OWR: Universal central extensions of gauge groups

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We indicate how to calculate the universal central extension of the gauge algebra $\Gamma(\mathrm{ad}(P))$, and how to obtain from this the corresponding universal central extension of the gauge group $\Gamma(\mathrm{Ad}(P))$.

Gauge groups occur as vertical symmetries of gauge theories, in which fields are connections on a principal G-bundle $P \to M$, and the action is invariant under vertical automorphisms of P. If we set $\mathrm{Ad}(P) := P \times_{\mathrm{Ad}} G$ and similarly $\mathrm{ad}(P) := P \times_{\mathrm{ad}} \mathfrak{g}$ (with $\mathfrak{g}$ the Lie algebra of G), we can identify the group of vertical automorphisms with $\Gamma(\mathrm{Ad}(P))$, and its Lie algebra with $\Gamma(\mathrm{ad}(P))$.

In the case that P admits a *flat* equivariant connection, these gauge algebras closely resemble equivariant map algebras and (twisted multi) loop algebras. Using the flat connection, one finds a cover $N \to M$, a monodromy group $\Delta < \pi_1(M)$, and a homomorphism $\Delta \to G$ such that $P = N \times_\Delta G$. The adjoint bundle then takes the shape $\mathrm{ad}(P) = N \times_\Delta \mathfrak{g}$, so that the gauge algebra is just $\Gamma(\mathrm{ad}(P)) = (C^\infty(N, \mathbb{R}) \otimes_{\mathbb{R}} \mathfrak{g})^\Delta$, the Lie algebra of smooth equivariant maps from N to $\mathfrak{g}$.

If X is an affine variety over $\mathbb{R}$ with an action of a discrete group Δ, and Δ acts by automorphisms on a real Lie algebra $\mathfrak{g}$, then the equivariant map algebra $(\mathbb{C}[X] \otimes_{\mathbb{R}} \mathfrak{g})^\Delta$ is the Lie algebra of equivariant regular maps from X to $\mathfrak{g}_{\mathbb{C}}$.

The set $X_{\mathbb{R}}^{\mathrm{reg}}$ of regular real points constitutes a smooth manifold, and under suitable conditions (see prop. 3), the homomorphism $\mathbb{C}[X] \to C^\infty(X_{\mathbb{R}}^{\mathrm{reg}}, \mathbb{C})$ is injective with dense image. If the action of Δ restricts to $X_{\mathbb{R}}^{\mathrm{reg}}$, then we obtain an inclusion $(\mathbb{C}[X] \otimes_{\mathbb{R}} \mathfrak{g})^\Delta \hookrightarrow (C^\infty(X_{\mathbb{R}}^{\mathrm{reg}}, \mathbb{C}) \otimes_{\mathbb{R}} \mathfrak{g})^\Delta$ of Lie algebras with dense image. If moreover $X_{\mathbb{R}}^{\mathrm{reg}}/\Delta$ is a manifold, then we have realised the equivariant map algebra as a dense subalgebra of the complexification of the gauge algebra $\Gamma(\mathrm{ad}(P))$, with $P \to X_{\mathbb{R}}^{\mathrm{reg}}/\Delta$ the principal $\mathrm{Aut}(\mathfrak{g})$-bundle $P = X_{\mathbb{R}}^{\mathrm{reg}} \times_\Delta \mathrm{Aut}(\mathfrak{g})$.

