Multiplicity and regularity of periodic solutions for a class of degenerate semilinear wave equations.

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Abstract

We prove the existence of infinitely many classical periodic solutions for a class of degenerate semilinear wave equations:

$$u_{tt} - u_{xx} + |u|^{s-1}u = f(x,t),$$

for all $s > 1$. In particular we prove the existence of infinitely many classical solutions for the case $s = 3$ posed by Brézis in [Brézis83]. The proof relies on a new approach and new upper a priori estimates for mini-max values of ,a pertubed from symmetry, strongly indefinite functional.[1]

[1]AMS classification: *35A15 35Jxx 35B45 35L05 35B10 42B35 34C25*

1 Introduction

In this paper we construct infinitely many classical time-periodic solutions for the following semilinear degenerate wave equation with time-dependent forcing term f:

$$u_{tt} - u_{xx} + g(u) - f(t,x) = 0 \tag{1.1}$$

$$u(0,t) = u(\pi,t) = 0. \tag{1.2}$$

where $g(u) = |u|^{s-1}u$ and $F(x,t,u) = g(u) + f(x,t)$, where f is of class C^2 and satisfies the Dirichlet boundary conditions.

Brézis problem[Brézis83]:*It seems reasonable to conjecture that when $g(u) = u^3$ problem (1.1),(1.2) possesses a solution -even infinitely many solutions- for every f(or at least a dense set of f's.)*

Theorem 1.1. *If $f \in C^2$ then there exists infinitely many classical solutions of (1.1),(1.2) for all $s > 1$.*

Theorem 1.1 also prove the existence of classical solutions for a question of Bahri-Berestycki in [BB84] on the existence of infinitely many solutions of (1.1),(1.2) for the class of $g(u) = |u|^{s-1}u$.

The weak version of the conjecture of Brézis, the existence of weak solutions for a dense set of f's has been shown to be true by Tanaka in [Tanaka86]. The problem (1.1),(1.2), for a given f, has been studied by Tanaka [Tanaka88], Bartsch-Ding-Lee [BDL99], for arbitrary $s > 1$, and Bolle-Ghoussoub-Tehrani [BGT2000], Ollivry [Ollivry83] for the case $1 < s < 2$ however only *weak solutions* have been obtained. As already noticed in [Rabinowitz71] there are two classes of monotone functions for problem (1.1),(1.2), the strongly monotone F, $\frac{\partial F}{\partial u} \geq \alpha > 0$ which can be compared to the uniformly elliptic case and the degenerate monotone case which allows $\frac{\partial F}{\partial u} = 0$. These two classes of monotone functions have been extensively studied by Torelli[Torelli69],Rabinowitz [Rabinowitz71], Hall[Hall70], Hale[Hale66], in the small perturbative case, i.e. with a smallness assumption on f . No such a smallness assumption is assumed here and the result we prove is a global one.

The difficulty in proving the regularity of the weak solutions obtained by [Tanaka88],[BDL99],[BGT2000] lies in the strong monotonicity assumption which is required by the regularity approach of Brézis-Nirenberg, [BN78-2]. In [BN78-2] Brézis and Nirenberg show that an L^∞ weak solution is smooth as long as F is smooth and satisfies the strong monotonicity assumption $\frac{\partial F}{\partial u} \geq \varepsilon > 0$ which fails here as $g(u)$ has a vanishing derivative. Note that in the highly degenerate case where F vanishes in an interval, weak solutions in L^∞ need not to be smooth, see [BN78-2] or [BN78-1] theorem I.8. Therefore, to find classical periodic solutions we will proceed differently. In [Rabinowitz78] Rabinowitz developed a regularity theory for this type of degeneracy where $\frac{\partial F}{\partial u} = 0$ is allowed but g strictly monotone ($z_1 > z_2$ implies $g(z_1) > g(z_2)$) for equations of the type (1.1),(1.2) and with $f = 0$. The approach in [Rabinowitz78] consisted in seeking viscous approximative solutions, studying a modified equation analogue of (1.4) with $f = 0$:

$$w_{tt}(\beta) - w_{xx}(\beta) = -|u|^{s-1}u(\beta) + \beta v_{tt}(\beta) \tag{1.3}$$

(Here $u(\beta) = v(\beta) + w(\beta)$ and $v(\beta)$ is the component of $u(\beta)$ in the direction of the infinite dimensional kernel of $\Box$, with the Dirichlet-periodic boundary conditions. The solution u is split in such a way to tackle the problem stemming from the infinite dimensional kernel of $\Box$.) with the parameter β and obtaining compactness via upper priori estimates independently of β of the critical values of the modified problem (1.3), enabling him to send β to 0 and then finding classical solutions. However the problem here contains the forcing term f and the natural functional associated with the problem (1.1) is no longer even thus the minimax sets for finding critical values in [Rabinowitz78] do not apply for forced vibrations.

In the eighties and nineties a perturbation theory for this type of problems *-perturbation from symmetry-* was developed,by Bahri-Berestycki [BB81],Bahri-Lions [BahriLions88],Tanaka [Tanaka89] Struwe [Struwe90], Rabinowitz [Rabinowitz82] and Bolle [Bolle99]. The approaches consist in finding growth estimates on some minimax values,b_n, and if they grow fast enough, will imply the existence of critical values of the perturbed functional. Hence it may seem natural to try to implement these approaches, to tackle the regularity issues stemming from the degenerate monotone semilinear term $g(u)$ and the infinite dimensional kernel of $\Box$ under Dirichlet boundary conditions, to the modified equation, seeking viscous approximative solutions:

$$w_{tt}(\beta) - w_{xx}(\beta) = -|u|^{s-1}u(\beta) + \beta v_{tt}(\beta) + f(t, x). \tag{1.4}$$

However the approaches by [BB81],[BahriLions88],[Bolle99],[Struwe90],[Rabinowitz82],*do not provide an upper explicit upper estimates* on the critical values, and this lead to serious unresolved difficulties to obtain compactness of $u(\beta)$, as $\beta \to 0$.

To overcome these difficulties we modify the minimax sets of Rabinowitz [Rabinowitz82], and introduce a class of sets for which we are able to find explicit a priori estimates for critical values of an appropriate functional J_β.

The main obstacle in finding upper estimates on the critical values of [Rabinowitz82], is the lack of an explicit function in the set of the maps in the minimax procedure which produces critical values. Our approach here allows for the construction of a function, in the sets of maps in the minimax procedure, whose energy is controlled. This map leads to, upper a priori estimates, which are independent of the small parameter β. This will imply compactness properties of the approximating sequence $u(\beta)$ as $\beta \to 0$.
The minimax sets which we introduce here can be adapted to semilinear elliptic equations of the type:

$$-\Delta u = g(u) + f(x) \tag{1.5}$$

$$u\,|_{\partial Q} = 0, \tag{1.6}$$

where $g(u) = |u|^{s-1}u$ satisfies the hypothesis Theorem 1 in Tanaka [Tanaka89] for $s < \frac{N}{N-2}$. Tanaka [Tanaka89], Bahri-Lions [BahriLions88] have shown multiplicity of solutions for the equation (1.5) without providing upper a priori on the location of critical values. The advantage of our approach is that, when combined with lower estimates on minimax numbers obtained by Tanaka in [Tanaka89], it will lead to upper a priori estimates on the critical values thence obtained.

Having constructed minimax values $c_n^m(\delta)$ with upper a priori estimates independently of, the Galerkin parameter m and β, we need information on the growth of some minimax values b_n^m to show that the $c_n^m(\delta)$ are critical values. To obtain the lower estimates of the growth of the b_n^m we employ the functional K introduced by Tanaka in [Tanaka88] and the Borsuk-Ulam lemma of Tanaka [Tanaka88], see lemma 2.5.

