Fractal curvature measures and Minkowski content for one-dimensional self-conformal sets

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Abstract

We investigate intrinsic geometric properties of invariant sets of one-dimensional conformal iterated function systems. We show that for such a set F the fractal curvature measures exist, if and only if the geometric potential function associated to F is nonlattice. In this case we obtain that the fractal curvature measures are constant multiples of the δ-conformal measure, where δ is the Minkowski dimension of F. Moreover, for the first fractal curvature measure, this constant factor coincides with the Minkowski content of F. We show that the existence of the fractal curvature measures implies the existence of the Minkowski content but that the converse is not true in general. That is, the Minkowski content may exist although the geometric potential function associated to F is lattice. Nevertheless, average versions of the fractal curvature measures always exist and are also constant multiples of the δ-conformal measure. We give explicit formulae for the (average) fractal curvature measures and further investigate the particular situations of self-similar sets and $\mathcal{C}^{1+\alpha}$ images of self-similar sets.

1 Brief Introduction

Notions of curvature are an important tool to describe the geometric structure of sets and have been introduced and intensively studied for broad classes of sets. However, for sets of a fractal nature, the classical notions of curvature are not fitting. Nevertheless, for these sets it is desirable to have such a notion at hand.

Originally, the idea to characterise sets in terms of their curvature stems from the study of smooth manifolds as well as from the theory of convex bodies with sufficiently smooth boundaries. In Federer's foundational text, *Curvature Measures* [Fed59], Federer localises, extends and unifies the existing notions of curvature for the afore mentioned sets to sets of positive reach. This is where he introduces curvature measures, which can be viewed as a measure theoretical substitute for the notion of curvature in the non differentiability situation. Federer's curvature measures were studied and generalised in various ways. An extension to finite unions of convex bodies is given in [Gro78] and [Sch80] and to finite unions of sets of positive reach in [Zäh84]. In [Win08] Winter extends the curvature measures to

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"fractal" sets in $\mathbb{R}^d$, which typically cannot be expressed as finite unions of sets with positive reach. These measures are referred to as fractal curvature measures and are defined as weak limits of rescaled versions of the curvature measures introduced by Federer, Groemer and Schneider. Winter also examines conditions for their existence in the self similar case. However, fractal sets arising in geometry (for instance as limit sets of Fuchsian groups) or in number theory (for instance as sets defined by Diophantine inequalities) are typically non self-similar but rather self-conformal. In order to make use of the notion of curvature also for this important class of fractal sets, we extend Winter's examinations to the conformal setting. In this way we contribute to the ongoing research on defining notions of curvature for fractal sets. Our objects of study are nonempty compact sets which occur as the invariant sets of finite conformal iterated function systems satisfying the open set condition as considered in [MU96]. We refer to these sets as self-conformal sets (see Definition 3.2).

The paper is organised as follows. In Section 2 we state the main results and provide in this way a complete answer to the question on the existence of the fractal curvature measures for self-conformal sets. The precise definitions and background information as well as the relevant properties and auxiliary results will be presented in Section 3. In Section 4 the proofs of our main theorems for self-conformal sets (Theorem 2.6 and Theorem 2.8) are provided. Finally, in Section 5, we conclude the paper by considering the special cases of self-similar sets and $\mathcal{C}^{1+\alpha}$ images of self-similar sets and thus proving Theorem 2.9, Theorem 2.11, Proposition 2.13, and Corollary 2.14.

2 Main Results

The introduction of the fractal curvature measures (see Section 3.1) relies on the definition of scaling exponents, for which we require the following notation. Let λ^0 and λ^1 respectively denote the zero- and one-dimensional Lebesgue measure. For $\varepsilon > 0$ we define $Y_\varepsilon := \{x \in \mathbb{R} \mid \inf_{y\in Y}|x-y| \leq \varepsilon\}$ to be the *ε-parallel neighbourhood* of $Y \subset \mathbb{R}$ and let ∂Y denote the boundary of Y.

Definition 2.1 For a compact set $Y \subset \mathbb{R}$ the *0-th* and *1-st curvature scaling exponents* of Y are respectively defined as

$$\begin{aligned} s_0(Y) &:= \inf\{t \in \mathbb{R} \mid \varepsilon^t \lambda^0(\partial Y_\varepsilon) \to 0 \text{ as } \varepsilon \to 0\} \quad \text{and} \\ s_1(Y) &:= \inf\{t \in \mathbb{R} \mid \varepsilon^t \lambda^1(Y_\varepsilon) \to 0 \text{ as } \varepsilon \to 0\}. \end{aligned}$$

Definition 2.2 Let $Y \subset \mathbb{R}$ denote a compact set. Provided, the weak limit

$$C_0^f(Y,\cdot) := \operatorname*{w-lim}_{\varepsilon\to 0} \varepsilon^{s_0(Y)}\lambda^0(\partial Y_\varepsilon \cap \cdot)/2$$

exists, we call it the *0-th fractal curvature measure* of Y. Likewise the weak limit

$$C_1^f(Y,\cdot) := \operatorname*{w-lim}_{\varepsilon\to 0} \varepsilon^{s_1(Y)}\lambda^1(Y_\varepsilon \cap \cdot)$$

is called the *1-st fractal curvature measure*, if it exists. Moreover, for a Borel set $B \subseteq \mathbb{R}$ we set

$$\overline{C}_0^f(Y,B) := \limsup_{\varepsilon\to 0} \varepsilon^{s_0(Y)}\lambda^0(\partial Y_\varepsilon \cap B)/2, \qquad \underline{C}_0^f(Y,B) := \liminf_{\varepsilon\to 0} \varepsilon^{s_0(Y)}\lambda^0(\partial Y_\varepsilon \cap B)/2$$
$$\overline{C}_1^f(Y,B) := \limsup_{\varepsilon\to 0} \varepsilon^{s_1(Y)}\lambda^1(Y_\varepsilon \cap B), \quad \text{and} \quad \underline{C}_1^f(Y,B) := \liminf_{\varepsilon\to 0} \varepsilon^{s_1(Y)}\lambda^1(Y_\varepsilon \cap B).$$

The central question arising in this context is to identify those sets $Y \subset \mathbb{R}$ for which the fractal curvature measures exist. In [Win08] it has been shown that the fractal curvature measures exist for self-similar sets with positive Lebesgue measure as well as for self-similar sets satisfying the open set condition which are nonlattice (see Definition 3.8). In the lattice case, Winter shows that an average version of the fractal curvature measures exists, which are defined as follows.

Definition 2.3 Let $Y \subset \mathbb{R}$ denote a compact set. Provided the weak limit exists, we let

$$\widetilde{C}_0^f(Y, \cdot) := \operatorname*{w-lim}_{T \to 0} |\ln T|^{-1} \int_T^1 \varepsilon^{\delta-1} \lambda^0(\partial Y_\varepsilon \cap \cdot) \mathrm{d}\varepsilon / 2$$

denote the *0-th average fractal curvature measure* of Y. Likewise, the weak limit

$$\widetilde{C}_1^f(Y, \cdot) := \operatorname*{w-lim}_{T \to 0} |\ln T|^{-1} \int_T^1 \varepsilon^{\delta-2} \lambda^1(Y_\varepsilon \cap \cdot) \mathrm{d}\varepsilon$$

is called the *1-st average fractal curvature measure* of Y, if it exists.

We are able to provide a complete characterisation of the self-conformal sets for which the (average) fractal curvature measures exist, generalising in this respect the results in [Win08].

As we will see, a self-conformal set is either a nonempty compact interval or has zero one-dimensional Lebesgue measure (Proposition 3.4). To determine the fractal curvature scaling exponents we have to distinguish these two cases.

Proposition 2.4 *Let δ denote the Minkowski dimension of a self-conformal set F. If $\lambda^1(F) = 0$, then $s_0(F) = \delta$ and $s_1(F) = \delta - 1$. If F is a nonempty compact interval, then $s_0(F) = s_1(F) = 0$.*

Let us first consider the latter situation of the above proposition. As an immediate consequence of Proposition 2.4 we obtain the following complete description.

Corollary 2.5 *If $Y \subset \mathbb{R}$ is a nonempty compact interval, then both the 0-th and 1-st fractal curvature measures exist and satisfy*

$$C_0^f(Y, \cdot) = \lambda^0(\partial Y \cap \cdot)/2 \quad \text{and} \quad C_1^f(Y, \cdot) = \lambda^1(Y \cap \cdot).$$

Let us now focus on self-conformal sets with zero one-dimensional Lebesgue measure. Fix an iterated function system $\Phi := \{\phi_1, \ldots, \phi_N\}$, $N \geq 2$, acting on a compact connected set $X \subset \mathbb{R}$ which satisfies the open set condition and let F denote the unique nonempty compact invariant set of Φ. For $\Sigma := \{1, \ldots, N\}$ let (Σ^∞, σ) denote the full shift-space on N symbols and let $\pi \colon \Sigma^\infty \to F$ be the natural code map as defined in Section 3. It turns out that the fractal curvature measures of F exist, if and only if the *geometric potential function* $\xi \colon \Sigma^\infty \to \Sigma^\infty$ given by $\xi(\omega) := -\ln|\phi'_{\omega_1}(\pi(\sigma\omega))|$ for $\omega := \omega_1\omega_2 \cdots \in \Sigma^\infty$, is nonlattice (see Definition 3.7). In this case we call Φ (resp. F) nonlattice, otherwise Φ (resp. F) is called lattice (see Definition 3.8). Before stating the main results we make the following observation and definitions.

By applying Φ to the convex hull of F one obtains a family of $Q-1$ gap intervals $L^1, \ldots, L^{Q-1}$, which we call the *primary gaps* of F, where we have $2 \leq Q \leq N$ since $\lambda^1(F) = 0$. Given an $n \in \mathbb{N}$ and an $\omega := \omega_1 \cdots \omega_n \in \Sigma^n$, let $L_\omega^1, \ldots, L_\omega^{Q-1}$ respectively denote the images of the primary gaps under the map $\phi_\omega := \phi_{\omega_1} \circ \cdots \circ \phi_{\omega_n}$ and call these sets the *main gaps* of $\phi_\omega F$.

Further, letting δ denote the Minkowski dimension of F, we call the unique probability measure ν supported on F, which satisfies

$$\nu(\phi_i X \cap \phi_j X) = 0 \text{ for } i \neq j \in \Sigma \quad \text{and} \quad \nu(\phi_\omega B) = \int_B |\phi'_\omega|^\delta \mathrm{d}\nu \tag{2.1}$$

for all $\omega \in \bigcup_{n\in\mathbb{N}} \Sigma^n =: \Sigma^*$ and for all Borel sets $B \subseteq F$ the *δ-conformal measure* associated to Φ. The statement on the uniqueness and existence is shown in [MU96] and goes back to the work of [Pat76], [Sul79], and [DU91].
Finally, let $H_{\mu_{-\delta\xi}}$ denote the measure theoretical entropy of the shift-map with respect to the unique shift-invariant Gibbs measure for the potential function $-\delta\xi$ (see (3.2)).

The following theorem gives the complete answer to the question concerning the existence of the fractal curvature measures for self-conformal sets.

Theorem 2.6 (Self-Conformal Sets) *Let F denote a self-conformal set associated to the iterated function system Φ. Assume that Φ satisfies the open set condition and that $\lambda^1(F) = 0$. Let δ denote the Minkowski dimension of F and let ξ denote the geometric potential function associated to Φ. Then the fractal curvature measures exist if and only if ξ is nonlattice. Moreover, the following more specific results hold.*

(i) The average fractal curvature measures always exist and are both constant multiples of the δ-conformal measure ν associated to F, that is

$$\widetilde{C}_0^f(F,\cdot) = \frac{2^{-\delta}c}{H_{\mu_{-\delta\xi}}} \cdot \nu(\cdot) \quad \text{and} \quad \widetilde{C}_1^f(F,\cdot) = \frac{2^{1-\delta}c}{(1-\delta)H_{\mu_{-\delta\xi}}} \cdot \nu(\cdot),$$

where the constant c is given by the well-defined limit

$$c := \lim_{n\to\infty} \sum_{\omega\in\Sigma^n} \sum_{i=1}^{Q-1} |L_\omega^i|^\delta. \tag{2.2}$$

(ii) If Φ is nonlattice, then both the 0-th and 1-st fractal curvature measures exist and satisfy $C_k^f(F,\cdot) = \widetilde{C}_k^f(F,\cdot)$ for $k \in \{0,1\}$.

(iii) If Φ is lattice, then neither the 0-th nor the 1-st fractal curvature measure exists. Nevertheless, there exists a constant $\overline{c} \in \mathbb{R}$ such that $\overline{C}_k^f(F,B) \leq \overline{c}$ for every Borel set $B \subseteq \mathbb{R}$ and $k \in \{0,1\}$. Additionally, $\underline{C}_k^f(F,\mathbb{R})$ is positive for $k \in \{0,1\}$.

Part (iii) in particular shows that the scaling exponents of F can alternatively be characterised by $s_0(F) = \sup\{t \in \mathbb{R} \mid \varepsilon^t \lambda^0(\partial F_\varepsilon) \to \infty$ as $\varepsilon \to 0\}$ and $s_1(F) = \sup\{t \in \mathbb{R} \mid \varepsilon^t \lambda^1(F_\varepsilon) \to \infty$ as $\varepsilon \to 0\}$ respectively.

For $\lambda^1(F) = 0$, it follows from Proposition 2.4 that the Minkowski content $\mathcal{M}(F)$ of F (see Definition 3.1) is obtained as the total mass of the 1-st fractal curvature measure, namely $\mathcal{M}(F) = C_1^f(F,\mathbb{R})$. Similarly, the average Minkowski content $\widetilde{\mathcal{M}}(F)$ is given by the total mass of the 1-st average fractal curvature measure, namely $\widetilde{\mathcal{M}}(F) = \widetilde{C}_1^f(F,\mathbb{R})$. Moreover, for the upper and lower Minkowski content we have that $\overline{\mathcal{M}}(F) = \overline{C}_1^f(F,\mathbb{R})$ and $\underline{\mathcal{M}}(F) = \underline{C}_1^f(F,\mathbb{R})$. Therefore, Theorem 2.6 immediately implies the following interesting observation.

Corollary 2.7 (Self-Conformal Sets – Minkowski Content) *Under the conditions of Theorem 2.6 the following holds.*

(i) The average Minkowski content exists and equals

$$\widetilde{\mathcal{M}}(F) = \frac{2^{1-\delta}c}{(1-\delta)H_{\mu_{-\delta\xi}}},$$

where c is the constant given in Equation (2.2).

(ii) If Φ is nonlattice, then the Minkowski content $\mathcal{M}(F)$ of F exists and coincides with $\widetilde{\mathcal{M}}(F)$.

(iii) In both the lattice and nonlattice case we have that

$$0 < \underline{\mathcal{M}}(F) \leq \overline{\mathcal{M}}(F) < \infty.$$

It is important to remark that the fact that the fractal curvature measures do not exist in the lattice case does not imply that the Minkowski content does not exist in this case. Indeed, for a general self-conformal set, which is lattice, the Minkowski content may or may not exist as we will see later. A sufficient condition under which the Minkowski content exists, or more generally $\underline{C}_k^f(F,B) = \overline{C}_k^f(F,B)$ for a Borel set $B \subseteq \mathbb{R}$ and $k \in \{0,1\}$, is given in the following theorem. For a Hölder continuous function $f \in \mathcal{C}(\Sigma^\infty)$ we let ν_f denote the unique eigenmeasure corresponding to the eigenvalue 1 of the dual of the Perron-Frobenius operator for the potential function f (see Section 3.4).

Theorem 2.8 (Self-Conformal Sets – Lattice Case) *Assume that we are in the setting of Theorem 2.6 and that ξ is lattice. Let $\zeta, \psi \in \mathcal{C}(\Sigma^\infty)$ denote the functions satisfying $\xi - \zeta = \psi - \psi \circ \sigma$, where ζ takes values in a discrete subgroup of $\mathbb{R}$ which is generated by $a \in \mathbb{R}$. Moreover, choose $N_1, N_2 \in \mathbb{Z}$ such that $\psi(\Sigma^\infty) \subseteq [N_1 a, N_2 a)$ and $N_2 - N_1$ is minimal. Let $\widetilde{B} \subseteq \Sigma^\infty$ be a Borel set which can be represented as a finite union of cylinder sets and set $B := \pi\widetilde{B} \subset \mathbb{R}$. If, for every $t \in [0,a)$, we have*

$$\sum_{n=N_1}^{N_2-1} e^{-\delta a n}\nu_{-\delta\zeta} \circ \psi^{-1}(\psi(\widetilde{B}) \cap [na, na+t)) = \frac{e^{\delta t}-1}{e^{\delta a}-1}\sum_{n=N_1}^{N_2-1} e^{-\delta a n}\nu_{-\delta\zeta} \circ \psi^{-1}(\psi(\widetilde{B}) \cap [na, (n+1)a)), \tag{2.3}$$

then it follows that $\underline{C}_k^f(F,B) = \overline{C}_k^f(F,B)$ for $k \in \{0,1\}$.

An example of a self-conformal set F, which satisfies Condition (2.3) for $B = \mathbb{R}$ is given in Example 2.15. Theorem 2.8 then implies that F is Minkowski measurable. However, in the special case when F is a self-similar set, Condition (2.3) cannot be satisfied. In this case it even turns out, that F is Minkowski measurable if and only if F is nonlattice. This is also reflected in the following theorem, where also $\underline{C}_k^f(F,B)$ and $\overline{C}_k^f(F,B)$ for $k \in \{0,1\}$ and a Borel set $B \subseteq \mathbb{R}$ are considered in the lattice case.

Theorem 2.9 (Self-Similar Sets) *Let F denote a self-similar set associated to the iterated function system $\Phi := \{\phi_1, \ldots, \phi_N\}$. Assume that Φ satisfies the open set condition and $\lambda^1(F) = 0$. Further, let $r_1, \ldots, r_N$ denote the respective similarity ratios of the maps $\phi_1, \ldots, \phi_N$. Then the fractal curvature measures exist if and only if Φ is nonlattice. Specifically, the following holds.*

(i) The average curvature measures of F exist and are given by

$$\widetilde{C}_0^f(F,\cdot) = \frac{2^{-\delta}\sum_{i=1}^{Q-1}|L^i|^\delta}{-\delta\sum_{i\in\Sigma}\ln(r_i)r_i^\delta}\cdot\nu(\cdot) \quad \textit{and} \quad \widetilde{C}_1^f(F,\cdot) = \frac{2^{1-\delta}\sum_{i=1}^{Q-1}|L^i|^\delta}{(\delta-1)\delta\sum_{i\in\Sigma}\ln(r_i)r_i^\delta}\cdot\nu(\cdot),$$

where δ denotes the Minkowski dimension of F and ν denotes the δ-conformal measure associated to F.

