# A NEW OSTROWSKI-TYPE INEQUALITY FOR DOUBLE INTEGRALS

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Abstract. In this paper, we established a new Ostrowski-type inequality involving functions of two independent variables.

## 1. INTRODUCTION

In [6], Ostrowski proved the following inequality.

**Theorem 1.** *Let $f : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, be a mapping differentiable in the interior of $I$ and $a, b \in I^o, a < b$. If $|f'| \leq M, \forall t \in [a, b]$, then we have*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \left[ \frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a)\, M, \tag{1.1}$$

*for $x \in [a, b]$.*

In [7], Özdemir et al. proved inequalities as above for $(\alpha, m)-$convex functions. In [1], Cheng proved the following inequality,

**Theorem 2.** *Let $I \subset \mathbb{R}$ be an open interval, $a, b \in I, a < b$. $f : I \to \mathbb{R}$ is a differentiable function such that there exist constants $\gamma, \Gamma \in \mathbb{R}$ with $\gamma \leq f'(x) \leq \Gamma, x \in [a, b]$. Then we have*

$$\left| \frac{1}{2} f(x) - \frac{(x-b)\, f(b) - (x-a)\, f(a)}{2(b-a)} - \frac{1}{b-a} \int_a^b f(t)dt \right|$$
$$\leq \frac{(x-a)^2 + (b-x)^2}{8(b-a)} (\Gamma - \gamma)$$

*for all $x \in [a, b]$.*

Similarly, in [3], Ujevic established some double integral inequalities and in [2], Liu et.al. proved two sharp inequalities of perturbed Ostrowski-type. Recently, in [4], Sarıkaya established following integral inequality of Ostrowski-type involving functions of two independent variables;

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**Theorem 3.** *Let $f:[a,b]\times[c,d]\to\mathbb{R}$ be an absolutely continuous function such that the partial derivative of order 2 exists and supposes that there exist constants $\gamma,\Gamma\in\mathbb{R}$ with $\gamma\leq\frac{\partial^2 f(t,s)}{\partial t\partial s}\leq\Gamma$ for all $(t,s)\in[a,b]\times[c,d]$. Then, we have*

$$\begin{aligned}
&\left|\frac{1}{4}f(x,y)+\frac{1}{4}H(x,y)-\frac{1}{2(b-a)}\int_a^b f(t,y)dt-\frac{1}{2(d-c)}\int_c^d f(x,s)ds\right.\\
&-\frac{1}{2(b-a)(d-c)}\int_a^b\left[(y-c)f(t,c)+(d-y)f(t,d)\right]dt\\
&\left.-\frac{1}{2(b-a)(d-c)}\int_c^d\left[(x-a)f(a,s)+(b-x)f(b,s)\right]ds+\frac{1}{2(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|\\
\leq\quad&\frac{\left[(x-a)^2+(b-x)^2\right]\left[(y-c)^2+(d-y)^2\right]}{32(b-a)(d-c)}(\Gamma-\gamma)
\end{aligned}$$

*for all $(x,y)\in[a,b]\times[c,d]$ where*

$$\begin{aligned}
&H(x,y)\\
=\quad&\frac{(x-a)\left[(y-c)f(a,c)+(d-y)f(a,d)\right]+(b-x)\left[(y-c)f(b,c)+(d-y)f(b,d)\right]}{(b-a)(d-c)}\\
&+\frac{(x-a)f(a,y)+(b-x)f(b,y)}{b-a}+\frac{(y-c)f(x,c)+(d-y)f(x,d)}{d-c}
\end{aligned}$$

