A NOTE ON A MATRIX VERSION OF THE FARKAS LEMMA

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Abstract. A linear polyomial non-negative on the non-negativity domain of finitely many linear polynomials can be expressed as their non-negative linear combination. Recently, under several additional assumptions, Helton, Klep, and McCullough extended this result to matrix polynomials. The aim of this paper is to study which of these additional assumptions are really necessary.

1. Introduction

We are interested in matrix generalizations of the following variant of the Farkas lemma.

Theorem 1. *Let $f_1, f_2, \ldots, f_k$ be linear polynomials in n variables, i.e. $f_i(x_1, \ldots, x_n) = a_0^{(i)} + a_1^{(i)} x_1 + \ldots + a_n^{(i)} x_n$, where $a_j^{(i)} \in \mathbb{R}$ for $i \in \{1, \ldots, k\}$, $j \in \{0, \ldots, n\}$. Let $K = \{x \in \mathbb{R}^n | f_i(x) \geq 0 \; \forall i \in \{1, \ldots, k\}\}$. If f is another linear polynomial in n variables, for which $f|_K \geq 0$ holds, then there exist non-negative constants c_i, such that*

$$f = c_0 + c_1 f_1 + c_2 f_2 + \ldots + c_k f_k.$$

The following generalization was obtained by Helton, Klep, and McCullough in [2], see their Theorem 6.1. We write $\mathbb{R}^{d \times d}[x]$ (resp. $SS\mathbb{R}^{d \times d}[x]$) for the set of all polynomials whose coefficients are $d \times d$ (resp. symmetric $d \times d$) matrices.

Theorem 2. *Suppose $L_1 = I + \sum_{i=1}^n P_i x_i \in SS\mathbb{R}^{d \times d}[x]$ is a monic linear polynomial whose non-negativity domain $D_{L_1}(1) := \{x \in \mathbb{R}^n \mid L_1(x) \succeq 0\}$ is bounded. Then for every linear polynomial $L_2 = R_0 + \sum_{i=1}^n R_i x_i \in SS\mathbb{R}^{l \times l}[x]$ such that $L_2|_{D_L(1)} \succ 0$, there are $A_j \in \mathbb{R}^{l \times l}[x]$ and $B_k \in \mathbb{R}^{d \times l}[x]$ satisfying*

$$L_2 = \sum_j A_j^* A_j + \sum_k B_k^* L_1 B_k.$$

Note that this result also covers the case of several constraints; simply take L_1 to be their direct sum. The aim of this paper is to study the necessity of the following assumptions in Theorem 2:

(1) **Boundedness of $D_{L_1}(1)$:** Example 1 shows, that this assumption cannot be removed.
(2) **Monicity of L_1:** In Theorem 3 we prove, that for diagonal L_1 monicity can be removed from Theorem 2. The general case remains open.

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(3) **Strict positivity of $L_2|_{D_{L_1}(1)}$:** In Section 4 we show that this assumption can be replaced with $L_2|_{D_{L_1}(1)} \succeq 0$ in the following special cases:
- In the one-variable case (even if $D_{L_1}(1)$ is unbounded).
- When the span of the coefficients of L_1 is closed for multiplication.

The general case remains open. However, Example 2 shows that simultaneous generalization to non-monic L_1 and non-strict $L_2|_{D_{L_1}(1)}$ is not possible.

For the sake of completeness we also note that polynomials A_j and B_k in Theorem 2 need not be constant as shown by Example 3.1 and Theorem 3.5 in [2].

2. Boundedness of $D_{L_1}(1)$

The assumption of boundedness in Theorem 2 cannot be removed, because of the following example:

Example 1. *For linear polynomials*

$$L_1 = \begin{bmatrix} 1+x_1 & 0 & 0 \\ 0 & 1+x_1+x_2 & 0 \\ 0 & 0 & 1+x_2 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 1+\frac{1}{3}x_1 & \frac{3}{4} \\ \frac{3}{4} & 1+\frac{1}{3}x_2 \end{bmatrix}$$

we have $D_{L_1}(1) \subseteq \tilde{D}_{L_2}(1) = \{(x_1,x_2) \in \mathbb{R}^2 | L_2(x_1,x_2) \succ 0\}$, but L_2 cannot be expressed as $\sum_j A_j^* A_j + \sum_k B_k^* L_1 B_k$. This implies that we need boundedness of $D_{L_1}(1)$ in Theorem 2.

Proof. We have $D_{L_1}(1) = \{(x_1,x_2) \in \mathbb{R}^2 | x_1 \geq -1, x_1+x_2 \geq -1, x_2 \geq -1\}$. On the other hand, the fact $1+\frac{1}{3}x_1 > 0$ on $D_{L_1}(1)$ together with

$$\begin{aligned} \det(L_2) &= \left(1+\frac{1}{3}x_1\right)\left(1+\frac{1}{3}x_2\right) - \left(\frac{3}{4}\right)^2 = \\ &= \left(\frac{2}{3}+\frac{1}{3}(1+x_1)\right)\left(\frac{2}{3}+\frac{1}{3}(1+x_2)\right) - \left(\frac{3}{4}\right)^2 = \\ &= \left(\frac{2}{3}\right)^2 + \frac{2}{9}(1+x_1+1+x_2) + \frac{1}{9}(1+x_1)(1+x_2) - \left(\frac{3}{4}\right)^2 = \\ &= \left(\frac{15}{144}\right) + \frac{2}{9}(1+x_1+x_2) + \frac{1}{9}(1+x_1)(1+x_2), \end{aligned}$$

where the second two sumands in the last line are non-negative on $D_{L_1}(1)$, gives $L_2|_{D_{L_1}(1)} \succ 0$.

