

CONSTRUCTIONS OF SURFACE BUNDLES WITH RANK TWO FUNDAMENTAL GROUPS

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ABSTRACT. We give a construction of hyperbolic 3-manifolds with rank two fundamental groups and report an experimental search to find such manifolds. Our manifolds are all surface bundles over the circle with genus two surface fiber. For the manifolds so obtained, we then examine whether they are of Heegaard genus two or not. As a byproduct, we give an infinite family of knots in the 3-sphere whose knot groups are of rank two.

1. INTRODUCTION

The *Heegaard genus* and the *rank* of the fundamental group are well-known and well-studied complexities of 3-manifolds (see the next section for precise definitions).

As an extension of the famous Poincaré conjecture, Waldhausen had asked in [27] whether the Heegaard genus of a compact orientable 3-manifold M is equal to the rank of its fundamental group $\pi_1(M)$ or not. The Poincaré conjecture states that it would be answered affirmatively in the simplest case; when M is closed, the rank of $\pi_1(M)$ is zero if and only if the Heegaard genus of M is zero.

In general, the question was negatively answered in [3]. In fact, they gave a family of closed 3-manifolds of Heegaard genus three with rank two fundamental groups. Those 3-manifolds are Seifert fibered spaces, and the family was extended to more wider one in [17]. Also such examples of graph manifolds were obtained in [29]. However, as far as the authors know, no such examples are known for either hyperbolic 3-manifolds or 3-manifolds with non-empty boundary [13, Problem 3.92].

In this paper, we give systematical and experimental constructions of hyperbolic 3-manifolds with rank two fundamental groups, and examine whether they are of Heegaard genus two or not.

First, we will give a construction of surface bundles over the circle with genus two fiber each of which has the fundamental group of rank two. With some exceptions, they will be shown to have pseudo-Anosov monodromies, and so, they admit hyperbolic structure by [26] (see [19] for a detailed proof). Actually, for these manifolds, we will verify that all of them are of Heegaard genus two.

We here remark that, in [11], a characterization of the monodromy maps for closed orientable surface bundles with genus two Heegaard splittings is obtained.

As a byproduct, our construction yields an infinite family of knots in the 3-sphere S^3 whose knot groups are of rank two. Precisely we will give an infinite family of hyperbolic, genus two, fibered knots in S^3 with rank two knot groups. Also they

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all can be checked to have Heegaard genus two; that is, they are tunnel number one knots. Thus they give supporting evidence for the conjecture raised in [21]: A knot group has two generators and one relator if and only if the knot is tunnel number one. See [13, Problem 1.10], [13, Problem 1.73] for related problems and [2] for a partial solution. There are also known examples of knots in S^3 which are tunnel number one, genus two, fibered knots. They are given by two-bridge knots, i.e., knots with bridge index 2. Here a *bridge index* of a knot in S^3 is defined as the minimal number of local maxima (or local minima) up to ambient isotopy. Precisely, the knots are 5_1 , 6_2 , 6_3 , 7_6 , 7_7 , 8_{12} , in the knot table [20], and also are composite knots coming from trefoil and figure-eight knots. Actually it is easy to see that two-bridge knots are tunnel number one by definition, and it is shown from [8, Proposition 2] that these are only fibered knots of genus two among two-bridge knots. Moreover, in [12, Corollary 5.4], Jong showed that they are only genus two fibered knots among alternating knots.

Second, we report on computer experiments for finding examples of surface bundles with rank two fundamental groups. In fact, we did generate over 100,000 hyperbolic surface bundles with rank two fundamental groups by computer, however we could not find examples of Heegaard genus more than two.

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2. PRELIMINARY

Throughout the paper, unless otherwise stated, all manifolds are assumed to be connected and orientable. In this section, we use F to denote a closed surface and M a compact 3-manifold with boundary ∂M .

2.1. A *compression body* C is defined as a compact 3-manifold obtained from the product $F \times [0, 1]$ by attaching 2-handles on $F \times \{1\}$ and then capping off the resulting 2-sphere boundary components by 3-handles. The subsurface of the boundary ∂C corresponding to $F \times \{0\}$ is denoted by $\partial_+ C$, and then the residual set $\partial C - \partial_+ C$ is denoted by $\partial_- C$. Under this setting, a *handlebody* is defined as a compression body C with $\partial_- C = \emptyset$.

By a *Heegaard surface* in M , we mean an embedded surface S in M which separates M into a handlebody C_1 and a compression body C_2 with $\partial M = \partial_- C_2$ and $S = \partial_+ C_1 = \partial_+ C_2$. Such a splitting of M is called a *Heegaard splitting*. It is well-known that every compact 3-manifold admits a Heegaard splitting. Thus one can consider the minimal genus of Heegaard surfaces in M , which is called the *Heegaard genus* of M .