In [5], Qiaoling et.al. derived a new inequality of Ostrowski-type as following

**Theorem 4.** *Let $f:[a,b]\times[c,d]\to\mathbb{R}$ be an absolutely continuous function such that the partial derivative of order 2 exists and suppose that there exist constants $\gamma,\Gamma\in\mathbb{R}$ with $\gamma\leq\frac{\partial^2 f(t,s)}{\partial t\partial s}\leq\Gamma$ for all $(t,s)\in[a,b]\times[c,d]$. Then, we have*

$$\begin{aligned}
&\left|(1-\lambda)^2f(x,y)+\frac{\lambda}{2}(1-\lambda)\left[f(a,y)+f(b,y)+f(x,c)+f(x,d)\right]\right.\\
&+\left(\frac{\lambda}{2}\right)^2\left[f(a,c)+f(b,c)+f(a,d)+f(b,d)\right]\\
&-\frac{1}{b-a}\left\{(1-\lambda)\int_a^b f(t,y)dt+\frac{\lambda}{2}\int_a^b\left[f(t,c)+f(t,d)\right]dt\right\}\\
&-\frac{1}{d-c}\left\{(1-\lambda)\int_c^d f(x,s)ds+\frac{\lambda}{2}\int_c^d\left[f(a,s)+f(b,s)\right]ds\right\}\\
&\left.-\frac{\Gamma+\gamma}{2}(1-\lambda)^2\left(x-\frac{a+b}{2}\right)\left(y-\frac{c+d}{2}\right)+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|\\
\leq\quad&\frac{\Gamma-\gamma}{2}\frac{1}{(b-a)(d-c)}\left[\left(\lambda^2+(1-\lambda)^2\right)\frac{(b-a)^2}{4}+\left(x-\frac{a+b}{2}\right)^2\right]\\
&\times\left[\left(\lambda^2+(1-\lambda)^2\right)\frac{(d-c)^2}{4}+\left(y-\frac{c+d}{2}\right)^2\right]
\end{aligned}$$

*for all* $(x,y) \in \left[a+\lambda\frac{b-a}{2}, b-\lambda\frac{b-a}{2}\right] \times \left[c+\lambda\frac{d-c}{2}, d-\lambda\frac{d-c}{2}\right]$ *and* $\lambda \in [0,1]$.

In this paper, we proved a new Ostrowski-type inequality involving functions of two independent variables as above.

## 2. MAIN RESULT

**Theorem 5.** *Let* $f : [a,b] \times [c,d] \to \mathbb{R}$ *be an absolutely continuous function such that the partial derivative of order 2 exists and supposes that there exist constants* $\gamma, \Gamma \in \mathbb{R}$ *with* $\gamma \leq \frac{\partial^2 f(t,s)}{\partial t \partial s} \leq \Gamma$ *for all* $(t,s) \in [a,b] \times [c,d]$. *Then, we have*

$$\begin{aligned} & \left| \frac{1}{16} K f(x,y) + \frac{1}{16} H(x,y) \right. \\ & - \frac{1}{4(b-a)(d-c)} \int_c^d \left[ 3(x-a) f(x,s) - (b-x) f(x,s) \right] ds \\ & - \frac{1}{4(b-a)(d-c)} \int_a^b \left[ 3(y-c) f(t,y) - (d-y) f(t,y) \right] dt \\ & - \frac{1}{4(b-a)(d-c)} \int_c^d \left[ 3(b-x) f(b,s) - (x-a) f(a,s) \right] ds \\ & - \frac{1}{4(b-a)(d-c)} \int_a^b \left[ 3(d-y) f(t,d) - (y-c) f(t,c) \right] dt \\ & - \frac{\left[ (y-c)^2 - (d-y)^2 \right] \left[ (x-a)^2 - (b-x)^2 \right]}{32(b-a)(d-c)} (\Gamma + \gamma) \\ & \left. + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s) ds dt \right| \\ \leq \; & \frac{25 \left[ (y-c)^2 + (d-y)^2 \right] \left[ (x-a)^2 + (b-x)^2 \right]}{512 (b-a)(d-c)} (\Gamma - \gamma) \end{aligned} \tag{2.1}$$

*for all* $(x,y) \in [a,b] \times [c,d]$, *where*

$$\begin{aligned} H(x,y) = \; & \frac{\left[ 3(b-x) f(b,y) - (x-a) f(a,y) \right] \left( 3(y-c) - (d-y) \right)}{(b-a)(d-c)} \\ & + \frac{\left[ 3(d-y) f(x,d) - (y-c) f(x,c) \right] \left( 3(x-a) - (b-x) \right)}{(b-a)(d-c)} \\ & + \frac{\left[ (y-c) f(a,c) - 3(d-y) f(a,d) \right] (x-a)}{(b-a)(d-c)} \\ & + \frac{\left[ 3(d-y) f(b,d) - (y-c) f(b,c) \right] (b-x)}{(b-a)(d-c)} \end{aligned}$$