Now we are going to show, that if L_2 can be expressed as $L_2 = \sum_j A_j^* A_j + \sum_k B_k^* L_1 B_k$, then A_j and B_k can be assumed to be constant matrices. Let us denote A_j, B_k as

$$\begin{bmatrix} P_1^{(j)}(x_1,x_2) & R_1^{(j)}(x_1,x_2) \\ P_2^{(j)}(x_1,x_2) & R_2^{(j)}(x_1,x_2) \end{bmatrix}, \begin{bmatrix} p_1^{(k)}(x_1,x_2) & r_1^{(k)}(x_1,x_2) \\ p_2^{(k)}(x_1,x_2) & r_2^{(k)}(x_1,x_2) \\ p_3^{(k)}(x_1,x_2) & r_3^{(k)}(x_1,x_2) \end{bmatrix}.$$

Comparing the entry (11) in $\sum_j A_j^* A_j + \sum_k B_k^* L_1 B_k$ and L_2 gives

$$\sum_j \left((P_1^{(j)}(x_1,x_2))^2 + (P_2^{(j)}(x_1,x_2))^2\right) + \sum_k \left((p_1^{(k)}(x_1,x_2))^2(1+x_1) + \right.$$

$$+(p_2^{(k)}(x_1,x_2))^2(1+x_1+x_2)+(p_3^{(k)}(x_1,x_2))^2(1+x_2)\Big)\overbrace{=}^{?}1+\frac{1}{3}x_1.$$

By observing the monomial of the form $Kx_i^n, n\in\mathbb{N}, K\neq 0, i=1,2$, of the highest degree on the left side, it can be seen, that monomials $Ax_i^n, n\in\mathbb{N}, A\neq 0, i=1,2$, do not appear in any $P_i^{(j)}$ nor $p_i^{(k)}$. With the same reasoning applied for entry (22) not even in $R_i^{(j)}$ and $r_i^{(k)}$. Further on, since monomials K and Kx_1, $K\neq 0$, do not come from products of monomials $Kx_1^m x_2^n$, $K\neq 0$, $m,n\in\mathbb{N}$ in $P_i^{(j)}$ and $p_i^{(k)}$, with some other monomials, they are not needed to satisfy the upper equality. Similar reasoning (regarding monomials $Kx_1^m x_2^n$) can be applied to other entries, hence WLOG A_j, B_k are constant matrices.
The comparison of coefficients in $\sum_j A_j^* A_j+\sum_k B_k^* L_1 B_k$ and L_2 in a constant-matrix case gives the following equalities:
Entry (11):

$$x_2:\quad \sum_k\Big((p_2^{(k)})^2+(p_3^{(k)})^2\Big)=0\quad\Rightarrow\quad p_2^{(k)}=p_3^{(k)}=0,\ \forall k \tag{1}$$

$$x_1:\quad \sum_k\Big((p_1^{(k)})^2+(p_2^{(k)})^2\Big)=\frac{1}{3} \tag{2}$$

$$1:\quad \sum_j\left(\sum_{i=1}^{2}(P_i^{(j)})^2\right)+\sum_k\Big((p_1^{(k)})^2+(p_2^{(k)})^2+(p_3^{(k)})^2\Big)=1 \tag{3}$$

Entry (12)(=entry (21)):

$$x_2:\quad \sum_k\Big(p_2^{(k)}r_2^{(k)}+p_3^{(k)}r_3^{(k)}\Big)=0 \tag{4}$$

$$x_1:\quad \sum_k\Big(p_1^{(k)}r_1^{(k)}+p_2^{(k)}r_2^{(k)}\Big)=0 \tag{5}$$

$$1:\quad \sum_j\left(\sum_{i=1}^{2}P_i^{(j)}R_i^{(j)}\right)+\sum_k\left(\sum_{i=1}^{3}p_i^{(k)}r_i^{(k)}\right)=\frac{3}{4} \tag{6}$$

Entry (22):

$$x_2:\quad \sum_k\Big((r_2^{(k)})^2+(r_3^{(k)})^2\Big)=\frac{1}{3} \tag{7}$$

$$x_1:\quad \sum_k\Big((r_1^{(k)})^2+(r_2^{(k)})^2\Big)=0\quad\Rightarrow\quad r_1^{(k)}=r_2^{(k)}=0,\ \forall l \tag{8}$$

$$1:\quad \sum_j\left(\sum_{i=1}^{2}(R_i^{(j)})^2\right)+\sum_k\Big((r_1^{(k)})^2+(r_2^{(k)})^2+(r_3^{(k)})^2\Big)=1 \tag{9}$$

We will see, that the upper equalities cannot be simultaneously satisfied. From 1 and 8 we conclude $\sum_k\left(\sum_{i=1}^{3}p_i^{(k)}r_i^{(k)}\right)=0$. We use this in 6 and get $\sum_j\left(\sum_{j=1}^{2}P_i^{(j)}R_i^{(j)}\right)=$

