DID A 1-DIMENSIONAL MAGNET DETECT A 248-DIMENSIONAL LIE ALGEBRA?

DAVID BORTHWICK AND SKIP GARIBALDI

Abstract. About a year ago, a team of physicists reported in *Science* that they had observed "evidence for E_8 symmetry" in the laboratory. This expository article is aimed at mathematicians and explains the chain of reasoning connecting measurements on a quasi-1-dimensional magnet with a 248-dimensional Lie algebra.

Recently, a team of physicists reported in *Science* that they had found evidence for the mathematical object E_8 in the laboratory, via measurements on a magnet; see the article [CTW$^+$10] by Coldea, et al. But what could that mean? It certainly sounds mystifying: if you think of E_8 as a 248-dimensional real manifold, then how is it possible to detect such a thing? In fact, the physicists only claim to give experimental corroboration of a theoretical model that involves E_8. We will attempt to explain the theoretical model, why it involves E_8, and in what sense the experiment can be viewed as corroborating the model. Along the way, we will explain some connections between the physical model and elementary properties of Lie algebras.

We should explain that we are mainly writing as journalists rather than mathematicians here. Our goal is to explain why the physicists' results point to E_8, by describing the various links in the chain of theories leading to the E_8 model. These links appear to be well accepted by theoretical physicists, but they are justified by arguments that are certainly not mathematical proofs.[1] We give pointers to the physics literature so that the adventurous reader can go directly to the words of the experts for the details.

1. The Ising model

The article in *Science* describes an experiment involving the magnetic material cobalt niobate ($CoNb_2O_6$). The material was chosen because the internal crystal structure is such that magnetic Co^{2+} ions are arranged into long chains running along the crystal's axis, and this could give rise to 1-dimensional magnetic behavior.[2] In particular, physicists expected that this material would provide a realization of the famous Ising model, which we now describe briefly.

DB was partially supported by the NSF under grant DMS-0901937. We are grateful to Richard Borcherds, Radu Coldea, Jacques Distler, Giuseppe Mussardo and others for their helpful remarks.

[1] This is an example of the philosophy of science truism: different sciences have different standards for what constitutes evidence for a claim. For physicists, the ultimate level of rigor is provided by experimental verification.

[2] Two additional practical constraints led to this choice of material: (1) large, high-quality single crystals of it can be grown as depicted in Figure 1, and (2) the strength of the magnetic interactions between the Co^{2+} spins is low enough that it can be matched by magnetic fields currently achievable in the laboratory.

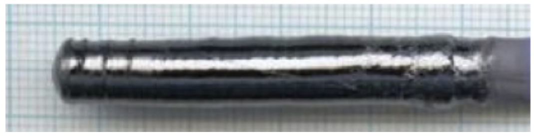


FIGURE 1. Photograph of an artificially-grown single crystal of $CoNb_2O_6$. The experiment involved a 2-centimeter-long piece of this crystal, weighing about 8 grams. (Image courtesy of Radu Coldea.)

The term *Ising model* refers generically to the original, classical model. This simple model for magnetic interactions was suggested by W. Lenz as a thesis problem for his student E. Ising, whose thesis appeared in Hamburg in 1922 [Isi25]. The classical form of the model consists of a square n-dimensional lattice for which each site j is assigned a spin $\sigma_j = \pm 1$, interpreted as the projection of the spin onto some preferential axis. The energy of a given configuration of spins is

$$H = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j, \tag{1.1}$$

where J is a constant and the sum ranges over pairs $\langle i, j \rangle$ of nearest-neighbor sites. This Hamiltonian gives rise to a statistical ensemble of states that is used to model the thermodynamic properties of actual magnetic materials. For $J > 0$, spins at neighboring sites tend to align in the same direction; this behavior is called *ferromagnetic*, because this is what happens with iron.

To describe the cobalt niobate experiment we actually want the quantum spin chain version of the Ising model. In this quantum model, each site in a 1-dimensional chain is assigned a 2-dimensional complex vector space. The Pauli spin matrices S^x, S^y, and S^z act separately on each of these vector spaces as spin observables. The ± 1 eigenvectors of S^z correspond to up and down spins, and a general spin state is a superposition of up and down, not exactly one or the other.

The Hamiltonian operator for the standard 1-dimensional quantum Ising model is given by

$$\hat{H} = -K \sum_j \left[S_j^z S_{j+1}^z + g_x S_j^x \right]. \tag{1.2}$$

In the quantum statistical ensemble one assigns probabilities to the eigenvectors of $\hat{H}$ weighted by the corresponding energy eigenvalues. The first term in the Hamiltonian has a ferromagnetic effect (assuming $K > 0$), just as in the classical case. That is, it causes spins of adjacent sites to align with each other along the z-axis, which we will refer to as the *preferential* axis. (Experimental physicists might call this the "easy" axis.) The second term represents the influence of an external magnetic field in the x-direction, perpendicular to the z-direction—we'll refer to this as the *transverse* axis. The effect of the second term is *paramagnetic*, meaning that it encourages the spins to align with the transverse field.

The 1-dimensional quantum Ising spin chain exhibits a phase transition at zero temperature. The phase transition (also called a critical point) is the abrupt transition between the ferromagnetic regime ($g_x < 1$, where spins tend to align along the z-axis) and the paramagnetic ($g_x > 1$). The critical point ($g_x = 1$) is distinguished by singular behavior of various macroscopic physical quantities, such as the correlation length.

One might wonder why an external magnetic field is included by default in the quantum case but not in the classical case. The reason for this is a correspondence between the classical models and the quantum models of one lower dimension. The quantum model includes a notion of time-evolution of an observable according to the Schrödinger equation, and the correspondence involves interpreting one of the classical dimensions as imaginary time in the quantum model. Under this correspondence, the classical interaction in the spatial directions gives the quantum ferromagnetic term, while interactions in the imaginary time direction give the external field term, see [Sac99] for details.