(ii) If Φ is nonlattice, then the fractal curvature measures of F exist and are given by $C_k^f(F,\cdot) = \widetilde{C}_k^f(F,\cdot)$ for $k \in \{0,1\}$.

(iii) If Φ is lattice, then the fractal curvature measures do not exist. What is more, for every Borel set $B \subseteq \mathbb{R}$ for which $F \cap B$ is nonempty and allows a representation as a finite union of sets of the form $\phi_\omega F$, where $\omega \in \Sigma^*$, we have that for $k \in \{0,1\}$

$$0 < \underline{C}_k^f(F,B) < \overline{C}_k^f(F,B) < \infty.$$

Note that ν coincides with the δ-dimensional Hausdorff measure normalised on F, that is with $\mathcal{H}^\delta(\cdot\cap F)/\mathcal{H}^\delta(F)$.

Parts (i) and (ii) of Theorem 2.9 are straightforward consequences of Parts (i) and (ii) of Theorem 2.6 and give handy formulae for computing the (average) fractal curvature measures provided they exist. The existence of the fractal curvature measures under the assumptions of Part (ii) of Theorem 2.9 and their average counterparts (Part (i) of Theorem 2.9) has also been shown in Theorem 1.2.6 of [Win08]. However, the formulae for the coefficients of the measures obtained in [Win08] are given by an integration over a certain "overlap function" and appear to be much harder to determine explicitely. Part (iii) of Theorem 2.9 is not covered by [Win08] and gives a new result.

For the Minkowski content Theorem 2.9 immediately implies the following corollary which we state without a proof.

Corollary 2.10 (Self-Similar Sets – Minkowski Content) *Under the conditions of Theorem 2.9 the following holds.*

(i) The average Minkowski content of F exists and is given by

$$\widetilde{\mathcal{M}}(F) = \frac{2^{1-\delta}\sum_{i=1}^{Q-1}|L^i|^\delta}{(\delta-1)\delta\sum_{i\in\Sigma}\ln(r_i)r_i^\delta}.$$

(ii) If Φ is nonlattice, its Minkowski content $\mathcal{M}(F)$ exists and is equal to $\widetilde{\mathcal{M}}(F)$.

(iii) If Φ is lattice, then

$$0 < \underline{\mathcal{M}}(F) < \overline{\mathcal{M}}(F) < \infty.$$

Part (ii) of Corollary 2.10 has been obtained in Proposition 4 of [Fal95] under the strong seperation condition. Part (iii) of Corollary 2.10 has also been addressed in Theorem 8.36 of [LvF06].

Another special case of self-conformal sets are $\mathcal{C}^{1+\alpha}$ images of self-similar sets. For these sets Parts (i) and (ii) of Theorem 2.6 yield interesting relationships between the (average) fractal curvature measures of the self-similar set and of its $\mathcal{C}^{1+\alpha}$ image which are stated in Parts (i) and (ii) of the following theorem.

Theorem 2.11 ($\mathcal{C}^{1+\alpha}$ Images) *Let $K \subset \mathbb{R}$ denote a self-similar set for the iterated function system Φ acting on X which satisfies the open set condition. Let δ denote its Minkowski dimension, denote by $\mathcal{U} \supset X$ a connected open neighbourhood of X in $\mathbb{R}$ and let $g\colon \mathcal{U} \to \mathbb{R}$ be a $\mathcal{C}^{1+\alpha}(\mathcal{U})$ map, $\alpha > 0$, for which $|g'|$ is bounded away from 0. Assume that $\lambda^1(K) = 0$ and set $F := g(K)$.*

(i) The average fractal curvature measures of both K and F exist. Moreover, they are absolutely continuous and for $k \in \{0,1\}$ their Radon-Nikodym derivatives are given by

$$\frac{\mathrm{d}\widetilde{C}_k^f(F,\cdot)}{\mathrm{d}\widetilde{C}_k^f(K,\cdot)\circ g^{-1}} = |g' \circ g^{-1}|^\delta.$$

(ii) If Φ is nonlattice, then the fractal curvature measures of both K and F exist and are absolutely continuous with Radon-Nikodym derivatives

$$\frac{\mathrm{d}C_k^f(F,\cdot)}{\mathrm{d}C_k^f(K,\cdot)\circ g^{-1}} = |g' \circ g^{-1}|^\delta,$$

for $k \in \{0,1\}$.

(iii) If Φ is lattice, then neither the curvature measures of K nor those of F exist.

As an immediate consequence of Theorem 2.11 we obtain the following.

Corollary 2.12 ($\mathcal{C}^{1+\alpha}$ Images – Minkowski Content) *In the setting of Theorem 2.11 and letting ν denote the δ-conformal measure associated to K, we have the following.*

(i) The average Minkowski content of both K and F exist and satisfy

$$\widetilde{\mathcal{M}}(F) = \widetilde{\mathcal{M}}(K) \cdot \int_K |g'|^\delta \mathrm{d}\nu.$$

(ii) If Φ is nonlattice, then the Minkowski content of both K and F exist and satisfy

$$\mathcal{M}(F) = \mathcal{M}(K) \cdot \int_K |g'|^\delta \mathrm{d}\nu.$$

(iii) Both in the lattice and nonlattice case we have that

$$0 < \underline{\mathcal{M}}(F) \leq \overline{\mathcal{M}}(F) < \infty.$$

The results stated in Corollary 2.12 have recently been obtained in [FK11] also for higher dimensions.

In the lattice case, $\mathcal{C}^{1+\alpha}$ images of self-similar sets play a crucial role as every lattice self-conformal set is in fact a $\mathcal{C}^{1+\alpha}$ image of a lattice self-similar set:

Proposition 2.13 *Let F denote a self-conformal set which is associated to the iterated function system Φ acting on X. Then there exists an open connected set $\mathcal{U} \subset \mathbb{R}$, a map $g \in \mathcal{C}^{1+\alpha}(\mathcal{U})$ for some $\alpha > 0$ and a lattice self-similar set $K \subset \mathcal{U}$ such that $F = g(K)$.*

In contrast to the self-similar setting, the Minkowski content of a $\mathcal{C}^{1+\alpha}$ image of a lattice self-similar set may or may not exist. It does exist if Condition (2.3) of Theorem 2.8 is satisfied. When F is a $\mathcal{C}^{1+\alpha}$ image of a self-similar set, we can simplify Condition (2.3) under certain normalisation assumptions and obtain the following corollary.

Corollary 2.14 ($\mathcal{C}^{1+\alpha}$ Images – Lattice Case) *Let $K \subset \mathbb{R}$ denote a self-similar set for the iterated function system Φ acting on X which satisfies the open set condition. Let δ be its Minkowski dimension and let ξ_K denote the geometric potential function associated to Φ. Assume that $\lambda^1(K) = 0$ and that ξ_K is lattice. Let $a > 0$ be maximal such that ξ_K takes values in $a\mathbb{Z}$. Denote by $\mathcal{U} \supset X$ a connected open neighbourhood of X in $\mathbb{R}$ and let $g\colon \mathcal{U} \to \mathbb{R}$ be a $\mathcal{C}^1(\mathcal{U})$ map. Take g to be normalised such that $g'(K) \subseteq (e^{-a}, 1]$ and set $F := g(K)$. Further, let $B \subseteq \mathbb{R}$ denote a Borel set and let ν denote the δ-conformal measure associated to K. Then*

(i) $\underline{C}_k^f(F, B) = \overline{C}_k^f(F, B)$ *for* $k \in \{0, 1\}$ *if for every* $t \in [0, a)$

$$\nu \circ (g')^{-1}(g'(B) \cap (e^{-t}, 1]) = \frac{e^{\delta t} - 1}{e^{\delta a} - 1} \nu \circ (g')^{-1}(g'(B)). \tag{2.4}$$

(ii) If we additionally assume that $K \subseteq [0, 1]$ and that g' is invertible, monotonically increasing and such that $g'(1) = 1$, then $\underline{C}_k^f(F, B) = \overline{C}_k^f(F, B)$ for $k \in \{0, 1\}$ if g' satisfies

$$g'(r) = \left(\frac{\nu(B \cap (r, 1])(e^{\delta a} - 1)}{\nu(B)} + 1 \right)^{-1/\delta} \tag{2.5}$$

for $r \in \mathcal{U}$. In fact, Equation (2.5) defines a positive δ-Hölder continuous function.

The above theorems enable us to construct examples for which the Minkowski content does or does not exist.

Example 2.15 Let K be the Middle Third Cantor Set and let ν denote the $\ln 2/\ln 3$-conformal measure associated to K. Let $B \subseteq \mathbb{R}$ denote a Borel set for which $B \cap K$ is nonempty and has a representation as a finite union of sets of the form $\phi_\omega F$ with $\omega \in \Sigma^*$.

(i) By Theorem 2.9, we have that $\underline{C}_k^f(K, B) < \overline{C}_k^f(K, B)$. In particular, the Minkowski content does not exist for K itself, where $k \in \{0, 1\}$.

(ii) A self-conformal set F which is not self-similar, but for which $\underline{C}_k^f(F, B) < \overline{C}_k^f(F, B)$ for $k \in \{0, 1\}$ can be constructed in the following way. Take $g\colon [-2^{-1}, 3 \cdot 2^{-1}] \to \mathbb{R}$ definied by $g(x) = (x + 1)^2$ and set $F := g(K)$. Then $\underline{C}_k^f(F, B) < \overline{C}_k^f(F, B)$ for $k \in \{0, 1\}$ follows directly from the proof of Part (iii) of Theorem 2.11 in Section 5.

(iii) For the following lattice self-conformal set the Minkowski content does exist. Let $f\colon \mathbb{R} \to \mathbb{R}$ denote the Devil staircase function defined by $f(r) := \nu((-\infty, r])$, define the function $g\colon \mathbb{R} \to \mathbb{R}$ by $g(x) := \int_{-\infty}^{x} (2 - f(y))^{-\ln 3/\ln 2} \mathrm{d}y$ and set $F := g(K)$. Then we have $\underline{\mathcal{M}}(F) = \overline{\mathcal{M}}(F)$, although $\underline{\mathcal{M}}(K) < \overline{\mathcal{M}}(K)$. This is a consequence of Corollary 2.14.

3 Preliminaries

3.1 Fractal Curvature Measures

The work of [Gro78] and [Sch80] plays a vital role in the introduction of Winter's fractal curvature measures. In what follows, we focus on the construction in the one-dimensional setting. For a set $Y \subset \mathbb{R}$ which is a finite union of compact convex sets, there exist two curvature measures, namely the 0-th and the 1-st curvature measure of Y. Originally, these measures were defined through a localised Steiner formula (see [Fed59] and [Sch80]), but an equivalent and simpler characterisation is the following. The 1-st curvature measure of Y equals $\lambda^1(Y \cap \cdot)$ and under the additional assumption that Y is the closure of its interior, the 0-th curvature measure is equal to $\lambda^0(\partial Y \cap \cdot)/2$.

If $Y \subset \mathbb{R}$ is not a finite union of compact convex sets, but an arbitrary compact set, we still have that the ε-parallel neighbourhood Y_ε of Y is a finite union of convex compact sets, for each $\varepsilon > 0$. Moreover, Y_ε is the closure of its interior, for each $\varepsilon > 0$. Thus, the 0-th and 1-st curvature measures are defined on Y_ε and are equal to the measures $\lambda^0(\partial Y_\varepsilon \cap \cdot)/2$ and $\lambda^1(Y_\varepsilon \cap \cdot)$. The fractal curvature measures now arise by taking the limit as $\varepsilon \to 0$. However, before taking the limit, we observe that for a fractal set $F \subset \mathbb{R}$ one typically obtains that the number of boundary points of F_ε tends to infinity as $\varepsilon \to 0$, whereas the volume of F_ε tends to zero as $\varepsilon \to 0$. In order to obtain nontrivial measures, we need to introduce the curvature scaling exponents $s_0(F)$ and $s_1(F)$ as in Definition 2.1. By taking the weak limits of the rescaled curvature measures $\varepsilon^{s_0(F)} \cdot \lambda^0(\partial F_\varepsilon \cap \cdot)/2$ and $\varepsilon^{s_1(F)} \cdot \lambda^1(F_\varepsilon \cap \cdot)$ as $\varepsilon \to 0$, we obtain the fractal curvature measures $C_0^f(F, \cdot)$ and $C_1^f(F, \cdot)$ (Definition 2.2), whenever the weak limits exist. The average fractal curvature measures are gained by taking the weak limit over the average rescaled curvature measures if these limits exist (Definition 2.3).

Besides extending the notions of curvature, the fractal curvature measures also provide a set of geometric characteristics of a fractal set which can be used to distinguish fractal sets of the same Minkowski dimension. More precisely, considering two fractal sets $F_1, F_2 \subseteq [0,1]$ with $\{0,1\} \subseteq F_1, F_2$ which are of the same Minkowski dimension, the 1-st fractal curvature measure compares the local rate of decay of the lengths of the ε-parallel neighbourhood of F_1 and F_2. In this way it can be interpreted as "local fractal length". Since, by the inclusion exclusion principle, the above mentioned rate of decay correlates with the length of the overlap of sets of the form $(\phi_\omega F_i)_\varepsilon$, where $\omega \in \Sigma^n$ for $n \in \mathbb{N}$ and $i \in \{0,1\}$, the value of the 1-st fractal curvature measure makes a statement on the distribution of the gaps. That is, the more equally spread the gaps are over the fractal, the smaller is the fractal curvature measure. Analogously, the value of the 0-th fractal curvature measure can be interpreted as the "local fractal number of boundary points" or "local fractal Euler number". For further information on the geometric interpretation in higher dimensions, we refer to [Win08], [LW07] and [Kom08].

3.2 Minkowski Content

Definition 3.1 Let $Y \subset \mathbb{R}$ and δ be its Minkowski dimension. The *upper Minkowski content* $\overline{\mathcal{M}}(Y)$ and the *lower Minkowski content* $\underline{\mathcal{M}}(Y)$ of Y are respectively defined as

$$\begin{aligned} \overline{\mathcal{M}}(Y) &:= \limsup_{\varepsilon \to 0} \varepsilon^{\delta-1} \lambda^1(Y_\varepsilon) \quad \text{and} \\ \underline{\mathcal{M}}(Y) &:= \liminf_{\varepsilon \to 0} \varepsilon^{\delta-1} \lambda^1(Y_\varepsilon). \end{aligned}$$

If the upper and lower Minkowski contents coincide, we denote the common value by $\mathcal{M}(Y)$ and call it the *Minkowski content* of Y. In case the Minkowski content exists, we call Y *Minkowski measurable*. The *average Minkowski content* of Y is defined as the following limit if it exists

$$\widetilde{\mathcal{M}}(Y) := \lim_{T \to 0} |\ln T|^{-1} \int_T^1 \varepsilon^{\delta-2} \lambda^1(Y_\varepsilon) \mathrm{d}\varepsilon.$$

The Minkowski content was proposed in [Man95] as a measure of "lacunarity" of a fractal set. Indeed, the value of the Minkowski content allows to compare the lacunarity of sets of the same Minkowski dimension.
Minkowski measurability has moreover attracted prominence in work related to the Weyl-Berry conjecture on the distribution of eigenvalues of the Laplacian on domains with fractal boundaries. We refer to Section 4 of [Fal95] for an overview and references concerning these studies.
An additional motivation for studying the Minkowski content of fractal sets arises from non-commutative geometry. In Connes' seminal book [Con94] the notion of a noncommutative fractal geometry is developed. There it is shown that the natural analogue of the volume of a compact smooth Riemannian spin manifold for a fractal set in $\mathbb{R}$ is that of the Minkowski content. This idea is also reflected in the works [GI03], [Sam10] and [FS11].

3.3 Self-Conformal Sets and the Shift-Space

Let $X \subset \mathbb{R}$ be a nonempty compact interval. We call $\Phi := \{\phi_i : X \to X \mid i \in \{1, \ldots, N\}\}$ a *conformal iterated function system (cIFS)* acting on X, provided $N \geq 2$ and $\phi_1, \ldots, \phi_N$ are differentiable contractions with α-Hölder continuous derivatives $\phi_1', \ldots, \phi_N'$, $\alpha > 0$, where $|\phi_1'|, \ldots, |\phi_N'|$ shall be bounded away from both 0 and 1. The cIFS $\Phi := \{\phi_1, \ldots, \phi_N\}$ is said to satisfy the *open set condition (OSC)* if there exists a nonempty open bounded set $O \subset \mathbb{R}$ such that $\bigcup_{i=1}^N \phi_i(O) \subseteq O$ and $\phi_i(O) \cap \phi_j(O) = \varnothing$ for $i, j \in \{1, \ldots, N\}$, $i \neq j$. Note, that in our context conformality in dimension one just means that the derivatives are Hölder continuous. This extra condition can be dropped in higher dimensions.

Definition 3.2 We call the unique nonempty compact invariant set F of a cIFS Φ the *self-conformal* set associated to Φ.

Remark 3.3 One easily verifies that our definition of a cIFS coincides with the definition of a finite conformal iterated function system in $\mathbb{R}$ given in [MU96].

Proposition 3.4 *Let Φ be a cIFS which satisfies the OSC and let F be the self-conformal set associated to Φ. Then F is either a nonempty compact interval or has zero one-dimensional Lebesgue measure.*

Proof. Let $\Phi := \{\phi_1, \ldots, \phi_N\}$, define $[a, b]$ to be the convex hull of F and assume without loss of generality that $\phi_1, \ldots, \phi_N$ are ordered such that $\phi_1(a) < \phi_2(a) < \ldots < \phi_N(a)$. If $\phi_i([a, b]) \cap \phi_{i+1}([a, b]) \neq \varnothing$ for all $i \in \{1, \ldots, N-1\}$, then clearly $F = [a, b]$. Now assume that there exists an $i \in \{1, \ldots, N-1\}$ such that $\phi_i([a, b]) \cap \phi_{i+1}([a, b]) = \varnothing$. Then Proposition 4.4 of [MU96] gives that F has zero Lebesgue measure. □

It turnes out to be useful to view a self-conformal set on a symbolic level. For the following, we fix a cIFS $\Phi := \{\phi_1, \ldots, \phi_N\}$ and let F denote the self-conformal set associated to Φ. We introduce the *shift-space* (Σ^∞, σ) as follows.

