CELLULAR AND QUASIHEREDITARY STRUCTURES OF GENERALIZED QUANTIZED SCHUR ALGEBRAS

STEPHEN DOTY AND ANTHONY GIAQUINTO

Abstract. We give a self-contained derivation of a new cellular basis of generalized quantized Schur algebras, independent of the theory of quantum groups. As a consequence, we obtain a new proof of the fact that these algebras are quasihereditary over a general ground field of characteristic zero, for any choice of parameter including roots of unity. Previous proofs of this result relied on a descent from a quantized enveloping algebra and the existence of the canonical basis of its positive part.

Introduction

We study a class $\mathbf{S}_q(\pi)$ of finite-dimensional algebras, the generalized quantized Schur algebras ("generalized q-Schur algebras" for short) associated to a given root datum, saturated set π of dominant weights, and choice of parameter q (a nonzero element of a field k). Since (see [D1, §5]) these algebras are quotients of the specialized quantized enveloping algebra $\mathbf{U}_q$ determined by the root datum, the study of representations of $\mathbf{S}_q(\pi)$ yields information on the representations of $\mathbf{U}_q$. The $\mathbf{S}_q(\pi)$ are q-deformations of the generalized Schur algebras $S(\pi)$ studied by Donkin [Do1], [Do2], [Do3] in the context of algebraic groups, where $S(\pi)$ is obtained as a certain quotient algebra of the algebra of distributions on the group, and its module category is equivalent to a certain full subcategory of the category of rational representations of the group.

Dipper and James [DJ] first studied the q-Schur algebras in Type A (see also Jimbo [Ji]). In previous work [DG1], [DG2], the authors discovered a presentation by generators and relations for the rational form of the Dipper–James–Jimbo q-Schur algebras. Actually, two such presentations were found, the latter of which involved a system $\{1_\mu\}$ of pairwise orthogonal idempotents which decomposes the identity. In [D1] the latter presentation was extended to an arbitrary generalized q-Schur algebra $\mathbf{S}(\pi)$ of arbitrary (finite) type and it was shown that specializations of $\mathbf{S}(\pi)$ in a field are quasihereditary. (The reader is referred to [CPS], [DR] for basic properties of the theory of quasihereditary algebras.) This result was proved by a descent argument using the canonical basis of the modified form of the corresponding quantized enveloping algebra. The main objective of the present paper is to obtain the same result by an *internal* approach, based solely on the defining generators and relations, and avoiding any appeal to the theory of quantum groups. In particular, the approach we present is not dependent on the existence of the canonical basis, and thus may be regarded as somewhat more elementary than the approach of [D1].

Our main new result is the elementary "integral" cellular (see Graham and Lehrer [GL]) basis of ${}_{\mathbf{A}}\mathbf{S}(\pi)$ attained in 4.1; this leads in 4.2 to a corresponding cellular basis of the k-form $\mathbf{S}_q(\pi)$, and it is applied in Theorem 4.5 to obtain a new proof that $\mathbf{S}_q(\pi)$ is quasihereditary.

Let us summarize our method of proof in greater detail. We start with the rational form of $\mathbf{S}(\pi)$ defined over the rational function field $\mathbf{Q}(v)$ where v is an indeterminate; see 1.5. First we establish a triangular decomposition $\mathbf{S} = \mathbf{S}^+\mathbf{S}^0\mathbf{S}^- = \mathbf{S}^-\mathbf{S}^0\mathbf{S}^+$ for $\mathbf{S} = \mathbf{S}(\pi)$, and apply it to study the left and right ideals $\Delta(\lambda) := \mathbf{S}1_\lambda = \mathbf{S}^-1_\lambda$, $\Delta^\sharp(\lambda) := 1_\lambda\mathbf{S} = 1_\lambda\mathbf{S}^+$ where λ is a *maximal* element of π. It turns out that these ideals are actually cell modules for $\mathbf{S}$, and the natural multiplication map

$$\mathbf{S}1_\lambda \otimes_{\mathbf{Q}(v)} 1_\lambda\mathbf{S} \to \mathbf{S}1_\lambda\mathbf{S}$$

is a vector space isomorphism. Furthermore, the two-sided ideal $\mathbf{S}1_\lambda\mathbf{S}$ is a cell ideal at the bottom of a cell chain. This implies that the kernel of the natural quotient map $\mathbf{S}(\pi) \to \mathbf{S}(\pi - \{\lambda\})$ is $\mathbf{S}1_\lambda\mathbf{S}$. By induction, we may assume that the cellularity of $\mathbf{S}(\pi - \{\lambda\})$ has already been established, and then the cellularity of $\mathbf{S}(\pi)$ itself follows. This approach yields the desired cellular basis of $\mathbf{S}$, and once that has been established, the quasihereditary structure on $\mathbf{S}_q(\pi)$ (Theorem 4.5) follows easily. It is interesting that in case the root datum is of rank one, the canonical basis can be easily derived from our new cellular basis (see 2.7).

Then we consider a natural "integral" form ${}_{\mathbf{A}}\mathbf{S} = {}_{\mathbf{A}}\mathbf{S}(\pi)$ of $\mathbf{S}(\pi)$, taken over the ring $\mathbf{A} = \mathbf{Q}[v, v^{-1}]$. Since $\mathbf{A}$ is a principal ideal domain, ${}_{\mathbf{A}}\mathbf{S}$ is free over $\mathbf{A}$, and by choosing an $\mathbf{A}$-basis of the corresponding integral form of each ideal ${}_{\mathbf{A}}\Delta(\lambda) = ({}_{\mathbf{A}}\mathbf{S})1_\lambda$, ${}_{\mathbf{A}}\Delta^\sharp(\lambda) = 1_\lambda({}_{\mathbf{A}}\mathbf{S})$ (for λ maximal) the above inductive construction yields a cellular $\mathbf{A}$-basis of ${}_{\mathbf{A}}\mathbf{S}$. Since a cellular structure is compatible with specialization [GL, (1.8)] this gives a cellular basis in every specialization

$$\mathbf{S}_q(\pi) := k \otimes_{\mathbf{A}} ({}_{\mathbf{A}}\mathbf{S}(\pi))$$

for every field k of characteristic zero and any choice of a non-zero element $q \in k$, where k is regarded as a $\mathbf{A}$-algebra by means of the natural homomorphism $\mathbf{A} \to k$ determined by sending v to q. It is then easy to check that a certain bilinear form defined on the cell modules is non-zero, and by a result of König and Xi this immediately implies that $\mathbf{S}_q(\pi)$ is quasihereditary. There are many important consequences of having a quasihereditary structure on a given finite-dimensional algebra. We list some of them in Section 4.8.

It would be desirable to extend the results of this paper to include specializations in fields of positive characteristic, in cases where q is not 1 (sometimes called the "mixed" case). To do that, we would need to work with "integral" forms taken not over the principal ideal domain $\mathbf{A} = \mathbf{Q}[v, v^{-1}]$ but rather over the ring $\mathcal{A} = \mathbf{Z}[v, v^{-1}]$. The trouble is that we do not know an elementary argument to establish that the $\mathcal{A}$-forms ${}_{\mathcal{A}}\Delta(\lambda)$ and ${}_{\mathcal{A}}\Delta^\sharp(\lambda)$ are free over $\mathcal{A}$, without appealing to properties of quantum groups, such as the existence of the canonical basis. The desired extension can be achieved if one is willing to take the freeness for granted.

It should be noted that the results of this paper extend to Donkin's original generalized Schur algebras $S(\pi)$. To do this one should take $v = 1$ in the defining presentation in 1.5 and repeat the arguments of the paper in that (simpler) context, replacing $\mathbf{A}$ by $\mathbf{Z}$; this allows specialization to any ground field k, including fields of positive characteristic. (In fact, k could more generally be taken to be an arbitrary commutative ring for many of the results.) We omit such a variation in order to keep this paper to a reasonable length.

1. The algebra $\mathbf{S} = \mathbf{S}(\pi)$

We define the rational form of the algebra $\mathbf{S}(\pi)$ by generators and relations (see 1.5), over the ground field $\mathbf{Q}(v)$ where v is an indeterminate. Later we shall specialize to a k-form $\mathbf{S}_q(\pi)$ of $\mathbf{S}(\pi)$, for an arbitrary ground field k of characteristic zero, by assigning v to a chosen nonzero element q of k.

1.1. We assume given a *root datum*, consisting of the following.

(a) A finite-dimensional $\mathbf{Q}$-vector space $\mathfrak{h}$ and two finite linearly independent subsets $\Pi = \{\alpha_i \colon i \in I\} \subset \mathfrak{h}^*$ (here $\mathfrak{h}^* = \mathrm{Hom}_{\mathbf{Q}}(\mathfrak{h}, \mathbf{Q})$ is the linear dual space of $\mathfrak{h}$) and $\Pi^\vee = \{\alpha_i^\vee \colon i \in I\} \subset \mathfrak{h}$ such that the square matrix $A = (a_{ij})_{i,j \in I}$ defined by

$$a_{ij} = \langle \alpha_i^\vee,\ \alpha_j \rangle = \alpha_j(\alpha_i^\vee)$$

is a symmetrizable generalized Cartan matrix. In other words, $a_{ii} = 2$, a_{ij} for $i \neq j$ is a non-positive integer, and $a_{ij} = 0$ if and only if $a_{ji} = 0$. Elements of Π are called simple roots, and elements of $\Pi^\vee$ simple coroots.

(b) An inner product $(\ ,\ )$ on $\mathfrak{h}^*$ such that (α_i, α_i) is a positive even integer for any $i \in I$, and

$$\langle \alpha_i^\vee,\ \lambda \rangle = \lambda(\alpha_i^\vee) = \frac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)} \qquad (i \in I, \lambda \in \mathfrak{h}^*).$$

(c) A lattice X in $\mathfrak{h}^*$ (the weight lattice) such that $\Pi \subset X$ and $\Pi^\vee \subset X^\vee := \{h \in \mathfrak{h} \colon \langle h,\ X \rangle \subseteq \mathbf{Z}\}$.