Another advantage of our approach is that it simplifies the weak solutions approach of [Tanaka88]. In [Tanaka88] some technical lemmas are employed to get information on the index of the weak solution u, obtained by passing to the limit in the Galerkin parameter m, the index of the critical value of the approximate solution u_n^m, obtained from the Galerkin scheme. Here the upper estimate on $c_n^m(\delta)$ is also independent of m thus it allows to simplify the passage to the limit as $m \to \infty$.

Once the compactness of the sequence $u(\beta)$ is obtained, the regularity will follow by the adapting the argument of [Rabinowitz78] to the problem considered here, in presence of a forcing term $f(x,t)$.

Remark: *Upper estimates for criticial values via the approach of [Bolle99] and under Dirichlet boundary conditions are in [CDHL2004] by Castro,Ding and Hernandez-Linares, and Castro and Clapp [CastroClapp2006], for perturbation of a differential operator,the Laplacian, the noncooperative elliptic system:*

$$-\Delta u = |u|^{p-1}u + f_u(x,u,v) \tag{1.7}$$

$$\Delta v = |v|^{q-1}v + f_v(x,u,v) \tag{1.8}$$

$$v\mid_{\partial Q} = u\mid_{\partial Q} = 0. \tag{1.9}$$

However the approaches in [CDHL2004],[CastroClapp2006] are incomplete as they rely on estimating $\int_Q |\nabla[\tau(u)u]|^2 dx$ *for* $u \in H_0^1(Q)$ *but the functional* $\tau : H_0^1(Q) \to \mathbb{R}$ *is not Fréchet differentiable and the authors do not define what they mean by* $\nabla[\tau(u)u]$,$\int_Q |\nabla[\tau(u)u]|^2 dx$, *for arbitrary* $u \in H_0^1(Q)$.

In Section 1:There is a functional I_β whose critical points correspond formally to solutions of (1.4). However as indicated by the approach of [Rabinowitz82], for technical reasons we will work with another functional J_β. We prove Palais-Smale conditions at large energies independently of β for the functional J_β and show implications for the functional I_β.

In Section 2: We introduce the minimax sets $\Lambda_n^m(\delta), \Lambda_n^m$ and Γ_n^m, and the minimax values $c_n^m(\delta), c_n^m, b_n^m$. Upper and lower estimates on $c_n^m(\delta)$ independently of β, m and consequences on the existence of critical values. The upper estimates will be obtained by the construction of a function whose energy is controlled and the lower estimates will follow by employing the functional K

introduced by Tanaka in [Tanaka88]. The upper estimates on the critical values independently of β, m imply compactness of the sequences $u(\beta)$ as $\beta \to 0$.

In Section 3 we adapt the arguments of [Rabinowitz78] and [Rabinowitz84] to end the proof. First we show that $u(\beta)$ is a classical solution of the modified equation (1.4) then we obtain a C^0 estimate for $w(\beta)$. This is followed by a C^0 on $v(\beta)$, and the existence of a C^0-solution u is proved. We then use the bootstrapping argument in [Rabinowitz78] to prove the existence of classical solutions. The multiplicity is deduced by noticing the lower estimates on the critical values $c_n^m(\delta)$ go to infinity as $n \to \infty$.

Functional I_β:

We define the functional I_β:

$$I_\beta(u) = \int_Q [\frac{1}{2}(u_t^2 - u_x^2 - \beta v_t^2) - \frac{1}{s+1}|u|^{s+1} - f(x,t)u]dxdt. \tag{1.10}$$

We seek time-periodic solutions satisfying Dirichlet boundary conditions so we seek functions $u \in \mathbb{R}$ with expansions of the form

$$u(x,t) = \sum_{(j,k)\in\mathbb{N}\times\mathbb{Z}} \widehat{u}(j,k) \sin jx e^{ikt}$$

and define the function space

$$||u||_{E^s} = \sum_{j\neq|k|} \frac{|Q|}{4}|k^2 - j^2|^s |\widehat{u}(j,k)|^2 + \sum_{j=\pm k} |\widehat{u}(j,k)|^2$$

where we denote by E the space E^s with $s = 1$. Define the functions spaces E^+, E^-, N as follows:

$$N = \{u \in E,\ \widehat{u}(j,k) = 0 \text{ for } j \neq |k|\}$$

$$E^+ = \{u \in E,\ \widehat{u}(j,k) = 0 \text{ for } |k| \leq j\}$$

$$E^- = \{u \in E,\ \widehat{u}(j,k) = 0 \text{ for } |k| \geq j\},$$

$w = w^+ + w^-$ where $w^+ \in E^+$,$w^- \in E^-$ and $v \in N$ and define the norm on $E \oplus N$

$$||u||^2_{\beta,E} = ||w^+||^2_E + ||w^-||^2_E + \beta||v_t||^2_{L^2}.$$

When u is trigonometric polynomial, I_β can also be represented as:

$$I_\beta(u) = \frac{1}{2}(||w^+||^2_E - ||w^-||^2_E - \beta||v_t||^2_{L^2}) - \frac{1}{s+1}||u||^{s+1}_{L^{s+1}} - \int_Q fudxdt. \tag{1.11}$$

The spectrum of the linear operator $\partial_t^2 - \partial_x^2$ under Dirichlet boundary conditions in space and time-periodicity consists of

$$-k^2 + j^2$$

where the eigenfunctions are the $\sin jx \cos kt, \sin jx \sin kt$. The eigenfunctions here are ordered as in [Tanaka88] i.e

$$... - \mu_3 \leq -\mu_2 \leq -\mu_1 < 0 < \mu_1 \leq \mu_2 \leq \mu_3 \leq ...$$

where the μ_l are the eigenvalues of $\partial_t^2 - \partial_x^2$ and have multiplicity one. Rearranging the eigenvalues this way is possible because all the non-zero eigenspaces of $\partial_t^2 - \partial_x^2$ have finite multiplicity. The $\mu_l \to +\infty$ as $l \to +\infty$ and denote by e_l the corresponding eigenfunctions, and we define the spaces

$$E^{+n} = span\{e_l, 1 \leq l \leq n\}.$$

For the Galerkin procedure we define the spaces

$$E^m = span\{\sin jx \cos kt, \sin jx \sin kt, \;\; j + k \leq m \;\; k \neq \pm j\},$$

$$E^{-m} = span\{\sin jx \cos kt, \sin jx \sin kt, \;\; j + k \leq m \;\; j < k\},$$

$$N^m = span\{\sin jx \cos jt, \sin jx \sin jt, \quad j \leq m\}$$

which are employed in the minimax procedure.

We start by following the procedure of [Rabinowitz82] for perturbation problems by proving some properties of the functional I_β. The difference here is that additionally we show that the constants involved in all the proof are independent of β to prepare for passing to the limit as $\beta \to 0$.