Although there are some important differences in the physical interpretation on each side, the classical-quantum correspondence allows various calculations to be carried over from one case to the other. For example, the critical behavior of the 1-dimensional transverse-field quantum Ising model (1.2) at zero temperature, with the transverse field parameter tuned to the critical value $g_x = 1$, can be "mapped" onto equivalent physics for the classical 2-dimensional Ising model (1.1), at its critical temperature $T = T_c$. The latter case is the famous phase transition of the 2-dimensional classical model, which was discovered by Peierls and later solved exactly by Onsager [Ons44].

2. Adapting the model to the magnet

The actual magnet used in the experiment is not quite modeled by the quantum Ising Hamiltonian (1.2). In the ferromagnetic regime ($g_x < 1$), weak couplings between the magnetic chains create an effective magnetic field pointing along the preferential axis [CT03]. The appropriate model for the experiment is thus

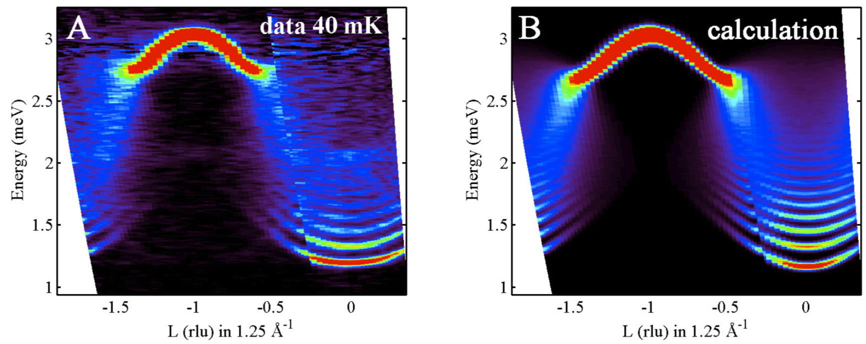
$$\hat{H} = -K \sum_j \left[S_j^z S_{j+1}^z + g_x S_j^x + g_z S_j^z \right], \tag{2.1}$$

which is just (1.2) with an additional term $g_z S_j^z$ representing this internal magnetic field.

The first phase of the cobalt niobate experiment tested the appropriateness of (2.1) as a model for the magnetic dynamics in the absence of an external magnetic field, i.e., with $g_x = 0$. The experimental evidence does support the claim that this 3-dimensional object is behaving as a 1-dimensional system. For example, Figure 2 shows a comparison of the experimental excitation energies (as a function of wave vector) to theoretical predictions from the 1-dimensional model. The presence of a sequence of well-defined and closely spaced energy levels, as shown in these pictures, is predicted only in dimension one.

3. What is E_8?

Before we explain how the rather simple quantum Ising model from the previous sections leads to a theory involving E_8, we had better nail down what it means to speak of "E_8". It's an ambiguous term, with at least the following six common meanings:

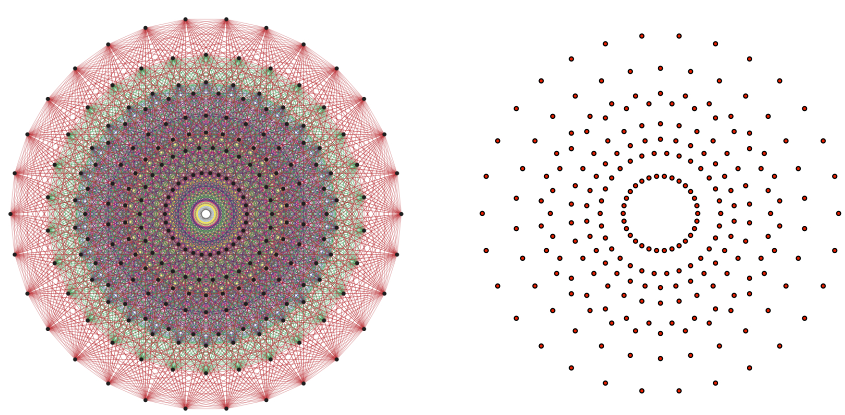


(A) Experimental data at 40mK

(B) Calculated

FIGURE 2. Comparison of excitations under no external magnetic field: experimental (left) versus predictions based on the 1-dimensional model (right). (Figure adapted from [CTW$^+$10].)

(1) The *root system* of type E_8. This is a collection of 240 points, called *roots*, in $\mathbb{R}^8$. The usual publicity photo for E_8 (reproduced in Figure 3a) is the orthogonal projection of the root system onto a copy of $\mathbb{R}^2$ in $\mathbb{R}^8$.
(2) The E_8 *lattice*, which is the subgroup of $\mathbb{R}^8$ (additively) generated by the root system.
(3) A *complex Lie group*–in particular, a closed subgroup of $\mathrm{GL}_{248}(\mathbb{C})$—that is simple and 248-dimensional.



(A) The popular picture

(B) Vertices only

FIGURE 3. The left panel (A) is the picture of E_8 that one finds in the popular press or the frontispiece of [Cox91]. (Image courtesy of John Stembridge [Ste].) The right panel (B) is the same picture with the edges removed; it is the image of the root system of E_8 in a Coxeter plane.

There are also three simple *real* Lie groups—meaning in particular that they are closed subgroups of $\mathrm{GL}_{248}(\mathbb{R})$—whose complexification is the complex Lie group from (3). The fact that there are exactly three is part of Elie Cartan's classification of simple real Lie groups.[3] They are:

(4) The *split* real E_8. This is the form of E_8 that one can define easily over any field or even over an arbitrary scheme [DG70, Th. XXV.1.1]. Its Killing form has signature 8.

(5) The *compact* real E_8, which is the unique largest subgroup of the complex E_8 that is compact as a topological space. Its Killing form has signature -248.

(6) The remaining real form of E_8 is sometimes called "quaternionic". Its Killing form has signature -24.