We put $d_i := (\alpha_i, \alpha_i)/2$ for each $i \in I$. Then the matrix $(d_i a_{ij})$ is symmetric.

1.2. The Kac–Moody algebra $\mathfrak{g}$ attached to the given root datum is the Lie algebra over $\mathbf{Q}$ generated by elements e_i, f_i $(i \in I)$ and $h \in X^\vee$ subject to the relations:

(a) $[h, h'] = 0$;
(b) $[e_i, f_j] = \delta_{ij} \alpha_i^\vee$;
(c) $[h, e_i] = \langle h,\ \alpha_i \rangle e_i, \quad [h, f_i] = -\langle h,\ \alpha_i \rangle f_i$;
(d) $(\operatorname{ad} e_i)^{1-a_{ij}}(e_j) = 0, \quad (\operatorname{ad} f_i)^{1-a_{ij}}(f_j) = 0 \quad$ (for $i \neq j$)

holding for all $i, j \in I$ and all $h, h' \in X^\vee$.

1.3. The universal enveloping algebra of $\mathfrak{g}$ is the associative $\mathbf{Q}$-algebra $U(\mathfrak{g})$ with 1 given by generators e_i, f_i $(i \in I)$ and $h \in X^\vee$ and satisfying the relations:

(a) $hh' = h'h$;
(b) $e_i f_j - f_j e_i = \delta_{ij} \alpha_i^\vee$;
(c) $he_i - e_i h = \langle h,\ \alpha_i \rangle e_i, \quad hf_i - f_i h = -\langle h,\ \alpha_i \rangle f_i$;
(d1) $\sum_{s=0}^{1-a_{ij}} (-1)^s \binom{1-a_{ij}}{s} e_i^{1-a_{ij}-s} e_j e_i^s = 0 \quad (i \neq j)$;
(d2) $\sum_{s=0}^{1-a_{ij}} (-1)^s \binom{1-a_{ij}}{s} f_i^{1-a_{ij}-s} f_j f_i^s = 0 \quad (i \neq j)$

for $i, j \in I$ and $h, h' \in X^\vee$.

1.4. Throughout this paper we assume that the root datum is of *finite* type; i.e., the Cartan matrix A is positive definite. This implies that $\mathfrak{g}$ is finite-dimensional and the Weyl group W is a finite group, where by definition W is the subgroup of $\mathrm{GL}(\mathfrak{h}^*)$ generated by the set of simple reflections s_i $(i \in I)$, where

$$s_i(\lambda) = \lambda - \langle \alpha_i^\vee,\ \lambda \rangle \alpha_i$$

for any $\lambda \in \mathfrak{h}^*$.

We have a partial order $\trianglerighteq$ on X (the dominance order) given by $\lambda \trianglerighteq \mu$ if and only if $\lambda - \mu = \sum_{i\in I} n_i\alpha_i$ where the $n_i \geqslant 0$ for all i. We write $\lambda \triangleright \mu$ if $\lambda \trianglerighteq \mu$ and $\lambda \neq \mu$. We write $\mu \trianglelefteq \lambda$ (respectively, $\mu \triangleleft \lambda$) if $\lambda \trianglerighteq \mu$ (resp., $\lambda \triangleright \mu$). Recall that the set X^+ of dominant weights is defined by

$$X^+ = \{\lambda \in X \colon \langle \alpha_i^\vee,\ \lambda\rangle \geqslant 0,\ \text{all } i \in I\}.$$

We say that a set π of dominant weights is *saturated* if π contains all dominant predecessors of all of its elements; i.e., $\mu' \triangleleft \mu$ for $\mu' \in X^+$ and $\mu \in \pi$ implies $\mu' \in \pi$.

Let $\mathbf{Q}(v)$ be the field of rational functions in an indeterminate v. Set $v_i = v^{d_i}$ for each $i \in I$. More generally, given any rational function $P \in \mathbf{Q}(v)$ we let P_i denote the rational function obtained from P by replacing v by v_i.

Set $\mathcal{A} = \mathbf{Z}[v, v^{-1}]$ and $\mathbf{A} = \mathbf{Q}[v, v^{-1}]$. Then $\mathcal{A}$ is an integral domain and $\mathbf{A}$ is a principal ideal domain, since it is a localization of the polynomial ring $\mathbf{Q}[v]$. Note that both $\mathcal{A}$ and $\mathbf{A}$ have field of fractions $\mathbf{Q}(v)$. For $a \in \mathbf{Z}$, $t \in \mathbf{N}$ we set

$$\begin{bmatrix} a \\ t \end{bmatrix} = \prod_{s=1}^{t} \frac{v^{a-s+1} - v^{-a+s-1}}{v^s - v^{-s}}.$$

By [Lu, 1.3.1(d)] this is an element of $\mathcal{A}$. We set

$$[n] = \begin{bmatrix} n \\ 1 \end{bmatrix} = \frac{v^n - v^{-n}}{v - v^{-1}} \qquad (n \in \mathbf{Z})$$

and

$$[n]^! = [1] \cdots [n-1]\,[n] \qquad (n \in \mathbf{N}).$$

Then it follows that

$$\begin{bmatrix} a \\ t \end{bmatrix} = \frac{[a]^!}{[t]^!\,[a-t]^!} \qquad \text{for all } 0 \leqslant t \leqslant a.$$

1.5. Let π be a finite saturated subset of the poset X^+ of dominant weights. The generalized q-Schur algebra $\mathbf{S}(\pi)$ is the algebra (associative with 1) over $\mathbf{Q}(v)$ given by generators E_i, F_i $(i \in I)$, 1_λ $(\lambda \in W\pi)$ together with the relations

(a) $1_\lambda 1_\mu = \delta_{\lambda\mu} 1_\lambda, \qquad \sum_{\lambda\in W\pi} 1_\lambda = 1;$

(b) $E_iF_j - F_jE_i = \delta_{ij} \sum_{\lambda\in W\pi} [\langle \alpha_i^\vee,\ \lambda\rangle]_i 1_\lambda;$

(c1) $E_i 1_\lambda = \begin{cases} 1_{\lambda+\alpha_i} E_i & \text{if } \lambda + \alpha_i \in W\pi \\ 0 & \text{otherwise;} \end{cases}$

(c2) $F_i 1_\lambda = \begin{cases} 1_{\lambda-\alpha_i} F_i & \text{if } \lambda - \alpha_i \in W\pi \\ 0 & \text{otherwise;} \end{cases}$

(c3) $1_\lambda E_i = \begin{cases} E_i 1_{\lambda-\alpha_i} & \text{if } \lambda - \alpha_i \in W\pi \\ 0 & \text{otherwise;} \end{cases}$

(c4) $1_\lambda F_i = \begin{cases} F_i 1_{\lambda+\alpha_i} & \text{if } \lambda + \alpha_i \in W\pi \\ 0 & \text{otherwise;} \end{cases}$

(d1) $\displaystyle\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_i E_i^{1-a_{ij}-s} E_j E_i^s = 0 \quad (i \neq j);$

(d2) $\displaystyle\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_i F_i^{1-a_{ij}-s} F_j F_i^s = 0 \quad (i \neq j)$

for any $i, j \in I$ and any $\lambda, \mu \in W\pi$.

We note that the defining relations immediately imply that the generators E_i, F_i are nilpotent in $\mathbf{S}(\pi)$, for any $i \in I$. To see this, observe that relation (c1) and the finiteness of $W\pi$ implies the existence of some natural number m such that $E_i^m 1_\mu = 0$ for any $\mu \in W\pi$, and hence $E_i^m = \sum_\mu E_i^m 1_\mu = 0$. The nilpotence of F_i is established similarly.

1.6. It is convenient to introduce additional generators 1_μ for any $\mu \in X$, with the stipulation $1_\mu = 0$ for any $\mu \in X$ such that $\mu \notin W\pi$. With this convention, the defining relations 1.5(a)–(c) for $\mathbf{S}(\pi)$ may be replaced by the following simplified version:

(a) $1_\lambda 1_\mu = \delta_{\lambda\mu} 1_\lambda, \qquad \sum_{\lambda \in X} 1_\lambda = 1;$

(b) $E_i F_j - F_j E_i = \delta_{ij} \sum_{\lambda \in X} [\langle \alpha_i^\vee,\ \lambda \rangle]_i 1_\lambda;$

(c) $E_i 1_\lambda = 1_{\lambda+\alpha_i} E_i; \quad F_i 1_\lambda = 1_{\lambda-\alpha_i} F_i$

holding for any $\lambda, \mu \in X$, and $i, j \in I$. Notice that the sums in (a), (b) are finite. Thus, $\mathbf{S}(\pi)$ is the algebra given by the generators E_i, F_i $(i \in I)$ and 1_λ $(\lambda \in X)$ subject to the relations 1.6(a), (b), (c) along with 1.5(d1), (d2) and the extra relation $1_\lambda = 0$ for all $\lambda \notin W\pi$.

Until further notice we fix π and write $\mathbf{S}$ for $\mathbf{S}(\pi)$. We introduce quantized divided powers $E_i^{(a)} = E_i^a/([a]^!_i)$, $F_i^{(a)} = F_i^a/([a]^!_i)$ for $i \in I$, $a \geqslant 0$. (For $a < 0$ we set $F_i^{(a)} = E_i^{(a)} = 0$.) Let ${}_{\mathbf{A}}\mathbf{S}$ be the $\mathbf{A}$-subalgebra of $\mathbf{S}$ generated by the $E_i^{(a)}$ and $F_i^{(b)}$, for $i \in I$ and $a, b \geqslant 0$, along with the idempotents 1_μ for $\mu \in W\pi$.

