Acta Univ. Sapientiae, Mathematica, **2**, 2 (2010) 160–167

Generalized GCD matrices

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Abstract. Let f be an arithmetical function. The matrix $[f(i,j)]_{n\times n}$ given by the value of f in greatest common divisor of (i,j), $f\big((i,j)\big)$ as its i, j entry is called the greatest common divisor (GCD) matrix. We consider the generalization of this matrix where the elements are in the form $f\big(i,(i,j)\big)$.

1 Introduction

The classical Smith determinant was introduced in 1875 by H. J. S. Smith [14] who also proved that

$$\det[(i,j)]_{n\times n} = \begin{vmatrix} (1,1) & (1,2) & \cdots & (1,n) \\ (2,1) & (2,2) & \cdots & (2,n) \\ \cdots & \cdots & \cdots & \cdots \\ (n,1) & (n,2) & \cdots & (n,n) \end{vmatrix} = \varphi(1)\varphi(2)\cdots\varphi(n), \qquad (1)$$

where (i,j) represents the greatest common divisor of i and j, and $\varphi(n)$ denotes the Euler's totient function.
The GCD matrix with respect to f is

$$[f(i,j)]_{n\times n} = \begin{bmatrix} f((1,1)) & f((1,2)) & \cdots & f((1,n)) \\ f((2,1)) & f((2,2)) & \cdots & f((2,n)) \\ \cdots & \cdots & \cdots & \cdots \\ f((n,1)) & f((n,2)) & \cdots & f((n,n)) \end{bmatrix}.$$

2010 Mathematics Subject Classification: 11C20, 11A25, 15A36
Key words and phrases: GCD matrix, Smith determinant, arithmetical function

If we consider the GCD matrix $[f(i,j)]_{n\times n}$, where

$$f(n)=\sum_{d|n} g(d),$$

H. J. Smith proved that

$$\det[f(i,j)]_{n\times n}=g(1)\cdot g(2)\cdots g(n).$$

For $g=\varphi$

$$f(i,j)=\sum_{d|(i,j)}\varphi(d)=(i,j),$$

this formula reduces to (1). Many generalizations of Smith determinants have been presented in literature, see [1, 5, 7, 10, 13].

If we consider the GCD matrix $[f(i,j)]_{n\times n}$ where $f(n)=\sum_{d|n} g(d)$ Pólya and Szegő [12] proved that

$$[f(i,j)]_{n\times n}=G\cdot C^T, \tag{2}$$

where G and C are lower triangular matrices given by

$$g_{ij}=\begin{cases} g(j), & j\mid i\\ 0, & \text{otherwise}\end{cases}$$

and

$$c_{ij}=\begin{cases} 1, & j\mid i\\ 0, & \text{otherwise}\end{cases}.$$

L. Carlitz [4] in 1960 gave a new form of (2)

$$[f(i,j)]_{n\times n}=C_n\ \mathrm{diag}\big(g(1),\ g(2),\ldots,g(n)\big)\ C_n^T, \tag{3}$$

where $C_n=[c_{ij}]_{n\times n}$,

$$c_{ij}=\begin{cases} 1, & j\mid i\\ 0, & j\nmid i\end{cases},$$

$D=[d_{ij}]_{n\times n}$ diagonal matrix

$$d_{ij}=\begin{cases} g(i), & i=j\\ 0, & i\neq j\end{cases}.$$

From (3) it follows that the value of the determinant is

$$\det[f(i,j)]_{n\times n}=g(1)g(2)\cdots g(n). \tag{4}$$

There are quite a few generalized forms of GCD matrices, which can be found in several references [2, 3, 6, 8, 9, 11].
In this paper we study matrices which have as variables the gratest common divisor and the indices:

$$[f(i,j)]_{n\times n} = \begin{bmatrix} f(1,(1,1)) & f(1,(1,2)) & \cdots & f(1,(1,n)) \\ f(2,(2,1)) & f(2(2,2)) & \cdots & f(2,(2,n)) \\ \cdots & \cdots & \cdots & \cdots \\ f(n,(n,1)) & f(n,(n,2)) & \cdots & f(n,(n,n)) \end{bmatrix}.$$