Lemma 1.1. *Suppose that u is a critical point of I_β. Then there is a constant a_6 depending on s, f but independent of β such that*

$$\int_Q |u|^{s+1} dxdt \leq a_6 \int_Q I_\beta^2(u) + 1 dxdt \tag{1.12}$$

$$\begin{aligned} I_\beta(u) &= I_\beta(u) - \frac{1}{2} I'(u)u \\ &= \frac{s}{s+1} \int_Q |u|^{s+1} dxdt - \frac{1}{2} \int_Q fudxdt. \end{aligned}$$

Now by applying Hausdorff-Young inequalities to $\int_Q fudxdt$ we deduce

$$I_\beta(u) \geq \frac{s}{s+1} \int_Q |u|^{s+1} dxdt - c_1(s) ||f||_{L^{\frac{s}{s+1}}}^{\frac{s}{s+1}} - \epsilon(s) ||u||_{L^{s+1}}^{s+1} \tag{1.13}$$

where $\epsilon(s) << 1$,$c_1(s)$ are both independent of β hence

$$I_\beta(u) \geq \frac{1}{2} \frac{s}{s+1} \int_Q |u|^{s+1} dxdt - c(f, s) \tag{1.14}$$

Lemma 1.2. If $u \in supp\psi$ then is a constant α_3 independent of β such that

$$|\int_Q fudxdt| \leq \alpha_3(\mathcal{I}_\beta^{\frac{1}{s+1}}(u)+1)$$

Proof:

$$|\int_Q fudxdt| \leq c(f,s)||u||_{L^{s+1}}$$

by Holder inequality, then if $u \in supp\psi$, then

$$\mathcal{I}_\beta(u)\int_Q |u|^{s+1}dxdt \leq 2$$

hence

$$c(f,s) \leq \alpha_3(I_\beta(u)+1)$$

and we conclude

$$|\int_Q fudxdt| \leq c(f,s)||u||_{L^{s+1}} \leq \alpha_3(I_\beta^{\frac{1}{s+1}}(u)+1)$$

We define the functional J_β which is amenable to minimax procedure. We start by defining a bump function χ. $\chi \in C^\infty(\mathbb{R},\mathbb{R})$:

$$\begin{cases} \chi(t)=1, \text{ if } t \leq 1 \\ \chi(t)=0 \text{ if } t > 2 \text{ .} \end{cases} \tag{1.15}$$

and $-2 < \chi' < 0$, for $1 < t < 2$. Then define

$$\mathcal{I}_\beta(u) = 2a_6(I^2(u)+1)$$

and

$$\psi(u) = \chi(\mathcal{I}_\beta^{-1}(u)\int_Q \frac{|u|^{s+1}}{s+1}dxdt)$$

$$J_\beta(u) = \int_Q [\frac{1}{2}(u_t^2 - u_x^2 - \beta v_t^2) - \frac{1}{s+1}|u|^{s+1} - \psi(u)f(x,t)u]dxdt, \tag{1.16}$$

which on $E^{+m} \oplus E^{-m} \oplus N^m$ can be rewritten as

$$J_\beta(u) = \frac{1}{2}(||w^+||_E^2 - ||w^-||_E^2 - \beta||v_t||_{L^2}^2) - \frac{1}{s+1}||u||_{L^{s+1}}^{s+1} - \int_Q \psi(u)fudxdt. \tag{1.17}$$

Lemma 1.3. There is a constant γ_1 depending on f, s but independent of β such that

$$|J_\beta(u) - J_\beta(-u)| \leq \gamma_1(|J_\beta(u)|^{\frac{s}{s+1}}+1) \tag{1.18}$$

Proof:

$$J_\beta(u) - J_\beta(-u) = -\psi(u)\int_Q fudxdt + \psi(-u)\int_Q fudxdt$$

and by the previous lemma 1.2 :

$$\psi(-u)\int_Q fudxdt \leq \alpha_3\psi(-u)\int_Q |I_\beta(u)|^{\frac{1}{s+1}} + 1dxdt$$

now

$$J_\beta(u) = I_\beta(u) + \int_Q fudxdt - \int_Q \psi(u)fudxdt$$

thus

$$|I_\beta(u)| \leq |J_\beta(u)| + 2|\int_Q fudxdt|$$

and

$$\psi(-u)|\int_Q fudxdt| \leq \alpha_3\psi(-u)(|J_\beta(u)^{\frac{1}{s+1}} + |\int_Q fudxdt|^{\frac{1}{s+1}} + 1)$$

and the lemma follows.

Lemma 1.4. There are constants $\alpha_0, M_0 > 0$ depending on f, s independent of β such that whenever $M \geq M_0$, then $J_\beta(u) \geq M$ and $u \in supp\psi$ then $I_\beta(u) \geq \alpha M_0$

Proof:

$$I_\beta(u) \geq J_\beta(u) - 2|\int_Q fudxdt| \tag{1.19}$$

while if $u \in supp\psi$ then

$$|I_\beta(u)|^{\frac{1}{s+1}} + 1 \geq \frac{1}{\alpha_1}|\int_Q fudxdt|$$

or

$$|I_\beta(u)|^{\frac{1}{s+1}} \geq \frac{1}{\alpha_1}|\int_Q fudxdt| - C \tag{1.20}$$

and adding (1.19) and (1.20)

$$I_\beta(u) + 2\alpha_1|I_\beta(u)|^{\frac{1}{s+1}} \geq J_\beta(u) - C \geq \frac{M}{2} \tag{1.21}$$

le facteur 2 doit multiplier $\alpha_1 I_\beta$?
for M_0 large enough. If $I_\beta(u) \leq 0$, then by Young inequality

$$\alpha_1|I_\beta(u)|^{\frac{1}{s+1}} \leq \frac{\alpha_1^{\frac{s+1}{s}}}{\frac{s+1}{s}} + \frac{1}{s+1}|I_\beta(u)|^{s+1} \tag{1.22}$$

while the inequality (1.21)

$$\alpha_1 |I_\beta(u)|^{\frac{1}{s+1}} \geq -I_\beta(u) + \frac{M}{2} \tag{1.23}$$

hence

$$\frac{\alpha_1^{\frac{s+1}{s}}}{\frac{s+1}{s}} + \frac{1}{s+1}|I_\beta(u)|^{s+1} \geq -I_\beta(u) + \frac{M}{2} = |I_\beta(u)| + \frac{M}{2} \tag{1.24}$$

thus there is $c(s) > 0$ such that

$$c(s)|I_\beta(u)| \leq -\frac{M}{4} < 0 \tag{1.25}$$

and we have a contradiction.

Lemma 1.5. *Lemma 1.29 [Rabinowitz82] In $E^{+m} \oplus E^{-m} \oplus N^m$,there is a constant $M_1 > 0$ independent of β, m such that $J_\beta(u) \geq M_1$ and $J'_\beta(u) = 0$ implies that $J_\beta(u) = I_\beta(u)$ and $I'_\beta(u) = 0$*

Proof:
We follow step by step the argument in [Rabinowitz82].
It suffices to show that

$$\mathcal{I}_\beta{}^{-1}(u) \int_Q \frac{1}{s+1}|u|^{s+1} dxdt \leq 1 \tag{1.26}$$

$$J'_\beta(u)u = \int_Q w_t^2 - w_x^2 - \beta v_t^2 - |u|^{s+1} dxdt - \psi(u) \int_Q fudxdt - \psi'(u)u \int_Q fudxdt \tag{1.27}$$

where

$$\begin{aligned} \psi'(u)u &= \chi'(\mathcal{I}_\beta{}^{-1}(u) \int_Q \frac{1}{s+1}|u|^{s+1} dxdt) \\ &\quad \times [-\mathcal{I}_\beta{}^{-3}(u) 2 I_\beta(u) I'_\beta(u)u \int_Q \frac{|u|^{s+1}}{s+1} dxdt + \mathcal{I}_\beta{}^{-1}(u) \int_Q |u|^{s+1} dxdt] \end{aligned}$$

and

$$J'_\beta(u) = (1+T_1(u)) \int_Q w_t^2 - w_x^2 - \beta v_t^2 dxdt - (1+T_2(u)) \int_Q |u|^{s+1} dxdt - (\psi(u)+T_1(u)) \int_Q fudxdt \tag{1.28}$$

where T_1, T_2 are exactly as in [Rabinowitz82]:

$$T_1(u) = \chi'(\mathcal{I}_\beta{}^{-1}(u) \int_Q \frac{1}{s+1}|u|^{s+1} + a_4 dxdt)(2a_6)^2 \mathcal{I}_\beta{}^{-3}(u) \int_Q \frac{|u|^{s+1}}{s+1} dxdt \int_Q fudxdt \tag{1.29}$$

and

$$T_2(u) = \chi'(\mathcal{I}_\beta{}^{-1}(u) \int_Q \frac{1}{s+1}|u|^{s+1} dxdt) \mathcal{I}_\beta{}^{-1}(u) \int_Q fudxdt + T_1(u) \tag{1.30}$$

and the conclusion follows just as in [Rabinowitz82].