The split real E_8 appears in supergravity [MS83] and in the Atlas project [Vog07], a source of many news stories back in 2007. The compact real form has played an important role in string theory [GHMR85]. These two appearances in physics, however, are purely theoretical; the models in which they are appear are not yet subject to experiment. It is the compact real E_8 (or, more precisely, the associated Lie algebra) that appears in the context of the cobalt niobate experiment, making this the first actual experiment to detect a phenomenon that is modeled using E_8.

Groups versus algebras. Throughout this note we conflate a real Lie group G, which is a manifold, with its Lie algebra $\mathfrak{g}$, which is the tangent space to G at the identity and is a real vector space endowed with a nonassociative multiplication. This identification is essentially harmless and is standard in physics. Even when physicists discuss symmetry "groups", they are frequently interested in symmetries that hold only in a a local sense, and so the Lie algebra is actually the more relevant object. Moreover, physicists typically compute within the complexification $\mathfrak{g} \otimes \mathbb{C}$ of $\mathfrak{g}$, with the view that one can recover $\mathfrak{g}$ as the set of vectors fixed by complex conjugation. (This is possibly the best concrete example of the theory of Galois descent as presented in [Ser02, §III.1].)

4. From the Ising model to E_8

What possible relevance could a 248-dimensional algebra have for a discrete one-dimensional statistical physics model? This is a long and interesting story, and we can only give a few highlights here.

As we mentioned above, the 1-dimensional quantum Ising model from (1.2) undergoes a phase transition at zero temperature at the critical value of the transverse magnetic field strength. Since any actual magnet contains huge numbers of spins, it is appropriate to consider the *continuum limit* where the lattice spacing goes to zero and the number of sites becomes infinite. In the continuum limit we have a quantum model with infinitely many degrees of freedom, and that is the realm of quantum field theory. For the 1-dimensional quantum Ising model, the continuum

[3] In modern language, this is proved by checking that the action of the Weyl group on the collection of elements of order 2 in a maximal torus in the complex E_8 has three orbits, see [Ser02, §II.4.5]. This check is surprisingly easy in the case of E_8 because by [Bou02, Ch. VI, Exercise 4.1], it amounts to verifying that the orthogonal group has three orbits in $(\mathbb{F}_2)^8$—the zero vector, vectors of length 1, and nonzero vectors of length zero—which is a special case of Witt's Extension Theorem [EKM08, Th. 8.3].

limit leads to a quantum field theory of free, spinless fermionic particles in 1+1 space-time dimensions.

One characteristic that defines the phase transition is the divergence of the correlation length (the distance scale below which spins would be expected to align with each other because of the ferromagnetic effect). This divergence occurs only in the continuum limit, because a system with finitely many sites does not allow arbitrarily large length scales. Polyakov famously demonstrated in [Pol70] that the divergence of correlation length at the phase transition in the statistical model corresponds to local conformal invariance in the field theory limit. This paper established the link between the study of phase transitions and *conformal field theory* (CFT).

In 1984, Belavin, Polyakov, and Zamolodchikov showed that certain simple CFT's called *minimal models* could be solved completely in terms the representation theory of the Virasoro algebra, a central extension of the algebra of local conformal transformations [BPZ84]. These representations are characterized by the eigenvalue c assigned to the central element, called the *central charge*. You can compute c directly from the continuum limit of the statistical model.

This works out beautifully in the case of the critical 1-dimensional quantum Ising model. In that case, the central charge is $c = 1/2$, and there is exactly one minimal model and this CFT exactly matches the Ising phase transition, see [DMS97] or [Mus10] for details.

Goddard, Kent, and Olive showed in [GKO85] that one can obtain — via the *coset construction* — the representations of the Virasoro appearing in the minimal models from representations of an affine Lie algebra, a central extension of the (infinite-dimensional) loop algebra of a compact Lie algebra $\mathfrak{g}$. Using the coset construction, there are two ways to obtain the $c = 1/2$ minimal model that applies to our zero-field Ising model: we could use either of the compact Lie algebras $\mathfrak{su}(2)$ or E_8 as the base $\mathfrak{g}$ for the affine Lie algebra. These two algebras are the only choices that lead to $c = 1/2$; see [Mus10, §14.2] for more details.

Of course, the appearance of E_8 here is somewhat incidental. We could just as well describe the minimal model without reference to either $\mathfrak{su}(2)$ or E_8. The actual evidence for E_8 comes only when we consider a perturbation of the critical Ising model.

5. Magnetic perturbation and Zamolodchikov's calculation

In a 1989 article [Zam89], Zamolodchikov investigated the field theory for the situation where the 1-dimensional quantum Ising model (1.2), in the vicinity of the critical point, was perturbed by a small magnetic field directed along the preferential spin axis. In other words, he considered the field theory model corresponding to (2.1) with $g_x \approx 1$ and g_z very small. Note the change of perspective: for Zamolodchikov g_x is fixed and the perturbation consists of a small change in the value of g_z. But in the cobalt niobate experiment, this magnetic "perturbation" is already built-in—it is the purely internal effect arising from the inter-chain interactions as we described in §2. The experimenters can't control the strength of the internal field, they only vary g_x. Fortunately, the internal magnetic field g_z turns out to be relatively weak, so when the external field g_x is tuned close to the critical value the experimental model matches the situation considered by Zamolodchidkov.

As we noted above, the $c = 1/2$ minimal model is the conformal field theory associated with the phase transition of the unperturbed quantum Ising model. The perturbed field theory is no longer a conformal field theory, but Zamolodchikov found six local integrals of motion for the perturbed field theory and conjectured that these were the start of an infinite series. On this basis, he made the fundamental conjecture:

(Z1) The perturbation gives an integrable field theory.