There is a unique $\mathbf{Q}(v)$-linear algebra anti-involution ι on $\mathbf{S}$ determined by the properties:

$$\iota(xy) = \iota(y)\iota(x) \text{ for all } x, y \in \mathbf{S},$$
$$\iota(E_i) = F_i,\ \iota(F_i) = E_i \text{ (any } i \in I\text{), and } \iota(1_\mu) = 1_\mu \text{ (any } \mu \in W\pi\text{).}$$

This is easily verified using the defining relations.

The following consequences of the defining relations will be needed later. Note that the sums in parts (ii) and (iii) below are finite.

1.7. **Lemma.** *Let $\mathbf{S} = \mathbf{S}(\pi)$. For any $a, b \geqslant 0$, $\mu \in X$, the following identities hold in $\mathbf{S}$:*

(i) $E_i^{(a)} 1_\mu = 1_{\mu+a\alpha_i} E_i^{(a)}$; $F_i^{(b)} 1_\mu = 1_{\mu-b\alpha_i} F_i^{(b)}$;

(ii) $E_i^{(a)} F_i^{(b)} 1_\mu = \sum_{t\geqslant 0} \left[\begin{smallmatrix} a-b+\langle \alpha_i^\vee,\, \mu\rangle \\ t \end{smallmatrix}\right]_i F_i^{(b-t)} E_i^{(a-t)} 1_\mu$;

(iii) $F_i^{(b)} E_i^{(a)} 1_\mu = \sum_{t\geqslant 0} \left[\begin{smallmatrix} b-a-\langle \alpha_i^\vee,\, \mu\rangle \\ t \end{smallmatrix}\right]_i E_i^{(a-t)} F_i^{(b-t)} 1_\mu$.

By applying (i) to each term of (ii) and (iii) to commute the idempotent to the middle of each product, and then replacing μ respectively by $-\lambda + b\alpha_i$ and $\lambda - a\alpha_i$, we obtain the following equivalent variants of formulas (ii) and (iii):

(ii′) $E_i^{(a)} 1_{-\lambda} F_i^{(b)} = \sum_{t\geqslant 0} \left[\begin{smallmatrix} a+b-\langle \alpha_i^\vee,\, \lambda\rangle \\ t \end{smallmatrix}\right]_i F_i^{(b-t)} 1_{-\lambda+(a+b-t)\alpha_i} E_i^{(a-t)}$;

(iii′) $F_i^{(b)} 1_\lambda E_i^{(a)} = \sum_{t\geqslant 0} \left[\begin{smallmatrix} a+b-\langle \alpha_i^\vee,\, \lambda\rangle \\ t \end{smallmatrix}\right]_i E_i^{(a-t)} 1_{\lambda-(a+b-t)\alpha_i} F_i^{(b-t)}$.

Proof. Part (i) is an obvious consequence of relation 1.6(c) and the definition of quantized divided powers.

We prove part (ii) by a double induction on a and b. First we observe that (ii) is vacuously true in case one of a or b is zero, so we may assume that both a and b are positive integers. Next one verifies by induction on a that from relation 1.6(b) we have

(a) $$E_i^{(a)} F_i 1_\mu = F_i E_i^{(a)} 1_\mu + [a - 1 + \langle \alpha_i^\vee,\ \mu\rangle]_i E_i^{(a-1)} 1_\mu$$

for all $a \geqslant 0$. This is elementary and left to the reader. A similar induction on b shows that

(b) $$E_i F_i^{(b)} 1_\mu = F_i^{(b)} E_i 1_\mu + [1 - b + \langle \alpha_i^\vee,\ \mu\rangle]_i F_i^{(b-1)} 1_\mu$$

for all $b \geqslant 0$. Now assume by induction that (ii) holds for some pair a, b of positive integers. Multiplying on the right by F_i and using the relation $1_\mu F_i = F_i 1_{\mu+\alpha_i}$ we obtain

$$E_i^{(a)} F_i^{(b)} F_i 1_{\mu+\alpha_i} = \sum_{t\geqslant 0} \left[\begin{smallmatrix} s-b \\ t \end{smallmatrix}\right]_i F_i^{(b-t)} E_i^{(a-t)} F_i 1_{\mu+\alpha_i}$$

where we have put $s := a + \langle \alpha_i^\vee,\ \mu\rangle$ for ease of notation. Now replace μ by $\mu - \alpha_i$ and use (a) and the definition of the divided powers to obtain

$$\begin{aligned}
&[b+1]_i E_i^{(a)} F_i^{(b+1)} 1_\mu \\
&\quad = \sum_{t\geqslant 0} \begin{bmatrix} s-b-2 \\ t \end{bmatrix}_i F_i^{(b-t)} \big(F_i E_i^{(a-t)} 1_\mu + [s-1-t]_i E_i^{(a-1-t)} 1_\mu \big) \\
&\quad = \sum_{t\geqslant 0} \begin{bmatrix} s-b-2 \\ t \end{bmatrix}_i [b+1-t]_i F_i^{(b+1-t)} E_i^{(a-t)} 1_\mu \\
&\qquad\qquad + \sum_{t\geqslant 0} \begin{bmatrix} s-b-2 \\ t \end{bmatrix}_i [s-1-t]_i F_i^{(b-t)} E_i^{(a-1-t)} 1_\mu
\end{aligned}$$

and by separating the first term in the first sum on the right hand side this takes the form

$$\begin{aligned}
[b+1]_i E_i^{(a)} F_i^{(b+1)} 1_\mu &= [b+1]_i F_i^{(b+1)} E_i^{(a)} 1_\mu \\
&+ \sum_{t\geqslant 1} \begin{bmatrix} s-b-2 \\ t \end{bmatrix}_i [b+1-t]_i F_i^{(b+1-t)} E_i^{(a-t)} 1_\mu \\
&\qquad + \sum_{t\geqslant 0} \begin{bmatrix} s-b-2 \\ t \end{bmatrix}_i [s-1-t]_i F_i^{(b-t)} E_i^{(a-1-t)} 1_\mu \,.
\end{aligned}$$

By shifting the index of summation in the first sum above we obtain

$$\begin{aligned}
[b+1]_i E_i^{(a)} F_i^{(b+1)} 1_\mu &= [b+1]_i F_i^{(b+1)} E_i^{(a)} 1_\mu \\
&+ \sum_{t\geqslant 0} \begin{bmatrix} s-b-2 \\ t+1 \end{bmatrix}_i [b-t]_i F_i^{(b-t)} E_i^{(a-1-t)} 1_\mu \\
&\qquad + \sum_{t\geqslant 0} \begin{bmatrix} s-b-2 \\ t \end{bmatrix}_i [s-1-t]_i F_i^{(b-t)} E_i^{(a-1-t)} 1_\mu \,.
\end{aligned}$$

Now we observe that by direct computation that

$$\begin{bmatrix} s-b-2 \\ t+1 \end{bmatrix}_i [b-t]_i + \begin{bmatrix} s-b-2 \\ t \end{bmatrix}_i [s-1-t]_i = [b+1]_i \begin{bmatrix} s-b-1 \\ t+1 \end{bmatrix}_i$$

and thus after dividing through by $[b+1]_i$ we obtain the equality

$$E_i^{(a)} F_i^{(b+1)} 1_\mu = F_i^{(b+1)} E_i^{(a)} 1_\mu + \sum_{t\geqslant 0} \begin{bmatrix} s-b-1 \\ t+1 \end{bmatrix}_i F_i^{(b-t)} E_i^{(a-1-t)} 1_\mu \,.$$

Upon shifting the index of summation again and recombining the first term this becomes

$$E_i^{(a)} F_i^{(b+1)} 1_\mu = \sum_{t\geqslant 0} \begin{bmatrix} s-b-1 \\ t \end{bmatrix}_i F_i^{(b+1-t)} E_i^{(a-t)} 1_\mu$$

which proves that (ii) holds for the pair a, $b+1$. The computation proving that (ii) for the pair a, b implies (ii) for the pair $a+1$, b is similar to the computation given above (using (b) in place of (a)) so we leave it to the reader.

Identity (iii) can be proved by a similar argument, but it is simpler to notice that there is an algebra automorphism ω of $\mathbf{S}$ of order two given on generators by

$$\omega(E_i) = F_i, \quad \omega(F_i) = E_i, \quad \omega(1_\mu) = 1_{-\mu}$$

and that by applying the involution ω to identity (ii), interchanging a and b, and then replacing $-\mu$ by μ, we obtain identity (iii). This completes the proof. □

1.8. The orthogonal idempotent decomposition of the identity given in defining relation 1.5(a) or its equivalent form 1.6(a) implies that if M is a left $\mathbf{S}$-module or a left ${}_{\mathbf{A}}\mathbf{S}$-module and if N is a right $\mathbf{S}$-module or a right ${}_{\mathbf{A}}\mathbf{S}$-module then there are direct sum decompositions of the form

(a) $$M = \bigoplus_{\mu \in W\pi} 1_\mu M; \qquad N = \bigoplus_{\mu \in W\pi} N 1_\mu.$$

We call $1_\mu M$ and $N1_\mu$ left and right *weight spaces* of M and N, respectively, and refer to the above decompositions as left and right *weight space decompositions*. Furthermore, if M is a $\mathbf{S}$-bimodule or a ${}_{\mathbf{A}}\mathbf{S}$-bimodule then we have a finer direct sum decomposition of the form