We now show that the functional J_β satisfies the Palais-Smale condition at large energies in $E^{+m} \oplus E^{-m} \oplus N^m$:

Lemma 1.6. There is a constant M_2 independent of β such that the Palais-Smale condition is satisfied on $A_{M_2} = \{u \in E^{+m} \oplus E^{-m} \oplus N^m, \quad J_\beta(u) \geq M_2\}$

Proof:
Let $u_l = w_l + v_l = w_l^+ + w_l^- + v_l$ a Palais-Smale sequence at large energies, there are M_2, K independent of β, m such that $M_2 \leq J_\beta(u_l) \leq K$ and $J'_\beta(u_l) \to 0$

$$\begin{aligned} J_\beta(u_l) - J'_\beta(u_l)(u_l) &= (\frac{1}{2} - \rho(1 + T_1(u_l))) \int_Q w_{lt}^2 - w_{lx}^2 - \beta v_{lt}^2 dxdt \\ &+ [\rho(1 + T_2(u_l) - \frac{1}{s+1})] \int_Q |u_l|^{s+1} dxdt \\ &(\rho(\psi(u_l) + T_1(u_l)) - \psi(u_l)) \int_Q f u_l dxdt \qquad (1.31) \end{aligned}$$

now we choose $\rho = \frac{1}{2(1+T_1(u_l))}$ then we have

$$\rho \to \frac{1}{2} \text{ independently of } \beta \text{ as } M_2 \to +\infty$$

$$\begin{aligned} J_\beta(u_l) - J'_\beta(u_l)(u_l) &= [\rho(1 + T_2(u_l) - \frac{1}{s+1})] \int_Q |u_l|^{s+1} dxdt \\ (\rho(\psi(u_l) + T_1(u_l)) - \psi(u_l)) \int_Q f u_l dxdt &\geq [\rho(1 + T_2(u_l) - \frac{1}{s+1}) - \frac{\epsilon(s)}{s+1}] \int_Q |u_l|^{s+1} dxdt - c(f, s) \end{aligned}$$

where $\epsilon(s)$ can be chosen to be a small positive constant by applying Young inequality, and $c(f,s)$ is another constant depending on f, s, both being independent of β. Now recall that $J'_\beta(u_l) \to 0$ and $\rho \to \frac{1}{2}$

$$J_\beta(u_l) - J'_\beta(u_l) \leq K + \rho ||u_l||_{E,\beta} \qquad (1.32)$$

so we have the inequalities:

$$K + \rho ||u_l||_E \geq J_\beta(u_l) - J'(u_l) u_l \geq c_3(f, s) - c_2(f, s) \qquad (1.33)$$

thus

$$\int_Q |u_l|^{s+1} dxdt \leq c_4(f, s) ||u_l||_{E,\beta} + K + c_2(f, s). \qquad (1.34)$$

Now

$$J'_\beta(u_l) v_l = (1 + T_1(u_l)) \int_Q \beta v_{lt}^2 dxdt - (1 + T_2(u_l)) \int_Q |u_l|^{s-1} u_l v_l dxdt - (\psi(u_l) + T_1(u_l)) \int_Q f v_l dxdt. \qquad (1.35)$$

u_l is a Palais-Smale sequence so there exists ϵ small such that
$J'_\beta(u_l)v_l \leq \epsilon||v_l||_{\beta,E}$ thus

$$(1+T_1(u_l))\beta||v_{lt}||^2_{L^2} \leq (1+T_2(u_l))\int_Q |u_l|^{s-1}u_lv_ldxdt+(\psi(u_l)+T_1(u_l))\int_Q fv_ldxdt+\epsilon||v_l||_{\beta,E}.$$

Now for M_2 large enough (independently of β) and we have

$$\frac{1}{2}\beta||v_{lt}||^2_{L^2} \leq (2\int_Q |u_l|^s|v_l|dxdt + 2\int_Q |f||v_l|dxdt + \epsilon||v_l||_{\beta,E} \tag{1.36}$$

and applying Hölder inequality we deduce:

$$\frac{\beta}{2}||v_{lt}||^2_{L^2} \leq c||u_l||^s_{L^{s+1}}||v_l||_{L^{s+1}} + 2||v_l||_{L^{s+1}}||f||_{L^{\frac{s+1}{s}}} + \epsilon||v_{lt}||_{L^2}.$$

A similar computation gives

$$||w_l^+||^2_{E,\beta} \leq c||u_l||^s_{L^{s+1}}||w_l^+||_{L^{s+1}} + 2||w_l^+||_{L^{s+1}}||f||_{L^{\frac{s+1}{s}}} + \epsilon||w_l^+||_E. \tag{1.37}$$

We now estimate $||v_l||_{L^{s+1}}$:$v_l = u_l - w_l^+ - w_l^-$ hence

$$\begin{aligned} ||v_l||_{L^{s+1}} &\leq ||u_l||_{L^{s+1}} + ||w_l^+||_{L^{s+1}} + ||w_l^-||_{L^{s+1}} \\ &\leq c||u_l||^{\frac{1}{s+1}}_{E,\beta} + c||w_l^-||_E + c||w_l^+||_E \qquad (1.38) \\ &\leq c||u_l||_{E,\beta} + D(f,s) \qquad (1.39) \end{aligned}$$

where the constants $c, D(f,s)$ are independent of β and (1.38) follows from (1.34) and the Sobolev inequality $||w_l||_{L^p} \leq c(p)||w_l||_E$ We can now deduce:

$$\begin{aligned} ||u_l||^2_{E,\beta} &\leq c(1+||u_l||^2_{L^{s+1}})(||v_l||_{L^{s+1}} + ||w_l^+||_{L^{s+1}}) + ||w_l^-||_{L^{s+1}}) + c||u_l||_{E,\beta} \\ &\leq c(1+c||u_l||^{\frac{1}{s+1}}_{E,\beta})(3c||u_l||_{E,\beta} + D(f,s)) + c||u_l||_{E,\beta} \qquad (1.40) \end{aligned}$$

so $||u_l||_{E,\beta} < +\infty$ and Palais-Smale is satisfied.