One implication of (Z1) is that the resulting scattering theory should be "purely elastic," meaning that the number of particles and their individual momenta would be conserved asymptotically. Zamolodchikov combined this purely elastic scattering assumption with three rather mild assumptions on the particle interactions of the theory [Zam89, p. 4236]:

(Z2) There are at least 2 particles, say p_1 and p_2.
(Z3) Both p_1 and p_2 appear as bound-state poles on the scattering amplitude for two p_1's.
(Z4) The particle p_1 appears as a bound-state pole in the scattering amplitude between p_1 and p_2.

Assumptions (Z3) and (Z4) merely assert that certain coupling constants that govern the interparticle interactions are non-zero, so they could be viewed as an assumption of some minimum level of interaction between the two particles.

The word "particle" bears some explaining here, because it is being used here in the sense of quantum field theory: a stable excitation of the system with distinguishable particle-like features such as mass and momentum. However, it is important to note that the continuum limit of the Ising model is made to look like a field theory only through the application of a certain transformation (Jordan-Wigner [JW28]), that makes "kink" states (boundaries between regions of differing spin) the basic objects of the theory. So Zamolodchikov's particles aren't electrons or ions. The field theory excitations presumably correspond to highly complicated aggregate spin states of the original system. On the statistical physics side the usual term for this kind of excitation is *quasiparticle*.

From the mild assumptions (Z2)–(Z4), he showed that the simplest purely elastic scattering theory consistent with the integrals of motion contains 8 particles with masses

$$\begin{gathered} m_1, \qquad m_2 = 2m_1 \cos\frac{\pi}{5}, \quad m_3 = 2m_1\cos\frac{\pi}{30}, \\ m_4 = 2m_2\cos\frac{7\pi}{30}, \qquad m_5 = 2m_2\cos\frac{2\pi}{15}, \quad m_6 = 2m_2\cos\frac{\pi}{30}, \\ m_7 = 4m_2\cos\frac{\pi}{5}\cos\frac{7\pi}{30}, \quad m_8 = 4m_2\cos\frac{\pi}{5}\cos\frac{2\pi}{15}, \end{gathered} \tag{5.1}$$

listed in increasing order. Here m_1 and m_2 are the masses of the two original particles p_1 and p_2.

Zamolodchikov's results give some indications of a connection with the algebra or root system E_8. The spins of the integrals of motion are the numbers relatively prime to 30,

$$s = 1, 7, 11, 13, 17, 19,$$

which are the exponents of E_8, and the number of particles equals the rank of E_8. Such a connection with E_8 had been conjectured by Fateev based on other theoretical considerations [Zam89, p. 4247, 4248].

6. Affine Toda field theory

Soon after Zamolodchikov's first paper appeared, Fateev and Zamolodchikov conjectured in [FZ90] that if you take a minimal model CFT constructed from a compact Lie algebra $\mathfrak{g}$ via the coset construction and perturb it in a particular way, then you obtain the *affine Toda field theory* (ATFT) associated with $\mathfrak{g}$, which is an integrable field theory. This was confirmed in [EY89] and [HM89].

If you do this with $\mathfrak{g} = \mathsf{E}_8$, you arrive at the conjectured integrable field theory investigated by Zamolodchikov and described in the previous paragraph. That is, if we take the E_8 ATFT as a starting point, then the assumptions (Z1)–(Z4) become deductions. This is the essential role of E_8 in the numerical predictions relevant to the cobalt niobate experiment. (In the next section, we will explain how the masses that Zamolodchikov found arise naturally in terms of the algebra structure. But this is just a bonus.)

What is the role of E_8 in the affine Toda field theory? To say the ATFT in question is "associated" with E_8 leaves open a range of possible interpretations, so we should perhaps spell out precisely what this means. The ATFT construction from a compact Lie algebra $\mathfrak{g}$ proceeds by choosing a Cartan subalgebra[4] $\mathfrak{h}$ in $\mathfrak{g}$—it is a real inner product space with inner product the Killing form (,), and is isomorphic to $\mathbb{R}^8$ in the case $\mathfrak{g} = \mathsf{E}_8$. Let ϕ be a scalar field in 2-dimensional Minkowski space-time, taking values in $\mathfrak{h}$. Then the Lagrangian density for the affine Toda field theory is

$$\frac{1}{2}(\partial_\mu \phi, \partial^\mu \phi) - (e^{\beta\phi} E e^{-\beta\phi}, \overline{E}), \tag{6.1}$$

where β is a coupling constant. Here E is a regular semisimple element of $\mathfrak{g} \otimes \mathbb{C}$ that commutes with its complex conjugate $\overline{E}$. More precisely, for $x \in \mathfrak{h}$ a principal regular element, conjugation by $e^{2\pi i x/h}$ with h the Coxeter number of $\mathfrak{g}$ gives a $\mathbb{Z}/h$-grading on $\mathfrak{g} \otimes \mathbb{C}$, and the element E belongs to the $e^{2\pi i/h}$-eigenspace. (Said differently, the centralizer of E is a Cartan subalgebra of $\mathfrak{g} \otimes \mathbb{C}$ in apposition to $\mathfrak{h} \otimes \mathbb{C}$ in the sense of [Kos59, p. 1018].)

The structure of E_8 thus enters into the basic definitions of the fields and their interactions. However, E_8 does not act by symmetries on this set of fields.

Why is it E_8 that leads to Zamolodchikov's theory? We opened this section by asserting that perturbing a minimal model CFT constructed from $\mathfrak{g}$ via the coset construction leads to an ATFT associated with $\mathfrak{g}$. For this association to make sense, the perturbing field is required to have "conformal dimension" $2/(h+2)$. The two coset models for the Ising model give us two possible perturbation theories. Starting from $\mathfrak{su}(2)$, which has $h = 2$, we could perturb using the field of conformal dimension $1/2$, which is the energy. This perturbation doesn't amount to much; the resulting field theory contains a single particle and the scattering theory is trivial.