*and*

$$K = \frac{\left[ \left( 3(x-a) - (b-x) \right) \left( 3(y-c) - (d-y) \right) \right]}{(b-a)(d-c)}.$$

*Proof.* We define the functions: $p:[a,b]^2\to\mathbb{R}$ and $q:[c,d]^2\to\mathbb{R}$ as following

$$p(x,t)=\begin{cases} t-\frac{3a+x}{4} & ,t\in[a,x] \\ t-\frac{3b+x}{4} & ,t\in(x,b] \end{cases}$$

and

$$q(y,s)=\begin{cases} s-\frac{3c+y}{4} & ,s\in[c,y] \\ s-\frac{3d+y}{4} & ,s\in(y,d] \end{cases}$$

From definitions of $p(x,t)$ and $q(y,s)$, we can write

$$\begin{aligned} &\int_a^b\int_c^d p(x,t)q(y,s)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt \\ =&\int_a^x\int_c^y\left(t-\frac{3a+x}{4}\right)\left(s-\frac{3c+y}{4}\right)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt \\ &+\int_a^x\int_y^d\left(t-\frac{3a+x}{4}\right)\left(s-\frac{3d+y}{4}\right)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt \\ &+\int_x^b\int_c^y\left(t-\frac{3b+x}{4}\right)\left(s-\frac{3c+y}{4}\right)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt \\ &+\int_x^b\int_y^d\left(t-\frac{3b+x}{4}\right)\left(s-\frac{3d+y}{4}\right)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt \end{aligned} \tag{2.2}$$

Computing each integral of right hand side of (2.2), we have

$$\begin{aligned} &\int_a^x\int_c^y\left(t-\frac{3a+x}{4}\right)\left(s-\frac{3c+y}{4}\right)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt \\ =&\frac{(x-a)(y-c)}{16}\left[9f(x,y)-3f(x,c)-3f(a,y)+f(a,c)\right] \\ &-\frac{(x-a)}{4}\int_c^y\left[3f(x,s)-f(a,s)\right]ds-\frac{(y-c)}{4}\int_a^x\left[3f(t,y)-f(t,c)\right]dt \\ &+\int_a^x\int_c^y f(t,s)dsdt \end{aligned} \tag{2.3}$$

$$\begin{aligned}
(2.4)\qquad & \int_a^x \int_y^d \left(t-\frac{3a+x}{4}\right)\left(s-\frac{3d+y}{4}\right)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt \\
= \quad & \frac{(x-a)(d-y)}{16}\left[9f(x,d)-3f(x,y)-3f(a,d)+f(a,y)\right] \\
& -\frac{(x-a)}{4}\int_y^d \left[3f(x,s)-f(a,s)\right]ds-\frac{(d-y)}{4}\int_a^x \left[3f(t,d)-f(t,y)\right]dt \\
& +\int_a^x \int_y^d f(t,s)dsdt
\end{aligned}$$

$$\begin{aligned}
(2.5)\qquad & \int_x^b \int_c^y \left(t-\frac{3b+x}{4}\right)\left(s-\frac{3c+y}{4}\right)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt \\
= \quad & \frac{(b-x)(y-c)}{16}\left[9f(b,y)-3f(b,c)-3f(x,y)+f(x,c)\right] \\
& -\frac{(b-x)}{4}\int_c^y \left[3f(b,s)-f(x,s)\right]ds-\frac{(y-c)}{4}\int_x^b \left[3f(t,y)-f(t,c)\right]dt \\
& +\int_x^b \int_c^y f(t,s)dsdt
\end{aligned}$$

$$\begin{aligned}
(2.6)\qquad & \int_x^b \int_y^d \left(t-\frac{3b+x}{4}\right)\left(s-\frac{3d+y}{4}\right)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt \\
= \quad & \frac{(b-x)(d-y)}{16}\left[9f(b,d)-3f(b,y)-3f(x,d)+f(x,y)\right] \\
& -\frac{(b-x)}{4}\int_y^d \left[3f(b,s)-f(x,s)\right]ds-\frac{(d-y)}{4}\int_x^b \left[3f(t,d)-f(t,y)\right]dt \\
& +\int_x^b \int_y^d f(t,s)dsdt
\end{aligned}$$