2 Minimax set-up

B_R the closed ball of radius R:

$$B_R = \{u \in E \oplus N \ ||u||_{E,\beta} \leq R\}$$

The ε-neighborhood of S in a space $W \subset E \oplus N$:

$$B_R(W,S,\varepsilon) = \{x \in W, ||x-y|| \leq \varepsilon \ \ y \in S\}.$$

$$D_n^m = \{u \in E^{+n} \oplus E^{-m} \oplus N^m \text{ and } ||u||_{E,\beta} \leq R_n\}$$

$$\Gamma_n^m = \{h : D_n^m \to E^{+m} \oplus E^{-m} \oplus N^m, \ h \,\text{odd}\ , h(x) = x, \,\text{for}\, x \in B(D_n^m, \partial D_n^m, \varepsilon(h)) \,\text{for some}\, \varepsilon(\text{h}) > 0\}$$

$$b_n^m = \inf_{h\in\Gamma_n^m} \max_{u\in D_n^m} J_\beta(h(u))$$

$$U_n^m = \{u_{n+1} = te_{n+1} + u_n,\ t \in [0, R_{n+1}], u_n \in B_{R_{n+1}} \cap (E^{+n} \oplus E^{-m} \oplus N^m), ||u_{n+1}||_{E,\beta} \le R_{n+1}\}$$

$$\Lambda_n^m = \left\{ \begin{array}{l} H \in C(U_n^m, E^{+m} \oplus E^{-m} \oplus N^m),\ \text{and}\ H(u) = u \\ \text{if} ||u||_{E,\beta} \ge R_{n+1} - \varepsilon(H)\,\text{for some}\,\varepsilon(H) > 0,\ \text{or if} \\ u \in B(U_n^m, (B_{R_{n+1}} \setminus B_{R_n}) \cap (E^{+n} \oplus E^{-m} \oplus N^m), \varepsilon(H)) \end{array} \right\}$$

where the constants R_n does not depend on β.

$$\Lambda_n^m(\delta) = \{H \in \Lambda_n^m,\ J_\beta(H(u)) \le b_n^m + \delta \text{ on } B(U_n^m, D_n^m, \varepsilon(H)), \text{ for some } \varepsilon(H) > 0\}$$

$$c_n^m = \inf_{H\in\Lambda_n^m} \max_{u\in U_n^m} J_\beta(H(u))$$

and

$$c_n^m(\delta) = \inf_{H\in\Lambda_n^m(\delta)} \max_{u\in U_n^m} J_\beta(H(u))$$

Our sets Γ_n^m, Λ_n^m differ from those defined by Rabinowitz in [Rabinowitz82] or Tanaka in [Tanaka88] in that we require that $H = Id$ not just on $(B_{R_{n+1}} \setminus B_{R_n}) \cap (E^{+n} \oplus E^{-m} \oplus N^m)$ but also in a small neighborhood in U_n^m of that set. This will allow for the construction of a bump functions χ_1, whose support is in $B(U_n^m, \partial U_n^m, \frac{\varepsilon}{2})$, for some $\epsilon > 0$. Now given an extension $H \in \Lambda_n^m$, of a map $h \in \Gamma_n^m$, we define another extension H_1 using the bump function and we will get an upper estimate of $J_\beta(H_1(u,t))$ independently of β. This will lead to an upper estimate of $c_n^m(\delta)$ explicit in n and independently of β, m.

Since we require some additional conditions on the maps in our Γ_n^m, Λ_n^m the b_n^m, c_n^m we define here are greater than or equal than the corresponding ones in [Rabinowitz82],[Tanaka88].

This approach can be adapted to obtain new estimates even for a class of semilinear elliptic equations considered by Tanaka [Tanaka89].

Lemma 2.1. *$\forall u \in D_n^m \cap E^{+n}$, there is a constant $C(n)$ independent of β, m such that*

$$J_\beta(u) \le C(n) \tag{2.41}$$

Proof:

Let $u \in E^{+n}$

$$\begin{aligned} J_\beta(u) &= \frac{1}{2}||w^+||_E^2 - \frac{1}{2}||w^-||_E^2 - \beta||v_t||_{L^2}^2 - \int_Q \frac{|u|^{s+1}}{s+1} dxdt - \psi(u) \int_Q fudxdt \\ &\le \frac{1}{2}||w^+||_E^2 - \frac{1}{2}||w^-||_E^2 - \beta||v_t||_{L^2}^2 - \frac{1}{2}\int_Q \frac{|u|^{s+1}}{s+1} dxdt + c(f,s) \quad (2.42) \\ &\le c(f,s) + \sup_{u\in E^{+n}} \frac{1}{2}||w^+||_E^2 - \frac{1}{2}\int_Q \frac{|u|^{s+1}}{s+1} dxdt \\ &\le c(f,s) + \sup_{u\in E^{+n}} \frac{1}{2}||w^+||_E^2 - c(s,Q)||u||_{L^2}^{s+1} \quad (2.43) \end{aligned}$$

Now in E^{+n}

$$||u||_E^2 \le \mu_n ||u||_{L^2}^2 \tag{2.44}$$

and on the other-hand

$$\sup_{u \in E^{+n}} \frac{1}{2}||w^+||_E^2 - c(s,Q)||u||_{L^2}^{s+1} > 0 \tag{2.45}$$

and is attained at say $\overline{u}$ hence we have

$$c(s,Q)||\overline{u}||_{L^2}^{s+1} \le \frac{1}{2}||\overline{u}||_E^2 \le \frac{1}{2}\mu_n||\overline{u}||_{L^2}^2 \tag{2.46}$$

and we can conclude there is $C(n)$ depending on n but independent of β such that

$$J_\beta(u) \le C(n) \tag{2.47}$$

Now lemma 1.57 in [Rabinowitz82]:

Lemma 2.2. *Suppose that* $c_n^m > b_n^m > M$. *Let* $0 < \delta < c_n^m - b_n^m$, *then* $c_n^m(\delta)$ *is a critical value of* J_β.

Note that in our case the sets $\Lambda_n^m(\delta)$ are more restrictive than the corresponding ones in [Rabinowitz82] and we have first to show they are nonempty. This will be done in the next lemma. An upper estimates on $c_n^m(\delta)$ will also be obtained independently of β which is our main contribution to obtain the needed compactness.

Lemma 2.3. $\Lambda_n^m(\delta) \ne \emptyset$, *and there is a map* $\chi_1 \in \Lambda_n^m(\delta)$ *such that*

$$J_\beta(\chi_1) \le C(n+1) \tag{2.48}$$

where $C(n+1)$ *is independent of* β, m

Proof:
Given $h \in \Gamma_n^m$ a minimizing map for b_n^m we assume without loss of generality that

$$J_\beta(h(u)) \le b_n^m + \frac{\delta}{2} \tag{2.49}$$

we construct and extension $H_1 \in \Lambda_n^m(\delta)$ such that $J_\beta(H_1)$ is bounded independently of β, m. Wlog we will assume that $R_{n+1} > 2R_n$.
Let $1 + R_n < R < \frac{R_{n+1}}{\sqrt{2}}$, and $v = u + te_{n+1}$, $u \in D_n^m$, writing v as (u,t) we define H :

$$H(u,t) = (1 - \frac{t}{R})h(u) + (\frac{t}{R}u, t) \tag{2.50}$$

for $t \le R$. If $||u||_{E,\beta} = R$ or $t = R$ $H(u,t) = Id$. By extending H as Id for the remaining values of $(u,t) \in U_n^m$ we obtain an $H \in \Lambda_n^m(\delta)$. We now construct an extension H_1 for which we can control $J_\beta(H_1)$.
By (2.47) and the uniform continuity of $J_\beta oH$ there is $\epsilon(\beta) > 0$ such that

$$J_\beta(H(u,t)) \le b_n^m + \delta \tag{2.51}$$

in $(u,t) \in B(U_n^m, D_n^m, \epsilon(\beta))$. Now since $H = Id$ on $\partial U_n^m \setminus D_n^m$ we also have