$\frac{3}{4}$. Using 1 and 2 in 3 gives $\sum_j \left(\sum_{i=1}^2 (P_i^{(j)})^2\right) = \frac{2}{3}$. Similarly using 7 and 8 in 9 gives $\sum_j \left(\sum_{i=1}^2 (R_i^{(j)})^2\right) = \frac{2}{3}$. The following chain of (in)equalities should hold:

$$\frac{2}{3} = \frac{\frac{2}{3}+\frac{2}{3}}{2} = \sum_j \sum_{i=1}^2 \frac{(P_i^{(j)})^2 + (R_i^{(j)})^2}{2} \geq \sum_j \sum_{i=1}^2 \left| P_i^{(j)} R_i^{(j)} \right| \geq \sum_j \sum_{i=1}^2 P_i^{(j)} R_i^{(j)} = \frac{3}{4},$$

where the first inequality follows from AG-inequalites applied to pairs $\left\{(P_i^{(j)})^2, (R_i^{(j)})^2\right\}$, i.e. $\frac{(P_i^{(j)})^2+(R_i^{(j)})^2}{2} \geq \left|P_i^{(j)} R_i^{(j)}\right|$, $\forall\ i,\ j$.
We conclude $\frac{2}{3} \geq \frac{3}{4}$, which is obviously a contradiction. □

3. Monicity of L_1

In this section we show, that for diagonal L_1 monicity in Theorem 2 can be removed. We first prove the case $D_{L_1}(1) = \{\vec{a}\}$ in Proposition 1 and then also the other cases of non-empty and bounded $D_{L_1}(1)$ in Theorem 3.

Proposition 1. *Suppose $L_1 \in SS\mathbb{R}^{d\times d}[x]$ is a diagonal linear polynomial and $D_{L_1}(1) = \{\vec{a}\}$. Then for every linear symmetric polynomial $L_2 \in SS\mathbb{R}^{l\times l}[x]$ with $L_2|_{D_L(1)} \succeq 0$, there are $A_j \in \mathbb{R}^{l\times l}[x]$ and $B_k \in \mathbb{R}^{d\times l}[x]$ satisfying*

$$L_2 = \sum_j A_j^* A_j + \sum_k B_k^* L B_k.$$

In the proof we will use the following proposition:

Proposition 2. *For $A \in SS\mathbb{R}^{l\times l}$ there exist $B_k \in SS\mathbb{R}^{2\times l}$, such that*

$$\sum_k B_k^* \begin{bmatrix} x & 0 \\ 0 & -x \end{bmatrix} B_k = Ax.$$

Proof. According to the well-known fact every real symmetric matrix is real congruent to a diagonal D with elements 1,−1 and 0 on the diagonal, i.e. $A = \sum_k \tilde{B}_k^* D \tilde{B}_k$, where D, $\tilde{B}_k \in SS\mathbb{R}^{l\times l}$. Dx can be constructed from $\begin{bmatrix} x & 0 \\ 0 & -x \end{bmatrix}$ with the aim of equalities:

$$E_{ii} = \begin{bmatrix} e_i & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} e_i^* \\ 0 \end{bmatrix},$$

$$-E_{ii} = \begin{bmatrix} 0 & e_i \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ e_i^* \end{bmatrix},$$

where e_i denotes the standard $\mathbb{R}^{l\times 1}$ vector. □

Proof. (Proposition 1) With translation we can assume $D_{L_1}(1) = \left\{\vec{0}\right\}$. For a polynomial

$$\tilde{L}_1 = \begin{bmatrix} x_1 & 0 \\ 0 & -x_1 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} x_n & 0 \\ 0 & -x_n \end{bmatrix}$$

we have $D_{\tilde{L}_1}(1) = \left\{\vec{0}\right\}$. After applying Theorem 1 for a tuple of diagonal entries of L_1 and each diagonal entry of $\tilde{L}_1$, it follows that $\tilde{L}_1 = \sum_j \tilde{A}_j^* \tilde{A}_j + \sum_l \tilde{B}_k^* L_1 \tilde{B}_k$ for some constant $\tilde{A}_j$, $\tilde{B}_k$. Therefore it suffices to find A_j, B_k, such that $L_2 =$

$\sum_j A_j^* A_j + \sum_k B_k^* \tilde{L}_1 B_k$.
If we write $L_2(x) = R_0 + \sum_{i=1}^n R_i x_i$, then $L_2(\vec{0}) = R_0 \succeq 0$. So there exists A, such that $R_0 = A^* A$. According to Proposition 2, $R_i x_i$ can be expressed in a desired way by $\begin{bmatrix} x_i & 0 \\ 0 & -x_i \end{bmatrix}$, hence also with $\tilde{L}_1$.
Since i was arbitrary, we are done. □

Now we will extend Proposition 1 to the general case. One additional lemma will be needed for that.

Lemma 1. *Suppose $L = P_0 + \sum_{i=1}^n P_i x_i \in SS\mathbb{R}^{d\times d}[x]$ is linear polynomial and let $\vec{0} \in D_{L_1}(1)$ be an interior point. Then there exists a monic linear polynomial $\tilde{L} = I + \sum_{i=1}^n \hat{P}_i x_i \in SS\mathbb{R}^{\tilde{d}\times\tilde{d}}[x]$, where $\tilde{d} \le d$ and $D_L(1) = D_{\tilde{L}}(1)$, such that $\tilde{L} = C^* L C$ in $L = D^* \tilde{L} D$, where $C \in \mathbb{R}^{d\times\tilde{d}}$, $D \in \mathbb{R}^{\tilde{d}\times d}$.*