Set $\Sigma := \{1, \ldots, N\}$ and call it the *alphabet*. Denote by Σ^n the set of words of length $n \in \mathbb{N}$ over Σ and by $\Sigma^* := \bigcup_{n \in \mathbb{N}_0} \Sigma^n$ the set of all finite words over Σ containing the empty word $\varnothing$. Further, let Σ^∞ be the *code space* which is the set of infinite words over Σ. The *shift-map* is then defined as the map $\sigma : \Sigma^* \cup \Sigma^\infty \to \Sigma^* \cup \Sigma^\infty$ given by $\sigma(\omega) := \varnothing$ for $\omega \in \{\varnothing\} \cup \Sigma^1$, $\sigma(\omega_1 \cdots \omega_n) := \omega_2 \cdots \omega_n \in \Sigma^{n-1}$ for $\omega_1 \cdots \omega_n \in \Sigma^n$, where $n \geq 2$ and $\sigma(\omega_1 \omega_2 \cdots) := \omega_2 \omega_3 \cdots \in \Sigma^\infty$ for $\omega_1 \omega_2 \cdots \in \Sigma^\infty$. For a finite word $\omega \in \Sigma^*$ we let $n(\omega)$ denote its length and define $\phi_\varnothing := \mathrm{id}|_X$ to be the identity map on X.

Note that Σ^∞ gives a coding of the self-conformal set F as can be seen as follows. For $\omega = \omega_1 \cdots \omega_n \in \Sigma^*$ we set $\phi_\omega := \phi_{\omega_1} \circ \cdots \circ \phi_{\omega_n}$ and for $\omega = \omega_1\omega_2 \cdots \in \Sigma^\infty$ and $n \in \mathbb{N}$ we denote the initial word by $\omega|_n := \omega_1\omega_2 \cdots \omega_n$. For each $\omega = \omega_1\omega_2 \cdots \in \Sigma^\infty$ the intersection $\bigcap_{n\in\mathbb{N}} \phi_{\omega|_n}(X)$ contains exactly one point $x_\omega \in F$ and gives rise to a surjection $\pi\colon \Sigma^\infty \to F$, $\omega \mapsto x_\omega$ which we call the *natural code map*. Let F^{unique} denote the set of points of F which have a unique preimage under π. Because of the open set condition $F \setminus F^{\text{unique}}$ is at most countable and thus F^{unique} is nonempty. Moreover, $x \in F^{\text{unique}}$ implies $\phi_i(x) \in F^{\text{unique}}$ for all $i \in \Sigma$. The map π allows to view points in F^{unique} as infinite words and vice versa. In order to have neater notation, we are going to omit the map π from now on. For example, by $\phi_\omega(u)$ we actually mean $\phi_\omega(\pi(u))$ for $\omega \in \Sigma^*$ and $u \in \Sigma^\infty$.

A key property of a cIFS is the bounded distortion property which is well used in the study of conformal iterated function systems (see for instance [MU96]). However, we need the following refinement of this statement, which we could not find in the literature and therefore give a short proof. For a set $Y \subset \mathbb{R}$ let $\langle Y\rangle$ denote the convex hull of Y.

Lemma 3.5 (bounded distortion) *There exists a sequence $(\rho_n)_{n\in\mathbb{N}}$ with $\rho_n > 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \rho_n = 1$ such that for all $\omega, u \in \Sigma^*$ and $x, y \in \langle \phi_\omega F\rangle$ we have*

$$\rho_{n(\omega)}^{-1} \leq \frac{|\phi_u'(x)|}{|\phi_u'(y)|} \leq \rho_{n(\omega)}.$$

Proof. Fix $\omega \in \Sigma^*$ and let $x, y \in \langle\phi_\omega F\rangle$ and $u = u_1 \cdots u_{n(u)} \in \Sigma^*$ be arbitrarily chosen. Then

$$|\phi_u'(x)| = \exp\Big(\sum_{k=1}^{n(u)} \ln|\phi_{u_k}'(\phi_{\sigma^k u}(x))|\Big) \leq |\phi_u'(y)| \exp\Big(\sum_{k=1}^{n(u)} \underbrace{\big|\ln|\phi_{u_k}'(\phi_{\sigma^k u}(x))| - \ln|\phi_{u_k}'(\phi_{\sigma^k u}(y))|\big|}_{=:A_k}\Big).$$

Since $|\phi_i'|$ is α-Hölder continuous and bounded away from 0, it follows that $\ln|\phi_i'|$ is α-Hölder continuous for each $i \in \{1, \ldots, N\}$. Let c_i be the corresponding Hölder constant and set $c := \max_{i\in\{1,\ldots,N\}} c_i$. Moreover, let $r < 1$ be a common upper bound for the contraction ratios of the maps $\phi_1, \ldots, \phi_N$. Without loss of generality we assume that $F \subseteq [0, 1]$. Then we have

$$A_k \leq c|\phi_{\sigma^k u}(x) - \phi_{\sigma^k u}(y)|^\alpha \leq c \cdot \Big(r^{n(u)-k}|x - y|\Big)^\alpha$$

and thus

$$\sum_{k=1}^{n(u)} A_k \leq \frac{c}{1 - r^\alpha}|x - y|^\alpha \leq \frac{c}{1 - r^\alpha} \max_{\omega\in\Sigma^n} \sup_{x,y\in\langle\phi_\omega F\rangle} |x - y|^\alpha =: \widetilde{\rho}_n.$$

Since $\widetilde{\rho}_n$ converges to 0 as $n \to \infty$, $\rho_n := \exp(\widetilde{\rho}_n)$ converges to 1 as $n \to \infty$. The estimate for the lower bound can be obtained by just interchanging the roles of x and y. □

3.4 Perron-Frobenius Theory and the Geometric Potential Function

In order to give a precise formulation of the constants occuring in Theorem 2.6 and its following theorems and corollaries and in order to set up some notation for the proofs in Sections 4 and 5, we now introduce to the Perron-Frobenius theory.

Let Σ^∞ be equipped with the topology of pointwise convergence and let $\mathcal{C}(\Sigma^\infty)$ denote the space of continuous real valued functions on Σ^∞. For $f \in \mathcal{C}(\Sigma^\infty)$, $0 < \alpha < 1$ and $n \in \mathbb{N}$

define

$$
\begin{aligned}
\mathrm{var}_n(f) &:= \sup\{|f(\omega)-f(u)| \mid \omega,u\in\Sigma^\infty \text{ and } \omega_i=u_i \text{ for all } i\in\{1,\ldots,n\}\},\\
|f|_\alpha &:= \sup_{n\geq 0}\frac{\mathrm{var}_n(f)}{\alpha^n} \qquad \text{and}\\
\mathcal{F}_\alpha(\Sigma^\infty) &:= \{f\in\mathcal{C}(\Sigma^\infty) \mid |f|_\alpha<\infty\}.
\end{aligned}
$$

Elements of $\mathcal{F}_\alpha(\Sigma^\infty)$ are called *α-Hölder continuous* functions on Σ^∞. For $f\in\mathcal{C}(\Sigma^\infty)$ define the *Perron-Frobenius operator* $\mathcal{L}_f\colon\mathcal{C}(\Sigma^\infty)\to\mathcal{C}(\Sigma^\infty)$ by

$$
\mathcal{L}_f\psi(x) := \sum_{y\colon\ \sigma y=x} e^{f(y)}\psi(y)
$$

for $x\in\Sigma^\infty$ and let $\mathcal{L}_f^*$ be the dual of $\mathcal{L}_f$ acting on the set of Borel probability measures on Σ^∞. By Theorem 2.16 and Corollary 2.17 of [Wal01] and Theorem 1.7 of [Bow08] for each real valued Hölder continuous $f\in\mathcal{C}(\Sigma^\infty)$ there exists a unique Borel probability measure ν_f on Σ^∞ such that $\mathcal{L}_f^*\nu_f=\gamma_f\nu_f$ for some $\gamma_f>0$. Moreover, γ_f is uniquely determined by this equation and satisfies $\gamma_f=\exp(P(f))$. Here $P\colon\mathcal{C}(\Sigma^\infty)\to\mathcal{C}(\Sigma^\infty)$ denotes the *topological pressure function* which for $f\in\mathcal{C}(\Sigma^\infty)$ is defined by

$$
P(f) := \lim_{n\to\infty} n^{-1}\ln\sum_{\omega\in\Sigma^n}\exp\sup_{u\in[\omega]}\sum_{k=0}^{n(\omega)} f\circ\sigma^k(u),
$$

(see Lemma 1.20 of [Bow08]), where $[\omega]:=\{u\in\Sigma^\infty \mid u_i=\omega_i \text{ for } 1\leq i\leq n(\omega)\}$ is the *ω-cylinder set.*

Further, there exists a unique strictly positive eigenfunction h_f of $\mathcal{L}_f$ satisfying $\mathcal{L}_f h_f=\gamma_f h_f$. We take h_f to be normalised so that $\int h_f \mathrm{d}\nu_f=1$. By μ_f we denote the σ-invariant probability measure defined by $\frac{\mathrm{d}\mu_f}{\mathrm{d}\nu_f}=h_f$. This is the unique σ-invariant Gibbs measure for the potential function f. Additionally, under some normalisation assumptions we have convergence of the iterates of the Perron-Frobenius operator to the projection onto its eigenfunction h_f. To be more precise we have

$$
\lim_{m\to\infty}\|\gamma_f^{-m}\mathcal{L}_f^m\psi-\textstyle\int\psi\mathrm{d}\nu_f\cdot h_f\|=0 \quad \forall\ \psi\in\mathcal{C}(\Sigma^\infty), \tag{3.1}
$$

where $\|\cdot\|$ denotes the supremum-norm on $\mathcal{C}(\Sigma^\infty)$.

Remark 3.6 The results on the Perron-Frobenius operator quoted above originate mainly from the work of [Rue68].

A central object of our investigations is the geometric potential function associated to the cIFS Φ and its property of being lattice or nonlattice, which we now define.

Definition 3.7 Two functions $f_1,f_2\in\mathcal{C}(\Sigma^\infty)$ are called *cohomologous*, if there exists a $\psi\in\mathcal{C}(\Sigma^\infty)$ such that $f_1-f_2=\psi-\psi\circ\sigma$. A function $f\in\mathcal{C}(\Sigma^\infty)$ is said to be a *lattice* function, if f is cohomologous to a function taking values in a discrete subgroup of $\mathbb{R}$. Otherwise, we say that f is a *nonlattice* function.

The notion of being lattice or not carries over to Φ and its self-conformal set F by considering the geometric potential function associated to Φ:

Definition 3.8 Let F denote the self-conformal set associated to the iterated function system $\Phi := \{\phi_1, \ldots, \phi_N\}$ with associated code space Σ^∞. Define the *geometric potential function* to be the map $\xi \colon \Sigma^\infty \to \mathbb{R}$ given by $\xi(\omega) := -\ln|\phi'_{\omega_1}(\sigma\omega)|$ for $\omega = \omega_1\omega_2 \cdots \in \Sigma^\infty$. If ξ is nonlattice, then we call Φ (and also F) *nonlattice*. On the other hand, if ξ is a lattice function, then we call Φ (and also F) *lattice*.

Remark 3.9 The geometric potential function ξ associated to a cIFS $\Phi := \{\phi_1, \ldots, \phi_N\}$ satisfies $\xi \in \mathcal{F}_{\widetilde{\alpha}}(\Sigma^\infty)$ for some $\widetilde{\alpha} \in (0,1)$. To see this, we let $r < 1$ be a common upper bound for the contraction ratios of $\phi_1, \ldots, \phi_N$. Because of the α-Hölder continuity of $\phi'_1, \ldots, \phi'_N$ we obtain that there exists a constant $c \in \mathbb{R}$ such that for every $n \in \mathbb{N}$ we have $\text{var}_n(\xi) \leq cr^{\alpha(n-1)}$. Thus, $\xi \in \mathcal{F}_{\widetilde{\alpha}}(\Sigma^\infty)$, where $\widetilde{\alpha} := r^\alpha \in (0,1)$.

For the geometric potential function $\xi \in \mathcal{C}(\Sigma^\infty)$ we introduce the *measure theoretical entropy* $H_{\mu_{-\delta\xi}}$ of the shift-map σ with respect to $\mu_{-\delta\xi}$ as the integral

$$H_{\mu_{-\delta\xi}} := \delta \int_{\Sigma^\infty} \xi \mathrm{d}\mu_{-\delta\xi}, \tag{3.2}$$

where δ is the Minkowski dimension of F. We remark that this is not the commonly used definition of measure theoretical entropy but that this characterisation for the measure theoretical entropy of σ with respect to $\mu_{-\delta\xi}$ follows from Theorem 1.22 of [Bow08] and the following result of [Bed88] which will also be needed in the proof of Theorem 2.6.

Theorem 3.10 *The Minkowski as well as the Hausdorff dimension of F is equal to the unique real number $t > 0$ such that $P(-t\xi) = 0$, where P denotes the topological pressure function.*

In what follows, we fix a cIFS $\Phi := \{\phi_1, \ldots, \phi_N\}$ acting on X and let $\alpha > 0$ denote the common Hölder exponent of $\phi'_1, \ldots, \phi'_N$. By δ we denote the corresponding Minkowski dimension and by ξ the geometric potential function. We are going to show that the eigenfunction $h_{-\delta\xi}$ of the Perron-Frobenius operator $\mathcal{L}_{-\delta\xi}$ can be extended to an α-Hölder continuous function on X. For that we let $\mathcal{C}(X)$ denote the set of real valued continuous functions on X and define the operator $\widetilde{\mathcal{L}} \colon \mathcal{C}(X) \to \mathcal{C}(X)$ by

$$\widetilde{\mathcal{L}}(g) := \sum_{i=1}^{N} |\phi'_i|^\delta \cdot g \circ \phi_i$$

for $g \in \mathcal{F}_\alpha(X)$, where $\mathcal{F}_\alpha(X)$ is the set of real valued α-Hölder continuous functions on X. We remark that $\widetilde{\mathcal{L}}$ is an extended version of the Perron-Frobenius operator given in (3.1) to functions which are defined on X.

Theorem 3.11 *Let ν be the δ-conformal measure and ξ the geometric potential function associated to the cIFS $\Phi := \{\phi_1, \ldots, \phi_N\}$. Let F denote the self-conformal set associated to Φ and let δ be its Minkowski dimension. Denote by π the natural code map and by α the Hölder exponent of the functions $\phi'_1, \ldots, \phi'_N$. Then there exists a unique $h \in \mathcal{F}_\alpha(X)$ such that*

$$\widetilde{\mathcal{L}}h = h, \quad \int_{\Sigma^\infty} h \mathrm{d}\nu = 1 \quad \text{and} \quad h|_F \circ \pi = h_{-\delta\xi},$$

where $h_{-\delta\xi} \in \mathcal{C}(\Sigma^\infty)$ is the unique eigenfunction of $\mathcal{L}_{-\delta\xi}$ to the eigenvalue 1.

Proof. We let 1 denote the constant one-function on X. By Lemma 6.1.1 of [MU03] the sequence $(\widetilde{\mathcal{L}}^n(1))_{n\in\mathbb{N}}$ is uniformly bounded and equicontinuous and thus so is the sequence

$(n^{-1}\sum_{i=0}^{n-1}\widetilde{\mathcal{L}}^i(1))_{n\in\mathbb{N}}$. Therefore, by Arzelà-Ascoli, the sequence of averages exhibits an accumulation point which we denote by h. Obviously $\widetilde{\mathcal{L}}h = h$ and $\int h\mathrm{d}\nu = 1$.

In order to show that $h \in \mathcal{F}_\alpha(X)$ it suffices to show that $f_n := n^{-1}\sum_{i=0}^{n-1}\widetilde{\mathcal{L}}^i(1)$ is α-Hölder continuous for every $n \in \mathbb{N}$ and that the Hölder constants are uniformly bounded.

$$\begin{aligned}|f_n(x) - f_n(y)| &= \left| n^{-1}\sum_{i=0}^{n-1}\sum_{\omega\in\Sigma^i} |\phi'_\omega(x)|^\delta - |\phi'_\omega(y)|^\delta \right| \\ &\leq n^{-1}\sum_{i=0}^{n-1}\sum_{\omega\in\Sigma^i}\left|\exp\left(\delta\sum_{k=1}^{i}\ln\phi'_{\omega_k}(\phi_{\sigma^k\omega}x)\right) - \exp\left(\delta\sum_{k=1}^{i}\ln\phi'_{\omega_k}(\phi_{\sigma^k\omega}y)\right)\right|.\end{aligned}$$

By hypotheses, $\ln\phi'_i$ is α-Hölder continuous for every $i \in \{1, \ldots, N\}$. Let $c_1, \ldots, c_N$ denote the respective Hölder constants of $\ln\phi'_1, \ldots, \ln\phi'_N$, set $c := \max_{i=1,\ldots,N} c_i$ and let $r < 1$ be a common upper bound for the contraction ratios of $\phi_1, \ldots, \phi_N$. Applying the Mean Value Theorem to exp and letting θ_ω denote the mean value corresponding to the ω-summand, we obtain the following set of inequalities.

$$\begin{aligned}|f_n(x) - f_n(y)| &\leq n^{-1}\sum_{i=0}^{n-1}\sum_{\omega\in\Sigma^i} e^{\theta_\omega}\cdot\delta\sum_{k=1}^{i} c|\phi_{\sigma^k\omega}x - \phi_{\sigma^k\omega}y|^\alpha \\ &\leq n^{-1}\sum_{i=0}^{n-1}\sum_{\omega\in\Sigma^i} e^{\theta_\omega}\cdot\frac{\delta c}{1-r^\alpha}|x-y|^\alpha.\end{aligned}$$

Since θ_ω lies between $\ln|\phi'_\omega(x)|^\delta$ and $\ln|\phi'_\omega(y)|^\delta$, there exists a $\widetilde{\theta}_\omega \in \mathbb{R}$ such that $|\phi'_\omega(\widetilde{\theta}_\omega)|^\delta = e^{\theta_\omega}$. By definition of the δ-conformal measure it can be easily seen that $|\phi'_\omega(\widetilde{\theta}_\omega)|^\delta \leq \rho_0\nu(\phi_\omega F)$. Thus,

$$|f_n(x) - f_n(y)| \leq \underbrace{\frac{\rho_0\delta c}{1-r^\alpha}}_{=:\widetilde{c}}|x-y|^\alpha.$$

Hence the Hölder constant of each function f_n is bounded by $\widetilde{c}$. The uniqueness of h and $h|_F \circ \pi = h_{-\delta\xi}$ have been shown in Theorem 6.1.2 of [MU03]. □

3.5 Renewal Theory and Geometric Measure Theory

In the proof of Theorem 2.6 we are going to make use of a renewal theory argument for counting measures in symbolic dynamics. For stating this we fix the following notations.