(b) $$M = \bigoplus_{\mu,\mu' \in W\pi} 1_\mu M 1_{\mu'}.$$

We call the subspaces $1_\mu M 1_{\mu'}$ in this decomposition *biweight* spaces. Elements of $1_\mu M 1_{\mu'}$ are said to have biweight (μ, μ'). In particular, $\mathbf{S}$ itself is a $\mathbf{S}$-bimodule under left and right multiplication, and similarly for ${}_{\mathbf{A}}\mathbf{S}$, so $\mathbf{S}$ and ${}_{\mathbf{A}}\mathbf{S}$ have such a decomposition:

(c) $$\mathbf{S} = \bigoplus_{\mu,\mu' \in W\pi} 1_\mu \mathbf{S} 1_{\mu'}; \qquad {}_{\mathbf{A}}\mathbf{S} = \bigoplus_{\mu,\mu' \in W\pi} 1_\mu ({}_{\mathbf{A}}\mathbf{S}) 1_{\mu'}$$

Put $\mathbf{Z}\Pi = \sum \mathbf{Z}\alpha_i \subseteq X$ (the root lattice). We have another direct sum decomposition

(d) $$\mathbf{S} = \bigoplus_{\nu \in \mathbf{Z}\Pi} \mathbf{S}_\nu$$

defined by the requirements: $\mathbf{S}_\nu \mathbf{S}_{\nu'} \subseteq \mathbf{S}_{\nu+\nu'}$, $E_i \in \mathbf{S}_{\alpha_i}$, $F_i \in \mathbf{S}_{-\alpha_i}$, and $1_\mu \in \mathbf{S}_0$. Putting ${}_{\mathbf{A}}\mathbf{S}_\nu = {}_{\mathbf{A}}\mathbf{S} \cap \mathbf{S}_\nu$ we also have

(e) $${}_{\mathbf{A}}\mathbf{S} = \bigoplus_{\nu \in \mathbf{Z}\Pi} {}_{\mathbf{A}}\mathbf{S}_\nu.$$

Combining (d), (e) with the biweight space decomposition (c) we have the finer decompositions

(f) $$\mathbf{S} = \bigoplus_{\nu,\mu,\mu'} 1_\mu \mathbf{S}_\nu 1_{\mu'}; \qquad {}_{\mathbf{A}}\mathbf{S} = \bigoplus_{\nu,\mu,\mu'} 1_\mu ({}_{\mathbf{A}}\mathbf{S}_\nu) 1_{\mu'}$$

It is easy to see that in these decompositions $1_\mu \mathbf{S}_\nu 1_{\mu'} = 0$ and $1_\mu ({}_{\mathbf{A}}\mathbf{S}_\nu) 1_{\mu'} = 0$ unless $\mu - \mu' = \nu$ in X.

Let $\mathbf{S}^+$ (respectively, $\mathbf{S}^-$) be the $\mathbf{Q}(v)$-subalgebra of $\mathbf{S}$ generated by the E_i (resp., the F_i) for $i \in I$. Let $\mathbf{S}^0$ be the $\mathbf{Q}(v)$-subalgebra of $\mathbf{S}$ generated by the 1_μ for $\mu \in W\pi$; this is the group algebra over $\mathbf{Q}(v)$ of the abelian group $\{1_\mu \colon \mu \in W\pi\}$. Let ${}_{\mathbf{A}}\mathbf{S}^+$ (respectively, ${}_{\mathbf{A}}\mathbf{S}^-$) be the $\mathbf{A}$-subalgebra of ${}_{\mathbf{A}}\mathbf{S}$ generated by all the $E_i^{(a)}$ (resp., all the $F_i^{(b)}$) for $i \in I$, $a, b \geqslant 0$. Let ${}_{\mathbf{A}}\mathbf{S}^0$ be the $\mathbf{A}$-subalgebra of $\mathbf{S}$ generated by the 1_μ for $\mu \in W\pi$.

We will refer to the following result as the *triangular decomposition* of $\mathbf{S}$ and ${}_{\mathbf{A}}\mathbf{S}$.

1.9. **Lemma.** *Let* $\mathbf{S} = \mathbf{S}(\pi)$. *We have* $\mathbf{S} = \mathbf{S}^-\mathbf{S}^0\mathbf{S}^+$ *and* ${}_{\mathbf{A}}\mathbf{S} = {}_{\mathbf{A}}\mathbf{S}^-\,{}_{\mathbf{A}}\mathbf{S}^0\,{}_{\mathbf{A}}\mathbf{S}^+$. *The same equalities hold if the three factors on the right hand side are permuted in any order.*

Proof. By the defining relations in 1.5 or 1.6 the algebra $\mathbf{S}$ is spanned by elements of the form

(a) $$P = x_1 \cdots x_m 1_\mu$$

where $x_1, \dots, x_m \in \{E_i, F_i \colon i \in I\}$ and $\mu \in W\pi$. For an element P_0 of such form, let $d(P_0)$ be the number of pairs (j, j') such that $1 \leqslant j \leqslant j' \leqslant m$ and $x_j \in \{E_i \colon i \in I\}$, $x_{j'} \in \{F_i \colon i \in I\}$. We claim that P_0 can be rewritten as a finite $\mathbf{Q}(v)$-linear combination of elements of the form (a) for which $d = 0$. This is proved by induction on d, using the following consequences of the defining relations 1.5(b):

(b) $$E_iF_j = F_jE_i \ (i \neq j); \quad E_iF_i1_\mu = F_iE_i1_\mu + [\langle \alpha_i^\vee,\ \mu\rangle]_i 1_\mu.$$

Thus it follows that all elements of $\mathbf{S}$ can be expressed as linear combinations of products P of the form (a) with $d(P) = 0$. Since elements of the form (a) with $d = 0$ have all occurrences of F_j appearing before any E_i, it follows that $\mathbf{S} = \mathbf{S}^-\mathbf{S}^+\mathbf{S}^0$. To get the equality $\mathbf{S} = \mathbf{S}^+\mathbf{S}^-\mathbf{S}^0$ just repeat the argument with the E's and F's interchanged. Finally, in any product of the form (a) with $d(P) = 0$ we can use relation 1.5(c) to commute the idempotent to the left of all the E_i's and the right of all the F_j's, so we obtain the equality $\mathbf{S} = \mathbf{S}^-\mathbf{S}^0\mathbf{S}^+$. The other variations are obtained similarly.

The corresponding claim for the integral forms now follows from the definitions. ☐

1.10. **Corollary.** *Let* $\mathbf{S} = \mathbf{S}(\pi)$. *Suppose* λ *is maximal in* π, *with respect to the dominance order* $\trianglelefteq$. *Then* $\mathbf{S}1_\lambda = \mathbf{S}^-1_\lambda$ *and* $1_\lambda\mathbf{S} = 1_\lambda\mathbf{S}^+$. *Similarly,* ${}_{\mathbf{A}}\mathbf{S}1_\lambda = {}_{\mathbf{A}}\mathbf{S}^-1_\lambda$ *and* $1_\lambda({}_{\mathbf{A}}\mathbf{S}) = 1_\lambda({}_{\mathbf{A}}\mathbf{S}^+)$.

Proof. Since λ is maximal, it follows from defining relation 1.6(c) that $E_i1_\lambda = 0$ for any $i \in I$, so $\mathbf{S}^+1_\lambda = \mathbf{Q}(v)1_\lambda$. Clearly $\mathbf{S}^01_\lambda = \mathbf{Q}(v)1_\lambda$, so the first equality follows from the triangular decomposition $\mathbf{S} = \mathbf{S}^-\mathbf{S}^0\mathbf{S}^+$. The proof of the second equality is similar. The corresponding results for integral forms follows from the definitions. $\square$

1.11. Let $\mathbf{N} = \mathbf{Z}_{\geqslant 0}$ be the set of non-negative integers, and let $\mathbf{N}\Pi$ be the submonoid of $\mathbf{Z}\Pi$ consisting of all linear combinations $\sum_i \nu_i\alpha_i$ with coefficients $\nu_i \in \mathbf{N}$. Then

(a) $$\mathbf{S}^+ = \bigoplus_{\nu\in\mathbf{N}\Pi} \mathbf{S}^+_\nu; \qquad \mathbf{S}^- = \bigoplus_{\nu\in\mathbf{N}\Pi} \mathbf{S}^-_{-\nu}$$

where for any $\nu = \sum_i \nu_i\alpha_i \in \mathbf{N}\Pi$, $\mathbf{S}^+_\nu$ is the $\mathbf{Q}(v)$-subspace of $\mathbf{S}^+$ spanned by all monomials of the form $E_{i_1}E_{i_2}\cdots E_{i_r}$ such that, for each $i \in I$, the number of occurrences of i in the sequence $i_1, i_2, \dots, i_r$ is ν_i and similarly, $\mathbf{S}^-_{-\nu}$ is the $\mathbf{Q}(v)$-subspace of $\mathbf{S}^-$ spanned by all monomials of the form $F_{i_1}F_{i_2}\cdots F_{i_r}$ such that, for each $i \in I$, the number of occurrences of i in the sequence $i_1, i_2, \dots, i_r$ is ν_i. Obviously $\mathbf{S}^+_\nu$ and $\mathbf{S}^-_{-\nu}$ are finite dimensional over $\mathbf{Q}(v)$, for each $\nu \in \mathbf{N}\Pi$. Put ${}_\mathbf{A}\mathbf{S}^+_\nu = {}_\mathbf{A}\mathbf{S}\cap\mathbf{S}^+_\nu$ and ${}_\mathbf{A}\mathbf{S}^-_\nu = {}_\mathbf{A}\mathbf{S} \cap \mathbf{S}^-_\nu$; then it follows that

(b) $${}_\mathbf{A}\mathbf{S}^+ = \bigoplus_{\nu\in\mathbf{N}\Pi} {}_\mathbf{A}\mathbf{S}^+_\nu; \qquad {}_\mathbf{A}\mathbf{S}^- = \bigoplus_{\nu\in\mathbf{N}\Pi} {}_\mathbf{A}\mathbf{S}^-_{-\nu}.$$