The other choice is to start from E_8, which has $h = 30$, and perturb using the field of conformal dimension $1/16$, which is the magnetic field along the preferential axis.[5] This is exactly the perturbation that Zamolodchikov considered in his original

[4] It doesn't matter which one you choose, because any one can be mapped to any other via some automorphism of $\mathfrak{g}$.

[5] The conformal dimension of the magnetic field is fixed by the model. It corresponds to the well-known critical exponent $1/8$ that governs the behavior of the spontaneous magnetization of the Ising model as the critical point is approached.

paper. This means that if an ATFT is used to describe the magnetically perturbed Ising model, we have no latitude in the choice of a Lie algebra: it must be E_8.

7. The Zamolodchikov masses and E_8's publicity photo

Translating Zamolodchikov's theory into the language of affine Toda field theory provides a way to transform his calculation of the particle masses in (5.1) into the solution of a rather easy system of linear equations, and that in turn is connected to the popular image of the E_8 root system from Figure 3a. These are connections that work for a general ATFT, and we write in that level of generality.

An ATFT is based on a compact semisimple real Lie algebra $\mathfrak{g}$, such as the Lie algebra of the compact real E_8. We assume further that this algebra is simple and is not $\mathfrak{su}(2)$. Then from $\mathfrak{g}$ we obtain a simple root system R spanning $\mathbb{R}^\ell$ for some $\ell \geq 2$; this is canonically identified with the dual $\mathfrak{h}^*$ of the Cartan subalgebra mentioned at the end of the previous section. (Conversely, one can start with such an R and construct $\mathfrak{g}$ as in [Car89, §4.2].)

We briefly explain how to make a picture like Figure 3b for R. (For background on the vocabulary used here, please see [Bou02] or [Car89].) Pick a set B of simple roots in R. For each $\beta \in R$, write s_β for the reflection in the hyperplane orthogonal to β. The product $w := \prod_{\beta \in B} s_\beta$ with respect to any fixed ordering of B is called a *Coxeter element* and its characteristic polynomial has $m(x) := x^2 - 2\cos(2\pi/h)x + 1$ as a simple factor [Bou02, VI.1.11, Prop. 30], where h is the Coxeter number of R. The primary decomposition theorem gives a uniquely determined plane P in $\mathbb{R}^\ell$ on which w restricts to have minimal polynomial $m(x)$, i.e., is a rotation through $2\pi/h$—we call P the *Coxeter plane* for w. The picture in Figure 3b is the image of R under the orthogonal projection $\pi\colon \mathbb{R}^\ell \to P$ in the case where $R = \mathsf{E}_8$. We remark that while P depends on the choice of w, all Coxeter elements are conjugate under the orthogonal group [Car89, 10.3.1], so none of the geometric features of $\pi(R)$ are changed if we vary w and we refer to P as simply a Coxeter plane for R.

In Figure 3b, the image of R lies on 8 concentric circles. This is a general feature of the projection in P and is not special to the case $R = \mathsf{E}_8$. Indeed, the action of w partitions R into ℓ orbits of h elements each [Bou02, VI.1.11, Prop. 33(iv)], and w acts on P as a rotation. So the image of R necessarily lies on ℓ (possibly non-distinct) circles.

The relationship between the circles in Figure 3b and physics is given by the following theorem.

Theorem 7.1. *Let $\mathfrak{g}$ be a compact* simple *Lie algebra that is not $\mathfrak{su}(2)$, and write R for its root system. For an affine Toda field theory constructed from $\mathfrak{g}$, the following multisets are the same, up to scaling by a positive real number:*

1. *The (classical) masses of the particles in the affine Toda theory.*
2. *The radii of the circles containing the projection of R in a Coxeter plane.*
3. *The entries in a Perron-Frobenius eigenvector for a Cartan matrix of R.*

The terms in (3) may need some explanation. The restriction of the inner product on $\mathbb{R}^\ell$ to R is encoded by an ℓ-by-ℓ integer matrix C, called the *Cartan matrix* of R. You can find the matrix for $R = \mathsf{E}_8$ in Figure 4a. We know a lot about the Cartan matrix, no matter which R one chooses—for example, its eigenvalues are all real and lie in the interval $(0, 4)$, see [BLM89, Th. 2]. Further, the matrix $2 - C$ has all non-negative entries and is irreducible in the sense of the Perron-Frobenius

Theorem, so its largest eigenvalue has a 1-dimensional eigenspace spanned by a vector $\vec{x}$ with all positive entries. (Such an eigenvector is exhibited in Figure 4b for the case $R = \mathsf{E}_8$.) This $\vec{x}$ is the vector in (3) and it is an eigenvector of C with eigenvalue $4\sin^2(\pi/2h)$, so calculating $\vec{x}$ amounts to solving an easy system of linear equations.

$$\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} \qquad \begin{pmatrix} m_2 \\ m_4 \\ m_6 \\ m_8 \\ m_7 \\ m_5 \\ m_3 \\ m_1 \end{pmatrix} \approx \begin{pmatrix} 1.618 \\ 2.405 \\ 3.218 \\ 4.783 \\ 3.891 \\ 2.956 \\ 1.989 \\ 1.000 \end{pmatrix} m_1$$

(A) (B)

FIGURE 4. The Cartan matrix (A) for the root system E_8 and a Perron-Frobenius eigenvector (B), where the m_i's are as in (5.1).

