_j^{t_{c+1}} \oplus \Sigma R_k^{s_c} \longrightarrow R_i^{(c+2)*} \longrightarrow \Sigma R_i^{(c+1)*}.$$

Complete the commutative square

$$\begin{tikzcd}
R_i^{(c+1)} \arrow[r, "u"] \arrow[d, "q"'] & R_j^{t_{c+1}} \oplus R_k^{s_{c+1}} \arrow[d, "\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]"] \\
R_i^{(c+1)*} \arrow[r, "\left[\begin{smallmatrix} m \\ r \end{smallmatrix}\right]"] & R_j^{t_{c+1}} \oplus \Sigma R_k^{s_c}
\end{tikzcd}$$

to a commutative diagram

$$\begin{tikzcd}
R_i^{(c+1)} \arrow[r, "u"] \arrow[d, "q"'] & R_j^{t_{c+1}} \oplus R_k^{s_{c+1}} \arrow[r] \arrow[d, "\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]"] & R_i^{(c+2)} \arrow[r] \arrow[d] & \Sigma R_i^{(c+1)} \\
R_i^{(c+1)*} \arrow[r, "\left[\begin{smallmatrix} m \\ r \end{smallmatrix}\right]"] \arrow[d] & R_j^{t_{c+1}} \oplus \Sigma R_k^{s_c} \arrow[r] \arrow[d] & R_i^{(c+2)*} \arrow[r] \arrow[d] & \Sigma R_i^{(c+1)*} \\
\Sigma X \arrow[r] \arrow[d] & \Sigma R_k^{s_{c+1}} \oplus \Sigma R_k^{s_c} \arrow[r] \arrow[d] & \Sigma Y \arrow[r, "\eta"] \arrow[d] & \Sigma^2 X \\
\Sigma R_i^{(c+1)} & \Sigma R_j^{t_{c+1}} \oplus \Sigma R_k^{s_{c+1}} & \Sigma R_i^{(c+2)} &
\end{tikzcd}$$

whose rows and columns are triangles. By construction, $R_i^{(c+2)*}$ belongs to ${}^{\perp}(\Sigma^2 R_k)$. Moreover, $\mathcal{C}(\Sigma R_i^{(c+2)}, \Sigma^2 R_k) \simeq \mathcal{C}(R_i^{(c+2)}, \Sigma R_k) = 0$. Thus ΣY also belongs to the extension-closed subcategory ${}^{\perp}(\Sigma^2 R_k)$, and the morphism η vanishes, since $X \in \operatorname{add} R_k$. As a consequence, the triangle in the third row splits and Y belongs to $\operatorname{add} R_k$. Define X_{c+2} to be Y. Then we see that (b) has been shown.

The statements (a') and (b') can be deduced from (a) and (b) by duality:

Consider in the category $\mathcal{C}^{\mathrm{op}}$ the object $R_i \oplus R_j \oplus \Sigma R_k$.

It is a rigid object:

$$\begin{array}{rcl} \mathcal{C}^{\mathrm{op}}(R_i, \Sigma^{\mathrm{op}}\Sigma R_k) & = & \mathcal{C}^{\mathrm{op}}(R_i, R_k) \\ & = & \mathcal{C}(R_k, R_i) \\ & = & 0, \end{array}$$

and similarly, $\mathcal{C}^{\mathrm{op}}(R_j, \Sigma^{\mathrm{op}}\Sigma R_k) = 0$. Moreover, it satisfies the assumptions we made to prove (a) and (b):

$$\begin{array}{rcl} \mathcal{C}^{\mathrm{op}}(\Sigma R_k, R_i) & = & \mathcal{C}(R_i, \Sigma R_k) \\ & = & 0. \end{array}$$

Note that $\mu^{\mathrm{op}}\Sigma R_k = R_k$, $R_i^{(c)\mathrm{op}} = R_i^{(-c)*}$ and $R_i^{(c)*\mathrm{op}} = R_i^{(-c)}$.

Therefore, there are triangles in $\mathcal{C}$

$$Y_{c+1} \longleftarrow R_i^{(-c-1)*} \longleftarrow R_i^{(-c-1)} \longleftarrow \Sigma^{-1} Y_{c+1}$$

with Y_{c+1} in $\operatorname{add}\Sigma R_k$. Let $X_{-c-1} = \Sigma^{-1} Y_{c+1}$, to get the triangles (b').

By (a) applied to $\mathcal{C}^{\mathrm{op}}$, there are minimal right $\operatorname{add} R_j$ approximations $R_i^{(-c)} \leftarrow R_j^{t_c^{\mathrm{op}}}$ in $\mathcal{C}_{R_k} = {}^{\perp}(\Sigma R_k)/(R_k)$. This proves that we have $t_c^{\mathrm{op}} = t_{-c-1}$. Written in $\mathcal{C}$, the exchange triangles in $\mathcal{C}^{\mathrm{op}}$ for R_i with respect to $R_j \oplus \Sigma R_k$ are of the form:

$$R_i^{(-c)*} \longleftarrow R_j^{t_c^{\mathrm{op}}} \oplus (\Sigma R_k)^{s_c^{\mathrm{op}}} \longleftarrow R_i^{(-c-1)*} \longleftarrow \Sigma^{-1} R_i^{(-c)*}.$$

By Lemma 3, we thus have minimal right $\operatorname{add} R_j$ approximations $R_i^{(-c)*} \leftarrow R_j^{t_{-c-1}}$ in $\underline{\mathcal{C}}$. √

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