$$J_\beta(H(u)) \leq C(n+1) \tag{2.52}$$

where the constant $C(n+1)$ is independent of β, m, in $B(U_n^m, \partial U_n^m \setminus D_n^m, \epsilon')$ for some $\epsilon' > 0$ and the argument follows as in lemma 2.1.
Now $B(U_n^m, D_n^m, \varepsilon(\beta)))$ is convex so if $u \in B(U_n^m, D_n^m, \varepsilon(\beta))$ then $\lambda u \in B(U_n^m, D_n^m, \varepsilon(\beta))$ for $0 \leq \lambda \leq 1$.
Let χ_1 a smooth bump function $0 \leq \chi_1(u,t) \leq 1$, supported in $B(U_n^m, \partial U_n^m, \epsilon(\beta))$, such that $\chi_1(u,t) = 1$ in a smaller $\varepsilon'' < \min(\epsilon(\beta), \epsilon')$ neighborhood of ∂U_n^m: $B(U_n^m, \partial U_n^m, \epsilon'')$ and

$$J_\beta(H(\chi_1(u,t)(u,t))) \leq b_n^m + \delta \tag{2.53}$$

in $B(U_n^m, D_n^m, \epsilon(\beta))$ as $\chi_1(u,t)(u,t) \in B(U_n^m, D_n^m, \epsilon(\beta))$ because of the convexity of $B(U_n^m, D_n^m, \epsilon(\beta))$. We define :

$$H_1(u,t) = H(\chi_1(u,t)(u,t)) \tag{2.54}$$

then $H_1 \in \Lambda_n^m(\delta)$. To estimate $J_\beta(H_1(u,t))$ we simply note that in $B(U_n^m, \partial U_n^m, \min(\epsilon', \epsilon(\beta)))$, is already bounded independently of β, m and that since χ_1 is supported in $B(U_n^m, \partial U_n^m, \min(\epsilon', \epsilon(\beta)))$, hence $J_\beta(H_1)$ is by a constant $C(n+1)$ independently of β, m in all of U_n^m.

Lemma 2.4. *There is a constant R_n such that for all $u \in E^{+n} \oplus E^{-m} \oplus N^m$ and $||u|| \geq R_n$*

$$J_\beta(u) \leq 0 \tag{2.55}$$

The proof is done by a standard argument. See for instance Proposition 2.37 in [Rabinowitz84] for a proof.

The proof that $c_n^m(\delta)$ is a critical value follows as the lemma 1.57 in [Rabinowitz82] step by step. We do not repeat it here. The a priori estimate is provided by the map H_1.

We recall the comparison functional K from lemma 2.2 in [Tanaka88]:

$$K(w^+) = \frac{1}{2}||w^+||_E - \frac{a_0(s)}{s+1}||w^+||_{L^{s+1}}^{s+1},$$

which satisfies the Palais-Smale condition. The functional K also satisfies the comparison property :

$$J_\beta(w^+) \geq K(w^+) - a_1(f,s)$$

for any $w^+ \in E^+$, $a_1(f,s)$ is a positive constant. We define the minimax sets:

$$A_n^m = \{\sigma \in C(S^{m-n}, E^{+m}), \sigma(-x) = \sigma(x)\}$$

where $S^{m-n} \subset E^{+m}$ is the unit sphere in $\mathbb{R}^{m-n+1}$, whose basis consists of eigenvectors $\{e_n, ..., e_m\}$. $x \in S^{m-n}$ if and only if

$$x = \sum_{i=n}^{m} x_i e_i \text{ and } \sum_{i=n}^{m} x_i^2 = 1 \tag{2.56}$$

and the minimax values

$$\beta_n^m = \sup_{\sigma \in A_n^m} \min_{x \in S^{m-n}} K(\sigma(x))$$

Properties of the minimax numbers β_n^m from [Tanaka88]: There exists sequences $\nu(n), \widetilde{\nu(n)}$

$$\nu(n) \leq \beta_n^m \leq \widetilde{\nu(n)} \tag{2.57}$$

such that $\nu(n), \widetilde{\nu(n)} \to \infty$ as $n \to \infty$(independently of m). the existence of the b_n^m must be done. They are finite and only the "sharp" lower bound established via Morse theory will be important however it seems natural to prove their existence before proving the preceding inequality.
Borsuk-Ulam type theorem:

Lemma 2.5. *[Tanaka88]Let $a, b \in \mathbb{N}$. Suppose that $h \in C(S^a, \mathbb{R}^{a+b})$, and $g \in C(\mathbb{R}^b, \mathbb{R}^{a+b})$ are continuous mappings such that*

$$h(x) = h(-x) \text{ for all x} \in \mathrm{S^a} \tag{2.58}$$

$$g(-y) = -g(y) \text{ for all y} \in \mathbb{R}^\mathrm{b} \tag{2.59}$$

and there is a r_0 such that $g(y) = y$ for all $r \geq r_0$. Then $h(S^a) \cap g(\mathbb{R}^b) \neq \emptyset$

Lemma 2.6. *[Tanaka88]Let $\gamma \in \Gamma_n^m$ and $\sigma \in A_n^m$, then*

$$\gamma(D_n^m) \cup \{u \in E^{+n} \oplus E^{-m} \oplus N^{-m}, ||u||_{\beta,E} \geq R_n\} \cap \sigma(S^{m-n}) \neq \emptyset \tag{2.60}$$

Proof: Apply the lemma above with $a = m - n$ and $b = dimension(E^{+n} \oplus E^{-m} \oplus N^{-m})$. Then extend γ to all of $E^{+n} \oplus E^{-m} \oplus N^{-m}$ by extending it by the identity map on ∂D_n^m and view $\sigma(S^{m-n})$ as embedded in $E^{+m} \oplus E^{-m} \oplus N^{-m}$, then apply the preceding lemma 2.5.

Lemma 2.7. *$\forall n \in \mathbb{N}$,*

$$b_n^m \geq \beta_n^m - a_1 \tag{2.61}$$

where a_1 is independent of n, m, β.

Proof:
Let $\sigma \in A_n^m$ and $\gamma \in \Gamma_n^m$. Then

$$\min_{x \in S^{m-n}} K(\sigma(x)) - a_1 < \min_{x \in S^{m-n}} J_\beta(\sigma(x)) \leq \sup_{u \in U_n^m} J_\beta(\gamma(u)) \tag{2.62}$$

as there exists x, u such that $\sigma(x) = \gamma(u)$. Then we can conclude that

$$\beta_n^m - a_1 \leq b_n^m \tag{2.63}$$

Note also that since $J_\beta(0) = 0$, $J_\beta \mid_{\partial U_n^m} \leq 0$ and tends to $-\infty$ uniformly as $R_n \to +\infty$,then

$$\sup_{u \in E^{+n} \oplus E^{-m} \oplus N^{-m}} J_\beta(\gamma(u)) = \sup_{u \in U_n^m} J_\beta(\gamma(u)) \tag{2.64}$$

Lemma 2.8. *(Proposition 4.1[Tanaka88])Suppose that $\beta_n^m < \beta_n^{m+1}$, $m > n+1$, then there exists a $u_n^m \in E^{+m}$ such that*

$$K(u_n^m) \leq \beta_n^m \tag{2.65}$$

$$K' \mid_{E^{+m}} (u_n^m) = 0 \tag{2.66}$$

$$index K'' \mid_{E^{+m}} (u_n^m) \geq n \tag{2.67}$$

Lemma 2.9. *(Proposition 5.1[Tanaka88]) For any $\varepsilon > 0$, there is a constant $C_\varepsilon > 0$, such that for $u \in E^+$*

$$index K''(u) \geq C_\varepsilon ||u||_{L^{(s-1)(1+\varepsilon)}}^{(s-1)(1+\varepsilon)} \tag{2.68}$$