2 Generalized GCD matrices

Theorem 1 *For a given arithmetical function* g *let*

$$f(n,m) = \sum_{d|n} g(d) - \sum_{d|(n,m)} g(d).$$

Then

$$[f(i,j)]_{n\times n} = C_n \text{ diag}[g(1), g(2), \ldots, g(n)]\, D_n^T,$$

where $C_n = [c_{ij}]_{n\times n}$,

$$c_{ij} = \begin{cases} 1, & j \mid i \\ 0, & j \nmid i \end{cases},$$

$D_n = [d_{ij}]_{n\times n}$,

$$d_{ij} = \begin{cases} 1, & j \nmid i \\ 0, & j \mid i \end{cases}.$$

Proof. After multiplication, the general element of

$$A = [a_{ij}]_{n\times n} = C \text{ diag}[g(1), g(2), \ldots, g(n)] D^T$$

is

$$a_{ij} = \sum_{\substack{k \mid i \\ k \nmid j}} g(k) = \sum_{d|n} g(d) - \sum_{d|(n,m)} g(d) = f(i,j).$$

□

Particular cases
1. If $g(n) = \varphi(n)$ then

$$f(n,m) = \sum_{d|n} \varphi(d) - \sum_{d|(n,m)} \varphi(d) = n - (n,m).$$

We have the following decomposition:

$$[i-(i,j)]_{n\times n}=\begin{bmatrix} 1-(1,1) & 1-(1,2) & \cdots & 1-(1,n)\\ 2-(2,1) & 2-(2,2) & \cdots & 2-(2,n)\\ \cdots & \cdots & \cdots & \cdots\\ n-(n,1) & n-(n,2) & \cdots & n-(n,n)\end{bmatrix}.$$

2. If $g(n)=1$ then

$$f(n,m)=\tau(n)-\tau(n,m)$$

and

$$[\tau(i)-\tau(i,j)]_{n\times n}=C_n \operatorname{diag}\big(1,\ 1,\ldots,1\big)\ D_n^T.$$

3. Let $g(n)=\mu(n)$. From

$$f(n,m)=\sum_{d|n}\mu(d)-\sum_{d|(n,m)}\mu(d)=\begin{cases} 0, & n=1\\ 0, & n>1, m>1, (n,m)>1\\ -1, & \text{othewise}\end{cases}.$$

we have

$$[f(i,j)]_{n\times n}=C_n \operatorname{diag}\big(\mu(1),\ \mu(2),\ldots,\mu(n)\big)\ D_n^T.$$

4. For $g(n)=n$, $f(n,m)=\sigma(n)-\sigma((n,m))$ and

$$[f(i,j)]_{n\times n}=C_n \operatorname{diag}\big(1,\ 2,\ldots,n\big)\ D_n^T.$$

Remarks

1. Due to the fact that the first line of the matrix $[f(i,j)]_{n\times n}$ contains only 0-s, the determinant of the matrix will always be 0.

2. We can determine the value of the matrix associated with f, if the function f is of the form

$$f(n,m)=h(n)-h\big((n,m)\big).$$

By using the Möbius inversion formula, we get

$$g(n)=\sum_{d|n}\mu(d)h\left(\frac{n}{d}\right),$$

consequently by using Theorem 1, the matrix can be decomposed according to the function $h(n)$:

$$[f(i,j)]_{n\times n}=C_n \operatorname{diag}[(\mu*h)(1),(\mu*h)(2),\ldots,(\mu*h)(n)]\, D_n^T.$$

Theorem 2 *For a given arithmetical function* g *let*

$$f(i,j)=\sum_{k=1}^{n} g(k)-\sum_{d|i} g(d)-\sum_{d|j} g(d)+\sum_{d|(i,j)} g(d).$$

Then

$$[f(i,j)]_{n\times n}=D_n \; \textit{diag}[g(1),g(2),\ldots,g(n)]\, D_n^T,$$

where $D_n=[d_{ij}]_{n\times n}$*,*

$$d_{ij}=\begin{cases} 1, & j \not| i \\ 0, & j \mid i \end{cases}.$$

Proof. After multiplication, the general element of the matrix

$$A=[a_{ij}]_{n\times n}=D_n \;\mathrm{diag}[g(1),g(2),\ldots,g(n)]\, D_n^T$$

is

$$\begin{aligned} a_{ij} &= \sum_{\substack{k \not| n \\ k \not| m}} g(k)=\sum_{k=1}^{n} g(k)-\sum_{k|n \text{ or } k|m} g(k)= \\ &= \sum_{k=1}^{n} g(k)-\sum_{k|n} g(k)-\sum_{k|m} g(k)+\sum_{k|(n,m)} g(k)=f(i,j). \end{aligned}$$