For a map $f\colon \Sigma^\infty \to \mathbb{R}$ and $n \in \mathbb{N}$ define the *n-th ergodic sum* to be $S_n f := \sum_{k=0}^{n-1} f\circ\sigma^k$ and $S_0 f := 0$. Moreover, we call a function $f_1\colon (0,\infty) \to \mathbb{R}$ *asymptotic* to a function $f_2\colon (0,\infty) \to \mathbb{R}$ as $\varepsilon \to 0$, in symbols $f_1(\varepsilon) \sim f_2(\varepsilon)$ as $\varepsilon \to 0$, if $\lim_{\varepsilon\to 0} f_1(\varepsilon)/f_2(\varepsilon) = 1$. Similarly, we say that f_1 is *asymptotic* to f_2 as $t \to \infty$, in symbols $f_1(t) \sim f_2(t)$ as $t \to \infty$, if $\lim_{t\to\infty} f_1(t)/f_2(t) = 1$.

The following Proposition is a well-known fact which is for example stated in Proposition 2.1 of [Lal89].

Proposition 3.12 *Let $f \in \mathcal{F}_\alpha(\Sigma^\infty)$ for some $0 < \alpha < 1$ be such that for some $n \geq 1$ the function $S_n f$ is strictly positive on Σ^∞. Then there exists a unique $s > 0$ such that*

$$\gamma_{-sf} = 1. \tag{3.3}$$

The following two theorems play a crucial role in the proof of Theorem 2.6. The first of the two theorems is Theorem 1 of [Lal89]. The second one is a refinement and generalisation of Theorem 3 in [Lal89] and hence we will state a proof.

Proposition 3.13 (Lalley) *Assume that f lies in $\mathcal{F}_\alpha(\Sigma^\infty)$ for some $0 < \alpha < 1$, is nonlattice and such that for some $n \geq 1$ the function $S_n f$ is strictly positive. Let $g \in \mathcal{F}_\alpha(\Sigma^\infty)$ be nonnegative but not identically zero and s as in Equation (3.3). Then we have that*

$$\sum_{n=0}^{\infty} \sum_{y\colon \sigma^n y = x} g(y) \mathbb{1}_{\{S_n f(y) \leq t\}} \sim \frac{\int g \mathrm{d}\nu_{-sf}}{s \int f \mathrm{d}\mu_{-sf}} h_{-sf}(x) e^{st}$$

as $t \to \infty$ uniformly for $x \in \Sigma^\infty$.

For $b \in \mathbb{R}$ we denote by $\lceil b \rceil$ the smallest integer which is greater than or equal to b. $\lfloor b \rfloor$ shall denote the greatest integer which is less than or equal to b and by $\{b\}$ we mean the fractional part of b, that is $\{b\} := b - \lfloor b \rfloor$.

Theorem 3.14 *Assume that f lies in $\mathcal{F}_\alpha(\Sigma^\infty)$ for some $0 < \alpha < 1$ is lattice and such that for some $n \geq 1$ the function $S_n f$ is strictly positive. Let $\zeta, \psi \in \mathcal{C}(\Sigma^\infty)$ denote functions which satisfy*

$$f - \zeta = \psi - \psi \circ \sigma,$$

where ζ is a function taking values in a discrete subgroup of $\mathbb{R}$. Let a denote the maximal positive real number such that $\{a^{-1}\zeta(x) \mid x \in \Sigma^\infty\} \subseteq \mathbb{Z}$. Further, let $g \in \mathcal{F}_\alpha(\Sigma^\infty)$ be nonnegative but not identically zero and s as in Equation (3.3). Then we have that

$$\sum_{n=0}^{\infty} \sum_{y\colon \sigma^n y = x} g(y) \mathbb{1}_{\{S_n f(y) \leq t\}} \sim \frac{a h_{-s\zeta}(x) \int g(y) e^{-sa \left\lceil \frac{\psi(y) - \psi(x)}{a} - \frac{t}{a} \right\rceil} \mathrm{d}\nu_{-s\zeta}(y)}{(1 - e^{-sa}) \int \zeta \mathrm{d}\mu_{-s\zeta}} \tag{3.4}$$

as $t \to \infty$ uniformly for $x \in \Sigma^\infty$.

Remark 3.15 Proposition 3.13 and Theorem 3.14 are also valid in the more general situation of (Σ^∞, σ) being a subshift of finite type.

Proof (of Theorem 3.14). For the proof we first assume that $a = 1$, which implies that ζ is integer valued and not cohomologous to any function taking its values in a proper subgroup of $\mathbb{R}$. We remark, that under these assumptions a similar result is stated in Theorem 3 of [Lal89]. However, there the exact asymptotic is not given. In order to obtain the exact asymptotic, we first follow the lines of the proofs of Theorem 2 and Theorem 3 of [Lal89] and then refine the last steps of the proof of Theorem 3 of [Lal89].

Lalley introduces the following functions for the definition of which we let $t \in \mathbb{R}$ and $x \in \Sigma^\infty$.

$$\begin{aligned} N_f(t,x) &:= \sum_{n=0}^{\infty} \sum_{y\colon \sigma^n y = x} g(y) \mathbb{1}_{\{S_n f(y) \leq t\}}, \\ N^*(t,x) &:= N_f(t - \psi(x), x) \end{aligned}$$

and for $\beta \in [0,1)$ and $z \in \mathbb{C}$ the Fourier-Laplace transform

$$\hat{N}^*_\beta(z,x) := \sum_{n=-\infty}^{\infty} e^{nz} N^*(n + \beta, x).$$

It is easy to verify that $N_f(t,x)$ satisfies a renewal equation (see Equation (2.2) in [Lal89])

$$N_f(t,x) = \sum_{y\colon \sigma y = x} N_f(t - f(y), y) + g(x)\mathbb{1}_{\{t\geq 0\}}$$

from which one can deduce that $\hat{N}^*_\beta$ satisfies the following equation.

$$\hat{N}^*_\beta(z,x) = (I - \mathcal{L}_{z\zeta})^{-1} g(x) \frac{e^{z\lceil \phi(x)-\beta\rceil}}{1-e^z}, \tag{3.5}$$

where I denotes the identity operator. We remark that Equation (3.5) differs slightly from the respective equation in [Lal89], in that Lalley obtains $z\lfloor \phi(x)+1-\beta\rfloor$ as the exponent of e whereas our calculations result in $z\lceil \phi(x)-\beta\rceil$ being the right exponent of e.

By arguments in the proof of Theorem 2 of [Lal89] the function $z \mapsto (I - \mathcal{L}_{z\zeta})^{-1} g(x)$ is meromorphic in $\{z \in \mathbb{C} \mid 0 \leq \mathrm{Im}(z) \leq \pi,\ \mathrm{Re}(z) < -s+\varepsilon\}$ for some $\varepsilon > 0$ and the only singularity in this region is a simple pole at $z = -s$ with residue

$$\frac{h_{-s\zeta}(x)\int g \mathrm{d}\nu_{-s\zeta}}{\int \zeta \mathrm{d}\mu_{-s\zeta}}.$$

Since $z \mapsto e^{z\lceil\psi(x)-\beta\rceil}$ and $z \mapsto (1-e^z)^{-1}$ are holomorphic in $\{z \in \mathbb{C} \mid \mathrm{Re}(z) < 0\}$ we deduce from this that $z \mapsto \hat{N}^*_\beta(z,x)$ is meromorphic in $\{z \in \mathbb{C} \mid 0 \leq \mathrm{Im}(z) \leq \pi,\ \mathrm{Re}(z) < -s+\varepsilon\}$ for some $\varepsilon > 0$ and that the only singularity in this region is a simple pole at $z=-s$ with residue

$$\frac{h_{-s\zeta}(x)\int g(y)e^{-s\lceil\psi(y)-\beta\rceil}\mathrm{d}\nu_{-s\zeta}(y)}{(1-e^{-s})\int \zeta \mathrm{d}\mu_{-s\zeta}} =: C(\beta, x).$$

Now, again following the lines of the proof of Theorem 2 of [Lal89], it follows that

$$N^*(n+\beta, x) \sim C(\beta,x)e^{sn}$$

as $n \to \infty$ uniformly for $x \in \Sigma^\infty$. Thus for $t \in (0,\infty)$

$$\begin{aligned} N_f(t,x) &= N_f(\underbrace{\lfloor\psi(x)+t\rfloor}_{=:n} + \underbrace{\{\psi(x)+t\}}_{=:\beta} - \psi(x), x) = N^*(n+\beta, x) \\ &\sim C(\beta,x)e^{sn} = \frac{h_{-s\zeta}(x)\int g(y)e^{-s\lceil\psi(y)-\psi(x)-t\rceil}\mathrm{d}\nu_{-s\zeta}(y)}{(1-e^{-s})\int\zeta\mathrm{d}\mu_{-s\zeta}} \end{aligned}$$

as $n \to \infty$ uniformly for $x \in \Sigma^\infty$. This proves the case $a=1$.

The case that $a \neq 1$ is not covered in [Lal89]. If $a > 0$ is arbitrary, then we consider the function $a^{-1}f = a^{-1}\zeta + a^{-1}\psi - \left(a^{-1}\psi\right)\circ\sigma$. Since by Proposition 3.12, s from Equation (3.3) is the unique positive real number such that $\gamma_{-sf} = 1$, $\widetilde{s} := as$ is the unique positive real number satisfying $\gamma_{-\widetilde{s}a^{-1}f} = 1$. Therefore, Equation (3.6) implies

$$\begin{aligned} \sum_{n=0}^{\infty}\sum_{y\colon\sigma^n y = x} g(y)\mathbb{1}_{\{S_n f(y)\leq t\}} &= \sum_{n=0}^{\infty}\sum_{y\colon\sigma^n y = x} g(y)\mathbb{1}_{\{S_n a^{-1}f(y)\leq ta^{-1}\}} \\ &\sim \frac{h_{-s\zeta}(x)\int g(y)e^{-sa\left\lceil\frac{\psi(y)-\psi(x)}{a}-\frac{t}{a}\right\rceil}\mathrm{d}\nu_{-s\zeta}(y)}{(1-e^{-as})\int a^{-1}\zeta\mathrm{d}\mu_{-s\zeta}} \end{aligned}$$

as $t \to \infty$ uniformly for $x \in \Sigma^\infty$. □

In view of the existence of the average fractal curvature measures the following corollary is essential.

Corollary 3.16 *Under the assumptions of Theorem 3.14*

$$\lim_{T\to\infty} T^{-1} \int_0^T e^{-st} \sum_{n=0}^{\infty} \sum_{y\colon \sigma^n y = x} g(y) \mathbb{1}_{\{S_n f(y) \le t\}} \mathrm{d}t \tag{3.6}$$

exists and equals

$$\frac{h_{-sf}(x) \int g \mathrm{d}\nu_{-sf}}{s \int f \mathrm{d}\mu_{-sf}}.$$

Proof. First, observe that for two functions $f_1,\ f_2\colon (0,\infty) \to \mathbb{R}$ which satisfy $f_1(t) \sim f_2(t)$ as $t \to \infty$, the existence of $G_1 := \lim_{T\to\infty} T^{-1} \int_0^T f_1(t)\mathrm{d}t$ implies the existence of $G_2 := \lim_{T\to\infty} T^{-1} \int_0^T f_2(t)\mathrm{d}t$ and $G_1 = G_2$. In view of Theorem 3.14, we hence consider the function $\eta\colon [0,\infty) \to \mathbb{R}$ given by

$$\eta(t) := e^{-st} \int_{\Sigma^\infty} g(y) e^{-sa\left\lceil \frac{\psi(y)-\psi(x)}{a} - \frac{t}{a} \right\rceil} \mathrm{d}\nu_{-s\zeta}(y).$$

Since $\eta(t+a) = \eta(t)$ for all $t \in (0,\infty)$, η is periodic with period a. As η is moreover locally integrable, this implies

$$\begin{aligned} \lim_{T\to\infty} T^{-1} \int_0^T \eta(t)\mathrm{d}t &= \lim_{T\to\infty} T^{-1} \Bigg(\sum_{k=0}^{\lfloor a^{-1}T \rfloor -1} \int_{T-a(k+1)}^{T-ak} \eta(t)\mathrm{d}t + \int_0^{T-a\lfloor a^{-1}T \rfloor} \eta(t)\mathrm{d}t \Bigg) \\ &= \lim_{T\to\infty} T^{-1} \lfloor a^{-1}T \rfloor \int_0^a \eta(t)\mathrm{d}t = a^{-1} \int_0^a \eta(t)\mathrm{d}t. \end{aligned}$$

Applying Fubini's theorem yields

$$\int_0^a \eta(t)\mathrm{d}t = \int_{\Sigma^\infty} \int_0^a e^{-st} g(y) e^{-sa\left\lceil \frac{\psi(y)-\psi(x)}{a} - \frac{t}{a} \right\rceil} \mathrm{d}t \mathrm{d}\nu_{-s\zeta}(y).$$

Define $E(y) := a\{a^{-1}(\psi(y)-\psi(x))\}$. This is the unique real number in $[0,a)$ such that $a^{-1}(\psi(y)-\psi(x)-E(y)) \in \mathbb{Z}$. Since $a^{-1}t \in [0,1)$ for $t \in [0,a)$, we hence have

$$\begin{aligned} &\int_0^a \eta(t)\mathrm{d}t \\ &\quad= \int_{\Sigma^\infty} \left(\int_0^{E(y)} e^{-st} g(y) e^{-sa\left\lceil \frac{\psi(y)-\psi(x)}{a} \right\rceil} \mathrm{d}t + \int_{E(y)}^a e^{-st} g(y) e^{-sa\left\lfloor \frac{\psi(y)-\psi(x)}{a} \right\rfloor} \mathrm{d}t \right) \mathrm{d}\nu_{-s\zeta}(y) \\ &\quad= \int_{\Sigma^\infty} \frac{g(y)}{s} \left(e^{-sa\left\lceil \frac{\psi(y)-\psi(x)}{a} \right\rceil} \left(1 - e^{-sE(y)}\right) + e^{-sa\left\lfloor \frac{\psi(y)-\psi(x)}{a} \right\rfloor} \left(e^{-sE(y)} - e^{-sa}\right) \right) \mathrm{d}\nu_{-s\zeta}(y) \\ &\quad= \frac{1-e^{-sa}}{s} e^{s\psi(x)} \int_{\Sigma^\infty} g(y) e^{-s\psi(y)} \mathrm{d}\nu_{-s\zeta}(y), \end{aligned}$$

where the last equality can be obtained by distinguishing the cases $E(y) \neq 0$ and $E(y) = 0$, that is $a^{-1}(\psi(y)-\psi(x)) \in \mathbb{Z}$. As by Theorem 3.14

$$e^{-st} \sum_{n=0}^{\infty} \sum_{y\colon \sigma^n y = x} g(y) \mathbb{1}_{\{S_n f(y) \le t\}} \sim \frac{a h_{-s\zeta}(x)}{(1-e^{-sa}) \int \zeta \mathrm{d}\mu_{-s\zeta}} \eta(t),$$

the entering remark of this proof now implies

$$\lim_{T\to\infty} T^{-1}\int_0^T e^{-st}\sum_{n=0}^{\infty}\sum_{y\colon \sigma^n y=x} g(y)\mathbb{1}_{\{S_nf(y)\le t\}}\mathrm{d}t = \frac{e^{s\psi(x)}h_{-s\zeta}(x)}{s\int\zeta\mathrm{d}\mu_{-s\zeta}}\int g(y)e^{-s\psi(y)}\mathrm{d}\nu_{-s\zeta}(y).$$

Finally, one easily verifies that $e^{s\psi}h_{-s\zeta} = h_{-sf}$, $e^{-s\psi}\mathrm{d}\nu_{-s\zeta} = \mathrm{d}\nu_{-sf}$ and $\int\zeta\mathrm{d}\mu_{-s\zeta} = \int f\mathrm{d}\mu_{-sf}$, which completes the proof. □

In order to prove Theorem 2.8 the following Lemma which is closely related to Theorem 3.14 is needed.

Lemma 3.17 *Assume that we are in the setting of Theorem 3.14. Further, let $B\subseteq\Sigma^\infty$ be a nonempty Borel set such that $\mathbb{1}_B\in\mathcal{F}_\alpha(\Sigma^\infty)$. Let $N_1,N_2\in\mathbb{Z}$ be such that $\psi(\Sigma^\infty)\subseteq[N_1a,N_2a)$ and N_2-N_1 is minimal. Define the function $\eta_B\colon(0,\infty)\to\mathbb{R}$ by*

$$\eta_B(t) := e^{-st}\int_{\Sigma^\infty}\mathbb{1}_B(y)e^{-sa\left\lceil\frac{\psi(y)-\psi(x)}{a}-\frac{t}{a}\right\rceil}\mathrm{d}\nu_{-s\zeta}(y).$$

Then $\lim_{t\to\infty}\eta_B(t)$ exists if and only if for every $t\in[0,a)$ we have

$$\sum_{n=N_1}^{N_2-1} e^{-san}\nu_{-s\zeta}\circ\psi^{-1}\big(\psi(B)\cap[na,na+t)\big)$$
$$= \frac{e^{st}-1}{e^{sa}-1}\sum_{n=N_1}^{N_2-1} e^{-san}\nu_{-s\zeta}\circ\psi^{-1}\big(\psi(B)\cap[na,(n+1)a)\big).$$

Proof. η_B is a periodic function with period a, meaning $\eta_B(t+a)=\eta_B(t)$ for all $t\in(0,\infty)$. Therefore, $\lim_{t\to\infty}\eta_B(t)$ exists if and only if η_B is a constant function. For $t\in[\psi(x),\psi(x)+a)$ we have

$$\begin{aligned}
\eta_B(t-\psi(x)) &= e^{s\psi(x)-st}\int_{\Sigma^\infty}\mathbb{1}_B(y)e^{-sa\left\lceil\frac{\psi(y)-t}{a}\right\rceil}\mathrm{d}\nu_{-s\zeta}(y)\\
&= e^{s\psi(x)-st}\sum_{n=N_1}^{N_2-1}\int_{na}^{(n+1)a}\mathbb{1}_{\psi(B)}(y)e^{-sa\left\lceil\frac{y-t}{a}\right\rceil}\mathrm{d}\nu_{-s\zeta}\circ\psi^{-1}(y)\\
&= e^{s\psi(x)-st+sa\left\lfloor\frac{t}{a}\right\rfloor}\sum_{n=N_1}^{N_2-1}e^{-san}\Big(\nu_{-s\zeta}\circ\psi^{-1}\big(\psi(B)\cap[na,na+a\{a^{-1}t\}]\big)\\
&\qquad +e^{-sa}\nu_{-s\zeta}\circ\psi^{-1}\big(\psi(B)\cap(na+a\{a^{-1}t\},(n+1)a)\big)\Big)\\
&= e^{s\psi(x)-sa\{\frac{t}{a}\}}\sum_{n=N_1}^{N_2-1}e^{-san}\Big((1-e^{-sa})\nu_{-s\zeta}\circ\psi^{-1}\big(\psi(B)\cap[na,na+a\{a^{-1}t\}]\big)\\
&\qquad +e^{-sa}\nu_{-s\zeta}\circ\psi^{-1}\big(\psi(B)\cap[na,(n+1)a)\big)\Big).
\end{aligned}$$

Thus, $\lim_{t\to\infty}\eta_B(t)$ exists if and only if there is a $\widetilde{c}\in\mathbb{R}$ such that for every $t\in[0,a)$ we have $\eta_B(t-\psi(x))=\widetilde{c}$, that is

$$\sum_{n=N_1}^{N_2-1}e^{-san}\nu_{-s\zeta}\circ\psi^{-1}\big(\psi(B)\cap[na,na+t]\big)$$
$$=(1-e^{-sa})^{-1}\Big(\widetilde{c}e^{st-s\psi(x)}-e^{-sa}\sum_{n=N_1}^{N_2-1}e^{-san}\nu_{-s\zeta}\circ\psi^{-1}\big(\psi(B)\cap[na,(n+1)a)\big)\Big).$$

Taking the limit as t tends to a we hence obtain

$$\widetilde{c} = e^{s\psi(x)-sa} \sum_{n=N_1}^{N_2-1} e^{-san} \nu_{-s\zeta} \circ \psi^{-1}\big(\psi(B) \cap [na,(n+1)a)\big)$$

which proves the statement. $\square$

Another important tool in the proofs of our results is a relationship between the 0-th and the 1-st (average) fractal curvature measures. In order to show that the existence of the 0-th fractal curvature measure implies the existence of the 1-st fractal curvature measure we use Corollary 3.2 of [RW] which is a higher dimensional and more general version of the following theorem.