Theorem 2.1. *There is a subsequence n_q and c independent of β, m, n such that*

$$b_{n_q} \geq c n_q^{\frac{s+1}{s}} \tag{2.69}$$

Proof:
The inequality (2.57) implies that there is a subsequence n_q such that $\beta_{n_q+1} > \beta_{n_q}$.

$$\begin{aligned} \beta_{n_q} &\geq K(u_{n_q}^m) - \frac{1}{2} K'(u_{n_q}^m) u_{n_q}^m \\ &\geq (\frac{1}{2} - \frac{1}{s+1}) a_0(s) ||u_{n_q}^m||_{s+1}^{s+1}. \end{aligned} \tag{2.70}$$

Then for $\varepsilon > 0$ small enough

$$\begin{aligned} ||u_{n_q}^m||_{s+1}^{s+1} &\geq c_\varepsilon ||u_{n_q}^m||_{s(1+\varepsilon)}^{s+1} \\ &\geq c_\varepsilon {n_q}^{\frac{s+1}{s(1+\varepsilon)}} \end{aligned} \tag{2.71}$$

by combining (2.68) and (2.67). Now recalling lemma 2.7 and that for ε small enough, $\frac{s+1}{s(1+\varepsilon)} > \frac{s+1}{s}$ the lemma follows.
To conclude we recall lemma 1.64 in [Rabinowitz82] which in our case implies that, for m large enough independently of β, if $c_n^m = b_n^m$ for all $n \geq n_1$ then $b_n \leq c n^{\frac{s+1}{s}}$. Then by lemma 2.2, $c_{n_q}^m(\delta)$ is a critical value of I_β in $E^{+m} \oplus E^{-m} \oplus N^m$.

3 Regularity

Theorem 3.1. Let f *be* C^2, for n large enough there is a classical solution $u = v + w$ of the modified problem *(1.4)* .

Proof:
In this proof the constants may dependent on β and f but are independent of m. The proof of this theorem here is slightly simpler from the one in [Rabinowitz84] as we take advantage of the polynomial growth of the nonlinear term and employ Galerkin approximation.
Let $u^m_{n_q} = w^m + v^m \in E^{+m} \oplus E^{-m} \oplus N^m$ a distributional solution corresponding to the critical value $c^m_{n_q}(\delta)$, and any $\phi \in E^{+m} \oplus E^{-m} \oplus N^m$:

$$I'(u^m_{n_q})\phi = 0 \tag{3.72}$$

now taking $\phi = v^m_{tt} \in N^m$ we have

$$(\beta v^m_{tt}, v^m_{tt})_{L^2} = (|u^m_{n_q}|^{s-1} u^m_{n_q} + f, v^m_{tt})_{L^2}$$

$$\beta ||v^m_{tt}||^2_{L^2} \leq ||u^2||_{L^2} ||v^m_{tt}||_{L^2} + ||f||_{L^2} ||v^m_{tt}||_{L^2}$$

$$\beta ||v^m_{tt}||_{L^2} \leq c||v^m_{tt}||_{L^2}$$

hence

$$||v^m_{tt}||_{L^2} \leq c(\beta, f)$$

we now have

$$w^m_{tt} - w^m_{xx} = \beta v^m_{tt} + |u^m_n|^{s-1} u^m + f^m(x,t) \in L^2$$

hence $w^m \in H^1 \cap C^1$ by [Rabinowitz67] and [BCN80]. This now implies $w^m \in H^2$, $w^m \to w(\beta)$ pointwise and $w(\beta) \in H^1 \cap C^1$. Then if $\phi = v^m_{tttt}$ then

$$(\beta v^m_{tt}, v^m_{tttt})_{L^2} = (|u^m_{n_q}|^{s-1} u^m_{n_q} + f, v^m_{tttt})_{L^2}$$

(here I need $f \in H^1$)

$$(\beta v^m_{ttt}, v^m_{ttt})_{L^2} = ([|u^m_{n_q}|^{s-1} u^m_{n_q} + f]_t, v^m_{ttt})_{L^2}$$

and we deduce $||v^m_{ttt}||_{L^2} \leq c(\beta, f)$ hence $v^m_{ttt} \to v_{tt}(\beta) \in C^0$ hence $v(\beta)$ is C^2 and $w(\beta)$ is C^1 by applying [BCN80] to (1.4) . We now have

$$u^m_{n_q} \to u(\beta) \in C^1 \text{ as } m \to \infty$$

and since (3.72) holds for any $\phi \in E^{+m} \oplus E^{-m} \oplus N^m$ we can deduce

$$I'(u(\beta))\phi = 0 \ \ \forall \phi \in E \oplus N, \tag{3.73}$$

and $u(\beta)$ is a weak solution of (1.4). Now for any $\phi \in C^\infty \cap L^2(S^1)$ we have

$$\begin{aligned} I'(u(\beta))[\phi(x+t) - \phi(x-t)] &= \int_Q [-\beta(p''(x+t) - p''(-x+t) + |u(\beta)|^{s-1}u(\beta)) + f(x,t)] \\ &\quad [\phi(x+t) - \phi(-x+t)]dxdt \end{aligned}$$

remarque: avoir $u \in C^1$ aide a definir the produit scalaire dans $E \oplus N, E, \beta$ pour definir les solutions faibles.
Denoting $\psi(x,t) := [-\beta(p''(x+t) + |u(\beta)|^{s-1}u(x,t) + f(x,t)]$ and noting that the functions ψ, ϕ are periodic we deduce as in [Rabinowitz78] that

$$\int_0^{2\pi}\int_0^{\pi} \psi(x,t)\phi(x+t)dxdt = \int_0^{\pi}\int_0^{2\pi} \psi(r, r-x)\phi(r)dxdr$$

and

$$\int_0^{\pi}\int_0^{2\pi} \psi(x,t)\phi(-x+t)dxdt = \int_0^{\pi}\int_0^{2\pi} \psi(x, r+x)\phi(r)dxdr$$

for all $\phi \in C^\infty \cap L^2(S^1)$ hence

$$\int_0^{\pi} \psi(x, r+x) - \psi(x, r-x)dxdr = 0$$

and we have

$$\pi p''(r) = \int_0^{\pi} (|u(\beta)|^{s-1}u(\beta)(x, r-x) - |u(\beta)|^{s-1}u(\beta)(x, r+x)) + f(x, r-x) - f(x, r+x)dx \tag{3.74}$$

so p is C^3 since $u(\beta) \in C^1$. Since RHS of (1.4) is C^1 then by [BCN80] $w \in C^2$ and $u(\beta)$ is a classical solution of (1.4).

Lemma 3.1. *There is a constant c independent of β, m such that*

$$||w(\beta)||_{C^0} \leq c \tag{3.75}$$

Proof:
By (1.12) there is a constant c independent of β, m such that $||u(\beta)||_{L^{s+1}} \leq c$. Then by (3.74) $||\beta v_{tt}||_{L^1}$ is bounded independently of β, m, hence by Lovicarova's formula [Lovicarova69] we conclude that there is a constant c

$$||w(\beta)||_{C^0} \leq c \tag{3.76}$$

which is independent of β, m.