If M admits a Heegaard surface of genus g , then, by the Van-Kampen's theorem, its fundamental group $\pi_1(M)$ admits a presentation with g generators. Since M admits at least one Heegaard splitting, $\pi_1(M)$ is always finitely generated. Among presentations of $\pi_1(M)$, one can consider the minimal number of generators, which we call the *rank* of the group $\pi_1(M)$. It then follows that the rank of $\pi_1(M)$ is less than or equal to the genus of any Heegaard surface in M .

As we stated in the previous section, we consider the question whether the rank of $\pi_1(M)$ is equal to the Heegaard genus of M or not. By the observation above, this question is equivalent to that whether a presentation of $\pi_1(M)$ with minimal number of generators is induced from a Heegaard splitting of M .

2.2. By a *surface bundle*, we mean a 3-manifold which fibers over the circle. Any surface bundle with fiber F can be regarded as

$$(F \times [0, 1]) / \{(x, 1) = (f(x), 0)\}_{x \in F}$$

with some orientation preserving homeomorphism f of F . This f is called the *monodromy* of the surface bundle. Throughout the paper, the surface bundle with monodromy f is denoted by M_f .

Note that if f and f' are isotopic, then M_f and $M_{f'}$ are homeomorphic. Thus for an element $[f]$ of the mapping class group of F , the surface bundle M_f is uniquely determined. Also note that for any f and its conjugate gfg^{-1} by some g , M_f and $M_{gfg^{-1}}$ are homeomorphic. In fact, it was shown in [16] that when the rank of the first homology is one, a surface bundle M_f is homeomorphic to another $M_{f'}$ if and only if f' is isotopic to a conjugate of f .

The question which we consider was answered affirmatively for torus bundles in [25]: The Heegaard genus of a torus bundle over the circle is equal to the rank of the fundamental group.

2.3. In the following, let F_2 denote a closed orientable surface of genus two. Let C_1, C_2, \dots, C_5 be the simple closed curves on F_2 depicted in Figure 1. For each i , $1 \leq i \leq 5$, let D_i denote the Dehn twist along C_i .

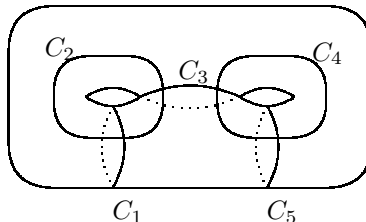


FIGURE 1.

By [15], every orientation preserving homeomorphism of F_2 is isotopic to a product of a finite number of the Dehn twists D_1, D_2, D_3, D_4, D_5 and their inverses.

3. CONSTRUCTION OF SURFACE BUNDLE

Our first theorem is the following.

Theorem 3.1. *Let n be an arbitrary integer and $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$ a quadruple of integers ε_i each of which is either $+1$ or -1 . Let $M_{\varepsilon, n}$ be the surface bundle over the circle with monodromy*

$$f_{\varepsilon, n} = D_2^{\varepsilon_2} \circ D_1^{\varepsilon_1} \circ D_3^{\varepsilon_3} \circ D_4^{\varepsilon_4} \circ D_5^n.$$

Then the rank of $\pi_1(M_{\varepsilon, n})$ is always two and the Heegaard genus of $M_{\varepsilon, n}$ is also two.

Of course, the second assertion implies the first assertion, but we will directly prove the first assertion without assuming the second assertion.

In fact, for the case of $n = 0$, the manifold $M_{\varepsilon,0}$ is that obtained from a 2-bridge knot in S^3 by 0-surgery. Thus it naturally admits a Heegaard surface of genus two and so the theorem follows immediately in this case. Our manifolds can be regarded as an extension of the class of such manifolds.

The first assertion of the theorem follows from the next lemma.

Lemma 3.2. *With the same notations as in Theorem 3.1, the rank of $\pi_1(M_{\varepsilon,n})$ is always two.*

Proof. Consider the oriented loops a_1, a_2, a_3, a_4 on F_2 illustrated in Figure 2.

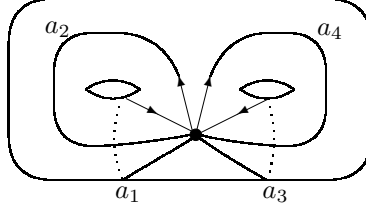


FIGURE 2.

We abuse the notations a_1, a_2, a_3, a_4 to denote the elements of the fundamental group $\pi_1(F_2)$ represented by the corresponding loops.