1.12. **Lemma.** *For any finite saturated subset π of X^+ let $\mathbf{S} = \mathbf{S}(\pi)$. Then the algebra $\mathbf{S}$ is finite-dimensional over $\mathbf{Q}(v)$.*

Proof. By the triangular decomposition, $\mathbf{S} = \mathbf{S}^-\mathbf{S}^0\mathbf{S}^+$, so there is a spanning set for $\mathbf{S}$ consisting of all elements of the form $F_A 1_\mu E_B$ where $\mu \in W\pi$ and

$$F_A = F_{i_1}\cdots F_{i_r}, \qquad E_B = E_{j_1}\cdots E_{j_s}$$

for various finite sequences $A = (i_1, \dots, i_r)$, $B = (j_1, \dots, j_s)$ of elements of I. Here $F_A \in \mathbf{S}^-_\nu$ and $E_B \in \mathbf{S}^+_{\nu'}$ for appropriate $\nu, \nu' \in \mathbf{N}\Pi$. But for ν sufficiently large F_A acts as zero on all 1_μ for $\mu \in W\pi$, hence is equal to zero in $\mathbf{S}$, and similarly, for ν sufficiently large E_B is zero as well. Thus there are only finitely many nonzero summands in 1.11(a), and the result follows. $\square$

1.13. **Lemma.** *The algebra ${}_\mathbf{A}\mathbf{S}$ is free over $\mathbf{A}$, and the natural map $\mathbf{Q}(v) \otimes_\mathbf{A} ({}_\mathbf{A}\mathbf{S}) \to \mathbf{S}$ is an isomorphism of $\mathbf{Q}(v)$-algebras. Similar statements apply to $\mathbf{S}^+$, $\mathbf{S}^-$ and to the subspaces $\mathbf{S}^+_\nu$ and $\mathbf{S}^-_{-\nu}$ appearing in the decomposition* 1.11(a).

Proof. Clearly ${}_\mathbf{A}\mathbf{S}$ is finitely generated and torsion free, hence free over the principal ideal domain $\mathbf{A}$. Since $\mathbf{Q}(v)$ is the field of fractions of $\mathbf{A}$, the natural map $\mathbf{Q}(v) \otimes_\mathbf{A} ({}_\mathbf{A}\mathbf{S}) \to \mathbf{S}$ is injective. The surjectivity is clear. The other cases are similar. $\square$

The lemma implies that an $\mathbf{A}$-basis of ${}_{\mathbf{A}}\mathbf{S}$ is also a $\mathbf{Q}(v)$-basis of $\mathbf{S}$ and similarly for $\mathbf{S}^+$ and $\mathbf{S}^-$.

1.14. Suppose that M is a left $\mathbf{S}$-module. Then M has weight space decomposition $M = \bigoplus_{\mu \in W\pi} 1_\mu M$. From relations 1.6(c), (a) we see that for any $i \in I$

(a) $$E_i(1_\mu M) \subseteq 1_{\mu+\alpha_i} M; \qquad F_i(1_\mu M) \subseteq 1_{\mu-\alpha_i} M$$

for $\mu \in W\pi$. So acting by E_i increases the weight by α_i and acting by F_i decreases the weight by α_i. We also have

(b) $$1_{\mu'}(1_\mu M) = \delta_{\mu,\mu'} 1_\mu M$$

for any $\mu, \mu' \in W\pi$. We make the following definitions.

Definition. Let M be an $\mathbf{S}$-module, for $\mathbf{S} = \mathbf{S}(\pi)$.

(i) If λ in $W\pi$ has the property that $1_\lambda M \neq 0$ but $1_\mu M = 0$ for all $\mu \rhd \lambda$, then we say that λ is a *highest weight* of M and we call any nonzero element of $1_\lambda M$ a *highest weight vector*.

(ii) If $0 \neq x_0 \in M$ is a weight vector such that $E_i \cdot x_0 = 0$ for every $i \in I$, then x_0 is called a *maximal vector* of M.

Obviously a highest weight vector must be a maximal vector. If $M \neq 0$ is a finite-dimensional $\mathbf{S}$-module then M has at least one highest weight vector, and thus has a maximal vector. We wish to study the submodules of M which are generated by a chosen maximal vector.

1.15. **Lemma.** *Let $\mathbf{S} = \mathbf{S}(\pi)$. Let x_0 be a maximal vector of weight λ in a finite-dimensional left $\mathbf{S}$-module M, and put $V = \mathbf{S}x_0$. Then:*

(a) *V is the $\mathbf{Q}(v)$-span of elements of the form $F_{i_1}F_{i_2}\cdots F_{i_r}x_0$ for various finite sequences $(i_1, \dots, i_r)$ (including the empty sequence) chosen from I.*

(b) *If $1_\mu V \neq 0$ then $\mu \trianglelefteq \lambda$, so x_0 is a highest weight vector of V.*

(c) $\dim_{\mathbf{Q}(v)} 1_\lambda V = 1$.

(d) *V is indecomposable, with a unique maximal submodule and a corresponding unique simple quotient.*

Proof. Part (a) follows from the triangular decomposition, which implies that $V = \mathbf{S}^- x_0$. Part (b) follows from part (a), part (c) is obvious, and to get part (d), let V' be the sum of all proper submodules and note that $V' \neq V$ since no proper submodule contains x_0. □

1.16. **Corollary.** *Let $\mathbf{S} = \mathbf{S}(\pi)$. Any simple left $\mathbf{S}$-module has a unique highest weight vector x_0, up to scalar multiple.*

Proof. Suppose M is simple; then M is finite dimensional and thus has a maximal vector. Any maximal vector necessarily generates M and is a highest weight vector. Suppose that there are two maximal vectors, of weight λ and λ', respectively. Then by Lemma 1.15(b), we must have $\lambda' \trianglelefteq \lambda$ and $\lambda \trianglelefteq \lambda'$, so $\lambda = \lambda'$. But Lemma 1.15(c) forces the two maximal vectors to be proportional. □

1.17. Inspired by prior work of Kahzdan and Lusztig and others, Graham and Lehrer in [GL] introduced the notion of a cellular algebra. This is an algebra given by an explicit basis (called a cellular basis) and an anti-involution, satisfying certain combinatorial properties. The motivating examples include Hecke algebras in Type A and Brauer algebras, but there are many other classes of examples.

In [KX1, Definition 3.2], [KX3, Definition 2.2] König and Xi gave an equivalent, basis-free, definition of cellularity for finite-dimensional algebras, as follows.

Definition (König and Xi)**.** Let A be a finite-dimensional k-algebra, where k is a field. Assume there is an anti-automorphism ι on A with $\iota^2 = \mathrm{id}$. A two-sided ideal J in A is called a *cell ideal* if $\iota(J) = J$ and there exists a left ideal $\Delta \subset J$ and an isomorphism of A-bimodules $\omega\colon J \to \Delta \otimes_k \iota(\Delta)$ making the following diagram commutative:

$$\begin{tikzcd} J \arrow[r, "\omega"] \arrow[d, "\iota"'] & \Delta \otimes_k \iota(\Delta) \arrow[d, "x\otimes y \mapsto \iota(y)\otimes\iota(x)"] \\ J \arrow[r, "\omega"] & \Delta \otimes_k \iota(\Delta) \end{tikzcd}$$

The algebra A (with the anti-involution ι) is called *cellular* if there is a vector space decomposition $A = J_1' \oplus J_2' \oplus \cdots \oplus J_n'$ (for some n) with $\iota(J_j') = J_j'$ for each j and such that setting $J_j = J_1' \oplus \cdots \oplus J_j'$ gives a chain of two-sided ideals of A: $0 = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = A$ (each of them fixed by ι) and for each j ($j = 1, \cdots, n$), the quotient $J_j' = J_j/J_{j-1}$ is a cell ideal (with respect to the involution induced by ι on the quotient) of A/J_{j-1}.

This will be needed in Section 3.

2. The rank one case

In this section we assume that the root datum has rank 1; i.e., there is just one simple root $\alpha = \alpha_i$, and single generators $E = E_i$, $F = F_i$ of the positive and negative parts. This implies that the Weyl group W has order 2, with a single generator $s = s_i$. In order to simplify the notation in this section, we assume throughout that $d = d_i = (\alpha, \alpha)/2$ is equal to 1. The general case can be recovered simply by replacing v by v^d and q by q^d throughout.

We wish to classify the simple representations of $\mathbf{S} = \mathbf{S}(\pi)$ in this case. This prepares the way for the classification of simple representations in higher ranks.

2.1. For any $\mu \in \pi$, we denote by $\mathbf{S}[\trianglerighteq\mu]$ the $\mathbf{Q}(v)$-subspace of $\mathbf{S}$ spanned by the collection of elements of the form

$$\text{(a)} \qquad F^{(b)} 1_{\mu+c\alpha} E^{(a)} \qquad (a, b, c \geqslant 0).$$

We also denote by $\mathbf{S}[\triangleright\mu]$ the $\mathbf{Q}(v)$-subspace of $\mathbf{S}$ spanned by the collection of elements of the form

$$\text{(b)} \qquad F^{(b)} 1_{\mu+c\alpha} E^{(a)} \qquad (a, b \geqslant 0,\ c > 0).$$

Note that $\mathbf{S}[\triangleright\mu] = \sum_{\mu' \triangleright \mu} \mathbf{S}[\trianglerighteq\mu']$, and in case μ is a maximal element of π (i.e., there exists no $\lambda \in \pi$ such that $\lambda \triangleright \mu$) we have $\mathbf{S}[\triangleright\mu] = (0)$. We observe the following.