Theorem 3.18 (Rataj, Winter) *Let $Y \subset \mathbb{R}$ be compact and such that $\lambda^1(Y) = 0$. Then*

$$\liminf_{\varepsilon\to 0} \frac{\varepsilon^{\delta}\lambda^0(\partial Y_\varepsilon)}{1-\delta} \leq \liminf_{\varepsilon\to 0} \varepsilon^{\delta-1}\lambda^1(Y_\varepsilon) \leq \limsup_{\varepsilon\to 0} \varepsilon^{\delta-1}\lambda^1(Y_\varepsilon) \leq \limsup_{\varepsilon\to 0} \frac{\varepsilon^{\delta}\lambda^0(\partial Y_\varepsilon)}{1-\delta}.$$

The proof is based on an interesting relationship between the derivative $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\lambda^1(F_\varepsilon)$ which exists Lebesgue almost everywhere and the quantity $\lambda^0(\partial F_\varepsilon)$ which was established in [Sta76] for arbitrary bounded subsets of $\mathbb{R}^d$ and builds on the work of [Kne51].

For the results on the average fractal curvature measures we use Part (ii) of Lemma 4.6 of [RW] which is a higher dimensional version of

Proposition 3.19 (Rataj, Winter) *Let $Y \subset \mathbb{R}$ be compact and such that its Minkowski dimension δ is less than 1, that is $\delta < 1$. If $\overline{\mathcal{M}}(Y) < \infty$, then*

$$\begin{aligned}
\limsup_{T\to 0} |\ln T|^{-1} \int_T^1 \varepsilon^{\delta-2}\lambda^1(Y_\varepsilon)\mathrm{d}\varepsilon &= (1-\delta)^{-1} \limsup_{T\to 0} |\ln T|^{-1} \int_T^1 \varepsilon^{\delta-1}\lambda^0(Y_\varepsilon)\mathrm{d}\varepsilon \quad \textit{and} \\
\liminf_{T\to 0} |\ln T|^{-1} \int_T^1 \varepsilon^{\delta-2}\lambda^1(Y_\varepsilon)\mathrm{d}\varepsilon &= (1-\delta)^{-1} \liminf_{T\to 0} |\ln T|^{-1} \int_T^1 \varepsilon^{\delta-1}\lambda^0(Y_\varepsilon)\mathrm{d}\varepsilon.
\end{aligned}$$

Finally, we also need the following result, which goes back to Stachó and is stated in Corollary 2.5 of [RW].

Proposition 3.20 (Stachó) *Let $Y \subset \mathbb{R}$ be compact. Then the function $\varepsilon \mapsto \lambda^1(Y_\varepsilon)$ is differentiable for all but a countable number of $\varepsilon > 0$ with differential*

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\lambda^1(Y_\varepsilon) = \lambda^0(\partial Y_\varepsilon).$$

4 Proofs of Theorem 2.6 and Theorem 2.8

We start by making the following observations which are needed in the proofs of all three parts of Theorem 2.6 and of Theorem 2.8.

Without loss of generality we assume that $\{0,1\} \subset F \subseteq [0,1]$ as otherwise the result follows by rescaling. We start by giving the proof for the 0-th fractal curvature measure. For that we fix an $\varepsilon > 0$ and consider the expression $\lambda^0(\partial F_\varepsilon \cap (-\infty, b])$ for some $b \in \mathbb{R}$. Since λ^0 is the counting measure, $\lambda^0(\partial F_\varepsilon \cap (-\infty, b])$ gives the number of endpoints of the connected components of F_ε in $(-\infty, b]$. This number can be obtained by looking at how many complementary intervals of lengths greater than or equal to 2ε exist in $(-\infty, b]$:

$$\lambda^0\big(\partial F_\varepsilon \cap (-\infty, b]\big)/2 = \underbrace{\sum_{i=1}^{Q-1} \#\{\omega \in \Sigma^* \mid L_\omega^i \subseteq (-\infty, b],\ |L_\omega^i| \geq 2\varepsilon\}}_{=:\Xi(\varepsilon)} + c_1/2, \tag{4.1}$$

where $c_1 \in \{0, 1, 2\}$ depends on the value of b. Next, we need to find appropriate bounds for $\Xi(\varepsilon)$. For this, we choose an $m \in \mathbb{N}$ such that for all $\omega \in \Sigma^m$ all main gaps $L_\omega^1, \ldots, L_\omega^{Q-1}$ of the sets $\phi_\omega(F)$ are greater than or equal to 2ε and set

$$\Xi_\omega^i(\varepsilon) \quad := \quad \#\{u \in \Sigma^* \mid L_{u\omega}^i \subseteq (-\infty, b],\ |L_{u\omega}^i| \geq 2\varepsilon\}$$

for each $\omega \in \Sigma^m$ and $i \in \{1, \ldots, Q-1\}$. We have the following connection.

$$\sum_{i=1}^{Q-1} \sum_{\omega \in \Sigma^m} \Xi_\omega^i(\varepsilon) \leq \Xi(\varepsilon) \leq \sum_{i=1}^{Q-1} \sum_{\omega \in \Sigma^m} \Xi_\omega^i(\varepsilon) + \sum_{j=1}^{m} (Q-1) \cdot N^{j-1}. \tag{4.2}$$

For the following, we fix $b \in \mathbb{R} \setminus F$. Then $F \cap (-\infty, b]$ can be expressed as a finite union of sets of the form $\phi_\kappa F$, where $\kappa \in \Sigma^*$. To be more precise, let $l \in \mathbb{N}$ be minimal such that there exist $\kappa_1, \ldots, \kappa_l \in \Sigma^*$ satisfying

(i) $F \cap (-\infty, b] = \bigcup_{j=1}^{l} \phi_{\kappa_j} F$ and

(ii) $\phi_{\kappa_i} F \cap \phi_{\kappa_j} F$ contains at most one point for all $i \neq j$, where $i, j \in \{1, \ldots, l\}$.

Then for $Z := \bigcup_{j=1}^{l} [\kappa_j]$ the function $\mathbb{1}_Z$ is Hölder continuous. Making use of the existence of the bounded distortion constant $\rho_{n(\omega)}$ of Φ on $\phi_\omega F$ (see Lemma 3.5), we can give estimates for $\Xi_\omega^i(\varepsilon)$, namely for an arbitrary $x \in F^{\text{unique}}$ we have

$$\Xi_\omega^i(\varepsilon) \quad \leq \quad \underbrace{\sum_{n=0}^{\infty} \sum_{u \in \Sigma^n} \mathbb{1}_Z(u\omega x) \mathbb{1}_{\{|\phi_u'(\phi_\omega x)| \cdot \rho_{n(\omega)} \cdot |L_\omega^i| \geq 2\varepsilon\}}}_{=:\overline{A}_\omega^i(x,\varepsilon,Z)} + \overline{c}_2(Z), \tag{4.3}$$

where we need to insert the constant $\overline{c}_2(Z)$ because of the following reason. $L_{u\omega}^i \subseteq (-\infty, b]$ does not necessarily imply $u\omega x \in Z$ for an arbitrary $x \in F^{\text{unique}}$. However, if $n(u) \geq \max_{j=1,\ldots,l} n(\kappa_j)$, either $[u\omega] \subseteq Z$ or $[u\omega] \cap Z = \varnothing$ obtains. Hence, there are only finitely many $u \in \Sigma^*$ such that $L_{u\omega}^i \subseteq (-\infty, b]$ does not imply $u\omega x \in Z$ for all $x \in F^{\text{unique}}$. Letting $\overline{c}_2(Z) \in \mathbb{R}$ denote this finite number shows that Equation (4.3) is true for all $\varepsilon > 0$. Likewise, there exists a constant $\underline{c}_2(Z) \in \mathbb{R}$ such that for all $\varepsilon > 0$

$$\Xi_\omega^i(\varepsilon) \quad \geq \quad \underbrace{\sum_{n=0}^{\infty} \sum_{u \in \Sigma^n} \mathbb{1}_Z(u\omega x) \cdot \mathbb{1}_{\{|\phi_u'(\phi_\omega x)| \cdot \rho_{n(\omega)}^{-1} \cdot |L_\omega^i| \geq 2\varepsilon\}}}_{=:\underline{A}_\omega^i(x,\varepsilon,Z)} - \underline{c}_2(Z). \tag{4.4}$$

Combining Equations (4.1)-(4.4) we obtain that for all $m \in \mathbb{N}$

$$\overline{C}_0^f(F, (-\infty, b]) \quad \leq \quad \limsup_{\varepsilon \to 0} \varepsilon^\delta \sum_{i=1}^{Q-1} \sum_{\omega \in \Sigma^m} \overline{A}_\omega^i(x, \varepsilon, Z) \quad \text{and} \tag{4.5}$$

$$\underline{C}_0^f(F, (-\infty, b]) \quad \geq \quad \liminf_{\varepsilon \to 0} \varepsilon^\delta \sum_{i=1}^{Q-1} \sum_{\omega \in \Sigma^m} \underline{A}_\omega^i(x, \varepsilon, Z). \tag{4.6}$$

We now want to apply Proposition 3.13 and Theorem 3.14 respectively to get asymptotics for both the expressions $\overline{A}^i_\omega(x,\varepsilon,Z)$ and $\underline{A}^i_\omega(x,\varepsilon,Z)$. For this note that

$$\begin{aligned}\sum_{u\in\Sigma^n}\mathbb{1}_Z(u\omega x)\cdot\mathbb{1}_{\{|\phi'_u(\phi_\omega x)|\cdot\rho^{\pm1}_{n(\omega)}\cdot|L^i_\omega|\geq 2\varepsilon\}} &= \sum_{y\colon\sigma^n y=\omega x}\mathbb{1}_Z(y)\cdot\mathbb{1}_{\{\sum_{k=1}^n -\ln|\phi'_{y_k}(\sigma^k y)|\leq -\ln\frac{2\varepsilon}{|L^i_\omega|\rho^{\pm1}_{n(\omega)}}\}}\\ &= \sum_{y\colon\sigma^n y=\omega x}\mathbb{1}_Z(y)\cdot\mathbb{1}_{\{S_n\xi(y)\leq -\ln\frac{2\varepsilon}{|L^i_\omega|\rho^{\pm1}_{n(\omega)}}\}}. \qquad (4.7)\end{aligned}$$

The hypotheses and Remark 3.9 imply that the geometric potential function ξ is Hölder continuous and strictly positive. The unique $s>0$ for which $\gamma_{-s\xi}=1$, is precisely the Minkowski dimension δ of F, which results by combining the fact that $\gamma_{-s\xi}=\exp(P(-s\xi))$ for each $s>0$ and Theorem 3.10.

Before we distinguish between the lattice and nonlattice case and give the proof of Theorem 2.6, we prove the following Lemma, which is needed in the proofs of all three parts of Theorem 2.6.

Lemma 4.1 *For an arbitrary $x\in F^{unique}$ and $\Upsilon\in\mathbb{R}$ we have that*

(i) $\Upsilon\leq\sum_{i=1}^{Q-1}\sum_{\omega\in\Sigma^m}h_{-\delta\xi}(\omega x)\big(|L^i_\omega|\rho_m\big)^\delta\ \forall m\in\mathbb{N}$ *implies* $\Upsilon\leq\liminf_{m\to\infty}\sum_{i=1}^{Q-1}\sum_{\omega\in\Sigma^m}|L^i_\omega|^\delta$

and

(ii) $\Upsilon\geq\sum_{i=1}^{Q-1}\sum_{\omega\in\Sigma^m}h_{-\delta\xi}(\omega x)\big(|L^i_\omega|\rho_m^{-1}\big)^\delta\ \forall m\in\mathbb{N}$ *implies* $\Upsilon\geq\limsup_{m\to\infty}\sum_{i=1}^{Q-1}\sum_{\omega\in\Sigma^m}|L^i_\omega|^\delta.$

Proof. We are first going to approximate the eigenfunction $h_{-\delta\xi}$ of the Perron-Frobenius operator $\mathcal{L}_{-\delta\xi}$. For that we claim that $\mathcal{L}^n_{-\delta\xi}1(x)=\sum_{u\in\Sigma^n}|\phi'_u(x)|^\delta$ for each $x\in\Sigma^\infty$, where 1 is the constant one-function. This can be easily seen by induction. Since $\mathcal{L}^n_{-\delta\xi}1$ converges uniformly to the eigenfunction $h_{-\delta\xi}$ when taking $n\to\infty$ (see (3.1)) we have that

$$\forall t>0\ \exists M\in\mathbb{N}\colon\forall n\geq M,\ \forall x\in\Sigma^\infty\colon\left|\sum_{u\in\Sigma^n}|\phi'_u(x)|^\delta-h_{-\delta\xi}(x)\right|<t.$$

Furthermore, through Lemma 3.5 we know that

$$\forall t'>0\ \exists M'\in\mathbb{N}\colon\forall m\geq M'\colon|\rho_m-1|<t'.$$

Thus, for all $n\geq M$ and $m\geq M'$

$$\begin{aligned}\Upsilon &\leq \sum_{i=1}^{Q-1}\sum_{\omega\in\Sigma^m}h_{-\delta\xi}(\omega x)\left(|L^i_\omega|\rho_m\right)^\delta\\ &\leq \sum_{i=1}^{Q-1}\sum_{\omega\in\Sigma^m}\left(\sum_{u\in\Sigma^n}|\phi'_u(\phi_\omega x)|^\delta+t\right)\left(|L^i_\omega|\rho_m\right)^\delta\\ &\leq \sum_{i=1}^{Q-1}\sum_{\omega\in\Sigma^m}\sum_{u\in\Sigma^n}|\phi_u(L^i_\omega)|^\delta\rho_m^{2\delta}+t\sum_{i=1}^{Q-1}\sum_{\omega\in\Sigma^m}\left(|L^i_\omega|\rho_m\right)^\delta\\ &\leq (1+t')^{2\delta}\sum_{i=1}^{Q-1}\sum_{\omega\in\Sigma^m}\sum_{u\in\Sigma^n}|L^i_{u\omega}|^\delta+t(1+t')^\delta\sum_{i=1}^{Q-1}\sum_{\omega\in\Sigma^m}|L^i_\omega|^\delta=:A_{m,n}\end{aligned}$$

Hence, for all $t, t' > 0$

$$\begin{aligned}\Upsilon &\leq \liminf_{m\to\infty} \liminf_{n\to\infty} A_{m,n} \\ &\leq (1+t')^{2\delta} \liminf_{m\to\infty} \liminf_{n\to\infty} \sum_{i=1}^{Q-1} \sum_{\omega\in\Sigma^m} \sum_{u\in\Sigma^n} |L^i_{u\omega}|^\delta + \; t(1+t')^\delta \limsup_{m\to\infty} \sum_{i=1}^{Q-1} \sum_{\omega\in\Sigma^m} |L^i_\omega|^\delta.\end{aligned}$$

Because we have $\sum_{i=1}^{Q-1} \sum_{\omega\in\Sigma^m} |L^i_\omega|^\delta \leq \sum_{i=1}^{Q-1} \sum_{\omega\in\Sigma^m} \|\phi'_\omega\|^\delta =: a_m$, where $\|\cdot\|$ denotes the supremum-norm on $\mathcal{C}(X)$, and the sequence $(a_m)_{m\in\mathbb{N}}$ is bounded by Lemma 4.2.12 of [MU03], letting t and t' tend to zero then gives the assertion.