Proof. Since 0 is an interior point, $P_0 \succeq 0$ and $\mathrm{Im}(P_i) \subseteq \mathrm{Im}(P_0)$ for $i = 1, \ldots, n$ (See [2, Proof of Proposition 2.1].). We have $P_0 = V^* D V$, where D is diagonal and V orthogonal. Further on $V^* L V = L|_{\mathrm{Im}(P_0)} \oplus 0_{d-\tilde{d}}$, $\tilde{d} = \dim(\mathrm{Im}(P_0))$. Hence $L|_{\mathrm{Im}(P_0)} = J^*(V^* L V)J$ with $J^* := [I_{\tilde{d}} \quad 0^{\tilde{d}\times(d-\tilde{d})}] \in \mathbb{R}^{\tilde{d}\times d}$. Defining $\tilde{P}_0 := P_0|_{\mathrm{Im}(P_0)} \succ 0$, gives $\tilde{P}_0 = B^* B$, where $\tilde{P}_0, B \in \mathbb{R}^{\tilde{d}\times\tilde{d}}$ and B is invertible. So $(B^{-1})^* L|_{\mathrm{Im}(P_0)} B^{-1} = (B^{-1})^* B^* B B^{-1} + \sum_i (B^{-1})^* P_i|_{\mathrm{Im}(P_0)} B^{-1} x_i = I + \sum_i (B^{-1})^* P_i|_{\mathrm{Im}(P_0)} B^{-1} x_i =: \tilde{L}$. $\tilde{L}$ is in $\mathbb{R}^{\tilde{d}\times\tilde{d}}$ and $D_L(1) = D_{\tilde{L}}(1)$. With $C^* := (B^{-1})^* J^* V^* \in \mathbb{R}^{\tilde{d}\times\tilde{d}}\mathbb{R}^{\tilde{d}\times d}\mathbb{R}^{d\times d} = \mathbb{R}^{\tilde{d}\times d}$ and $D^* := (V^{-1})^* J B^* \in \mathbb{R}^{d\times d}\mathbb{R}^{d\times\tilde{d}}\mathbb{R}^{\tilde{d}\times\tilde{d}} = \mathbb{R}^{d\times\tilde{d}}$, the lemma is proven. □

Theorem 3. *Suppose $L_1 \in SS\mathbb{R}^{d\times d}[x]$ is a diagonal linear polynomial and $D_{L_1}(1)$ is non-empty and bounded. Then for every linear symmetric polynomial $L_2 \in SS\mathbb{R}^{l\times l}[x]$ with $L_2|_{D_L(1)} \succ 0$, there are $A_j \in \mathbb{R}^{l\times l}[x]$ and $B_k \in \mathbb{R}^{d\times l}[x]$ satisfying*

$$L_2 = \sum_j A_j^* A_j + \sum_k B_k^* L B_k.$$

Proof. If $D_{L_1}(1) = \{\vec{a}\}$, then we can use Proposition 1 and we are done. If $\dim D_{L_1}(1) = n$, then by Lemma 1 WLOG L_1 is monic and Theorem 2 is used. Otherwise we have $1 \le \dim D_{L_1}(1) =: k \le n-1$. Since $D_{L_1}(1)$ is convex, it lies in some affine subspace of dimension k. With translation WLOG $\vec{0} \in D_{L_1}(1)$ and hence the affine subspace is actually a vector subspace of dimension k. Let $B = \{e_1', e_2', \ldots, e_k'\}$ be the basis of this subspace. B can be completed to the basis of $\mathbb{R}^n$, i.e. $B' = \{e_1', \ldots, e_k', e_{k+1}', \ldots, e_n'\}$. Standard basis $\{e_1, e_2, \ldots, e_n\}$ of $\mathbb{R}^n$ can be uniquelly expressed by B' and vice versa, i.e. $e_i = \sum_{j=1}^n \alpha_j^{(i)} e_j'$ and $e_i' = \sum_{j=1}^n \beta_j^{(i)} e_j$, for unique $\alpha_j^{(i)}, \beta_j^{(i)} \in \mathbb{R}$. Introducing new unknows x_i' as $x_i = \sum_{j=1}^n \alpha_j^{(i)} x_j'$, gives also $x_i' = \sum_{j=1}^n \beta_j^{(i)} x_j$. Putting expressed x_i-s into $L_1(x_1, \ldots, x_n)$, we get $\tilde{L}_1(x_1', \ldots, x_n')$. The map $\Phi : \mathbb{R}^n \to \mathbb{R}^n$, defined by $\Phi : (a_1, \ldots, a_n) \mapsto (\sum_{j=1}^n \beta_j^{(1)} a_j, \ldots, \sum_{j=1}^n \beta_j^{(n)} a_j)$, is bijective and $L_1((a_1, \ldots, a_n)) = \tilde{L}_1(\Phi(a_1, \ldots, a_n))$. Hence $\Phi(D_{L_1}(1)) = D_{\tilde{L}_1}(1)$. So $D_{L_1}(1)$ and $D_{\tilde{L}_1}(1)$ are in bijective correspondence. Similarly for $D_{L_2}(1)$ and $D_{\tilde{L}_2}(1)$. Therefore $D_{L_1}(1) \subseteq D_{L_2}(1) \Leftrightarrow D_{\tilde{L}_1}(1) \subseteq D_{\tilde{L}_2}(1)$.
From the construction of basis B', $x' \in D_{\tilde{L}_1}(1)$ is of the form $(x_1', \ldots, x_k', 0, \ldots, 0)$.