Let $\Phi_{\varepsilon,n}$ be the isomorphism of $\pi_1(F_2)$ induced by $f_{\varepsilon,n}$. Then it is well-known that $\pi_1(M_{\varepsilon,n})$ decomposes as a semidirect product of $\pi_1(F_2)$ and \mathbb{Z} . Precisely, $\pi_1(M_{\varepsilon,n})$ has the following presentation:

$$(1) \quad \langle a_1, a_2, a_3, a_4, t \mid t^{-1}a_it = \Phi_{\varepsilon,n}(a_i), (1 \leq i \leq 4), [a_1, a_2] = [a_3, a_4] \rangle,$$

where t denotes a generator of the infinite cyclic factor.

Claim 1. *For each $i \in \{1, 2, 3\}$, the image of a_i under $\Phi_{\varepsilon,n}$ is represented by a word which contains just one a_{i+1} (or $(a_{i+1})^{-1}$) and no letters a_j ($j > i + 1$).*

Proof. We first describe the isomorphism $\Phi_{\varepsilon,n} : \pi_1(F_2) \rightarrow \pi_1(F_2)$ in detail. Let Δ_i denote the isomorphism of $\pi_1(F_2)$ induced by the Dehn twists D_i for $1 \leq i \leq 4$. Then

$$\Phi_{\varepsilon,n} = \Delta_2^{\varepsilon_2} \circ \Delta_1^{\varepsilon_1} \circ \Delta_3^{\varepsilon_3} \circ \Delta_4^{\varepsilon_4} \circ \Delta_5^n$$

holds.

By Figure 1 and 2, the actions of Δ_i 's on a_j 's are described as follows.

$$\left(\Delta_i^{\varepsilon}(a_j) \right)_{1 \leq i \leq 5, 1 \leq j \leq 4} = \begin{pmatrix} a_1 & a_2(a_1)^{\varepsilon} & a_3 & a_4 \\ a_1(a_2)^{-\varepsilon} & a_2 & a_3 & a_4 \\ a_1 & ((a_3)^{-1}a_1)^{\varepsilon}a_2 & a_3 & ((a_3)^{-1}a_1)^{\varepsilon}a_4 \\ a_1 & a_2 & a_3(a_4)^{\varepsilon} & a_4 \\ a_1 & a_2 & a_3 & a_4(a_3)^{-\varepsilon} \end{pmatrix},$$

where $\varepsilon = \pm 1$.

For a_1 , this implies that

$$\Phi_{\varepsilon,n}(a_1) = \Delta_2^{\varepsilon_2} \circ \Delta_1^{\varepsilon_1} \circ \Delta_3^{\varepsilon_3} \circ \Delta_4^{\varepsilon_4} \circ \Delta_5^n(a_1) = \Delta_2^{\varepsilon_2}(a_1) = a_1(a_2)^{-\varepsilon_2}.$$

Thus the claim holds for the case that $i = 1$.

For a_2 , we have

$$\begin{aligned}\Phi_{\varepsilon,n}(a_2) &= \Delta_2^{\varepsilon_2} \circ \Delta_1^{\varepsilon_1} \circ \Delta_3^{\varepsilon_3} \circ \Delta_4^{\varepsilon_4} \circ \Delta_5^n(a_2) = \Delta_2^{\varepsilon_2} \circ \Delta_1^{\varepsilon_1} \circ \Delta_3^{\varepsilon_3}(a_2) \\ &= \Delta_2^{\varepsilon_2} \circ \Delta_1^{\varepsilon_1}(((a_3)^{-1}a_1)^{\varepsilon_3}a_2).\end{aligned}$$

From the matrix above, note that a_3 is invariant under the isomorphisms $(\Delta_1)^{\pm 1}$, $(\Delta_2)^{\pm 1}$ and the image of a_1, a_2 under $(\Delta_1)^{\pm 1}$, $(\Delta_2)^{\pm 1}$ are represented by the word with letters a_1, a_2 . Thus $\Phi_{\varepsilon,n}(a_2) = \Delta_2^{\varepsilon_2} \circ \Delta_1^{\varepsilon_1}(((a_3)^{-1}a_1)^{\varepsilon_3}a_2)$ is represented by the word with a single letter a_3 and the letters a_1, a_2 .

Finally for a_3 , we have

$$\begin{aligned}\Phi_{\varepsilon,n}(a_3) &= \Delta_2^{\varepsilon_2} \circ \Delta_1^{\varepsilon_1} \circ \Delta_3^{\varepsilon_3} \circ \Delta_4^{\varepsilon_4} \circ \Delta_5^n(a_3) \\ &= \Delta_2^{\varepsilon_2} \circ \Delta_1^{\varepsilon_1} \circ \Delta_3^{\varepsilon_3} \circ \Delta_4^{\varepsilon_4}(a_3) \\ &= \Delta_2^{\varepsilon_2} \circ \Delta_1^{\varepsilon_1}(a_3(((a_3)^{-1}a_1)^{\varepsilon_3}a_4)^{\varepsilon_4}).\end{aligned}$$