For example, let X be $T^n = \{(\vec{z}, \vec{w}) \in \mathbb{C}^{2n} \,|\, z_k^2 + w_k^2 = 1, 1 \le k \le n\}$, the complex n-torus. In this case, $\mathbb{C}[T^n] \hookrightarrow C^\infty(T^n_{\mathbb{R}}, \mathbb{C})$ is injective with dense image by Fourier theory. We now look for a regular group action on T^n that restricts to $T^n_{\mathbb{R}}$ and such that $M = T^n_{\mathbb{R}}/\Delta$ is a manifold. Although the Bieberbach groups spring to mind, the choice that is studied most is $\Delta = \prod_{k=1}^n \mathbb{Z}/r_k\mathbb{Z}$, with $\delta : (z_k \pm iw_k) \mapsto e^{\pm 2\pi i \delta_k / r_k}(z_k \pm iw_k)$. In this case, $M = T^n_{\mathbb{R}}/\Delta$ is again a torus. For any homomorphism $\Delta \to \mathrm{Aut}(\mathfrak{g})$, the twisted multiloop algebra $(\mathbb{C}[T^n] \otimes_{\mathbb{R}} \mathfrak{g})^\Delta$ forms a dense subalgebra of the complexification of $\Gamma(\mathrm{ad}(P))$, where P is the principal $\mathrm{Aut}(\mathfrak{g})$-bundle $P = T^n_{\mathbb{R}} \times_\Delta \mathrm{Aut}(\mathfrak{g})$ over $T^n_{\mathbb{R}}$.

The case of the circle is special in that every principal G-bundle over $M = T^1_{\mathbb{R}}$ is given by a twist $g \in G$ upon a full rotation, and therefore admits a flat connection. A smooth path connecting g to g' yields an isomorphism of the corresponding bundles, so principal G-bundles are classified by $\pi_0(G)$. Consequently, the complexified adjoint bundles are classified by $\pi_0(\mathrm{Aut}(\mathfrak{g}_{\mathbb{C}}))$, which for simple $\mathfrak{g}_{\mathbb{C}}$ amounts to diagram automorphisms of order 1, 2 or 3. Complexified gauge algebras over $T^1_{\mathbb{R}}$ are thus precisely the closures of twisted loop algebras.

We return to the case of smooth principal fibre bundles which do not necessarily have a flat connection, and sketch the universal 2-cocycle for the compactly supported gauge algebra $\Gamma_{\mathrm{c}}(\mathrm{ad}(P))$. We refer the interested reader to [1] for details.

For any Lie algebra $\mathfrak{g}$, set $V(\mathfrak{g}) := (\mathfrak{g} \otimes_s \mathfrak{g}) / \mathrm{der}(\mathfrak{g}) \cdot (\mathfrak{g} \otimes_s \mathfrak{g})$, and denote by $\kappa \colon \mathfrak{g} \times \mathfrak{g} \to V(\mathfrak{g})\,;\,(x,y) \mapsto [x \otimes_s y]$ the universal $\mathrm{der}(\mathfrak{g})$-invariant bilinear form on $\mathfrak{g}$. Any Lie connection ∇ on $\mathrm{ad}(P)$ induces a flat connection $\mathtt{d}$ on the vector bundle $V(\mathrm{ad}(P)) \to M$, which does not depend on ∇ as any two Lie connections differ by a pointwise derivation, which acts trivially on $V(\mathrm{ad}(P))$. Using the identities $\mathtt{d}\kappa(\xi,\eta) = \kappa(\nabla\xi,\eta) + \kappa(\xi,\nabla\eta)$ and $\nabla[\xi,\eta] = [\nabla\xi,\eta] + [\xi,\nabla\eta]$ for all sections $\xi,\eta \in \Gamma_{\mathrm{c}}(\mathrm{ad}(P))$, one checks that

$$\omega_{\nabla} \colon \wedge^2 \Gamma_{\mathrm{c}}(\mathrm{ad}(P)) \to \overline{\Omega}^1_{\mathrm{c}}(M, V(\mathrm{ad}(P)) \quad \xi \wedge \eta \mapsto [\kappa(\xi, \nabla\eta)] \tag{1}$$

defines a Lie algebra cocycle, where the subscript c denotes compact support, and we set $\overline{\Omega}^1_{\mathrm{c}} := \Omega^1_{\mathrm{c}}/\mathtt{d}\Omega^0_{\mathrm{c}}$. If $\mathfrak{g}$ is semisimple, then the cohomology class $[\omega_\nabla]$ does not depend on ∇. We equip our spaces of smooth forms and sections with the usual LF-topology, in terms of which the universality result is formulated as follows.