Let us write $\tilde{L}_1 = \left(P'_0 + \sum_{i=1}^k P'_i x'_i\right) + \sum_{i=k+1}^n P'_i x'_i = \tilde{L}_{1,1}(x'_1, \ldots, x'_k) +$ $+\tilde{L}_{1,2}(x'_{k+1}, \ldots, x'_n)$ ($\tilde{L}_1$ is still diagonal.). For

$$\tilde{L} = \tilde{L}_{1,1}(x'_1, \ldots, x'_k) \oplus \begin{bmatrix} x'_{k+1} & 0 \\ 0 & -x'_{k+1} \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} x'_n & 0 \\ 0 & -x'_n \end{bmatrix}$$

(which is obviously diagonal), $D_{\tilde{L}_1}(1) = D_{\tilde{L}}(1)$, and with the use of Theorem 1 for the tuple of diagonal entries of $\tilde{L}_1$ and each diagonal entry of $\tilde{L}$, $\tilde{L}$ can be expressed as $\sum_j A_j^* A_j + \sum_k B_k^* \tilde{L}_1 B_k$. Hence it suffices to prove the statement of the theorem for the pair $\tilde{L}, \tilde{L}_2$.

Analogously as for $\tilde{L}_1$ we write $\tilde{L}_2$ as $\tilde{L}_2 = \left(R'_0 + \sum_{i=1}^k R'_i x'_i\right) + \sum_{i=k+1}^n R'_i x'_i = \tilde{L}_{2,1}(x'_1, \ldots, x'_k) + \tilde{L}_{2,2}(x'_{k+1}, \ldots, x'_n)$. We have $\tilde{L}_{2,1}|_{D_{\tilde{L}_{1,1}}(1)} \succ 0$. Since there exists an interior point in $D_{\tilde{L}_{1,1}}(1)$, Lemma 1 allows us to regard $\tilde{L}_{1,1}$ as monic. Finally Theorem 2 is used for the pair $\tilde{L}_{1,1}, \tilde{L}_{2,1}$.

It remains to express $\tilde{L}_{2,2}(x'_{k+1}, \ldots, x'_n) = R'_{k+1} x'_{k+1} + \cdots + R'_n x'_n$ with $\tilde{L}$. According to Proposition 2, $R_i x'_i$ can be expressed with $\begin{bmatrix} x'_i & 0 \\ 0 & -x'_i \end{bmatrix}$. Hence also with $\tilde{L}$. Since i was arbitrary, we are done.

To conclude, we got the expression

$$\tilde{L}_2(x'_1, \ldots, x'_n) = \sum_j \tilde{A}_j^* \tilde{A}_j + \sum_k \tilde{B}_k^* \tilde{L}_1(x'_1, \ldots, x'_n) \tilde{B}_k.$$

Using $x'_i = \sum_{j=1}^n \beta_j^{(i)} x_j$, we finally get

$$L_2(x_1, \ldots, x_n) = \sum_j A_j^* A_j + \sum_k B_k^* L_1(x_1, \ldots, x_n) B_k.$$

□

4. Strict positivity of $L_2|_{D_L(1)}$

The next thing to be studied is the necessity of positive definiteness in Theorem 2, i.e. whether semidefiniteness suffices. We separately study the one-variable case from the general diagonal case.

4.1. **One-variable case.**

Theorem 4. *Suppose $L_1(x) = P_0 + P_1 x \in SS\mathbb{R}^{d\times d}[x]$ is a linear polynomial and $D_{L_1}(1)$ has an interior point. Then for every linear symmetric polynomial $L_2(x) = R_0 + R_1 x \in SS\mathbb{R}^{l\times l}[x]$ with $L_2|_{D_L(1)} \succeq 0$, there are $A_j \in \mathbb{R}^{l\times l}[x]$ and $B_k \in \mathbb{R}^{d\times l}[x]$ satisfying*

$$L_2 = \sum_j A_j^* A_j + \sum_k B_k^* L_1 B_k.$$

Proof. **Case 1:** If $D_{L_1}(1)$ is bounded and there exists an interior point in $D_{L_1}(1)$, then according to Lemma 1, we can assume L_1 and L_2 are monic, i.e. $P_0 = I, R_0 = I$. Since we have just one variable, we may interpret both L_1 and L_2 as NC polynomials, or precisely linear pencils. We will first show, that

$\underline{D_{L_1}(1) \subseteq D_{L_2}(1) \Rightarrow D_{L_1} \subseteq D_{L_2}}$: Let us take $X \in D_{L_1}$, $X \in \mathbb{R}^{s\times s}$, which means $L_1(X) = I \otimes I + P_1 \otimes X \succeq 0$. Or equivalently $I \otimes I + X \otimes P_1 \succeq 0$. We have to