Again, from the matrix above, note that a_4 is invariant under the isomorphisms $(\Delta_1)^{\pm 1}$, $(\Delta_2)^{\pm 1}$ and the image of a_1, a_2, a_3 under $(\Delta_1)^{\pm 1}$, $(\Delta_2)^{\pm 1}$ are represented by the word with letters a_1, a_2, a_3 . Thus

$$\Phi_{\varepsilon,n}(a_3) = \Delta_2^{\varepsilon_2} \circ \Delta_1^{\varepsilon_1}(a_3(((a_3)^{-1}a_1)^{\varepsilon_3}a_4)^{\varepsilon_4})$$

is represented by the word with single letter a_4 and the letters a_1, a_2 and a_3 . \square

The claim enables us to reduce the number of the generators of the presentation (1), and we see that $\pi_1(M_{\varepsilon,n})$ is generated by a_1 and t only. This completes the proof of the lemma. \square

Next we show that the Heegaard genus of $M_{\varepsilon,n}$ is also two independently from Lemma 3.2.

Lemma 3.3. *With the same notations as in Theorem 3.1, every $M_{\varepsilon,n}$ is of Heegaard genus two.*

Proof. To prove the lemma, we first create a surgery description of $M_{\varepsilon,n}$.

Let L be the link defined as follows. Prepare five copies of annulus embedded in S^3 and plumb them. Denote by S the surface so obtained. The boundary of S gives a two component link $l_0 \cup l'_0$ in S^3 . Add five trivial components l_1, l_2, l_3, l_4, l_5 piercing each plumbed annulus. Add more five trivial components $l'_1, l'_2, l'_3, l'_4, l'_5$ such that they correspond to the cores of plumbed annuli and l'_k is isotopic to the meridian of l_k ($1 \leq k \leq 5$) in the exterior of the link $l_0 \cup l'_0 \cup l_1 \cup l_2 \cup l_3 \cup l_4 \cup l_5$. See Figure 3.

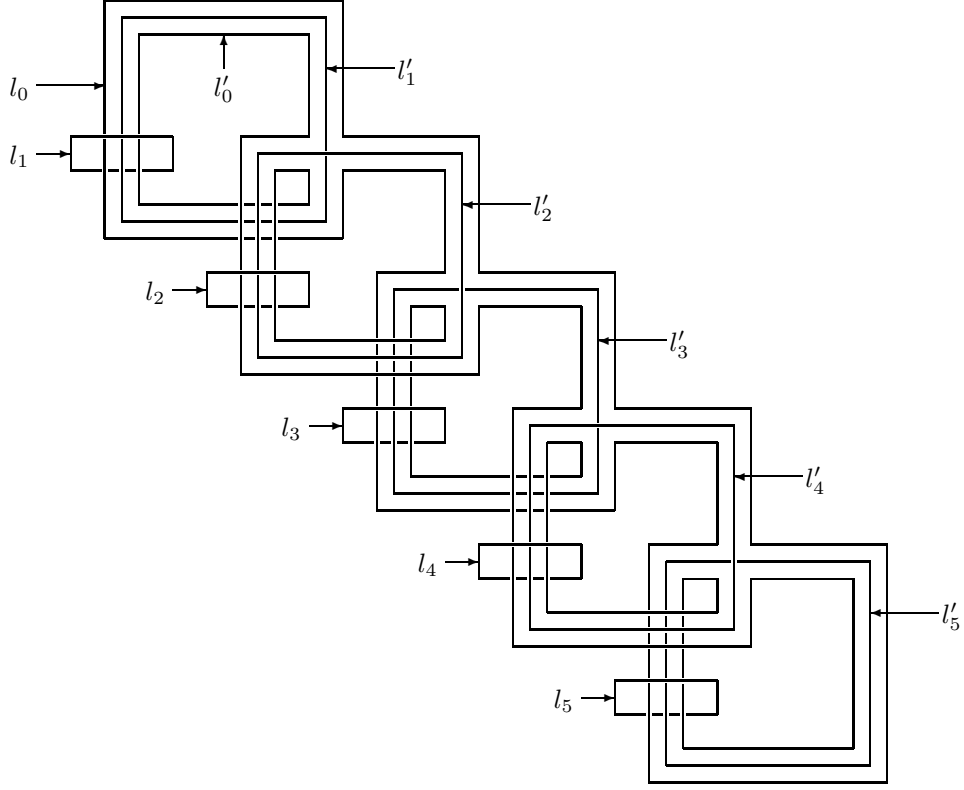
Let $L(r_0, r'_0, r_1, r_2, r_3, r_4, r_5, r'_1, r'_2, r'_3, r'_4, r'_5)$ denote the 3-manifold obtained by Dehn surgery along the link L with surgery coefficient r_i, r'_j for l_i, l'_j ($0 \leq i, j \leq 5$).