The same arguments can be used in order to show that $\limsup_{m\to\infty} \sum_{i=1}^{Q-1} \sum_{\omega\in\Sigma^m} |L^i_\omega|^\delta$ is a lower bound in the second case. □

The Nonlattice Case

Proof (of Part (ii) of Theorem 2.6). If $\mathbb{1}_Z$ is identically zero, we immediately obtain $C^f_0(F, (-\infty, b]) = 0 = \nu(F \cap (-\infty, b])$. Therefore, in the following, we assume that $\mathbb{1}_Z$ is not identically zero. Since $\mathbb{1}_Z$ is Hölder continuous, by combining equations (4.3), (4.4) and (4.7), we see that Proposition 3.13 can be applied to $\overline{A}^i_\omega(x,\varepsilon,Z)$ and $\underline{A}^i_\omega(x,\varepsilon,Z)$ giving the following asymptotics.

$$\overline{A}^i_\omega(x,\varepsilon,Z) \sim \frac{\int \mathbb{1}_Z \mathrm{d}\nu_{-\delta\xi}}{\delta \int \xi \mathrm{d}\mu_{-\delta\xi}} \cdot h_{-\delta\xi}(\omega x) \cdot (2\varepsilon)^{-\delta} \left(|L^i_\omega| \rho_{n(\omega)}\right)^\delta \quad \text{and} \tag{4.8}$$

$$\underline{A}^i_\omega(x,\varepsilon,Z) \sim \frac{\int \mathbb{1}_Z \mathrm{d}\nu_{-\delta\xi}}{\delta \int \xi \mathrm{d}\mu_{-\delta\xi}} \cdot h_{-\delta\xi}(\omega x) \cdot (2\varepsilon)^{-\delta} \left(|L^i_\omega| \rho^{-1}_{n(\omega)}\right)^\delta \tag{4.9}$$

as $\varepsilon \to 0$. We first put our focus on finding an upper bound for $\overline{C}^f_0(F, (-\infty, b])$. As in the statement of this theorem set $H_{\mu_{-\delta\xi}} := \delta \int \xi \mathrm{d}\mu_{-\delta\xi}$. Combining the Equations (4.5) and (4.8), we obtain for $x \in F^{\text{unique}}$ and all $m \in \mathbb{N}$

$$\overline{C}^f_0(F, (-\infty, b]) \leq \frac{2^{-\delta}}{H_{\mu_{-\delta\xi}}} \sum_{i=1}^{Q-1} \sum_{\omega\in\Sigma^m} h_{-\delta\xi}(\omega x) \left(|L^i_\omega| \rho_m\right)^\delta \int_{\Sigma^\infty} \mathbb{1}_Z \mathrm{d}\nu_{-\delta\xi}.$$

Now an application of Lemma 4.1 implies

$$\overline{C}^f_0(F, (-\infty, b]) \leq \frac{2^{-\delta}}{H_{\mu_{-\delta\xi}}} \liminf_{m\to\infty} \sum_{i=1}^{Q-1} \sum_{\omega\in\Sigma^m} |L^i_\omega|^\delta \int_{\Sigma^\infty} \mathbb{1}_Z \mathrm{d}\nu_{-\delta\xi}. \tag{4.10}$$

Analogously, one can conclude that

$$\underline{C}^f_0(F, (-\infty, b]) \geq \frac{2^{-\delta}}{H_{\mu_{-\delta\xi}}} \limsup_{m\to\infty} \sum_{i=1}^{Q-1} \sum_{\omega\in\Sigma^m} |L^i_\omega|^\delta \int_{\Sigma^\infty} \mathbb{1}_Z \mathrm{d}\nu_{-\delta\xi}. \tag{4.11}$$

Combining the inequalities (4.10) and (4.11) yields that all the limits occuring therein exist and are equal. Moreover, the δ-conformal measure introduced in (2.1) and $\nu_{-\delta\xi}$ satisfy the relation $\nu_{-\delta\xi}(\mathbb{1}_Z) = \nu((-\infty, b])$. Therefore,

$$C^f_0(F, (-\infty, b]) = \frac{2^{-\delta}}{H_{\mu_{-\delta\xi}}} \lim_{n\to\infty} \sum_{\omega\in\Sigma^n} \sum_{i=1}^{Q-1} |L^i_\omega|^\delta \cdot \nu(F \cap (-\infty, b])$$

holds for every $b \in \mathbb{R} \setminus F$. As $\mathbb{R} \setminus F$ is dense in $\mathbb{R}$ the assertion concerning the 0-th fractal curvature measure follows. The result on the 1-st fractal curvature measure now follows by applying Theorem 3.18, as for every $b \in \mathbb{R} \setminus F$ we have that $\partial F_\varepsilon \cap (-\infty, b] = \partial (F \cap (-\infty, b])_\varepsilon$ for sufficiently small $\varepsilon > 0$. $\square$

The Lattice Case

In this part, ξ is a lattice function. Therefore, there exist $\zeta, \psi \in \mathcal{C}(\Sigma^\infty)$ such that

$$\xi - \zeta = \psi - \psi \circ \sigma$$

and such that ζ is a function taking values in a discrete subgroup of $\mathbb{R}$. Let $a > 0$ be the maximal real number such that $\{a^{-1}\zeta(x) \mid x \in \Sigma^\infty\} \subseteq \mathbb{Z}$. As in the nonlattice case, the hypotheses and Remark 3.9 imply that ξ is Hölder continuous and strictly positive. Moreover, the unique $s > 0$ for which $\gamma_{-s\xi} = 1$ is the Minkowski dimension δ of F.

Since $\mathbb{1}_Z$ is Hölder continuous and since we can assume that $\mathbb{1}_Z$ is not identically zero, by combining equations (4.3), (4.4) and (4.7), we see that an application of Theorem 3.14 to $\overline{A}^i_\omega(x, \varepsilon, Z)$ and $\underline{A}^i_\omega(x, \varepsilon, Z)$ gives the following asymptotics.

$$\overline{A}^i_\omega(x,\varepsilon,Z) \sim W_\omega(x) \cdot \int_{\Sigma^\infty} \mathbb{1}_Z(y) e^{-\delta a \left\lceil \frac{\psi(y)-\psi(\omega x)}{a} + \frac{1}{a} \ln \frac{2\varepsilon}{|L^i_\omega| \rho_{n(\omega)}} \right\rceil} \mathrm{d}\nu_{-\delta\zeta}(y) \quad \text{and} \tag{4.12}$$

$$\underline{A}^i_\omega(x,\varepsilon,Z) \sim W_\omega(x) \cdot \int_{\Sigma^\infty} \mathbb{1}_Z(y) e^{-\delta a \left\lceil \frac{\psi(y)-\psi(\omega x)}{a} + \frac{1}{a} \ln \frac{2\varepsilon \rho_{n(\omega)}}{|L^i_\omega|} \right\rceil} \mathrm{d}\nu_{-\delta\zeta}(y) \tag{4.13}$$

as $\varepsilon \to 0$ uniformly for $x \in \Sigma^\infty$, where

$$W_\omega(x) := \frac{a h_{-\delta\zeta}(\omega x)}{(1-e^{-\delta a}) \int \zeta \mathrm{d}\mu_{-\delta\zeta}}. \tag{4.14}$$

Proof (of Part (iii) of Theorem 2.6). The statement on the nonexistence of the fractal curvature measures results by combining Part (iii) of Theorem 2.11 with Proposition 2.13. For the boundedness we first remark that $\overline{C}^f_0(F, \cdot)$ is monotonically increasing as a function in the second component. Therefore, in order to find an upper bound for $\overline{C}^f_0(F, \cdot)$ it suffices to consider $\overline{C}^f_0(F, \mathbb{R})$. For all $m \in \mathbb{N}$ we have

$$\begin{aligned}
\overline{C}^f_0(F,\mathbb{R}) &\overset{(4.5)}{\leq} \limsup_{\varepsilon\to 0} \varepsilon^\delta \sum_{i=1}^{Q-1} \sum_{\omega\in\Sigma^m} \overline{A}^i_\omega(x,\varepsilon,\Sigma^\infty) \\
&\overset{(4.12)}{=} \limsup_{\varepsilon\to 0} \varepsilon^\delta \sum_{i=1}^{Q-1} \sum_{\omega\in\Sigma^m} W_\omega(x) \cdot \int_{\Sigma^\infty} e^{-\delta a \left\lceil \frac{\psi(y)-\psi(\omega x)}{a} + \frac{1}{a} \ln \frac{2\varepsilon}{|L^i_\omega| \rho_m} \right\rceil} \mathrm{d}\nu_{-\delta\zeta}(y) \\
&\leq \limsup_{\varepsilon\to 0} \varepsilon^\delta \sum_{i=1}^{Q-1} \sum_{\omega\in\Sigma^m} W_\omega(x) \cdot \int_{\Sigma^\infty} e^{-\delta a \left(\frac{\psi(y)-\psi(\omega x)}{a} + \frac{1}{a} \ln \frac{2\varepsilon}{|L^i_\omega| \rho_m} \right)} \mathrm{d}\nu_{-\delta\zeta}(y) \\
&= \sum_{i=1}^{Q-1} \sum_{\omega\in\Sigma^m} \frac{a e^{\delta\psi(\omega x)} h_{-\delta\zeta}(\omega x)}{(1-e^{-\delta a}) \int \zeta \mathrm{d}\mu_{-\delta\zeta}} \left(\frac{|L^i_\omega| \rho_m}{2} \right)^\delta \int_{\Sigma^\infty} e^{-\delta\psi(y)} \mathrm{d}\nu_{-\delta\zeta}(y).
\end{aligned}$$

Note that $h_{-\delta\xi} = e^{\delta\psi} h_{-\delta\zeta}$ and $\mathrm{d}\nu_{-\delta\xi} = e^{-\delta\psi}\mathrm{d}\nu_{-\delta\zeta}$. Hence, by Lemma 4.1

$$\overline{C}^f_0(F,\mathbb{R}) \leq \liminf_{m\to\infty} \sum_{i=1}^{Q-1} \sum_{\omega\in\Sigma^m} |L^i_\omega|^\delta \frac{a 2^{-\delta}}{(1-e^{-\delta a}) \int \zeta \mathrm{d}\mu_{-\delta\zeta}} =: c_0.$$

$c_0 \in (0,\infty)$ because $\sum_{i=1}^{Q-1} \sum_{\omega\in\Sigma^m} |L^i_\omega|^\delta \le \sum_{i=1}^{Q-1} \sum_{\omega\in\Sigma^m} \|\phi'_\omega\|^\delta =: a_m$, where $\|\cdot\|$ denotes the supremum-norm on $\mathcal{C}(X)$ and the sequence $(a_m)_{m\in\mathbb{N}}$ is bounded by Lemma 4.2.12 of [MU03].

That $\underline{C}_0^f(F,\mathbb{R})$ is positive can be seen by the following.

$$\begin{aligned}
\underline{C}_0^f(F,\mathbb{R}) \quad &\ge \quad \liminf_{\varepsilon\to 0} \varepsilon^\delta \sum_{i=1}^{Q-1} \sum_{\omega\in\Sigma^m} \underline{A}^i_\omega(x,\varepsilon,\Sigma^\infty) \\
&\overset{(4.13)}{\ge} \quad \liminf_{\varepsilon\to 0} \varepsilon^\delta \sum_{i=1}^{Q-1} \sum_{\omega\in\Sigma^m} W_\omega(x) \cdot \int_{\Sigma^\infty} e^{-\delta a\left(\frac{\psi(y)-\psi(\omega x)}{a} + \frac{1}{a}\ln \frac{2\varepsilon\rho_{n(\omega)}}{|L^i_\omega|} + 1\right)} \mathrm{d}\nu_{-\delta\zeta}(y) \\
&= \quad \sum_{i=1}^{Q-1} \sum_{\omega\in\Sigma^m} \frac{a h_{-\delta\zeta}(\omega x)}{(1-e^{-\delta a})\int \zeta \mathrm{d}\mu_{-\delta\zeta}} e^{\delta\psi(\omega x)-\delta a} \left(\frac{|L^i_\omega|}{2\rho_m}\right)^\delta \int_{\Sigma^\infty} e^{-\delta\psi(y)} \mathrm{d}\nu_{-\delta\zeta}(y).
\end{aligned}$$

By using $h_{-\delta\xi} = e^{\delta\psi} h_{-\delta\zeta}$ and $\mathrm{d}\nu_{-\delta\xi} = e^{-\delta\psi}\mathrm{d}\nu_{-\delta\zeta}$ and Lemma 4.1, we hence obtain

$$\underline{C}_0^f(F,\mathbb{R}) \ge \limsup_{m\to\infty} \sum_{i=1}^{Q-1} \sum_{\omega\in\Sigma^m} |L^i_\omega|^\delta \frac{a 2^{-\delta} e^{-\delta a}}{(1-e^{-\delta a})\int \zeta \mathrm{d}\mu_{-\delta\zeta}} > 0.$$

The results on $\underline{C}_1^f(F,B)$ and $\overline{C}_1^f(F,B)$ are now a straightforward application of Theorem 3.18. □

Proof (of Theorem 2.8). By Lemma 3.17 we know that

$$A := \lim_{\varepsilon\to 0} \varepsilon^\delta \int_{\Sigma^\infty} \mathbb{1}_{\widetilde{B}}(y) e^{-\delta a\left\lceil \frac{\psi(y)-\psi(\omega x)}{a} + \frac{1}{a}\ln \frac{2\varepsilon}{|L^i_\omega|\rho_m} \right\rceil} \mathrm{d}\nu_{-\delta\zeta}(y)$$

exists for every $\omega \in \Sigma^m$ and $i \in \{1,\dots,Q-1\}$. Moreover, the limit is independent of ω and i. Therefore, applying the same arguments which lead to Equations (4.5) and (4.6) and using Equations (4.12) and (4.13) we conclude

$$\begin{aligned}
\overline{C}_0^f(F,B) \quad &\le \quad \sum_{i=1}^{Q-1} \sum_{\omega\in\Sigma^m} W_\omega(x) \left(\frac{|L^i_\omega|\rho_m}{2}\right)^\delta \cdot A \quad \text{and} \\
\underline{C}_0^f(F,B) \quad &\ge \quad \sum_{i=1}^{Q-1} \sum_{\omega\in\Sigma^m} W_\omega(x) \left(\frac{|L^i_\omega|}{2\rho_m}\right)^\delta \cdot A,
\end{aligned}$$

where $W_\omega(x)$ is as defined in (4.14). Applying Lemma 4.1 and Theorem 3.18 then completes the proof. □

Average Fractal Curvature Measures

Proof (of Part (i) of Theorem 2.6). If ξ is nonlattice, Part (i) of Theorem 2.6 immediately follows from Part (ii) of Theorem 2.6 and the fact that $f(\varepsilon) \sim c$ as $\varepsilon \to 0$ for some constant $c \in \mathbb{R}$ implies $\lim_{T\to 0} |\ln T|^{-1} \int_T^1 \varepsilon^{-1} f(\varepsilon) \mathrm{d}\varepsilon = c$ for every locally integrable function $f\colon (0,\infty) \to \mathbb{R}$.

Thus for the rest of the proof we assume that ξ is lattice. We begin with showing the result on the 0-th average fractal curvature measure.

Observe that $\lim_{T\to 0}|\ln T|^{-1}\int_T^1 c\varepsilon^{\delta-1}\mathrm{d}\varepsilon = \lim_{T\to\infty}|T|^{-1}\int_0^T ce^{-\delta t}\mathrm{d}t = 0$ for every constant $c \in \mathbb{R}$. For a fixed $m \in \mathbb{N}$ define $M := \min\{|L_\omega^i| \mid i \in \{1,\dots,Q-1\}, \omega \in \Sigma^m\}/2$. From Equations (4.2) and (4.3) we deduce the following.

$$\begin{aligned}\overline{D} &:= \limsup_{T\to 0}|2\ln T|^{-1}\int_T^1 \varepsilon^{\delta-1}\lambda^0(\partial F_\varepsilon \cap (-\infty,b])\mathrm{d}\varepsilon \\ &\leq \limsup_{T\to 0}|\ln T|^{-1}\Bigg(\int_T^M \varepsilon^{\delta-1}\sum_{i=1}^{Q-1}\sum_{\omega\in\Sigma^m}\overline{A}_\omega^i(x,\varepsilon,Z)\mathrm{d}\varepsilon + \frac{1}{2}\int_M^1 \varepsilon^{\delta-1}\lambda^0(\partial F_\varepsilon\cap(-\infty,b])\mathrm{d}\varepsilon\Bigg).\end{aligned}$$

Local integrability of the integrands implies that we have the following equation for all $m \in \mathbb{N}$.

$$\begin{aligned}\overline{D} &\leq \limsup_{T\to 0}|\ln T|^{-1}\int_T^1 \varepsilon^{\delta-1}\sum_{i=1}^{Q-1}\sum_{\omega\in\Sigma^m}\overline{A}_\omega^i(x,\varepsilon,Z)\mathrm{d}\varepsilon \\ &= \limsup_{T\to\infty} T^{-1}\sum_{i=1}^{Q-1}\sum_{\omega\in\Sigma^m}\int_0^T e^{-\delta t}\overline{A}_\omega^i(x,e^{-t},Z)\mathrm{d}t \\ &\overset{(4.7)}{=} \limsup_{T\to\infty} T^{-1}\sum_{i=1}^{Q-1}\sum_{\omega\in\Sigma^m}\int_0^T e^{-\delta t}\sum_{n=0}^{\infty}\sum_{y:\,\sigma^n y=\omega x}\mathbb{1}_Z(y)\cdot\mathbb{1}_{\{S_n\xi(y)\leq t-\ln\frac{2}{|L_\omega^i|\rho_m}\}}\mathrm{d}t \\ &\leq \limsup_{T\to\infty}\sum_{i=1}^{Q-1}\sum_{\omega\in\Sigma^m}\left(\frac{|L_\omega^i|\rho_m}{2}\right)^\delta \frac{T-\ln\frac{2}{|L_\omega^i|\rho_m}}{T}\cdot \\ &\qquad\left(T-\ln\frac{2}{|L_\omega^i|\rho_m}\right)^{-1}\int_0^{T-\ln\frac{2}{|L_\omega^i|\rho_m}} e^{-\delta t}\sum_{n=0}^{\infty}\sum_{y:\,\sigma^n y=\omega x}\mathbb{1}_Z(y)\cdot\mathbb{1}_{\{S_n\xi(y)\leq t\}}\mathrm{d}t \\ &= \sum_{i=1}^{Q-1}\sum_{\omega\in\Sigma^m}\left(\frac{|L_\omega^i|\rho_m}{2}\right)^\delta \frac{h_{-\delta\xi}(\omega x)\int\mathbb{1}_Z\mathrm{d}\nu_{-\delta\xi}}{\delta\int\xi\mathrm{d}\mu_{-\delta\xi}}. \end{aligned} \tag{4.15}$$

The last equality is an application of Corollary 3.16. Because (4.15) holds for all $m \in \mathbb{N}$, applying Lemma 4.1 yields

$$\begin{aligned}&\limsup_{T\to 0}|2\ln T|^{-1}\int_T^1 \varepsilon^{\delta-1}\lambda^0(\partial F_\varepsilon\cap(-\infty,b])\mathrm{d}\varepsilon \\ &\qquad\leq \frac{2^{-\delta}\int\mathbb{1}_Z\mathrm{d}\nu_{-\delta\xi}}{\delta\int\xi\mathrm{d}\mu_{-\delta\xi}}\liminf_{m\to\infty}\sum_{i=1}^{Q-1}\sum_{\omega\in\Sigma^m}|L_\omega^i|^\delta.\end{aligned} \tag{4.16}$$

Analogous estimates give

$$\begin{aligned}&\liminf_{T\to 0}|2\ln T|^{-1}\int_T^1 \varepsilon^{\delta-1}\lambda^0(\partial F_\varepsilon\cap(-\infty,b])\mathrm{d}\varepsilon \\ &\qquad\geq \frac{2^{-\delta}\int\mathbb{1}_Z\mathrm{d}\nu_{-\delta\xi}}{\delta\int\xi\mathrm{d}\mu_{-\delta\xi}}\limsup_{m\to\infty}\sum_{i=1}^{Q-1}\sum_{\omega\in\Sigma^m}|L_\omega^i|^\delta.\end{aligned} \tag{4.17}$$

Equations (4.16) and (4.17) together imply that for every $b \in \mathbb{R}\setminus F$

$$\lim_{T\to 0}|2\ln T|^{-1}\int_T^1 \varepsilon^{\delta-1}\lambda^0(\partial F_\varepsilon\cap(-\infty,b])\mathrm{d}\varepsilon = \frac{2^{-\delta}c}{H_{\mu_{-\delta\xi}}}\nu(F\cap(-\infty,b]),$$

where the constant $c := \lim_{m\to\infty} \sum_{i=1}^{Q-1} \sum_{\omega\in\Sigma^m} |L_\omega^i|^\delta$ is defined as in Equation (2.2). Since $\mathbb{R}\setminus F$ is dense in $\mathbb{R}$, the statement on the 0-th average fractal curvature measure in Part (i) of Theorem 2.6 follows.