2.2. **Lemma.** *For any $\mu \in \pi$, both $\mathbf{S}[\trianglerighteq\mu]$ and $\mathbf{S}[\triangleright\mu]$ are two-sided ideals of $\mathbf{S}$. Hence the quotient $\mathbf{S}[\trianglerighteq\mu]/\mathbf{S}[\triangleright\mu]$ is naturally an $\mathbf{S}$-bimodule.*

Proof. Put $m := \langle \alpha^\vee,\ \lambda \rangle$. From Lemma 1.7 it follows by a routine calculation that

$$E \cdot F^{(b)} 1_\lambda E^{(a)} = [a+1] F^{(b)} 1_{\lambda+\alpha} E^{(a+1)} + [1-b+m] F^{(b-1)} 1_\lambda E^{(a)}$$

and from the definition of quantized divided powers it follows that

$$F \cdot F^{(b)} 1_\lambda E^{(a)} = [b+1] F^{(b+1)} 1_\lambda E^{(a)}$$

for all $a, b \geqslant 0$. By applying the anti-involution ι to the formulas in Lemma 1.7 we obtain similar formulas with the idempotent on the left of each term, and it follows by calculations similar to the above that

$$F^{(b)} 1_\lambda E^{(a)} \cdot F = [b+1] F^{(b+1)} 1_{\lambda+\alpha} E^{(a)} + [1-a+m] F^{(b)} 1_\lambda E^{(a-1)}$$

and

$$F^{(b)} 1_\lambda E^{(a)} \cdot E = [a+1] F^{(b)} 1_\lambda E^{(a+1)}$$

for all $a, b \geqslant 0$. Moreover, multiplying $F^{(b)} 1_\lambda E^{(a)}$ on the left or right by some idempotent 1_μ either produces zero or gives the element $F^{(b)} 1_\lambda E^{(a)}$ back again. Since these equalities hold for arbitrary λ, the claims follow. $\square$

2.3. Let $\lambda \in \pi$. Setting $\mu = s(\lambda)$, $a = b = \langle \alpha^\vee,\ \lambda\rangle$ in Lemma 1.7(iii) we obtain

$$F^{(a)} E^{(a)} 1_{s(\lambda)} = 1_{s(\lambda)} \tag{a}$$

which by Lemma 1.7(i) implies immediately that

$$F^{(a)} 1_\lambda E^{(a)} = 1_{s(\lambda)}. \tag{b}$$

From this and the triangular decomposition (Lemma 1.9) it follows that $\mathbf{S}$ is spanned by the set of all elements of the form $F^{(a)} 1_\lambda E^{(b)}$ where $\lambda \in \pi$, $a, b \geqslant 0$. In particular, if we fix λ and define Q_λ to be the subspace of $\mathbf{S}$ spanned by the nonzero products of the form $F^{(a)} 1_\lambda E^{(b)}$ $(a, b \geqslant 0)$ then we have a vector space decomposition

$$\mathbf{S} = \textstyle\bigoplus_{\lambda \in \pi} Q_\lambda. \tag{c}$$

The set of nonzero products of the form $F^{(a)} 1_\lambda E^{(b)}$ $(a, b \geqslant 0)$ is linearly independent, since the elements have distinct biweights, hence the set is a basis for Q_λ. This set is also a set of coset representatives for the quotient space $\mathbf{S}[\trianglerighteq\lambda]/\mathbf{S}[\triangleright\lambda]$.

Since $\mathbf{S}[\trianglerighteq\lambda]$ and $\mathbf{S}[\triangleright\lambda]$ are two-sided ideals in $\mathbf{S}$, the quotient space $\mathbf{S}[\trianglerighteq\lambda]/\mathbf{S}[\triangleright\lambda]$ has an $\mathbf{S}$-bimodule structure. For any $\lambda \in \pi$, we let $\Delta(\lambda)$ be the left $\mathbf{S}$-submodule of $\mathbf{S}[\trianglerighteq\lambda]/\mathbf{S}[\triangleright\lambda]$ generated by $1_\lambda + \mathbf{S}[\triangleright\lambda]$, and let $\Delta^\sharp(\lambda)$ be the right $\mathbf{S}$-submodule of $\mathbf{S}[\trianglerighteq\lambda]/\mathbf{S}[\triangleright\lambda]$ generated by $1_\lambda + \mathbf{S}[\triangleright\lambda]$. For ease of notation, we put $x_0 := 1_\lambda + \mathbf{S}[\triangleright\lambda]$; then we have $E \cdot x_0 = 0 = x_0 \cdot F$.

2.4. **Lemma.** *Put $x_t = F^{(t)} \cdot x_0 \in \Delta(\lambda)$ and $x'_t = x_0 \cdot E^{(t)} \in \Delta^\sharp(\lambda)$, for $t \geqslant 0$. For $t < 0$ put $x_t = 0$. Then with $m := \langle \alpha^\vee,\ \lambda\rangle$ we have $x_t = 0 = x'_t$ for all $t > m$, and*

$$F \cdot x_t = [t+1]\, x_{t+1}, \qquad E \cdot x_t = [m-(t-1)]\, x_{t-1}$$
$$x'_t \cdot E = [t+1]\, x'_{t+1}, \qquad x'_t \cdot F = [m-(t-1)]\, x'_{t-1}.$$

Thus $\{x_0, \dots, x_m\}$ is a basis for $\Delta(\lambda)$ and $\{x'_0, \dots, x'_m\}$ is a basis for $\Delta^\sharp(\lambda)$, and both $\Delta(\lambda)$ and $\Delta^\sharp(\lambda)$ are simple $\mathbf{S}$-modules.

Proof. The formulas are proved by elementary calculations similar to those appearing in the proof of Lemma 2.2. They imply that $\Delta(\lambda)$ and $\Delta^\sharp(\lambda)$ are respectively generated, as $\mathbf{S}$-modules, by x_t and x'_t, for any $0 \leqslant t \leqslant m$. The simplicity of $\Delta(\lambda)$ and $\Delta^\sharp(\lambda)$ follows. Since the elements $x_0, \dots, x_m$ have distinct left weights, they are linearly independent, and hence form a basis of $\Delta(\lambda)$. Similarly, the elements $x'_0, \dots, x'_m$ form a basis for $\Delta^\sharp(\lambda)$. □

2.5. **Lemma.** *The natural map $\Delta(\lambda) \otimes_{\mathbf{Q}(v)} \Delta^\sharp(\lambda) \to \mathbf{S}[\trianglerighteq\lambda]/\mathbf{S}[\triangleright\lambda]$, defined by sending $x \otimes x'$ to xx', is surjective.*

Proof. This is clear, since $\mathbf{S}[\trianglerighteq\lambda]/\mathbf{S}[\triangleright\lambda]$ is spanned by the set of cosets of the form $F^{(b)}1_\lambda E^{(a)} + \mathbf{S}[\triangleright\lambda]$ such that $a, b \geqslant 0$. □

2.6. **Proposition.** *In the rank one case, the algebra $\mathbf{S} = \mathbf{S}(\pi)$ is semisimple, and a complete set of isomorphism classes of simple $\mathbf{S}$-modules is given by $\{\Delta(\lambda)\colon \lambda \in \pi\}$.*

Proof. By Lemma 2.5 and the decomposition 2.3(c) it follows that

$$\dim \mathbf{S} = \sum_{\lambda\in\pi} \dim \mathbf{S}[\trianglerighteq\lambda]/\mathbf{S}[\triangleright\lambda] \leqslant \sum_{\lambda\in\pi} (\dim \Delta(\lambda))^2$$

(all dimensions are over $\mathbf{Q}(v)$). On the other hand, since the $\Delta(\lambda)$ are pairwise non-isomorphic simple modules (indeed, no two of them has the same highest weight), the standard theory of finite-dimensional algebras implies that

$$\dim \mathbf{S} \geqslant \sum_{\lambda\in\pi} (\dim \Delta(\lambda))^2.$$

It follows that

$$\dim \mathbf{S} = \sum_{\lambda\in\pi} (\dim \Delta(\lambda))^2$$

and thus that $\mathbf{S}$ is semisimple, with simple modules as stated. □

2.7. From the above results it follows that the multiplication map $\Delta(\lambda) \otimes_{\mathbf{Q}(v)} \Delta^\sharp(\lambda) \to \mathbf{S}[\trianglerighteq\lambda]/\mathbf{S}[\triangleright\lambda]$ is actually an isomorphism of $\mathbf{S}$-bimodules, for each $\lambda \in \pi$. Moreover, the set

(a) $$\textstyle\bigsqcup_{\lambda\in\pi} \{F^{(b)}1_\lambda E^{(a)}\colon 0 \leqslant a, b \leqslant \langle \alpha^\vee,\ \lambda\rangle\}$$

is a basis for $\mathbf{S}$ over $\mathbf{Q}(v)$. This basis is actually a cellular basis, in the sense of Graham and Lehrer [GL].