Claim 2. *The manifold $M_{\varepsilon,n}$ is homeomorphic to $L(0, 0, 0, 0, 0, 0, 0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, 1/n)$.*

Proof. First we consider the sublink $L_0 = l_0 \cup l'_0 \cup l_1 \cup l_2 \cup l_3 \cup l_4 \cup l_5$. Then it is easily seen that $L_0(0, 0, 0, 0, 0, 0, 0)$ is homeomorphic to the product bundle $F_2 \times S^1$. See [10] for example.

The remaining components $l'_1, l'_2, l'_3, l'_4, l'_5$ are regarded as in the surgered manifold. These can be isotoped to lie on surface fibers, and we can assume that l'_k is projected to C_k on F_2 by the natural projection $F_2 \times S^1 \rightarrow F_2$.

Now we regard $F_2 \times S^1$ as a surface bundle M_{id} with trivial monodromy. Remark that the preferred longitudes of the components $l'_1, l'_2, l'_3, l'_4, l'_5$ in S^3 are coincident

FIGURE 3. The link L

with the longitudes induced by the bundle structure of $F_2 \times S^1$. Then it is known that the $1/n$ -surgery on such a component l'_k yields the manifold $M_{D_k^n}$ (see [24, 10]).

Note that the components l'_1, l'_3, l'_5 are regarded as lying on the same fiber $F_2 \times \{1/3\}$ and also l'_2, l'_4 as lying on $F_2 \times \{2/3\}$. Thus $L(0, 0, 0, 0, 0, 0, 0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, 1/n)$ is homeomorphic to $M_{D_1^{\varepsilon_1} \circ D_3^{\varepsilon_3} \circ D_5^n \circ D_2^{\varepsilon_2} \circ D_4^{\varepsilon_4}}$. We note that

$$\begin{aligned}
 D_1^{\varepsilon_1} \circ D_3^{\varepsilon_3} \circ D_5^n \circ D_2^{\varepsilon_2} \circ D_4^{\varepsilon_4} &\sim D_1^{\varepsilon_1} \circ D_3^{\varepsilon_3} \circ D_5^n \circ D_4^{\varepsilon_4} \circ D_2^{\varepsilon_2} \\
 &\sim D_2^{\varepsilon_2} \circ D_1^{\varepsilon_1} \circ D_3^{\varepsilon_3} \circ D_5^n \circ D_4^{\varepsilon_4} \\
 &\sim D_5^n \circ D_2^{\varepsilon_2} \circ D_1^{\varepsilon_1} \circ D_3^{\varepsilon_3} \circ D_4^{\varepsilon_4} \\
 &\sim D_2^{\varepsilon_2} \circ D_1^{\varepsilon_1} \circ D_3^{\varepsilon_3} \circ D_4^{\varepsilon_4} \circ D_5^n \sim f_{\varepsilon, n},
 \end{aligned}$$

where \sim denotes suitable isotopy or conjugates. Thus $M_{D_1^{\varepsilon_1} \circ D_3^{\varepsilon_3} \circ D_5^n \circ D_2^{\varepsilon_2} \circ D_4^{\varepsilon_4}}$ is homeomorphic to our manifold $M_{\varepsilon, n}$. \square

By using the well-known modifications (see [20] for example), we can simplify the surgery description obtained above to the one depicted in Figure 4.

To find a Heegaard surface, we perform surgeries on the four components with surgery coefficients $-\varepsilon_i$ ($1 \leq i \leq 4$) to obtain a two component link. For example, the link corresponding to the case $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = (-1, 1, -1, -1)$ is illustrated in Figure 5.

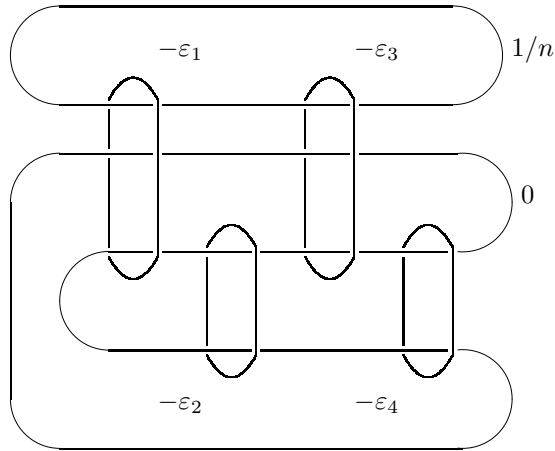


FIGURE 4.