□

Particular cases

1. If $g(n)=\varphi(n)$ then

$$f(i,j)=\sum_{k=1}^{n} \varphi(k)-i-j+(i,j),$$

$$[f(i,j)]_{n\times n}=D_n \;\mathrm{diag}[\varphi(1),\varphi(2),\ldots,\varphi(n)]\, D_n^T.$$

2. If $g(n)=1$ then

$$f(i,j)=n-\tau(i)-\tau(j)+\tau(i,j)$$

and

$$[f(i,j)]_{n\times n}=D_n \;\mathrm{diag}\big(1,\; 1,\ldots,1\big)\; D_n^T.$$

3. $g(n) = n$. Then

$$f(i,j) = \frac{n(n+1)}{2} - \sigma(n) - \sigma(m) + \sigma((n,m))$$

and

$$[f(i,j)]_{n\times n} = D_n \operatorname{diag}\big(1,\ 2,\ldots,n\big)\ D_n^T.$$

Another generalization is the following:

Theorem 3 *For a given arithmetical function* g *let*

$$f(i,j) = \sum_{k=1}^{n} g(k) - \sum_{d|i} g(d) - \sum_{d|j} g(d) + \sum_{d|(i,j)} g(d).$$

We define the following $A = [a_{ij}]_{n\times n}$ *matrix*

$$a_{ij} = \begin{cases} f(i,j), & i,j > 1 \\ g(1) + f(i,j), & i = 1 \ \textit{or}\ j = 1 \end{cases}.$$

Then

$$A = D'_n \operatorname{diag}[g(1), g(2), \ldots, g(n)]\, D_n'^T,$$

where $D'_n = [d'_{ij}]_{n\times n}$,

$$d'_{ij} = \begin{cases} 1, & i = j = 1 \\ d_{ij}, & ij \neq 1 \end{cases}.$$

Proof.
We calculate the general element of the matrix

$$B = [a_{ij}]_{n\times n} = D'_n \operatorname{diag}[g(1), g(2), \ldots, g(n)]\, D_n'^T.$$

If $i > 1$ or $j > 1$ we have

$$\begin{aligned} b_{ij} &= \sum_{\substack{k \nmid n \\ k \nmid m}} g(k) = \sum_{k=1}^{n} g(k) - \sum_{k|n \text{ or } k|m} g(k) = \\ &= \sum_{k=1}^{n} g(k) - \sum_{k|n} g(k) - \sum_{k|m} g(k) + \sum_{k|(n,m)} g(k) = a_{ij}. \end{aligned}$$

If $i = j = 1$

$$b_{11} = g(1) = a_{11}.$$

□

Particular cases

1. If $g(n) = \varphi(n)$ then

$$a_{ij} = \begin{cases} \sum_{k=1}^{n} \varphi(k) - i - j + (i,j), & i,j > 1 \\ \sum_{k=1}^{n} \varphi(k) - i - j + (i,j) + 1, & i = 1 \text{ or } j = 1 \end{cases}.$$

2. If $g(n) = 1$ then

$$a_{ij} = \begin{cases} n - \tau(i) - \tau(j) + \tau(i,j), & i,j > 1 \\ n - \tau(i) - \tau(j) + \tau(i,j) + 1, & i = 1 \text{ or } j = 1 \end{cases}.$$

The following problems remain open:

Problem 1 *Let $F(n,m)$ be an arithmetical function with two vriables. Determine the structure and the determinant of modified GCD matrices $A = [a(i,j)]_{n\times n}$, where*

$$a_{ij} = F(i,(i,j)).$$

Problem 2 *Determine the structure and the determinant of modified GCD matrices $A = [a(i,j)]_{n\times n}$, where*

$$a_{ij} = F(n,i,j,(i,j)).$$

Acknowledgement

This research was supported by the grant of Sapientia Foundation, Institute of Scientific Research.

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Received: June 12, 2010