By using these inequalities in (2.2), we get

$$
\begin{aligned}
&\int_a^b \int_c^d p(x,t)\, q(y,s)\, \frac{\partial^2 f(t,s)}{\partial t \partial s} ds dt \\
=\ & \frac{1}{16}\{[(3(x-a)-(b-x))(3(y-c)-(d-y))] f(x,y) \\
&+[3(b-x)f(b,y)-(x-a)f(a,y)](3(y-c)-(d-y)) \\
&+[3(d-y)f(x,d)-(y-c)f(x,c)](3(x-a)-(b-x)) \\
&+[(y-c)f(a,c)-3(d-y)f(a,d)](x-a) \\
&+[3(d-y)f(b,d)-(y-c)f(b,c)](b-x)\} \\
&-\frac{1}{4}\int_c^d [3(x-a)f(x,s)-(b-x)f(x,s)]\, ds \\
&-\frac{1}{4}\int_a^b [3(y-c)f(t,y)-(d-y)f(t,y)]\, dt \\
&-\frac{1}{4}\int_c^d [3(b-x)f(b,s)-(x-a)f(a,s)]\, ds \\
&-\frac{1}{4}\int_a^b [3(d-y)f(t,d)-(y-c)f(t,c)]\, dt \\
&+\int_a^b \int_c^d f(t,s) ds dt
\end{aligned}
\tag{2.7}
$$

We also have

$$
\int_a^b \int_c^d p(x,t)\, q(y,s)\, ds dt = \frac{\left[(y-c)^2-(d-y)^2\right]\left[(x-a)^2-(b-x)^2\right]}{16} \tag{2.8}
$$

Let $M = \frac{\Gamma+\gamma}{2}$. From (2.7) and (2.8), we can write

$$
\begin{aligned}
&\int_a^b \int_c^d p(x,t)\, q(y,s) \left[\frac{\partial^2 f(t,s)}{\partial t \partial s} - M\right] ds dt \\
=\ & \int_a^b \int_c^d p(x,t)\, q(y,s)\, \frac{\partial^2 f(t,s)}{\partial t \partial s} ds dt \\
&-\frac{\Gamma+\gamma}{2} \frac{\left[(y-c)^2-(d-y)^2\right]\left[(x-a)^2-(b-x)^2\right]}{16}
\end{aligned}
\tag{2.9}
$$

On the other hand, we have

$$\left|\int_a^b\int_c^d p(x,t)\,q(y,s)\left[\frac{\partial^2 f(t,s)}{\partial t\partial s}-M\right]dsdt\right| \tag{2.10}$$
$$\leq \max_{(t,s)\in[a,b]\times[c,d]}\left|\frac{\partial^2 f(t,s)}{\partial t\partial s}-M\right|\int_a^b\int_c^d |p(x,t)\,q(y,s)|\,dsdt$$

Since

$$\max_{(t,s)\in[a,b]\times[c,d]}\left|\frac{\partial^2 f(t,s)}{\partial t\partial s}-M\right|\leq\frac{\Gamma-\gamma}{2} \tag{2.11}$$

and

$$\int_a^b\int_c^d |p(x,t)\,q(y,s)|\,dsdt=\frac{25\left[(y-c)^2+(d-y)^2\right]\left[(x-a)^2+(b-x)^2\right]}{256} \tag{2.12}$$

By using (2.11) and (2.12) in (2.10), we get

$$\left|\int_a^b\int_c^d p(x,t)\,q(y,s)\left[\frac{\partial^2 f(t,s)}{\partial t\partial s}-M\right]dsdt\right| \tag{2.13}$$
$$\leq \frac{25\left[(y-c)^2+(d-y)^2\right]\left[(x-a)^2+(b-x)^2\right]}{512}(\Gamma-\gamma)$$

From (2.9) and (2.13), we get the required result. □

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