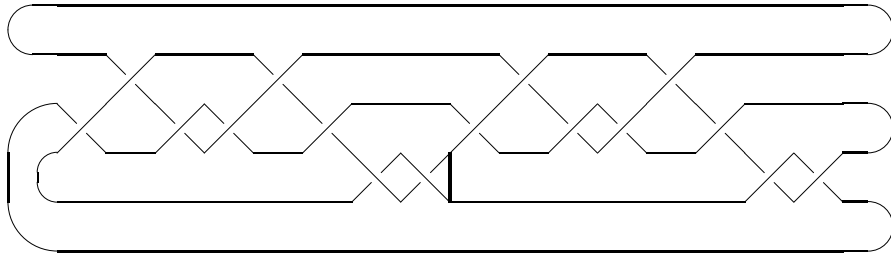


FIGURE 5.

Consider the boundary of the regular neighborhood of the non-trivial component of the link together with the thickened arc illustrated in Figure 5. Then, by manipulating the figure, one can check that the surface separates the link complement into two homeomorphic compression bodies. This implies the surface becomes a genus two Heegaard surface of the surgered manifold, i.e., of $M_{\varepsilon,n}$. Thus the Heegaard genus of $M_{\varepsilon,n}$ is at most two.

On the other hand, since our manifold $M_{\varepsilon,n}$ fibers over the circle with genus two fiber, its fundamental group contains a non-abelian surface subgroup. This implies that $M_{\varepsilon,n}$ fails to be S^3 , $S^2 \times S^1$, lens spaces; in particular, it fails to have Heegaard genus one. Therefore the Heegaard genus of $M_{\varepsilon,n}$ is shown to be two. \square

This completes the proof of Theorem 3.1.

As we stated in Section 1, most of our manifolds $\{M_{\varepsilon,n}\}$ are shown to be hyperbolic.

Proposition 3.4. *With the same notations as in Theorem 3.1,*

- (1) *for $n = 0$, the manifolds $\{M_{\varepsilon,0}\}$ are hyperbolic unless $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4$,*
- (2) *for each ε , the manifolds $\{M_{\varepsilon,n}\}$ are hyperbolic with at most five exceptions.*

Proof. We can see directly from Figure 4 that $M_{\varepsilon,0}$ is obtained by 0-surgery along the two-bridge knot with Conway's normal form $[2\varepsilon_4, -2\varepsilon_3, 2\varepsilon_2, -2\varepsilon_1]$. Then, by the classification of exceptional surgeries on two-bridge knots [6], we see that $M_{\varepsilon,0}$

is hyperbolic other than the corresponding knot is of genus one or a torus knot. All our knots are of genus two, and so, the knot we have to exclude is the (2,5)-torus knot. This corresponds to the case where $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4$.

Next, note that the homeomorphism $f_{\varepsilon,0} = D_2^{\varepsilon_2} \circ D_1^{\varepsilon_1} \circ D_3^{\varepsilon_3} \circ D_4^{\varepsilon_4}$ of F_2 is irreducible. Because, as we claimed above, each manifold $M_{\varepsilon,0}$ is obtained by 0-surgery along a two-bridge knot of genus two, but such a 3-manifold is always atoroidal by [6]. Thus $(D_2^{\varepsilon_2} \circ D_1^{\varepsilon_1} \circ D_3^{\varepsilon_3} \circ D_4^{\varepsilon_4}, C_5)$ fills F_2 up, and so, we can apply [5, Theorem 2.6]. This concludes the second assertion of the proposition. \square

4. FIBERED KNOTS

As a byproduct of Lemma 3.2 and Lemma 3.3, we obtain an infinite family of hyperbolic knots in the 3-sphere S^3 whose knot groups are of rank two.

Proposition 4.1. *Let n be an arbitrary integer and ε_i either $+1$ or -1 for $i = 1, 2, 4$ and $\varepsilon_3 = -\varepsilon_1$. Then the Montesinos knot $M\left(\frac{1}{2\varepsilon_1}, \frac{2\varepsilon_2}{4\varepsilon_2n-1}, \frac{2\varepsilon_4}{4\varepsilon_3\varepsilon_4-1}\right)$ is always hyperbolic, genus two, fibered knots in S^3 with rank two knot group.*

Proof. We consider the once-punctured surface bundles $M'_{\varepsilon,n}$ defined in the same way as the manifold $M_{\varepsilon,n}$ in Theorem 3.1. Then its fundamental group is shown to be of rank two in the same way as proving Lemma 3.2. As we explained in the proof of Lemma 3.3, the manifold $M'_{\varepsilon,n}$ has obtained by surgery on the link depicted in Figure 3. The surgery coefficients are the same as for $M_{\varepsilon,n}$ except for that on l_0 . On l_0 , no filling is performed, i.e., it corresponds to a toral boundary. This surgery description can be simplified, under the assumption that $\varepsilon_3 = -\varepsilon_1$, to the one illustrated in Figure 6.