Proposition 1. *If $\mathfrak{g}$ is semisimple, then $[\omega_\nabla]$ is universal; every continuous 2-cocycle ψ with values in a trivial real topological module W can be written up to coboundary as $\psi = \phi \circ \omega_\nabla$, for some continuous $\mathbb{R}$-linear $\phi : \overline{\Omega}^1_{\mathrm{c}}(M, V(\mathrm{ad}(P))) \to W$.*

In [1], this is proved by noting that 2-cocycles are automatically diagonal, so that the second cohomology in fact constitutes a sheaf. The result can then be reduced to the well known local one, described e.g. in [2]. Using the results of [3, 4, 5], this can be used (see [1]) to prove the following theorem.

Theorem 2. *Let $P \to M$ be a principal fibre bundle with compact connected base, and with a semisimple structure group with finitely many connected components. Then the cocycle (1) integrates to a central extension of $\Gamma(\mathrm{Ad}(P))$ that is universal for abelian Lie groups modelled on Mackey-complete locally convex spaces.*

Although the application of differential geometric techniques in an algebraic context has intrinsic drawbacks, it is perhaps worth while to briefly explore the ramifications of proposition 1 to equivariant map algebras. We start by substantiating our claim as to the injectivity and denseness of $\mathbb{C}[X] \to C^\infty(X^{\mathrm{reg}}_{\mathbb{R}}, \mathbb{C})$.

Proposition 3. *Let X be an affine variety over $\mathbb{R}$ such that every connected component of X^{an} possesses a nonsingular real point. Then the ring homomorphism $\mathbb{C}[X] \hookrightarrow C^\infty(X^{\mathrm{reg}}_{\mathbb{R}}, \mathbb{C})$ is injective, with dense image in the topology of uniform convergence of derivatives on compact subsets.*

Proof. First, we prove that the image is dense. As every smooth function on $X^{\mathrm{reg}}_{\mathbb{R}}$ can be approximated by compactly supported smooth functions, and every compactly supported (in $X^{\mathrm{reg}}_{\mathbb{R}}$) smooth function on $X^{\mathrm{reg}}_{\mathbb{R}}$ extends to a compactly supported (in $\mathbb{R}^n$) smooth function on $\mathbb{R}^n$, it is enough to show that every smooth function f on a compact subset K of $\mathbb{R}^n$ can be approximated by polynomials. Now by Weierstraß' theorem, there exist, for any multi-index $\vec{\mu}$, polynomials p with $\sup_K |\partial_{\vec{\mu}} f - p|$ arbitrarily small. By integrating these, one can produce polynomials

p such that $\sup_K |\partial_{\vec{\nu}} f - \partial_{\vec{\nu}} p|$ is arbitrarily small for all $\vec{\nu} < \vec{\mu}$. A sequence p_k of such polynomials for $\vec{\mu}_k \to \infty$ (in the sense that for every fixed $\vec{\nu}$, we eventually have $\vec{\nu} < \vec{\mu}_k$) will converge to f uniformly on K for every derivative.