show, that $X \in D_{L_2}$. Since X is symmetric, it can be real ortogonally diagonalized, i.e. $UXU^T = D$, $U \in \mathbb{R}^{s\times s}$. After multiplying with invertible matrix $U \otimes I$ we get $(U\otimes I)(I\otimes I + X\otimes P_1)(U\otimes I)^T = I\otimes I + D\otimes P_1 \succeq 0$. Hence $X \in D_{L_1} \Leftrightarrow D \in D_{L_1}$. $I\otimes I + D\otimes P_1 \succeq 0$ is a block-diagonal matrix with the blocks of the form $I + d_iP_1$. It follows $I\otimes I + D\otimes P_1 \succeq 0 \Leftrightarrow I + d_iP_1 \succeq 0, \forall i \Leftrightarrow d_i \in D_{L_1}(1), \forall i$. According to the assumption $d_i \in D_{L_2}(1), \forall i$. Therefore $X \in D_{L_2}$.
To be able to use LP-satz for the pair L_1, L_2, D_{L_1} must be bounded. But by [2, Proposition 2.4] this is equivalent to $D_{L_1}(1)$ being bounded. So by [2, Corollary 3.7] there exist B_k, such that $L_2 = \sum_k B_k^* L_1 B_k$.
Case 2: If $D_{L_1}(1)$ is unbounded, then it is an interval of the form $[a,\infty)$, $(-\infty,a]$, $(-\infty,\infty)$, $a \in \mathbb{R}$. With translation we may assume $a = 0$.
First we study the case $D_{L_1}(1) = [0,\infty)$. Since $0 \in D_{L_1}(1)$, we have $P_0 \succeq 0$. We can also show, that $P_1 \succeq 0$. To explain: $u^*L_1(x)u = u^*P_0u + u^*(P_1x)u = u^*P_0u + xu^*P_1u$. In the case that $u^*P_1u \neq 0$, we have $x\,|u^*P_1u| > |u^*P_0u|$, for x great enough. Therefore, if there exists u, such that $u^*P_1u < 0$, then $\lim\limits_{x\to\infty} x \notin D_{L_1}(1)$. Contradiction.
Since P_0 and P_1 are positive semidefinite, we can use Newcomb's theorem [5, Theorem 20.2.2] (It is actually made for complex matrices but with a slight modification of the proof it holds for real as well.) to simultaneously diagonalize them with invertible S, i.e. S^*P_0S, S^*P_1S are both diagonal. So WLOG L_1 is diagonal. Analogously for L_2. Now we just use Theorem 1 for diagonal entries of L_1 and each diagonal entry of L_2 and we are done.
In the case $D_{L_1}(1) = (-\infty,0]$, we have again $P_0 \succeq 0$. As above we show, that $P_1 \preceq 0$. Since P_0, P_1 are semidefinite, Newcomb's theorem [5, Theorem 20.2.2] can be used and we proceed as above.
In the case $D_{L_1}(1) = (-\infty,\infty)$, we have $P_0 \succeq 0$ and it is easy to show, that $P_1 = 0$. Therefore $L_1(x) = P_0$ and analogously $L_2(x) = R_0$, where $R_0 \succeq 0$. Hence $L_2(x) = C^*C$. □

The following example shows, that $D_{L_1}(1)$ must have an interior point in Theorem 4.

Example 2. *For non-monic, non-diagonal polynomial* $L_1(x) = \begin{bmatrix} 1 & x \\ x & 0 \end{bmatrix}$, *we have* $D_{L_1}(1) = \{0\}$. *Therefore* $D_{L_1}(1)$ *is non-empty and bounded. It also holds that* $L_2(x) = x$ *is non-negative on* $D_{L_1}(1)$, *but there do not exist* $A_j \in \mathbb{R}[x], B_k \in \mathbb{R}^{2\times 1}[x]$, *such that* $L_2 = \sum_j A_j^*A_j + \sum_k B_k^*L_1B_k$.

Proof. Since $\det(L_1) = -x^2$, $D_{L_1}(1) = \{0\}$. It is obvious, that $L_2|_{D_{L_1}(1)} \geq 0$. The proof will be by contradiction. Let us say there exist A_j, B_k, such that $\sum_j A_j^*A_j + \sum_k B_k^*L_1B_k = x$. Let B_k be of the form $[b_1^{(k)}, b_2^{(k)}]^T$, where $b_i^{(k)} \in \mathbb{R}[x]$. Similarly $A_j \in \mathbb{R}[x]$. Comparing the expression $\sum_j A_j^*A_j + \sum_k B_k^*L_1B_k$ with x:

$$\sum_j A_j^2 + \sum_k \left((b_1^{(k)})^2 + 2(b_1^{(k)})(b_2^{(k)})x \right) \overbrace{=}^{?} x.$$

The coefficient at 1 on LHS equals $\sum_j A_{j,0}^2 + \sum_k (b_{1,0}^{(k)})^2$, where $A_{j,0}$ denotes the free monomial in A_j and $b_{1,0}^{(k)}$ the free monomial in $b_1^{(k)}$. Since on *RHS* it is 0,

$A_{j,0} = b_{1,0}^{(k)} = 0$, $\forall j,k$. But then the coefficient at x on LHS is 0, while on RHS 1. Contradiction. □

4.2. **General diagonal case.** For diagonal L_1 and $L_2|_{D_{L_1}(1)} \succeq 0$ in Theorem 2, we need two additional assumptions, to conclude, that unital linear map from the continuation, which is positive, is actually completely positive. This is done by:

Theorem 5. [1*, Theorem* 2] *Let A be commutative C^*-algebra and $\tau : A \to S$ positive linear function, where S is operator vector space. Then τ is completely positive.*

Definition Let $L_1 = P_0 + \sum_{i=1}^n P_i x_i \in SS\mathbb{R}^{d\times d}[x]$, $S_1 = \mathrm{Lin}\{P_0, P_1, \dots, P_n\}$ and $v = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$. A point v is an *invertible interior point* of L_1, if it is interior point in $D_{L_1}(1)$ and $v_{-1} := \left(L_1(v)|_{\mathrm{Im}(L_1(v))}\right)^{-1} \oplus 0_k \in S_1$ $(v_{-1}^* = v_{-1})$, where $k = n - \dim(\mathrm{Im}(L_1(v)))$.