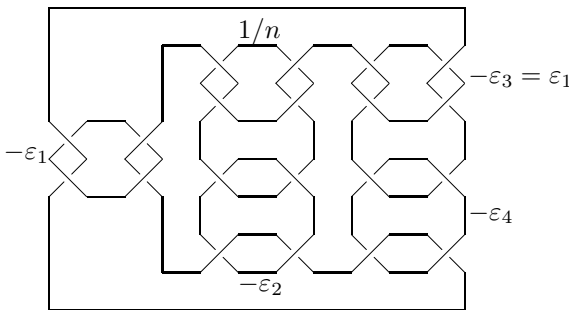
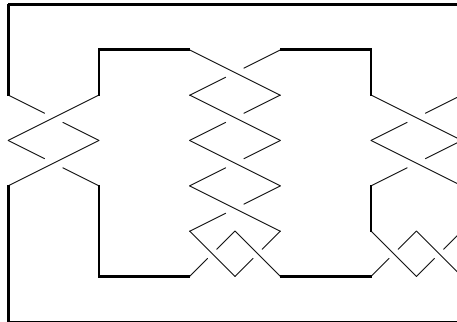


FIGURE 6.

By performing surgeries, we obtain the Montesinos knot $M\left(\frac{1}{2\varepsilon_1}, \frac{2\varepsilon_2}{4\varepsilon_2n-1}, \frac{2\varepsilon_4}{4\varepsilon_3\varepsilon_4-1}\right)$ from the unlabeled component.

Consequently the knot groups of our knots, which are isomorphic to the fundamental groups of $M'_{\varepsilon,n}$, are shown to be of rank two.

The fact that the knots are of genus two and fibered can be checked directly. Consider the diagram of the knot which is naturally obtained from Figure 6 by performing surgeries (an example is depicted in Figure 7). Let S be the Seifert surface obtained from this diagram by Seifert's algorithm. One can see that two Hopf bands are deplumbed from S , and the surface so obtained is modified to the connected sum of two Hopf bands by Stallings twists. This shows that S is a fiber

FIGURE 7. $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, n) = (1, 1, -1, 1, 2)$

surface. See [9] for example. Since S is of genus two and any fiber surface is minimal genus (see [7] for example), all our knots are of genus two.

While it is known which Montesinos knots are hyperbolic [4, 18], here we show that our knots are hyperbolic directly. First note that our knots are all unknotting number one as the diagram considered above shows. By [14], no torus knot of genus two has unknotting number one, and so, none of our knots are torus knots. Also note that our knots are all at most three bridge as the diagram shows. This implies that if our knots are satellite, then they must be composite ([23]), but it is impossible by [22]. \square

In fact, they are all *tunnel number one knots*, meaning that, the exteriors admit a genus two Heegaard surface. Their unknotting tunnels place in the corresponding position to the thickened arc illustrated in Figure 5.

5. COMPUTER EXPERIMENTS

In this section, we show some examples of 3-manifolds with 2-generator fundamental groups which are found by computer experiments. All examples are surface bundles over the circle with genus two surface fiber.

Recall that the Dehn twists D_1, D_2, \dots, D_5 generate the mapping class group of a closed surface of genus two. We fix this generating system and describe the monodromy of surface bundles in terms of the word of these generators.

In the first experiment, we compute the rank of the fundamental group for all closed/once-punctured surface bundles with monodromy up to word length 5. There is a table of the representative elements of all conjugacy classes of monodromies up to word length 4 in [1]. As an extension of the result, we can obtain such a list up to length 5, which is now available at <http://www.is.titech.ac.jp/~takasawa/MCG/>. In fact, there are 172 nontrivial representative elements. To compute the rank of the fundamental group of the manifold, we implement the algorithm in [10], which constructs a surgery description of the surface bundle from the monodromy. Then we use SnapPea [28] to compute the fundamental group of the manifold. Note that in general, it is difficult to determine the rank of the fundamental group. However, in this case, if there is a presentation which has just 2-generator, we can say the rank is 2. The result of the experiment is shown in Table 1. There are 30 distinct examples which have 2-generator fundamental groups. For some examples the first Betti number β_1 equals to 2, however they are torus sums of two Seifert manifolds.