Next, we prove injectivity. Denote by $\mathcal{O}^{\mathrm{an}}_Y$ and C^{∞}_Y the sheaves of analytic and smooth functions on Y. Choose a nonsingular real point x_i in each connected component (in the analytic topology) of X^{an}, so that $\mathbb{C}[X] \to \bigoplus_i \mathcal{O}^{\mathrm{an}}_{X,x_i}$ is injective. Using the inverse function theorem, we find analytic charts $\phi_i : \mathbb{C}^d \supset U_i \to V_i \subset X^{\mathrm{an}}$ around x_i in which $U_i \cap \mathbb{R}^d$ corresponds to $V_i \cap X^{\mathrm{reg}}_{\mathbb{R}}$. In those coordinates, the map $\mathcal{O}^{\mathrm{an}}_{X,x_i} \to C^{\infty}_{X^{\mathrm{reg}}_{\mathbb{R}},x_i}$ corresponds to the injective map $\mathcal{O}^{\mathrm{an}}_{\mathbb{C}^d,0} \to C^{\infty}_{\mathbb{R}^d,0}$, and is therefore injective. Since the injective map $\mathbb{C}[X] \to \bigoplus_i C^{\infty}_{X^{\mathrm{reg}}_{\mathbb{R}},x_i}$ factors through $\mathbb{C}[X] \to C^{\infty}(X^{\mathrm{reg}}_{\mathbb{R}}, \mathbb{C})$, the latter must be injective itself. □

Consider X, $\mathfrak{g}$ and Δ as before, but now with $X^{\mathrm{reg}}_{\mathbb{R}}/\Delta$ a *compact* manifold, and $\mathfrak{g}$ *semisimple*. Since $\iota\colon (\mathbb{C}[X] \otimes_{\mathbb{R}} \mathfrak{g})^{\Delta} \hookrightarrow \Gamma(\mathrm{ad}(P)_{\mathbb{C}})$ is a dense inclusion, we conclude with [7, Lem. 2] that $\iota^*\colon H^2_{\mathrm{ct}}(\Gamma(\mathrm{ad}(P))_{\mathbb{C}}, W) \to H^2_{\mathrm{ct}}((\mathbb{C}[X] \otimes_{\mathbb{R}} \mathfrak{g})^{\Delta}, W)$ is an isomorphism, where W is a complex Fréchet space considered as a trivial module, and continuity is in the C^{∞}-topology on *both* sides. Restricted to the equivariant map algebra, our canonical cocycle takes values in the space $\overline{\Omega}^1_{\mathrm{alg}}(\mathbb{C}[X])$ of Kähler differentials modulo closed ones, and can be written

$$\omega_{\mathrm{alg}} : \wedge^2(\mathbb{C}[X] \otimes_{\mathbb{C}} \mathfrak{g}_{\mathbb{C}})^{\Delta} \to (\overline{\Omega}^1_{\mathrm{alg}}(\mathbb{C}[X]) \otimes_{\mathbb{C}} V(\mathfrak{g}_{\mathbb{C}}))^{\Delta} \quad : \quad \xi \wedge \eta \mapsto [\kappa(\xi, d\eta)]\,. \tag{2}$$

It is universal in the sense that every *continuous* $\mathbb{C}$-valued cocycle τ on the equivariant map algebra can be written up to coboundary as $\tau = \phi \circ \omega_{\mathrm{alg}}$ for some continuous $\mathbb{C}$-linear functional ϕ on $(\overline{\Omega}^1_{\mathrm{alg}}(\mathbb{C}[X]) \otimes_{\mathbb{C}} V(\mathfrak{g}_{\mathbb{C}}))^{\Delta}$.

In the case of twisted multiloop algebras, a cocycle is continuous if it is of polynomial growth in the $\mathbb{Z}^n$-grading of $\mathbb{C}[T^n]$. If $\mathfrak{g}_{\mathbb{C}}$ is simple, then κ is just the $\mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})$-invariant Killing form, so that $V(\mathfrak{g}_{\mathbb{C}}) \simeq \mathbb{C}$ is a trivial Δ-representation. The universal cocycle thus takes values in $\overline{\Omega}^1_{\mathrm{alg}}(\mathbb{C}[T^n])^{\Delta}$, in agreement with the purely algebraic result [6]. It might not be overly optimistic to hope for universality of (2) for equivariant map algebras with semisimple $\mathfrak{g}_{\mathbb{C}}$ in a more general context.

References

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