For the statement on the 1-st average fractal curvature measure, we use Part (iii) of Theorem 2.6 which says that $\overline{C}_0^f(F,(-\infty,b]) < \infty$ for every $b\in\mathbb{R}\setminus F$. Applying Theorem 3.18 hence yields that $\overline{\mathcal{M}}(F\cap(-\infty,b]) < \infty$ for every $b\in\mathbb{R}\setminus F$. By the same arguments that were used in the end of the proof of Part (ii), we can thus apply Proposition 3.19 to $F\cap(-\infty,b]$ and obtain the desired statement. □

5 Special Cases

We start this section by a Lemma which is used in the proofs of Part (iii) of Theorem 2.9 and Theorem 2.11.

Lemma 5.1 *Let F denote a self-conformal set associated to the cIFS $\Phi := \{\phi_1,\dots,\phi_N\}$. Let δ denote the Minkowski dimension of F and let $B\subseteq\mathbb{R}$ denote a Borel set. Assume that there exists a positive bounded periodic Borel-measurable function $f\colon\mathbb{R}^+\to\mathbb{R}^+$ which has the following properties.*

(i) f is not equal to an almost everywhere constant function.

(ii) There exists a constant $c\in\mathbb{R}$ such that for all $t>0$ and $m\in\mathbb{N}$ there exists an $M\in\mathbb{N}$ such that for all $T\geq M$

$$\begin{aligned}(1-t)\rho_m^{-\delta} f(T-\ln\rho_m) - ce^{-\delta T}\\ \leq e^{-\delta T}\lambda^0(\partial F_{e^{-T}}\cap B) \leq (1+t)\rho_m^{\delta} f(T+\ln\rho_m) + ce^{-\delta T}.\end{aligned} \quad (5.1)$$

Then for $k\in\{0,1\}$ we have

$$\underline{C}_k^f(F,B) < \overline{C}_k^f(F,B).$$

Proof. We first cover the case $k=0$. Since f is positive and not equal to an almost everywhere constant function, there exist $\widetilde{T}_1,\widetilde{T}_2>0$ such that $R := f(\widetilde{T}_2)/f(\widetilde{T}_1) > 1$. Choose $m\in\mathbb{N}$ so that $\rho_m^{2\delta} < \sqrt{R}$ and choose $t>0$ such that $(1+t)/(1-t) < \sqrt{R}$. Then $\widetilde{R} := (1-t)\rho_m^{-\delta} f(\widetilde{T}_2) - (1+t)\rho_m^{-\delta} f(\widetilde{T}_2) > 0$. By Condition (ii) we can find an $M\in\mathbb{N}$ for these t and m such that for all $T\geq M$ Equation (5.1) is satisfied. Because of the periodicity of f we can find $T_1,T_2\geq M$ such that $f(\widetilde{T}_1) = f(T_1+\ln\rho_m)$ and $f(\widetilde{T}_2) = f(T_2-\ln\rho_m)$. Moreover, we can assume that T_1,T_2 are so large that $ce^{-\delta T_1}+ce^{-\delta T_2}\leq \widetilde{R}/2$. Then

$$\begin{aligned} e^{-\delta T_1}\lambda^0(\partial F_{e^{-T_1}}\cap B) &\leq (1+t)\rho_m^{\delta} f(T_1+\ln\rho_m) + ce^{-\delta T_1}\\ &\leq (1-t)\rho_m^{-\delta} f(T_2-\ln\rho_m) - \widetilde{R}/2 - ce^{-\delta T_2}\\ &< e^{-\delta T_2}\lambda^0(\partial F_{e^{-T_2}}\cap B).\end{aligned}$$

Because of the periodicity of f this proves the case $k=0$. For $k=1$ observe that the function $g\colon\mathbb{R}^+\to\mathbb{R}^+$ defined by

$$g(T) := \int_0^\infty f(s+T)e^{(\delta-1)s}\mathrm{d}s$$

is also periodic. Moreover g is not a constant function. Because if it was, then $0 = g(0)-g(T)$ for all $T\geq 0$. This would imply $\int_T^\infty f(s)e^{(\delta-1)s}\mathrm{d}s = e^{(\delta-1)T}\int_0^\infty f(s)e^{(\delta-1)s}\mathrm{d}s$ for all $T\geq 0$.

Differentiating with respect to T would imply that f itself is constant almost everywhere which is a contradiction. Using Stachó's Theorem (Proposition 3.20), we obtain

$$\begin{aligned} e^{-T(\delta-1)}\lambda^1(F_{e^{-T}}\cap B) &= e^{-T(\delta-1)}\int_T^\infty \lambda^0(\partial F_{e^{-s}}\cap B)e^{-s}\mathrm{d}s \\ &\le e^{-T(\delta-1)}(1+t)\rho_m^\delta\int_T^\infty f(s+\ln\rho_m)e^{s(\delta-1)}\mathrm{d}s + ce^{-T\delta} \\ &= (1+t)\rho_m^\delta g(T+\ln\rho_m)+ce^{-\delta T}. \end{aligned}$$

Analoguously we obtain

$$e^{-T(\delta-1)}\lambda^1(F_{e^{-T}}\cap B) \ge (1-t)\rho_m^{-\delta}g(T-\ln\rho_m) - ce^{-\delta T}.$$

Therefore, the same arguments which were used in the proof of the case $k=0$ imply that

$$\liminf_{\varepsilon\to 0}\varepsilon^{\delta-1}\lambda^1(F_\varepsilon\cap B) < \limsup_{\varepsilon\to 0}\varepsilon^{\delta-1}\lambda^1(F_\varepsilon\cap B).$$

□

5.1 Self-Similar Sets; Proof of Theorem 2.9

Self-similar sets satisfying the open set condition form a special class of self-conformal sets, namely those which are generated by an iterated function system Φ consisting of similarities $\phi_1,\dots,\phi_N$. We let $r_1,\dots,r_N$ denote the respective similarity ratios of $\phi_1,\dots,\phi_N$ and set $r_\omega := r_{\omega_1}\cdots r_{\omega_n}$ for a finite word $\omega=\omega_1\cdots\omega_n\in\Sigma^n$. When considering self-similar sets some of the formulae simplify significantly:

(i) The geometric potential function is constant on the one-cylinders meaning $\xi(\omega) = -\ln r_{\omega_1}$ for $\omega=\omega_1\omega_2\cdots\in\Sigma^\infty$.

(ii) The unique σ-invariant Gibbs measure $\mu_{-\delta\xi}$ for the potential function $-\delta\xi$ coincides with the δ-dimensional normalised Hausdorff measure on F. Thus, $\mu_{-\delta\xi}([i]) = r_i^\delta$, where $[i]$ shall denote the cylinder of $i\in\Sigma$. Therefore we have that $H_{\mu_{-\delta\xi}} = -\sum_{i\in\Sigma}\ln(r_i)r_i^\delta$.

(iii) The lengths of the main gaps of $\phi_\omega F$ are just multiples of the lengths of the primary gaps of F, that is $|L_\omega^i| = r_\omega|L^i|$ for each $i\in\{1,\dots,Q-1\}$ and $\omega\in\Sigma^*$.

(iv) By the Moran-Hutchinson formula (see e. g. Theorem 9.3 of [Fal03]) we have that $\sum_{\omega\in\Sigma^n} r_\omega^\delta = 1$ for each $n\in\mathbb{N}$.

Combining (i)-(iv) with Theorem 2.6, we obtain Parts (i) and (ii) of Theorem 2.9.

In order to prove Part (iii) of Theorem 2.9 which actually is a stronger result than that of Part (iii) of Theorem 2.6, we are going to make use of the asymptotics (4.12) and (4.13) that we obtained for self-conformal sets.

Proof (of Part (iii) of Theorem 2.9). As $F\cap B$ has got a representation as a finite nonempty union of sets of the form $\phi_\omega F$ with $\omega\in\Sigma^*\setminus\{\varnothing\}$, there is a set $Z\subseteq\Sigma^\infty$ which is a finite union of cylinder sets and which satisfies $\pi Z = F\cap B$. Then $\mathbb{1}_Z$ is Hölder continuous which allows an application of Theorem 3.14 to $\overline{A}_\omega^i(x,e^{-T},Z)$ and $\underline{A}_\omega^i(x,e^{-T},Z)$ which leads to Equations (4.12) and (4.13).

The geometric potential function of a lattice self-similar set itself takes values in a discrete subgroup of $\mathbb{R}$. Thus, ψ is a constant function and $\zeta = \xi$. Moreover, one easily verifies that $h_{-\delta\xi} \equiv 1$ and $|L^i_\omega| = r_\omega |L^i|$. For these reasons, the formulae (4.12) and (4.13) simplify to

$$\begin{aligned} \overline{A}^i_\omega(x, e^{-T}, Z) &\sim \frac{a r^\delta_\omega}{(1-e^{-\delta a}) H_{\mu_{-\delta\xi}}} \cdot e^{-\delta a \left\lceil \frac{1}{a} \ln \frac{2e^{-T}}{|L^i| \rho_{n(\omega)}} \right\rceil} \int_{\Sigma^\infty} \mathbb{1}_Z(y) \mathrm{d}\nu_{-\delta\xi}(y) \\ &= \frac{a r^\delta_\omega \nu(B)}{(1-e^{-\delta a}) H_{\mu_{-\delta\xi}}} \cdot e^{-\delta a \left\lceil \frac{1}{a} \ln \frac{2e^{-T}}{|L^i| \rho_{n(\omega)}} \right\rceil} \qquad \text{and} \qquad (5.2) \\ \underline{A}^i_\omega(x, e^{-T}, Z) &\sim \frac{a r^\delta_\omega \nu(B)}{(1-e^{-\delta a}) H_{\mu_{-\delta\xi}}} \cdot e^{-\delta a \left\lceil \frac{1}{a} \ln \frac{2e^{-T} \rho_{n(\omega)}}{|L^i|} \right\rceil}. \qquad (5.3) \end{aligned}$$

Here, we have also used that $\ln r_\omega \in a\mathbb{Z}$ for every $\omega \in \Sigma^*$. Now, by arguments of the beginning of Section 4, there exists a constant $c \in \mathbb{R}$ such that for every $m \in \mathbb{N}$ and $T \geq M$, where $M \in \mathbb{N}$ is chosen such that all main gaps of the sets $\phi_\omega F$ for $\omega \in \Sigma^m$ in B are greater than or equal to $2e^{-M}$, we have

$$e^{-\delta T} \sum_{i=1}^{Q-1} \sum_{\omega \in \Sigma^m} \underline{A}^i_\omega(x, e^{-T}, Z) \leq e^{-\delta T} \lambda^0(\partial F_{e^{-T}} \cap B) \leq e^{-\delta T} \sum_{i=1}^{Q-1} \sum_{\omega \in \Sigma^m} \overline{A}^i_\omega(x, e^{-T}, Z) + c e^{-\delta T}.$$

Note that there is no need of substracting $ce^{-\delta T}$ on the left hand side of the above equation as we are in the self-similar situation. We introduce the function $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ given by

$$f(T) := e^{-\delta T} \frac{a \nu(B)}{(1-e^{-\delta a}) H_{\mu_{-\delta\xi}}} \sum_{i=1}^{Q-1} e^{-\delta a \left\lceil \frac{1}{a} \ln \frac{2e^{-T}}{|L^i|} \right\rceil}.$$

Because of the asymptotics in (5.2) and (5.3) we know that for all $t > 0$ there exists an $\widetilde{M} \in \mathbb{N}$ such that for all $T \geq \max\{\widetilde{M}, M\}$ we have

$$(1-t) \rho_m^{-\delta} f(T - \ln \rho_m) \leq e^{-\delta T} \lambda^0(\partial F_{e^{-T}} \cap B) \leq (1+t) \rho_m^\delta f(T + \ln \rho_m) + c e^{-\delta T}.$$

Clearly, f is a periodic function with period a. Moreover, f is piecewise continuous with a finite number of discontinuities in an interval of length a. Additionally, on every interval where f is continuous, f is stricly decreasing. Therefore f is not equal to an almost everywhere constant function. Thus, all conditions of Lemma 5.1 are satisfied which finishes the proof. □

5.2 $\mathcal{C}^{1+\alpha}$ Images of Self-Similar Sets; Proofs of Theorem 2.11, Proposition 2.13 and Corollary 2.14

In this subsection we consider the case that F is an image of a self-similar set $K \subseteq Y$ under a conformal map $g \in \mathcal{C}^{1+\alpha}(\mathcal{U})$, where $\alpha > 0$ and $\mathcal{U}$ is a convex neighbourhood of Y. We assume that $|g'|$ is bounded away from 0 on its domain of definition. Thus, g is bi-Lipschitz and therefore the Minkowski dimension of F coincides with the Minkowski dimension of K (see e. g. Corollary 2.4 of [Fal03]). We denote the common value by δ.

The similarities $R_1, \dots, R_N$ generating K and the mappings $\phi_1, \dots, \phi_N$ generating F are connected through the equations $\phi_i = g \circ R_i \circ g^{-1}$ for each $i \in \Sigma$. If we let $\mathcal{H}^\delta_K$ denote the normalised δ-dimensional Hausdorff measure on K, that is $\mathcal{H}^\delta_K(\cdot) := \mathcal{H}^\delta(\cdot \cap K)/\mathcal{H}^\delta(K)$, and let $r_1, \dots, r_N$ denote the respective similarity ratios of $R_1, \dots, R_N$, we have the following list of observations.

(i) ϕ_i is differentiable for every $i \in \Sigma$ with differential

$$\phi_i'(x) = \frac{g'(R_i \circ g^{-1}(x))}{g'(g^{-1}(x))} \cdot r_i,$$

where $x \in X$ and X is the nonempty compact interval which each ϕ_i is defined on.

(ii) The geometric potential function ξ_F associated to F is given by $\xi_F(\omega) = -\ln|g'(g^{-1}(\omega))| + \ln|g'(g^{-1}(\sigma\omega))| - \ln|r_{\omega_1}|$, where $\omega = \omega_1\omega_2 \cdots \in \Sigma^*$. The geometric potential function ξ_K associated to K is given by $\xi_K(\omega) = -\ln|r_{\omega_1}|$. Thus ξ_K is nonlattice, if and only if ξ_F is nonlattice.

(iii) The unique σ-invariant Gibbs measure for the potential function $-\delta\xi_F$ is $\mu_{-\delta\xi_F} = \mathcal{H}_K^\delta \circ g^{-1}$, the one associated with $-\delta\xi_K$ is $\mu_{-\delta\xi_K} = \mathcal{H}_K^\delta$.

(iv) From (ii) and (iii) we obtain

$$H_{-\delta\xi_F} = \int_{\Sigma^\infty} \xi_F \mathrm{d}\mu_{-\delta\xi_F} = -\sum_{i\in\Sigma} \ln|r_i| r_i^\delta = \int_{\Sigma^\infty} \xi_K \mathrm{d}\mu_{-\delta\xi_K} = H_{-\delta\xi_K}.$$

Further, let $\widetilde{L}^1, \dots, \widetilde{L}^{Q-1}$ denote the primary gaps of K and $\widetilde{L}^1_\omega, \dots, \widetilde{L}^{Q-1}_\omega$ the main gaps of $R_\omega K$ for each $\omega \in \Sigma^*$. Similarly, $L^1, \dots, L^{Q-1}$ and $L^1_\omega, \dots, L^{Q-1}_\omega$ shall respectively denote the primary gaps of F and the main gaps of $\phi_\omega F$. Then

(v) $L^i_\omega = g(\widetilde{L}^i_\omega)$ for $i \in \{1, \dots, Q-1\}$ and $\omega \in \Sigma^*$. Since furthermore $|\widetilde{L}^i_\omega| = r_\omega |\widetilde{L}^i|$, we have

$$\lim_{n\to\infty} \sum_{\omega\in\Sigma^n} \sum_{i=1}^{Q-1} |L^i_\omega|^\delta = \lim_{n\to\infty} \sum_{i=1}^{Q-1} \sum_{\omega\in\Sigma^n} \left(r_\omega |\widetilde{L}^i| \cdot |g'(x_\omega)| \right)^\delta = \sum_{i=1}^{Q-1} |\widetilde{L}^i|^\delta \int_K |g'|^\delta \mathrm{d}\mathcal{H}_K^\delta,$$

where $x_\omega \in [\omega]$ for each $\omega \in \Sigma^*$. Note that the above line can be rigorously proven by using the bounded distortion lemma (Lemma 3.5).

(vi) The δ-conformal measure ν_F associated to F and the δ-conformal measure ν_K associated to K are absolutely continuous with Radon-Nikodym derivative

$$\frac{\mathrm{d}\nu_F}{\mathrm{d}\nu_K \circ g^{-1}} = |g' \circ g^{-1}|^\delta \left(\int_K |g'|^\delta \mathrm{d}\mathcal{H}_K^\delta \right)^{-1}.$$

(vii) Let $\widetilde{Y} \subset \mathcal{U}$ be a compact set which contains an open neighbourhood of X. Since $|g'|$ is bounded away from 0 on $\mathcal{U}$, we obtain that $\theta := 2 \cdot \max\{|\phi_i'(x)| \mid x \in \widetilde{Y},\ i \in \Sigma\}$ is finite and positive. Under the assumption that K satisfies $\lambda^1(K) = 0$ and the OSC, $\Phi_\theta := \{\theta^{-1}\phi_1, \dots, \theta^{-1}\phi_N\}$ is a cIFS which satisfies the OSC. Hence, the invariant set F_θ of Φ_θ is a self-conformal set.