TABLE 1.

| monodromy | closed | | | once-punctured | | |
|----------------------------------|-----------|-----------------|-----------|----------------|-----------------|-----------|
| | volume | rank of π_1 | β_1 | volume | rank of π_1 | β_1 |
| $D_1D_2D_3$ | 0.000002 | 2 | 2 | 0.000077 | 2 | 2 |
| $D_1D_2D_3^{-1}$ | 3.663862 | 2 | 2 | 3.663872 | 2 | 2 |
| $D_1D_2^{-1}D_3$ | 5.333490 | 2 | 2 | 5.333605 | 2 | 2 |
| $D_1D_2D_3D_4$ | 0.000000 | 2 | 1 | 0.000000 | 2 | 1 |
| $D_1D_2D_3D_4^{-1}$ | 4.400833 | 2 | 1 | 3.770830 | 2 | 1 |
| $D_1D_2D_3^{-1}D_4^{-1}$ | 5.693021 | 2 | 1 | 4.059766 | 2 | 1 |
| $D_1D_2^{-1}D_3D_4$ | 7.084926 | 2 | 1 | 6.180274 | 2 | 1 |
| $D_1D_2^{-1}D_3D_4^{-1}$ | 8.935857 | 2 | 1 | 7.646593 | 2 | 1 |
| $D_1D_2^{-1}D_3^{-1}D_4$ | 7.643375 | 2 | 1 | 6.332667 | 2 | 1 |
| $D_1^2D_2D_3D_4$ | 0.000000 | 2 | 1 | 0.000000 | 2 | 1 |
| $D_1^2D_2D_3D_4^{-1}$ | 4.921483 | 2 | 1 | 3.970290 | 2 | 1 |
| $D_1^2D_2D_3^{-1}D_4$ | 7.967261 | 3 | 1 | 6.788090 | 2 | 1 |
| $D_1^2D_2D_3^{-1}D_4^{-1}$ | 6.577943 | 2 | 1 | 4.765940 | 2 | 1 |
| $D_1^2D_2^{-1}D_3D_4$ | 8.118328 | 2 | 1 | 7.023949 | 2 | 1 |
| $D_1^2D_2^{-1}D_3D_4^{-1}$ | 9.951958 | 2 | 1 | 8.550620 | 2 | 1 |
| $D_1^2D_2^{-1}D_3^{-1}D_4$ | 8.554563 | 2 | 1 | 7.180680 | 2 | 1 |
| $D_1^2D_2^{-1}D_3^{-1}D_4^{-1}$ | 5.231154 | 2 | 1 | 4.725229 | 2 | 1 |
| $D_1D_2D_1^{-1}D_2D_3^{-1}$ | 5.333490 | 2 | 2 | 5.333490 | 2 | 2 |
| $D_1D_2^2D_3D_4$ | 4.464659 | 2 | 1 | 3.853456 | 2 | 1 |
| $D_1D_2^2D_3D_4^{-1}$ | 6.746042 | 3 | 1 | 5.137941 | 2 | 1 |
| $D_1D_2D_3D_4D_5$ | 0.000000 | 2 | 1 | 0.000000 | 2 | 1 |
| $D_1D_2D_3D_4D_5^{-1}$ | 4.056860 | 2 | 1 | 3.177293 | 2 | 1 |
| $D_1D_2D_3D_4^{-1}D_5$ | 6.771750 | 2 | 1 | 5.563668 | 2 | 1 |
| $D_1D_2D_3D_4^{-1}D_5^{-1}$ | 4.124903 | 2 | 1 | 0.790429 | 2 | 1 |
| $D_1D_2D_3^{-1}D_4D_5$ | 7.746275 | 2 | 1 | 6.551743 | 2 | 1 |
| $D_1D_2D_3^{-1}D_4D_5^{-1}$ | 8.602031 | 2 | 1 | 6.965760 | 2 | 1 |
| $D_1D_2D_3^{-1}D_4^{-1}D_5$ | 7.406768 | 3 | 1 | 5.333490 | 2 | 1 |
| $D_1D_2^{-1}D_3D_4D_5^{-1}$ | 9.250556 | 2 | 1 | 7.517690 | 3 | 1 |
| $D_1D_2^{-1}D_3D_4^{-1}D_5$ | 10.649781 | 3 | 1 | 8.793346 | 2 | 1 |
| $D_1D_2^{-1}D_3^{-1}D_4^{-1}D_5$ | 6.783714 | 2 | 1 | 5.333490 | 3 | 1 |

In the second experiment, we searched over 100,000 randomly generated words up to word length 20 and we found huge number of examples of hyperbolic surface bundles with rank two fundamental groups. Unfortunately, for many of such manifolds, we have no practical algorithm to detect their Heegaard genus, and at present, we do not have examples of Heegaard genus more than two.

Here we exhibit single example we found, which seems independently interesting. The closed surface bundle M_f with monodromy $f = D_1^2D_3^3D_5^{-1}D_4^{-1}D_3^{-1}D_2$ has 2-generator fundamental group and the first Betti number of M_f is also 2 and M_f admits a hyperbolic structure. The fact that the first Betti number is two implies that it admits more than one surface bundle structures. Unfortunately, we could not determined the Heegaard genus of M_f .

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