Theorem 6. *Suppose $L_1 = P_0 + \sum_{i=1}^n P_i x_i \in SS\mathbb{R}^{d\times d}[x]$ is diagonal, $D_{L_1}(1)$ is bounded and has an interior point. Let a vector space $S_1 = \mathrm{Lin}\{P_0, P_1, \dots, P_n\}$ be algebra and there exists an invertible interior point v for L_1. If for $L_2 = R_0 + \sum_{i=1}^n R_i x_i \in SS\mathbb{R}^{l\times l}[x]$, where $L_2|_{D_{L_1}(1)} \succeq 0$, then there exist $A_j \in \mathbb{R}^{l\times l}, B_k \in \mathbb{R}^{d\times l}$, such that*

$$L_2 = \sum_j A_j^* A_j + \sum_k B_k^* L_1 B_k.$$

Proof. With substitutions $x_i = \tilde{x}_i + v_i$, the invertible interior point v of L_1 becomes interior point $\vec{0}$ for $\tilde{L}_1 = \tilde{P}_0 + \sum_{i=1}^n \tilde{P}_i \tilde{x}_1$. $\tilde{L}_1$ is also diagonal, $\tilde{P}_0 = L_1(v)$ and $D_{\tilde{L}_1}(1) = D_{L_1}(1) - v$. The same is true for L_2 and $\tilde{L}_2$. Since $D_{L_1}(1) \subseteq D_{L_2}(1) \Leftrightarrow D_{L_1}(1) - v \subseteq D_{L_2}(1) - v \Leftrightarrow D_{\tilde{L}_1}(1) \subseteq D_{\tilde{L}_2}(1)$, we have $\tilde{L}_2|_{D_{\tilde{L}_1}(1)} \succeq 0$.
Since 0 is an interior point for $\tilde{L}_1$, we have $\mathrm{Ker}(\tilde{P}_i) \subseteq \mathrm{Ker}(\tilde{P}_0)$ (See [2, Part of the proof of Proposition 2.1] for details.). Let $k = \dim(\mathrm{Ker}(\tilde{P}_0)))$.
$\left((v_{-1})^{\frac{1}{2}}\right)^* \tilde{L}_1 (v_{-1})^{\frac{1}{2}} = \left((v_{-1})^{\frac{1}{2}}\right)^* \tilde{P}_0 (v_{-1})^{\frac{1}{2}} + \sum_i \left((v_{-1})^{\frac{1}{2}}\right)^* (\tilde{P}_i \tilde{x}_i)(v_{-1})^{\frac{1}{2}} == I \oplus 0_k + \sum_i (\hat{P}_i \oplus 0_k)\tilde{x}_i$.
Defining $\hat{L}_1 := I + \sum_i \hat{P}_i \tilde{x}_i$, we see, that $\hat{L}_1 = [I\ 0]\tilde{L}_1[I\ 0]^*$, where $I \in \mathbb{R}^{(n-k)^2}, 0 \in \mathbb{R}^{(n-k)\times k}$, $D_{\hat{L}_1}(1) = D_{\tilde{L}_1}(1)$ and $\hat{L}_1$ is diagonal. Since $\vec{0}$ an interior point in $D_{\tilde{L}_2}(1)$, by Lemma 1 there exists monic $\hat{L}_2$, such that $D_{\tilde{L}_2}(1) = D_{\hat{L}_2}(1)$ and $\hat{L}_2 = C^* \tilde{L}_2 C$, $\tilde{L}_2 = D^* \hat{L}_2 D$. Therefore it suffices to prove the statement for $\hat{L}_1, \hat{L}_2$.
Now we define vector spaces $\hat{S}_1 := \mathrm{Lin}\left\{I, \hat{P}_1, \dots, \hat{P}_n\right\}$ and $\hat{S}_2 := \mathrm{Lin}\left\{I, \hat{R}_1, \dots, \hat{R}_n\right\}$.
Since $D_{\hat{L}_1}(1)$ is bounded, generators for $\hat{S}_1$ are lineary independent by [2, Proposition 2.6] . Therefore $\tau : \hat{S}_1 \to \hat{S}_2$, where $I \mapsto I$ and $\hat{P}_i \mapsto \hat{R}_i$, is well-defined unital linear map. By [2, Theorem 3.5] it is also positive.
By the assumption of the theorem $S_1 = \mathrm{Lin}\{P_0, P_1, \dots, P_n\}$ is algebra. We will show, that $\hat{S}_1$ is also algebra. First we have

$$\begin{aligned}\tilde{S}_1 &= \mathrm{Lin}\left\{\tilde{P}_0, \tilde{P}_1, \dots, \tilde{P}_n\right\} = \mathrm{Lin}\left\{P_0 + \sum_i v_i P_i, P_1, \dots, P_n\right\} = \\ &= \mathrm{Lin}\{P_0, P_1, \dots, P_n\} = S_1.\end{aligned}$$