Sketch of proof. Here's how to assemble a proof of Theorem 7.1 from the literature. Freeman showed that (1) and (3) are equivalent in [Fre91]. We omit his argument, which amounts to computations in the complex Lie algebra $\mathfrak{g} \otimes \mathbb{C}$, but it is worth noting that his proof does rely on $\mathfrak{g}$ being compact. The equivalence of (2) and (3) can be proved entirely in the language of root systems and finite reflection groups. One picks w to be a "bicolored" Coxeter element as in [Car89, §10.4]. Then there is a function $\sigma\colon B \to \{\pm 1\}$ so that the elements $\sigma(\beta)\beta$ for $\beta \in B$ are representatives of the orbits of w on R [FLO91, p. 84]. The function σ is easy to understand and it is an elementary computation to find the inner products of $\pi(\sigma(\beta)\beta)$ with the basis vectors for P given in [Car89, §10.4], hence the radius of the circle containing $\pi(\sigma(\beta)\beta)$. The entries of the Perron-Frobenius eigenvector appear naturally, because these entries are part of the expressions for the basis vectors for P. □

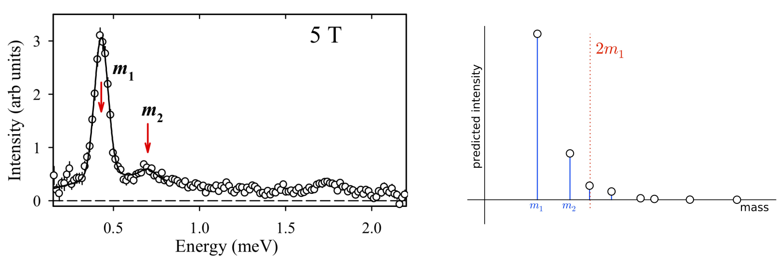
Theorem 7.1 was known to physicists by the early 1990s, as outlined above. An alternative view via Lie algebras can be found in [Kos10].

There is a deeper connection between the particles in the ATFT and the roots in the root system. Physicists identify the w-orbits in the root system with particles in the ATFT. The rule for the coupling of particles in a scattering experiment (called a "fusing" rule) is that the scattering amplitude for two particles Ω_1 and Ω_2 has a bound-state pole corresponding to Ω_3 if and only if there are roots $\rho_i \in \Omega_i$ so that $\rho_1 + \rho_2 + \rho_3 = 0$ in $\mathbb{R}^\ell$, see [Dor91] and [FLO91]. This leads to a "Clebsch-Gordan" necessary condition for the coupling of particles, see [Bra92]. We remark that these fusing rules are currently only theoretical—it is not clear how they could be tested experimentally.

8. Back to the experiment

Let's get back to the cobalt niobate experiment. As we noted above, when the external magnetic field is very close to the critical value that induces the phase transition, it was expected that the experimental system would be modeled by the critical 1-dimensional quantum Ising model perturbed by a small magnetic field directed along the preferential axis. This model is the subject of Zamolodchikov's perturbation theory, and the resulting field theory has been identified as the E_8 ATFT.

To test this association, the experimenters conducted neutron scattering experiments on the magnet. Figure 5a shows an intensity plot of scattered neutrons averaged over a range of scattering angles. Observations were actually made at a series of external field strengths, from 4.0 T to 5.0 T, with the second peak better resolved at the lower energies. Both peaks track continuously as the field strength is varied. Figure 5a shows the highest field strength at which the second peak could be resolved.



(A) Masses detected (B) Predictions

FIGURE 5. The left panel (A) is an example intensity plot, exhibiting the two detected masses under a transverse magnetic field of 5 tesla, 90% of the critical strength. (Figure adapted from [CTW+10].) The right panel (B), shows the relative intensities obtained from the form factors computed in [DM95, p. 741, Table 3]. The axes have the same labels as in the left panel. The dotted vertical line marks the onset of the incoherent continuum.

The two peaks give evidence of the existence of at least two particles in the system, which was one of Zamolodchikov's core assumptions. And indeed, the ratio of the masses appear to approach the golden ratio—see Figure 6—as the critical value (about 5.5 T) is approached, just as Zamolodchikov predicted twenty years earlier.

We can also compare the relative intensities of the first two mass peaks to the theoretical predictions exhibited in Figure 5b. Here again we see approximate agreement between the observations and theoretical predictions. The figure shows a threshold at $2m_1$, where a continuous spectrum is generated by the scattering of the lightest particle with itself. Particles with masses at or above this threshold will be very difficult to detect, as their energy signature is expected to consist of rather small peaks that overlap with the $2m_1$ continuum.[6] Hence the fact that only two particles out of eight were observed is again consistent with the theoretical model.

[6]Possibly because of this, the region above this threshold has been called the "incoherent continuum", a suggestive and Lovecraftian term.

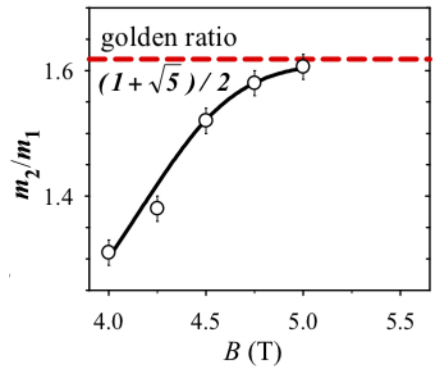


FIGURE 6. The ratio m_2/m_1 of the masses of the two lightest particles approaches the golden ratio as the transverse magnetic field approaches critical strength of 5.5 tesla. (Figure adapted from [CTW+10].)

9. Experimental evidence for E_8 symmetry?

We can now finally address the question from the title of this paper, slightly rephrased: Did the experimenters detect E_8? First we should say that they themselves do not claim to have done so. Rather, they claim to have found experimental evidence for the theory developed by Zamolodchikov, et al, and described above—which we shall call below simply *Zamolodchikov's theory*—and that this in turn means giving evidence for E_8 symmetry.

The argument for these claims goes as follows. The E_8 ATFT is an integrable field theory describing the magnetically perturbed Ising model (2.1) and satisfying (Z1)–(Z4). In that situation, Zamolodchikov and Delfino-Mussardo made some numerical predictions regarding the relative masses of the particles and relative intensities of the scattering peaks. The experimental data show two peaks, but the second peak is only resolved at lower energies. The ratios of masses and intensities are certainly consistent with the theoretical predictions, although the ratios appear to be measured only rather roughly.