(viii) By the definition of the 0-th and 1-st fractal curvature measures and using the homogenity property of the Lebesgue measure, we have that $C_0^f(F, \cdot) = \theta^\delta C_0^f(F_\theta, \cdot/\theta)$ and $C_1^f(F, \cdot) = \theta^\delta C_1^f(F_\theta, \cdot/\theta)$, whenever they exist. The same equalities also obtain for the average versions $\widetilde{C}_0^f(F, \cdot)$ and $\widetilde{C}_1^f(F, \cdot)$.

Using (i)-(viii) an application of Parts (i) and (ii) of Theorem 2.6 and Theorem 2.9 to K and F_θ proves Parts (i) and (ii) of Theorem 2.11.

Proof (of Part (iii) of Theorem 2.11). In order to show that there exists a Borel set $B \subseteq \mathbb{R}$ for which $\underline{C}_k^f(F,B) < \overline{C}_k^f(F,B)$ for $k \in \{0,1\}$ we want to apply Lemma 5.1. To show that the assumptions of Lemma 5.1 are satisfied, we make the following estimates which are closely linked to the ones given in the beginning of Section 4.

Let $\kappa \in \Sigma^* \setminus \{\varnothing\}$ be arbitrary. Fix an $m \in \mathbb{N}$ and choose $M \in \mathbb{N}$ so that for every $\omega \in \Sigma^m$ all primary gaps of the sets $\phi_\omega F$ which lie in $\langle \phi_\kappa F \rangle$ are of length greater than or equal to $2e^{-M}$. Then for all $T \geq M$ we have

$$\begin{aligned} \lambda^0\left(\partial F_{e^{-T}} \cap \langle \phi_\kappa F \rangle\right) &= 2\sum_{i=1}^{Q-1} \#\{\omega \in \Sigma^* \mid L_\omega^i \subseteq \langle \phi_\kappa F\rangle,\ |L_\omega^i| \geq 2e^{-T}\} \\ &\leq 2\sum_{\omega\in\Sigma^m}\sum_{i=1}^{Q-1} \Xi_\omega^i(e^{-T}) + \underbrace{2\sum_{j=1}^{m-n(\kappa)-1} (Q-1)\cdot N^{j-1}}_{=: c_m}. \end{aligned}$$

where we agree that $\sum_{j=1}^{m-n(\kappa)-1}(Q-1)\cdot N^{j-1} = 0$ if $m - n(\kappa) - 1 < 1$ and where

$$\Xi_\omega^i(e^{-T}) := \#\{u \in \Sigma^* \mid L_{u\omega}^i \subseteq \langle \phi_\kappa F\rangle,\ |L_{u\omega}^i| \geq 2e^{-T}\}.$$

Likewise

$$\lambda^0\left(\partial F_{e^{-T}} \cap \langle \phi_\kappa F \rangle\right) \geq 2\sum_{\omega\in\Sigma^m}\sum_{i=1}^{Q-1} \Xi_\omega^i(e^{-T}).$$

As before, we let $\widetilde{L}^i$ and $\widetilde{L}_\omega^i$ denote the gaps of the self-similar set K and recall that $g \in \mathcal{C}^{1+\alpha}(\mathcal{U})$. Let c_g be the Hölder constant of g' and $k_g > 0$ such that $|g'| \geq k_g$. Then for every $\omega \in \Sigma^n$, where $n \in \mathbb{N}$, and $x, y \in R_\omega K$ we have that

$$\left|\frac{g'(x)}{g'(y)}\right| \leq \left|\frac{g'(x)-g'(y)}{g'(y)}\right| + 1 \leq \frac{c_g\cdot|x-y|^\alpha}{k_g} + 1 \leq \max_{\omega\in\Sigma^n}\frac{c_g\cdot r_\omega^\alpha}{k_g} + 1 =: p_n.$$

Clearly, $p_n \to 1$ as $n \to \infty$. Thus, for $u \in \Sigma^*$, $\omega \in \Sigma^m$ and $i \in \{1,\ldots,Q-1\}$ we have for an arbitrary $x \in K^{\text{unique}}$

$$\begin{aligned} |L_{u\omega}^i| &= |g\widetilde{L}_{u\omega}^i| \leq |g'(R_{u\omega}x)|p_{n(\omega)}|\widetilde{L}_\omega^i|r_u = |(g\circ R_u)'(R_\omega x)|p_{n(\omega)}|\widetilde{L}_\omega^i| \\ &= |(\phi_u\circ g)'(R_\omega x)|p_{n(\omega)}|\widetilde{L}_\omega^i| = |\phi_u'(gR_\omega x)||g'(R_\omega x)|p_{n(\omega)}|\widetilde{L}_\omega^i| \\ &\overset{(ii)}{=} \exp\left(-S_{n(u)}\xi(u\omega x) - \psi(\omega x) + \ln(p_{n(\omega)}|\widetilde{L}_\omega^i|)\right). \end{aligned}$$

Therefore, for $x \in K^{\text{unique}}$

$$\Xi_\omega^i(e^{-T}) \leq \#\{u \in \Sigma^* \mid L_{u\omega}^i \subseteq \langle\phi_\kappa F\rangle,\ S_{n(u)}\xi(u\omega x) \leq -\ln(2e^{-T}) + \ln(p_{n(\omega)}|\widetilde{L}_\omega^i|) - \psi(\omega x)\}.$$

Hence, by Theorem 3.14

$$\begin{aligned} \lambda^0(\partial F_{e^{-T}} \cap \langle\phi_\kappa F\rangle) &\leq 2\sum_{\omega\in\Sigma^m}\sum_{i=1}^{Q-1}\sum_{n=0}^{\infty}\sum_{y\colon \sigma^n y = \omega x} \mathbb{1}_{[\kappa]}(y)\cdot\mathbb{1}_{\{S_n\xi(y)\leq -\ln(2e^{-T})+\ln(p_m|\widetilde{L}_\omega^i|)-\psi(\omega x)\}} + c_m \\ &\sim 2\sum_{\omega\in\Sigma^m}\sum_{i=1}^{Q-1} \frac{ah_{-\delta\zeta}(\omega x)\int \mathbb{1}_{[\kappa]}(y) e^{-\delta a\left\lceil \frac{\psi(y)-\psi(\omega x)}{a} + \frac{1}{a}\ln\frac{2e^{-T}}{p_m|\widetilde{L}_\omega^i|} + \frac{\psi(\omega x)}{a}\right\rceil} \mathrm{d}\nu_{-\delta\zeta}(y)}{(1-e^{-\delta a})\int\zeta\mathrm{d}\mu_{-\delta\zeta}} + c_m. \end{aligned} \tag{5.4}$$

Define $W := a\left(1-e^{-\delta a}\right)^{-1}\left(\int \zeta \mathrm{d}\mu_{-\delta\zeta}\right)^{-1}$ and note that $h_{-\delta\zeta} \equiv 1$, since ζ is the geometric potential function of a self-similar set. Thus, using $|\widetilde{L}^i_\omega| = r_\omega|\widetilde{L}^i|$ and that $\ln r_\omega \in a\mathbb{Z}$ for every $\omega \in \Sigma^*$, Equation (5.4) simplifies to

$$\sum_{\omega\in\Sigma^m}\sum_{i=1}^{Q-1} W r_\omega^\delta \int_{\Sigma^\infty} \mathbb{1}_{[\kappa]}(y) e^{-\delta a\left\lceil \frac{\psi(y)}{a}+\frac{1}{a}\ln\frac{2e^{-T}}{p_m|\widetilde{L}^i|}\right\rceil} \mathrm{d}\nu_{-\delta\zeta}(y)$$
$$= \sum_{i=1}^{Q-1} W \int_{\Sigma^\infty} \mathbb{1}_{[\kappa]}(y) e^{-\delta a\left\lceil \frac{\psi(y)}{a}+\frac{1}{a}\ln\frac{2e^{-T}}{p_m|\widetilde{L}^i|}\right\rceil} \mathrm{d}\nu_{-\delta\zeta}(y).$$

Hence, for all $t>0$ there exists an $\widetilde{M} \geq M$ such that for all $T \geq \widetilde{M}$ we have

$$e^{-\delta T}\lambda^0(\partial F_{e^{-T}} \cap \langle\phi_\kappa F\rangle)$$
$$\leq (1+t)e^{-\delta T} 2 \sum_{i=1}^{Q-1} W \int_{\Sigma^\infty} \mathbb{1}_{[\kappa]}(y) e^{-\delta a\left\lceil \frac{\psi(y)}{a}+\frac{1}{a}\ln\frac{2e^{-T}}{p_m|\widetilde{L}^i|}\right\rceil} \mathrm{d}\nu_{-\delta\zeta}(y) + c_m e^{-\delta T}.$$

Defining the function $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ given by

$$f(T) := e^{-\delta T} 2 \sum_{i=1}^{Q-1} W \int_{\Sigma^\infty} \mathbb{1}_{[\kappa]}(y) e^{-\delta a\left\lceil \frac{\psi(y)}{a}+\frac{1}{a}\ln\frac{2e^{-T}}{|\widetilde{L}^i|}\right\rceil} \mathrm{d}\nu_{-\delta\zeta}(y)$$

we thus have

$$e^{-\delta T}\lambda^0(\partial F_{e^{-T}} \cap \langle\phi_\kappa F\rangle) \leq (1+t)p_m^\delta f(T+\ln p_m) + c_m e^{-\delta T}.$$

Likewise,

$$e^{-\delta T}\lambda^0(\partial F_{e^{-T}} \cap \langle\phi_\kappa F\rangle) \geq (1-t)p_m^{-\delta} f(T-\ln p_m).$$

Clearly, f is periodic with period a. In order to apply Lemma 5.1 it remains to show that f is not equal to an almost everywhere constant function. For that we let $\underline{\beta} := 1-\max\{\{-a^{-1}\ln|\widetilde{L}^i|\} \mid i=1,\ldots,Q-1\}$ and $\overline{\beta} := \max\{\{a^{-1}\ln|\widetilde{L}^i|\} \mid i=1,\ldots,Q-1\}$ and consider the following three cases, of which at least one always obtains.

<u>CASE 1:</u> $\underline{D} := \{y \in \Sigma^\infty \mid \{a^{-1}\psi(y)\} < \underline{\beta}\} \neq \varnothing$.
Since $\psi \in \mathcal{C}(\Sigma^\infty)$ and thus $\underline{D}$ is open, there exists a $\kappa \in \Sigma^* \setminus \{\varnothing\}$ such that $[\kappa] \subseteq \underline{D}$. For $n \in \mathbb{N}$ and $r \in (0, 1-\overline{\beta})$ define $T_n(r) := a(n-\overline{\beta}-r)+\ln 2$. Then

$$f(T_n(r)) = e^{\delta ar}\cdot e^{\delta a\overline{\beta}} 2^{1-\delta} \sum_{i=1}^{Q-1} W \int_{\Sigma^\infty} \mathbb{1}_{[\kappa]}(y) e^{-\delta a\left\lceil \frac{\psi(y)}{a}-\frac{1}{a}\ln|\widetilde{L}^i|\right\rceil - \delta a} \mathrm{d}\nu_{-\delta\zeta}(y).$$

This shows that f is strictly decreasing on $(a(n-1)+\ln 2, a(n-\overline{\beta})+\ln 2)$ for every $n \in \mathbb{N}$. Therefore, f is not equal to an almost everywhere constant function.

<u>CASE 2:</u> $\overline{D} := \{y \in \Sigma^\infty \mid \{a^{-1}\psi(y)\} > \overline{\beta}\} \neq \varnothing$.
Like in CASE 1, there exists a $\kappa \in \Sigma^* \setminus \{\varnothing\}$ such that $[\kappa] \subseteq \overline{D}$. For $n \in \mathbb{N}$ and $r \in (0, \underline{\beta})$ set $T_n(r) := a(n+1-r)+\ln 2$. Then

$$f(T_n(r)) = e^{\delta ar}\cdot 2^{1-\delta} \sum_{i=1}^{Q-1} W \int_{\Sigma^\infty} \mathbb{1}_{[\kappa]}(y) e^{-\delta a\left\lceil \frac{\psi(y)}{a}-\frac{1}{a}\ln|\widetilde{L}^i|\right\rceil} \mathrm{d}\nu_{-\delta\zeta}(y).$$

This shows that f is strictly decreasing on $(a(n+1-\underline{\beta})+\ln 2, a(n+1)+\ln 2)$ for every $n \in \mathbb{N}$. Therefore, f is not equal to an almost everywhere constant function.

If neither CASE 1 nor CASE 2 obtains, then CASE 3 obtains which is the following.

CASE 3: $\{y \in \Sigma^\infty \mid \underline{\beta} \leq \{a^{-1}\psi(y)\} \leq \overline{\beta}\} = \Sigma^\infty$.
Here, we can use $T_n(r) := a(n-1+r)+\ln 2$ for $n \in \mathbb{N}$ and $r \in (0, \underline{\beta})$ to obtain

$$f(T_n(r)) = e^{-\delta a r} \cdot W \sum_{i=1}^{Q-1} \int_{\Sigma^\infty} \mathbb{1}_{[\kappa]}(y) e^{-\delta a \lceil \frac{\psi(y)}{a} \rceil + 2\delta a} \mathrm{d}\nu_{-\delta\zeta}(y).$$

This shows that f is strictly decreasing on $(a(n-1)+\ln 2, a(n-1+\underline{\beta})+\ln 2)$ for every $n \in \mathbb{N}$. Therefore, f is not equal to an almost everywhere constant function.

Thus, we can apply Lemma 5.1 in all three cases to obtain the desired statement. □

PROOF (OF PROPOSITION 2.13). Without loss of generality we may assume $\{0,1\} \subseteq F \subseteq [0,1]$. We let ξ denote the geometric potential function of Φ and let $\zeta, \psi \in \mathcal{C}(\Sigma^\infty)$ be the functions satisfying $\xi - \zeta = \psi - \psi \circ \sigma$, where ζ takes values in a discrete subgroup of $\mathbb{R}$. Moreover, $\pi\colon \Sigma^\infty \to F$ shall denote the natural code map. We define $\widetilde{\psi} := \delta^{-1} \ln h$, where h is the function which is uniquely defined through Theorem 3.11. $\widetilde{\psi}$ satisfies the equation $\widetilde{\psi} \circ \pi = \psi$ since h satisfies

$$h \circ \pi = h_{-\delta\xi} = \frac{\mathrm{d}\mu_{-\delta\xi}}{\mathrm{d}\nu_{-\delta\xi}} = \frac{\mathrm{d}\mu_{-\delta\zeta}}{e^{-\delta\psi}\mathrm{d}\nu_{-\delta\zeta}} = e^{\delta\psi}.$$

We define the function $f\colon X \to \mathbb{R}$ by $f(x) := \int_0^x e^{-\widetilde{\psi}(y)} \mathrm{d}y$ for $x \in X$. As $\widetilde{\psi}$ is α-Hölder continuous, the Fundamental Theorem of Calculus implies that $\widetilde{\psi} = -\ln f'$. Moreover, the continuity of $\widetilde{\psi}$ implies that $\widetilde{\psi}$ is bounded on X. Therefore, f' is bounded away from both 0 and ∞ and thus f is invertible. Define $Y := f(X)$, set $g := f^{-1}$ and extend g to a $\mathcal{C}^{1+\alpha}(\mathcal{U})$ function on an open neighbourhood $\mathcal{U}$ of the compact set Y such that $g' > 0$ on $\mathcal{U}$. Define $R_i := g^{-1} \circ \phi_i \circ g$ for $i \in \{1, \ldots, N\}$. Then setting $\pi_K := \pi^{-1} g$, we have

$$\begin{aligned} -\ln R'_{\omega_1}(\pi_K \sigma\omega) &= -\ln f'(\phi_{\omega_1} g \pi_K \sigma\omega) - \ln \phi'_{\omega_1}(g\pi_K\sigma\omega) - \ln g'(\pi_K\sigma\omega) \\ &= \widetilde{\psi}(\phi_{\omega_1}\pi\sigma\omega) + \xi(\omega) + \ln f'(g\pi_K\sigma\omega) \\ &= \xi(\omega) + \psi(\omega) - \psi \circ \sigma(\omega). \end{aligned}$$

Thus, $\zeta(\omega) = -\ln R'_{\omega_1}(\pi_K\sigma\omega)$ for $\omega \in \Sigma^\infty$. Since ζ takes values in a discrete subgroup of $\mathbb{R}$ and ξ and ψ are bounded on Σ^∞, ζ in fact takes a finite number of values. Therefore, R_i is piecewise linear for every $i \in \{1, \ldots, N\}$ with only a finite number of nondifferentiability points. As furthermore every R'_i is α-Hölder continuous on Σ^∞, we obtain a system of similarities by iterating the system $\{R_1, \ldots, R_N\}$ a finite number of times. Thus, the invariant set K of the iterated function system $\{R_1, \ldots, R_N\}$ is self-similar and lattice. Moreover, $F = g(K)$. □

PROOF (OF COROLLARY 2.14). Let π_K and π_F respectively denote the natural code map from Σ^∞ to K and F respectively. Observe that $\pi_F^{-1} \circ g = \pi_K^{-1}$ and $\nu_{-\delta\xi_K} \circ \pi_K^{-1} = \nu$ and define $B := \pi_K(\widetilde{B})$. By Property (ii) of the beginning of this subsection, we see that $\psi = -\ln g' \circ \pi_K$. Using this and the normalisation conditions given in Corollary 2.14, the left hand side of Condition (2.3) simplifies to

$$\nu_{-\delta\xi_K}\pi_K^{-1} \circ (g')^{-1}\left(g' \circ \pi_K(\widetilde{B}) \cap \exp(-t, 0]\right) = \nu \circ (g')^{-1}\left(g'(B) \cap (e^{-t}, 1]\right).$$

The right hand side of Condition (2.3) simplifies to

$$\frac{e^{\delta t}-1}{e^{\delta a}-1}\nu_{-\delta\xi_K}\pi_K^{-1}\circ (g')^{-1}\left(g'\circ \pi_K(\widetilde{B})\cap \exp(-a,0]\right)=\frac{e^{\delta t}-1}{e^{\delta a}-1}\nu\circ (g')^{-1}(g'(B)).$$

This shows that Part (i) of Corollary 2.14. Part (ii) can be easily deduced from this by substituting $e^{-t}=g'(r)$. □

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