It is interesting to compare this cellular basis to Lusztig's canonical basis in the rank 1 case. In order to make the comparison, let us assume temporarily that $X = \mathbf{Z} = X^\vee$ with $\alpha = 2$ and $\alpha^\vee = 1$. Then a saturated set of dominant weights is just a subset π of the positive integers such that $n \in \pi$ and $n - 2 \geqslant 0$ imply that $n - 2 \in \pi$. For any $n \in \pi$, if $a + b \geqslant n$ then we have

(b) $$F^{(b)}1_n E^{(a)} = \textstyle\sum_{t\geqslant 0} \left[\begin{smallmatrix} a+b-n \\ t \end{smallmatrix}\right] E^{(a-t)} 1_{n-2(a+b-t)} F^{(b-t)}.$$

This follows from Lemma 1.7 by an easy calculation. This shows that for all $a + b \geqslant n$ the element $F^{(b)}1_n E^{(a)}$ is equal to the element $E^{(n-b)}1_{-n}F^{(n-a)}$ modulo terms in $\mathbf{S}[\triangleright n]$. Notice that if we put $a' = n - b$, $b' = n - a$ then $a' + b' \leqslant n$. Thus, we obtain a different basis of $\mathbf{S}$ of the form

(c) $$\textstyle\bigsqcup_{n\in\pi} \{F^{(b)}1_n E^{(a)}\colon a + b \leqslant n\} \cup \{E^{(a')}1_{-n}F^{(b')}\colon a' + b' \leqslant n\}$$

which has a unitriangular relation with the original basis. Note that in (c) there is overlap between the two sets $\{F^{(b)}1_n E^{(a)} : a + b \leqslant n\}$, $\{E^{(a')}1_n F^{(b')} : a' + b' \leqslant n\}$ since when $a + b = n$ we have from (b) the equality $F^{(b)}1_n E^{(a)} = E^{(a)}1_{-n}F^{(b)}$. The basis in (c) is the canonical basis; compare with [Lu, 29.4.3]. (One should recall that, by [D1], there is a natural quotient map from the modified form $\dot{\mathbf{U}}$ of the quantized enveloping algebra determined by the given root datum onto $\mathbf{S}(\pi)$, taking generators onto generators.)

2.8. We now consider the issue of specialization, in the rank 1 case. Recall the "integral" form ${}_{\mathbf{A}}\mathbf{S}$ of $\mathbf{S}$, which by definition is the $\mathbf{A}$-subalgebra of $\mathbf{S}$ generated by all idempotents 1_μ along with the divided powers $E^{(a)}$, $F^{(b)}$ for $a, b \geqslant 0$. It is clear that the set in (a) above is also a (cellular) $\mathbf{A}$-basis for ${}_{\mathbf{A}}\mathbf{S}$. Moreover, we define ${}_{\mathbf{A}}\Delta(\lambda)$ to be the ${}_{\mathbf{A}}\mathbf{S}$-submodule of $\Delta(\lambda)$ generated by x_0; then the ${}_{\mathbf{A}}\Delta(\lambda)$ are the left cell modules for ${}_{\mathbf{A}}\mathbf{S}$. Notice that if we put $m = \langle \alpha^\vee,\ \lambda\rangle$ as above then ${}_{\mathbf{A}}\Delta(\lambda)$ is the $\mathbf{A}$-span of $\{x_0, \dots, x_m\}$, and this set is an $\mathbf{A}$-basis of ${}_{\mathbf{A}}\Delta(\lambda)$. Furthermore, it follows by induction from the formulas in Lemma 2.4 and the definitions that

(a) $$F^{(b)} \cdot x_t = \begin{bmatrix} t+b \\ b \end{bmatrix} x_{t+b}\,, \quad E^{(a)} \cdot x_t = \begin{bmatrix} m-t+a \\ a \end{bmatrix} x_{t-a}$$

for any $a, b, t \geqslant 0$.

Now suppose that k is any field of characteristic zero, and fix a nonzero element $q \in k$. We regard k as an $\mathbf{A}$-algebra by means of the algebra homomorphism $\mathbf{A} \to k$ such that $v \to q$. Then by the general theory of cellular algebras given in [GL], the k-algebra

(b) $$\mathbf{S}_q := k \otimes_{\mathbf{A}} ({}_{\mathbf{A}}\mathbf{S})$$

is again a cellular algebra, with cellular basis given by

(c) $$\textstyle\bigsqcup_{\lambda\in\pi} \{1 \otimes F^{(b)}1_\lambda E^{(a)} : 0 \leqslant a, b \leqslant \langle \alpha^\vee,\ \lambda\rangle\}.$$

Moreover, the $\mathbf{S}_q$-modules $\Delta_q(\lambda)$ defined by

(d) $$\Delta_q(\lambda) := k \otimes_{\mathbf{A}} ({}_{\mathbf{A}}\Delta(\lambda))$$

for $\lambda \in \pi$ are the left cell modules for $\mathbf{S}_q$.

If q is a primitive lth root of unity, we put $l' = l$ if l is odd and $l' = l/2$ if l is even. We shall denote the images of the elements $[n]$, $[n]^!$, $\begin{bmatrix} a \\ t \end{bmatrix}$ (defined in 1.4) under the map $\mathbf{A} \to k$ by the corresponding symbols $[n]_q$, $[n]^!_q$, $\begin{bmatrix} a \\ t \end{bmatrix}_q$. Then we have $[l']_q = 0$ and more generally $[nl']_q = 0$ for any $n \geqslant 0$.

2.9. **Proposition.** *Assume the root datum has rank* 1.

(a) *If $q \in k$ is not a root of unity then each $\Delta_q(\lambda)$ is a simple $\mathbf{S}_q$-module, for any $\lambda \in \pi$, and thus $\mathbf{S}_q$ is semisimple. This also holds if $q = 1$.*

(b) *If $1 \neq q \in k$ is a primitive lth root of unity, put $l' = l$ if l is odd and $l' = l/2$ if l is even. If $0 \leqslant \langle \alpha^\vee, \lambda \rangle < l'$ for $\lambda \in \pi$ then $\Delta_q(\lambda)$ is simple as an $\mathbf{S}_q$-module. Otherwise, write $1 + \langle \alpha^\vee, \lambda \rangle$ in the form $nl' + r$ with $0 \leqslant r < l'$; then $\{1 \otimes x_t \colon r \leqslant t \leqslant \langle \alpha^\vee, \lambda \rangle - r\}$ spans an $\mathbf{S}_q$-submodule of $\Delta_q(\lambda)$, and the corresponding quotient module is simple as an $\mathbf{S}_q$-module.*

Proof. For convenience of notation, put $m = \langle \alpha^\vee, \lambda \rangle$. From 2.8(b) we see that

$$(1 \otimes F^{(b)}) \cdot (1 \otimes x_t) = \left[{t+b \atop b} \right]_q (1 \otimes x_{t+b}),$$
$$(1 \otimes E^{(a)}) \cdot (1 \otimes x_t) = \left[{m-t+a \atop a} \right]_q (1 \otimes x_{t-a})$$

for any $a, b, t \geqslant 0$.

In case q is not a root of unity, all the q-binomial coefficients above are nonzero in k, so the assertions in part (a) follow, since it follows that each basis vector $1 \otimes x_t$ of $\Delta_q(\lambda)$ generates $\Delta_q(\lambda)$. If $q = 1$ then the q-binomial coefficients become ordinary binomial coefficients, and the same conclusion holds.

So suppose that $q \neq 1$ is a primitive lth root of unity. Then the simplicity of the $\Delta_q(\lambda)$ for any λ such that $0 \leqslant m < l'$ follows by the same argument as above.

Moreover, if $m \geqslant l'$ then one checks from the above formulas that $(1 \otimes E^{(a)}) \cdot (1 \otimes x_r) = 0$ for any $a \geqslant 0$, and similarly that $(1 \otimes F^{(b)}) \cdot (1 \otimes x_{m-r}) = 0$ for any $b \geqslant 0$. This implies that the span of all $1 \otimes x_t$ for $r \leqslant t \leqslant m - r$ is an $\mathbf{S}_q$-submodule of $\Delta_q(\lambda)$. This submodule is, in fact, the unique maximal submodule. Otherwise, there would be a nonzero weight vector in $\Delta_q(\lambda)$, different from any multiple of $1 \otimes x_0$, which is not in the submodule and which is killed by all $1 \otimes E^{(a)}$, for $a \geqslant 1$, and inspection shows there are no such vectors. The simplicity of the corresponding quotient follows. □

3. The general case

Now we return to the case of a general root datum. We concentrate in this section on the rational form of $\mathbf{S} = \mathbf{S}(\pi)$ for some fixed saturated set $\pi \subset X^+$, and leave consideration of what happens under specialization to the last section of the paper. We are going to extend many of the results of the preceding section to the general case. The argument is nearly self-contained, but we do need two well known facts from the representation theory of complex semisimple Lie algebras: the classification of the finite-dimensional simple modules, and Weyl's theorem on complete reducibility. We use no results from the established theory of quantum groups.

3.1. Let s_i be the generating reflection in the Weyl group W corresponding to a simple root α_i. For any fixed $i \in I$, let $\mathbf{S}_i$ be the subalgebra of $\mathbf{S}$ generated by E_i, F_i along with all 1_λ for $\lambda \in W\pi$. Then $\mathbf{S}_i$ is a generalized q-Schur algebra of rank 1, since $W\pi = W_i\pi_i$ where $W_i = \langle s_i \rangle = \{1, s_i\}$ and $\pi_i = \{\lambda \in W\pi \colon \langle \alpha_i^\vee,\ \lambda \rangle \geqslant 0\}$. (Note that π_i is saturated with respect to a rank 1 root datum determined by α_i, $\alpha_i^\vee$.) Thus, any of the results proved in the rank 1 case may be applied to $\mathbf{S}_i$. In particular, any simple left $\mathbf{S}_i$-module is generated by an $\mathbf{S}_i$-maximal vector of weight $\lambda \in W\pi$, where $\langle \alpha_i^\vee,\ \lambda \rangle \geqslant 0$. (In this case a maximal vector is just a vector killed by E_i.)

3.2. **Proposition.** *Let $\mathbf{S} = \mathbf{S}(\pi)$. If M is a simple left $\mathbf{S}$-module then the weight of a highest weight vector must be a dominant weight $\lambda \in \pi$.*

Proof. Let x_0 be a highest weight vector, of weight some $\lambda \in W\pi$. For each $i \in I$ we can restrict M to $\mathbf{S}_i$. Of course x_0 is an $\mathbf{S}_i$-maximal vector, so by Lemma 1.15 the $\mathbf{S}_i$-submodule it generates has a unique simple quotient. But, by Proposition 2.6, $\mathbf{S}_i$ is a semisimple algebra, so in fact that submodule is already simple as a $\mathbf{S}_i$-module, and thus $\langle \alpha_i^\vee,\ \lambda \rangle \geqslant 0$. Since this holds for each $i \in I$, we have shown that λ is a dominant weight. In other words, $\lambda \in W\pi \cap X^+ = \pi$, as desired. □

We wish to classify the simple $\mathbf{S}$-modules, by showing that for each λ in π there exists a unique (up to isomorphism) simple $\mathbf{S}$-module of highest weight λ. The uniqueness part is easy to establish, as follows.