What is evidence? Let's talk about the first claim, that they found experimental evidence for Zamolodchikov's theory. Suddenly we have come up against a—possibly the—central problem in philosophy of science: what does it mean to provide evidence for a scientific claim?

While this is not the place to survey the various competing theories of evidence, we do need to dispel one bugbear that is naturally attractive to mathematicians, namely falsificationism. This view can be summarized by saying: experiments can never provide evidence for a scientific theory, they can only provide evidence against it. For mathematics, this is reasonable in that "proof by example" is no proof at all. But for the experimental sciences this is not reasonable, it is preposterous. Science only progresses through the acceptance of theories that have survived enough good experimental tests, even if the words "enough" and "good" are open to subjective interpretations. As a consequence, in contrast to mathematical theorems, all physical laws are provisional, limited by the range of our observation.

With regard to the experimental results from [CTW$^+$10], we must first stress that this only a single experiment. One might also question whether the second peak is sufficiently resolved or the measured ratios sufficiently accurate to really provide evidence. We're not qualified to address such issues here; those questions are more properly addressed to the experimental physics community (or to the editors of *Science*). What we can say is that the first experimental test of Zamolodchikov's theory produced data that seems to be consistent with the predictions made twenty years earlier.

Evidence for E_8 symmetry? As we explained in §5, the role of E_8 in the theory is in some sense a separate issue. Zamolodchikov's theory rests on his conjecture (Z1) that the field theory is integrable (which was suggested by the presence of a set of integrals of motion). So far, the only candidate anyone has discovered for this integrable field theory is the E_8 ATFT (see §6). This connection may seem fishy, in that the absence of a simpler explanation doesn't rule one out. However, if one at least accepts the conjecture that the correct field theory is integrable, then the coincidences between the masses and the spins of the integrals of motion from §5 on the one hand and numerology of the E_8 root system on the other strongly suggest that E_8 ought to appear in the theory somehow.

Finally, we should address the distinction between "detecting E_8" and "finding evidence for E_8 symmetry". While the former is pithier, we're only talking about the latter here. The reason is that, as far as we know, there is no direct correspondence between E_8 and any physical object. This is in contrast, for example, to the case of the gauge group SU(3) of the strong force in the Standard Model in particle physics. One can meaningfully identify basis vectors of the Lie algebra $\mathfrak{su}(3)$ with gluons, the mediators of the strong force, which have been observed in the laboratory. With this distinction in mind, our view is that the experiment cannot be said to have detected E_8, but that it has provided evidence for Zamolodchikov's theory and hence for E_8 symmetry as claimed in [CTW$^+$10].

10. Summary

The experiment with the cobalt niobate magnet consisted of two phases. In the first phase, the experimenters verified that in the absence of an external magnetic field, the 1-dimensional quantum Ising model (2.1) accurately describes the spin dynamics, as predicted by theorists. In the second phase, the experimenters added an external magnetic field directed transverse to the spins' preferred axis, and tuned this field close to the value required to reach the quantum critical regime. In that situation, Zamolodchikov, et al, had predicted the existence of 8 distinct types of particles in a field theory governed by the compact Lie algebra E_8. The experimenters observed the two smallest particles and confirmed two numerical predictions: the ratio of the masses of the two smallest particles (predicted by Zamolodchikov) and the ratio of the intensities corresponding to those two particles (predicted by Delfino-Mussardo). The E_8 model happens to be the only known theoretical framework available to justify these observations.

Why does E_8 play a special role here? Why don't we find other algebras? This presentation so far naturally prompts the question: Why E_8? What if we replace E_8 in the affine Toda field theory with other algebras, even exceptional algebras like E_6 or E_7? In fact, the field theories based on these other algebras do have interesting connections to statistical models. For example, E_7 Toda field theory describes the thermal perturbation of the tricritical Ising model, and the E_6 theory the thermal deformation of the tricritical three-state Potts model. These other models are easily distinguished from the magnetically perturbed Ising model by their central charges. It will be interesting to see if physicists can come up with ways to probe these other models experimentally. The E_7 model might be easiest—the unperturbed, CFT version has already been realized, for example, in the form of helium atoms on krypton-plated graphite [TFV80].

Why the compact form? As Folland noted recently in [Fol10] physicists tend to think of Lie algebras in terms of generators and relations, without even specifying a background field if they can help it. So it can be difficult to judge from the appearance of a Lie algebra in the physics literature if any particular form of the algebra is being singled out.

Nevertheless, the algebras appearing here are the compact ones. The reason is that the minimal model realization of CFT's involves unitary representations of the Virasoro algebra. The coset construction shows that these come from representations of affine Lie algebras which are themselves constructed from compact finite-dimensional algebras. And it is these finite-dimensional Lie algebras that appear in the affine Toda field theory.

Conclusion. In this article, we have focused on the E_8 side of the story, both because E_8 is a mathematical celebrity and because of the incongruity between 1 and 248 dimensions. But there are two serious scientific reasons to be interested in the experiment. We have already mentioned one: it is the first experimental test of the perturbed conformal field theory constructed by Zamolodchikov around 1990. Another is that, from the view of statistical mechanics, the main achievement of the cobalt niobate experiment is the realization in the laboratory, for the very first time, of the critical state of the quantum 1-dimensional Ising model. Since the Ising model is the fundamental model for quantum phase transitions, the opportunity to probe experimentally its very rich physics represents a breakthrough.[7]

References

[BLM89] S. Berman, Y.S. Lee, and R.V. Moody, *The spectrum of a Coxeter transformation, affine Coxeter transformations, and the defect map*, J. Algebra **121** (1989), no. 2, 339–357.

[Bou02] N. Bourbaki, *Lie groups and Lie algebras: Chapters 4–6*, Springer-Verlag, Berlin, 2002.