Then

$$\begin{aligned}\hat{S}_1 \oplus 0_k &= \operatorname{Lin}\left\{\left((v_{-1})^{\frac{1}{2}}\right)^* \tilde{P}_0 (v_{-1})^{\frac{1}{2}}, \left((v_{-1})^{\frac{1}{2}}\right)^* \tilde{P}_1 (v_{-1})^{\frac{1}{2}}, \ldots, \left((v_{-1})^{\frac{1}{2}}\right)^* \tilde{P}_n (v_{-1})^{\frac{1}{2}}\right\} = \\ &= \left((v_{-1})^{\frac{1}{2}}\right)^* \tilde{S}_1 (v_{-1})^{\frac{1}{2}} = \left((v_{-1})^{\frac{1}{2}}\right)^* \tilde{S}_1 (v_{-1})^{\frac{1}{2}}.\end{aligned}$$

For $\hat{S}_1$ to be algebra, it must be closed for multiplication. Equivalently $\hat{S}_1 \oplus 0_k$ must be closed for multiplication. Let us take $s_1, s_2 \in \hat{S}_1 \oplus 0_k$ and prove $s_1 s_2 \in \hat{S}_1 \oplus 0_k$.

$$s_1 = \left((v_{-1})^{\frac{1}{2}}\right)^* s (v_{-1})^{\frac{1}{2}},\ s_2 = \left((v_{-1})^{\frac{1}{2}}\right)^* s' (v_{-1})^{\frac{1}{2}},$$

where $s, s' \in S_1$.

$$s_1 s_2 = \left((v_{-1})^{\frac{1}{2}}\right)^* s (v_{-1})^{\frac{1}{2}} \left((v_{-1})^{\frac{1}{2}}\right)^* s' (v_{-1})^{\frac{1}{2}} = \left((v_{-1})^{\frac{1}{2}}\right)^* s v_{-1} s' (v_{-1})^{\frac{1}{2}}.$$

Since S_1 is algebra and v_{-1} by assumption in S_1, $sv_{-1}s' \in S_1$, $\hat{S}_1 \oplus 0_k$ is algebra. Since all matrices in $\left\{I, \hat{P}_1, \ldots, \hat{P}_n\right\}$ are diagonal, $\hat{S}_1$ is commutative algebra. Let $\hat{S}_1^{\mathbb{C}}$ be complex linear span of $\left\{I, \hat{P}_1, \ldots, \hat{P}_n\right\}$. Similarly for $\hat{S}_2, \hat{S}_2^{\mathbb{C}}$. Now we extend τ to $\tau^{\mathbb{C}} : \hat{S}_1^{\mathbb{C}} \to \hat{S}_2^{\mathbb{C}}$, where $\tau^{\mathbb{C}}(I/\hat{P}_i) = I/\hat{R}_i$. Since positive elements from $\hat{S}_1^{\mathbb{C}}$ are in $\hat{S}_1$ and $\tau^{\mathbb{C}}|_{\hat{S}_1} = \tau$, $\tau^{\mathbb{C}}$ is positive. Taking $\hat{S}_1^{\mathbb{C}}$ as A and $\hat{S}_2^{\mathbb{C}}$ as S in Theorem 5, $\tau^{\mathbb{C}}$ is in fact completely positive. Also $\tau^{\mathbb{C}}|_{\hat{S}_1} = \tau$ is completely positive. By [2, Theorem 3.5] $D_{L_1} \subseteq D_{L_2}$. By LP-satz [2, Corollary 3.7] for the pair $\hat{L}_1, \hat{L}_2$ there exist $V_j \in \mathbb{R}^{d\times m}$ and $\mu \in \mathbb{N}$, such that $\hat{L}_2 = \sum_{j=1}^{\mu} V_j^* \hat{L}_1 V_j$. The theorem is proven. □

Remark: A_j, B_k in Theorem 6 are constant matrices and not matrix polynomials, such as in Theorem 2.

Corollary 1. *Suppose $L_1 = P_0 + \sum_{i=1}^n P_i x_i \in SS\mathbb{R}^{d\times d}[x]$ is diagonal and $D_{L_1}(1)$ is n-simplex. If for $L_2 = R_0 + \sum_{i=1}^n R_i x_i \in SS\mathbb{R}^{l\times l}[x]$, where $L_2|_{D_{L_1}(1)} \succeq 0$, then there exist $A_j \in \mathbb{R}^{l\times l}, B_k \in \mathbb{R}^{d\times l}$, such that*

$$L_2 = \sum_j A_j^* A_j + \sum_k B_k^* L_1 B_k.$$

Proof. n-simplex in $\mathbb{R}^n$ is an intersection of $n+1$ halfspaces. Therefore it can be defined as $D_L(1)$ of $L = \bigoplus_{i=1}^{n+1}\left(a_0^{(i)} + \sum_{j=1}^n a_j^{(i)} x_j\right) = \tilde{P}_0 + \sum_{i=1}^n \tilde{P}_i x_i \in SS\mathbb{R}^{(n+1)\times(n+1)}$, for appropriate $a_j^{(i)} \in \mathbb{R}$. By Theorem 1, $L = \sum_k A_k^* L_1 A_k$. So it suffices to prove the statement for the pair L, L_2. Since $D_{L_1}(1)$ is bounded, $\left\{\tilde{P}_0, \tilde{P}_1, \ldots, \tilde{P}_n\right\}$ is lineary independent set in $\mathbb{DR}^{(n+1)^2} = \left\{\text{diagonal } \mathbb{R}^{(n+1)^2} \text{ matrices}\right\}$. Hence also its basis, which firstly means S_1 is algebra and secondly $v_{-1} \in S_1$ in Theorem 6 for any interior point $v \in D_{L_1}(1)$. By the latter the statement follows. □

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