Lemma 3.2. *There is a constant c, independent of β such that*

$$||v(\beta)||_{C^0} \leq c. \tag{3.77}$$

Proof:
$\forall \phi \in N$,

$$\int_0^{\pi}\int_0^{2\pi} (-\beta v_{tt}(\beta) + (g(u(\beta)) + f(x,t))\phi dxdt = 0$$

$$\int_0^{\pi}\int_0^{2\pi} \beta v_t(\beta)\phi_t + (g(v(\beta) + w(\beta)) - g(w))\phi dxdt = -\int_0^{\pi}\int_0^{\pi} (f(x,t)) + g(w))\phi dxdt \tag{3.78}$$

Define q:

$$\begin{cases} q(s) = 0, \text{ if } |\text{s}| \leq \text{M}. \\ q(s) = s + M \text{ if s} \geq \text{M and q(s) = s} - \text{M if s} \leq \text{M}. \end{cases} \tag{3.79}$$

Now define the function $\psi_K(z)$:

$$\begin{cases} \psi_K(z) = \max_{|\xi| \leq M_5} f_K(z+\xi) - f_K(\xi) \text{ if z} > 0. \\ \psi_K(z) = -\min_{|\xi| \leq M_5}(f_K(\xi) - f_K(z+\xi)) \text{ if z} < 0 \end{cases} \tag{3.80}$$

ψ_K is monotonically increasing and $\lim_{z \to \pm\infty} \psi_K(z) = \pm\infty$. For $z \geq 0$, $\mu(z) = \min(\psi(z), \psi(-z))$. Define

$$T_\delta = \{(x,t) \in [0,\pi] \times [0, 2\pi] \; |v(\beta)| \geq \delta\}.$$

By taking the test function $\phi = q(v^+) - q(v^-) = v^+ - v^-$ and noting that g is strictly increasing we have the estimate following lemma 3.7 in [Rabinowitz78]:

$$\int_{T_\delta} (g(v+w) - g(v))(q^+ - q^-) dxdt \geq \frac{M - \delta}{||v||_{C^0}} \mu(\delta) \int_{T_\delta} (|q^+| + |q^-|) dxdt \tag{3.81}$$

hence:

$$(||g(w)||_{C^0} + ||f||_{C^0}) \int_T |q^+| + |q^-| dxdt \geq \frac{M - \delta}{||v||_{C^0}} \mu(\delta) \int_{T_\delta} (|q^+| + |q^-|) dxdt. \tag{3.82}$$

Denoting $\max(||v^+||_{C^0}, ||v^-||_{C^0}) = ||v^\pm||_{C^0}$ we have

$$\mu(\frac{1}{2}||v^\pm||_{C^0}) \leq 4(||f||_{C^0} + ||g(w)||_{C^0}) \tag{3.83}$$

and we can conclude that there is a constant c independent of β such that

$$||v(\beta)||_{C^0} \leq c. \tag{3.84}$$

Lemma 3.3. *The family $v(\beta)$ is equicontinuous.*

Proof: $u = v + w$. Define $\widehat{v}(x,t) = v(x, t+h)$,$\widehat{w}(x,t) = w(x, t+h)$ and $\widehat{u} = \widehat{v} + \widehat{w}$,$\widehat{f} = f(x, t+h)$,$U = V + W$, where $V = \widehat{v} - v$,$W = \widehat{w} - w$, $q(V^+) = Q^+$,$q(V^-) = Q^-$

$$\int_T \beta V_t \phi_t dxdt + \int_T g(\widehat{v}+w) - g(u) dxdt = -\int_T g(\widehat{u}) - g(\widehat{v}+w) + \widehat{f} - f dxdt \tag{3.85}$$

For $\phi = q(V^+) - q(V^-)$ and $V^+ = \widehat{v^+} - v^+$, we have

$$\int_T [g(V+u) - g(u) + \widehat{f} - f][Q^+ - Q^-] dxdt \leq (||f(\widehat{u}) - f(\widehat{v}+w)||_{C^0} + ||\widehat{f} - f||_{C^0}) \int_T (|Q^+| + |Q^-|) dxdt \tag{3.86}$$

and

$$\int_T [g(V+u)-g(u)][Q^+-Q^-]dxdt \geq \frac{\mu(\delta)(M-\delta)}{||V||_{C^0}} \int_T [|Q^+|+|Q^-|]dxdt. \quad (3.87)$$

Since $w(\beta) \in C^1$ and $f \in C^1$ we deduce

$$||f(\widehat{u}) - f(\widehat{v}+w)||_{C^0} + ||\widehat{f} - f||_{C^0}) \leq c|h| \quad (3.88)$$

where c is independent of β, thus

$$\mu(\frac{1}{2}||V^{\pm}||_{C^0}) \leq c|h| \quad (3.89)$$

and the modulus of continuity of $v(\beta)$ is independent of β.

Theorem 3.2. *The problem (1.1),(1.2) has an infinite number of weak solutions $u = w + v$ where $w \in C^1$ and $v \in C^0$.*

Proof:
$||\beta v_{tt}||_{L^1} \to 0$ as $\beta \to 0$: Recalling the interpolation inequalities [Rabinowitz78],[Nirenberg59] and (3.74):

$$\beta||v_{tt}||_{L^1} \leq \beta||v_{tt}||^{\frac{1}{2}}_{C^0}||v(\beta)||^{\frac{1}{2}}_{C^0} \to 0 \quad (3.90)$$

and Lovicarova fundamental solution in [Lovicarova69] implies that $w \in C^1$.
Case 1:
If $\exists \overline{r}$ such that $u(x, \overline{r}-x) = \alpha$ for $\forall x \in [0,\pi]$ then the boundary conditions imply $\alpha = 0$ and $p(\overline{r}-2x) = p(\overline{r}) + w(x, \overline{r}-x)$, thus

$$||v||_{C^1} \leq ||w||_{C^1}. \quad (3.91)$$

Case 2:
There is no $\overline{r}$ such that $u(x, \overline{r}-x) = 0$, then there is $\gamma > 0$ such that $\int_0^\pi s|u|^{s-1}(x, r-x)dx > \gamma$, $\forall r \in [0, 2\pi]$. Now since $u(\beta) \to$ as $\beta \to 0$ we have

$$\int_0^\pi s|u|^{s-1}(\beta)(x, r-x)dx > \frac{\gamma}{2} \quad (3.92)$$

Differentiating (3.74) with refer to r and using the boundary conditions for u as in [Rabinowitz78] we obtain:

$$\begin{aligned} -\pi\beta p'''(r) + a(r)p'(r) \quad = \quad & \int_0^\pi s|u|^{s-1}(x, r-x)[-\frac{1}{2}w_x(x, r-x) - w_r(x, r-x)] + \\ & s|u|^{s-1}(x, r+x)[-\frac{1}{2}w_x(x, r+x) + w_r(x, r+x)] + \\ & f_r(x, r+x) - f_r(x, r-x)dx, \quad (3.93) \end{aligned}$$

where $a(r) = \int_0^\pi s|u|^{s-1}(\beta)(x, r-x) + s|u|^{s-1}(\beta)(x, r+x)dx$. Now by writing $\phi(r) = p'(r)$ we have:

$$-\pi\beta\phi''(r) + a(r)\phi(r) = h(r) \quad (3.94)$$

where $h \in C^0(S^1)$ and since $f \in C^1$ we deduce as in [Rabinowitz78] that $\lim_{\beta\to 0}\phi(\beta)$ exists and is in $H^1(S^1)$. Denoting this limit by $\phi(0)$ we deduce that $v \in C^1$. This implies $w \in C^2$ and $h \in C^1$, as $f \in C^2$. Now (3.94) is valid a.e at $\beta = 0$ which implies $\phi \in C^1$ and $u \in C^2$ is a classical solution of (1.1),(1.2).

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