[BPZ84] A.A. Belavin, A.M. Polyakov, and A.B. Zamolodchikov, *Infinite conformal symmetry of critical fluctuations in two dimensions*, J. Statist. Phys. **34** (1984), no. 5-6, 763–774.

[Bra92] H.W. Braden, *A note on affine Toda couplings*, J. Phys. A **25** (1992), L15–L20.

[Car89] R.W. Carter, *Simple groups of Lie type*, Wiley, 1989.

[Cox91] H.S.M. Coxeter, *Regular complex polytopes*, Cambridge University Press, 1991.

[CT03] S.T. Carr and A.M. Tsvelik, *Spectrum and correlation functions of a quasi-one-dimensional quantum Ising model*, Physical Rev. Letters **90** (2003), no. 17, 177206.

[CTW+10] R. Coldea, D.A. Tennant, E.M. Wheeler, E. Wawrzynska, D. Prabhakaran, M. Telling, K. Habicht, P. Smibidl, and K. Kiefer, *Quantum criticality in an Ising chain: experimental evidence for emergent E_8 symmetry*, Science **327** (2010), 177–180.

[7] A crucial factor here is the ability to vary g_x across a wide range while preserving the 1-dimensional character of the system, allowing the experimenters to tune the system very near to the critical point.

[DG70] M. Demazure and A. Grothendieck, *Schémas en groupes III: Structure des schemas en groupes reductifs*, Lecture Notes in Mathematics, vol. 153, Springer, 1970.

[DM95] G. Delfino and G. Mussardo, *The spin-spin correlation function in the two-dimensional Ising model in a magnetic field at $T = T_c$*, Nuclear Phys. B (1995), 724–758.

[DMS97] P. Di Francesco, P. Mathieu, and D. Sénéchal, *Conformal field theory*, Graduate Texts in Contemporary Physics, Springer-Verlag, New York, 1997.

[Dor91] P. Dorey, *Root systems and purely elastic S-matrices*, Nuclear Phys. B **358** (1991), no. 3, 654–676.

[EKM08] R.S. Elman, N. Karpenko, and A. Merkurjev, *The algebraic and geometric theory of quadratic forms*, Colloquium Publications, vol. 56, Amer. Math. Soc., 2008.

[EY89] T. Eguchi and S.-K. Yang, *Deformations of conformal field theories and soliton equations*, Phys. Lett. B **224** (1989), no. 4, 373–378.

[FLO91] A. Fring, H.C. Liao, and D.I. Olive, *The mass spectrum and coupling in affine Toda theories*, Phys. Lett. B **266** (1991), 82–86.

[Fol10] G. B. Folland, *Speaking with the natives: reflections on mathematical communication*, Notices Amer. Math. Soc. **57** (2010), no. 9, 1121–1124.

[Fre91] M.D. Freeman, *On the mass spectrum of affine Toda field theory*, Phys. Lett. B **261** (1991), no. 1-2, 57–61.

[FZ90] V.A. Fateev and A.B. Zamolodchikov, *Conformal field theory and purely elastic S-matrices*, Internat. J. Modern Phys. A **5** (1990), no. 6, 1025–1048.

[GHMR85] D.J. Gross, J.A. Harvey, E. Martinec, and R. Rohm, *Heterotic string*, Physical Review Letters **54** (1985), no. 6, 502–505.

[GKO85] P. Goddard, A. Kent, and D. Olive, *Virasoro algebras and coset space models*, Phys. Letters B **152** (1985), no. 1–2, 88–92.

[HM89] T.J. Hollowood and P. Mansfield, *Rational conformal field theories at, and away from, criticality as Toda field theories*, Phys. Lett. B **226** (1989), no. 1–2, 73–79.

[Isi25] E. Ising, *Beitrag zur Theorie des Ferromagnetismus*, Z. Phys. **31** (1925), 253–258.

[JW28] P. Jordan and E. Wigner, *Uber das Paulische Äquivalenzverbot*, Z. Phys. **47** (1928), 631.

[Kos59] B. Kostant, *The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group*, Amer. J. Math. **81** (1959), 973–1032.

[Kos10] ______, *Experimental evidence for the occurrence of E_8 in nature and the radii of the Gosset circles*, Selecta Math. (NS) **16** (2010), no. 3, 419–438.

[MS83] N. Marcus and J.H. Schwarz, *Three-dimensional supergravity theories*, Nuclear Phys. B **228** (1983), no. 1, 145–162.

[Mus10] G. Mussardo, *Statistical field theory*, Oxford Graduate Texts, Oxford University Press, Oxford, 2010.

[Ons44] L. Onsager, *Crystal statistics. I. A two-dimensional model with an order-disorder transition*, Phys. Rev. (2) **65** (1944), 117–149.

[Pol70] A.M. Polyakov, *Conformal symmetry of critical fluctuations*, JETP Lett. **12** (1970), 381.

[Sac99] S. Sachdev, *Quantum phase transitions*, Cambridge University Press, Cambridge, 1999.

[Ser02] J-P. Serre, *Galois cohomology*, Springer-Verlag, 2002, originally published as *Cohomologie galoisienne* (1965).

[Ste] J. Stembridge, *Coxeter planes*, http://www.math.lsa.umich.edu/˜jrs/coxplane.html.

[TFV80] M.J. Tejwani, O. Ferreira, and O.E. Vilches, *Possible Ising transition in a ^{4}He monolayer adsorbed on Kr-plated graphite*, Phys. Rev. Letters **44** (1980), no. 3, 152–155.

[Vog07] D. Vogan, *The character table for E_8*, Notices of the AMS **54** (2007), no. 9, 1022–1034.

[Zam89] A.B. Zamolodchikov, *Integrals of motion and S-matrix of the (scaled) $T = T_c$ Ising model with magnetic field*, Int. J. Modern Physics **4** (1989), no. 16, 4235–4248.

E-mail address: davidb at mathcs.emory.edu

E-mail address: skip at mathcs.emory.edu