3.3. **Lemma.** *Let $\mathbf{S} = \mathbf{S}(\pi)$. Let L, L' be two simple left $\mathbf{S}$-modules, each of highest weight λ, for some $\lambda \in \pi$. Then L is isomorphic to L'.*

Proof. (Compare with the proof of Theorem A of [Hu, §20.3].) Let $M = L \oplus L'$. Suppose that y_0, y_0' are maximal vectors in L, L' respectively. The left weight of both y_0, y_0' is λ. Put $x_0 = (y_0, y_0')$. Then x_0 is a maximal vector in M, of left weight λ. Let N be the submodule of M generated by x_0. Lemma 1.15 implies that N has a unique simple quotient. But the natural projections $N \to L$, $N \to L'$ are $\mathbf{S}$-module epimorphisms, so $L \simeq L'$, as desired. □

It remains to establish the existence of a simple $\mathbf{S}$-module of highest weight λ for each $\lambda \in \pi$. The following result will enable us to construct the needed simple modules.

3.4. **Lemma.** *Let $\mathbf{S} = \mathbf{S}(\pi)$. For each $\lambda \in \pi$ we define ι-invariant subspaces of $\mathbf{S}$ as follows:*

$$\mathbf{S}[\rhd\lambda] = \textstyle\sum_{\mu\rhd\lambda} \mathbf{S}^- 1_\mu \mathbf{S}^+; \qquad \mathbf{S}[\unrhd\lambda] = \textstyle\sum_{\mu\unrhd\lambda} \mathbf{S}^- 1_\mu \mathbf{S}^+.$$

Then both $\mathbf{S}[\rhd\lambda]$ and $\mathbf{S}[\unrhd\lambda]$ are two sided ideals of $\mathbf{S}$. We have $\mathbf{S}[\rhd\lambda] \subset \mathbf{S}[\unrhd\lambda]$ and $\mathbf{S}[\unrhd\lambda] \subseteq \mathbf{S}[\unrhd\lambda']$ whenever $\lambda \unrhd \lambda'$. Moreover, for any j, r we have, modulo terms in $\mathbf{S}[\rhd\lambda]$

(a) $E_j \cdot F_{i_1} \dots F_{i_r} 1_\lambda = \sum \delta_{j,i_s} [\langle \alpha_j^\vee,\ \lambda - \gamma_{s,r}\rangle]_j F_{i_1} \cdots F_{i_{s-1}} F_{i_{s+1}} \cdots F_{i_r} 1_\lambda$

and

(b) $1_\lambda E_{i_r} \dots E_{i_1} \cdot F_j = \sum \delta_{j,i_s} [\langle \alpha_j^\vee,\ \lambda - \gamma_{s,r}\rangle]_j 1_\lambda E_{i_r} \cdots E_{i_{s+1}} E_{i_{s-1}} \cdots E_{i_1}$

where $\gamma_{s,r} := \alpha_{i_{s+1}} + \cdots + \alpha_{i_r}$. (Both sums are over s.)

Proof. The proof is by a double induction. First assume that λ is maximal in π. We establish (a) in case $r = 1$. From the defining relations 1.6 it follows that

$$E_j F_i 1_\lambda = \delta_{j,i}(F_i E_j 1_\lambda + [\langle \alpha_j^\vee,\ \lambda\rangle]_j 1_\lambda)$$

for any $i, j \in I$. Since $E_j 1_\lambda = 1_{\lambda+\alpha_j} E_j$ and since λ is maximal, it follows that

$$E_j F_i 1_\lambda = \delta_{j,i}(F_i 1_{\lambda+\alpha_j} E_j + [\langle \alpha_j^\vee,\ \lambda\rangle]_j 1_\lambda) = [\langle \alpha_j^\vee,\ \lambda\rangle]_j 1_\lambda$$

which proves (a) in the case $r = 1$ and λ maximal. (Note that $\mathbf{S}[\rhd\lambda] = 0$ for maximal λ.) Assuming now that (a) holds for λ maximal and for all words of length at most r, we have

$$\begin{aligned} E_j \cdot F_{i_1} \dots F_{i_{r+1}} 1_\lambda &= (F_{i_1} E_j + \delta_{j,i_1}[\langle \alpha_j^\vee,\ \lambda - \gamma_{1,r+1}\rangle]_j) F_{i_2} \cdots F_{i_{r+1}} 1_\lambda \\ &= F_{i_1} E_j \cdot F_{i_2} \cdots F_{i_{r+1}} 1_\lambda + \delta_{j,i_1}[\langle \alpha_j^\vee,\ \lambda - \gamma_{1,r+1}\rangle]_j F_{i_2} \cdots F_{i_{r+1}} 1_\lambda. \end{aligned}$$

Expanding the sub-expression $E_j \cdot F_{i_2} \cdots F_{i_{r+1}} 1_\lambda$ appearing in the first term of the last equality above by the inductive hypothesis, we obtain (a) in case λ is maximal. Now (b) follows by applying the anti-involution ι to (a).

Still assuming that λ is maximal in π, let $u = F_{i_1} \dots F_{i_r} 1_\lambda E_{j_s} \dots E_{j_1} \in \mathbf{S}[\unrhd\lambda]$. Then it follows from (a) and (b) that $E_j \cdot u \in \mathbf{S}[\unrhd\lambda]$ and $u \cdot F_j \in \mathbf{S}[\unrhd\lambda]$ for any $j \in I$. It follows from the defining relations 1.5 that $F_j \cdot u, 1_\mu \cdot u \in \mathbf{S}[\unrhd\lambda]$ and $u \cdot E_j, u \cdot 1_\mu \in \mathbf{S}[\unrhd\lambda]$ for any $j \in I$, $\mu \in W\pi$. Thus $\mathbf{S}[\unrhd\lambda]$ is a two-sided ideal of $\mathbf{S}$.

Now we fix some λ which is not maximal, and assume that all assertions have been established for all $\lambda' \rhd \lambda$. Then $\mathbf{S}[\unrhd\lambda']$ and $\mathbf{S}[\rhd\lambda']$

are two-sided ideals for every $\lambda' \rhd \lambda$. Thus $\mathbf{S}[\rhd\lambda] = \sum_{\lambda'\rhd\lambda} \mathbf{S}[\unrhd\lambda']$ is a subring of $\mathbf{S}$. Furthermore, given any $s \in \mathbf{S}[\rhd\lambda]$ there exist $s_{\lambda'} \in \mathbf{S}[\unrhd\lambda']$ such that $s = \sum_{\lambda'\rhd\lambda} s_{\lambda'}$. Thus $as = \sum_{\lambda'\rhd\lambda} as_{\lambda'} \in \mathbf{S}[\rhd\lambda]$ and $sa = \sum_{\lambda'\rhd\lambda} s_{\lambda'}a \in \mathbf{S}[\rhd\lambda]$, and $\mathbf{S}[\rhd\lambda]$ is a two-sided ideal of $\mathbf{S}$.

We can now repeat the induction on r in the first paragraph of the proof to obtain (a) for λ, modulo terms in $\mathbf{S}[\rhd\lambda]$, and then obtain (b) by applying ι. It then follows from (a) and (b) that $\mathbf{S}[\unrhd\lambda]$ is a two-sided ideal of $\mathbf{S}$, and the proof is complete. □

3.5. **The modules $\Delta(\lambda)$, $\Delta^\sharp(\lambda)$.** For any $\lambda \in \pi$, we regard the quotient $M = \mathbf{S}[\unrhd\lambda]/\mathbf{S}[\rhd\lambda]$ as a left $\mathbf{S}$-module. Note that $x_0 = 1_\lambda + \mathbf{S}[\rhd\lambda]$ is a maximal vector in M. Let $\Delta(\lambda)$ be the left submodule of M generated by x_0. Clearly $\Delta(\lambda) \neq 0$, since its generating maximal vector is not zero. The maximal vector is a highest weight vector in $\Delta(\lambda)$ of weight λ, so $\Delta(\lambda)$ has a unique simple quotient $L(\lambda)$ of highest weight λ.

We have now established that the set of isomorphism classes of simple left $\mathbf{S}$-modules is given by $\{L(\lambda)\colon \lambda \in \pi\}$. This completes the classification of simple $\mathbf{S}$-modules.

For later use we define $\Delta^\sharp(\lambda)$ to be the *right* $\mathbf{S}$-submodule of $M = \mathbf{S}[\unrhd\lambda]/\mathbf{S}[\rhd\lambda]$ generated by x_0.

As already noted, if λ is a maximal element in the poset π then $\mathbf{S}[\rhd\lambda] = 0$. Thus $\mathbf{S}[\unrhd\lambda] = \mathbf{S}^-1_\lambda\mathbf{S}^+$ and $\Delta(\lambda)$ is in this case just the left ideal $\mathbf{S}1_\lambda = \mathbf{S}^-1_\lambda$ of $\mathbf{S}$. In this special case, we can prove that $\Delta(\lambda) = \mathbf{S}1_\lambda$ is actually simple as an $\mathbf{S}$-module.

3.6. **Theorem.** *Let $\mathbf{S} = \mathbf{S}(\pi)$. If λ is a maximal element in π (with respect to the partial order $\unlhd$) then the left ideal $\Delta(\lambda) = \mathbf{S}1_\lambda$ is a simple $\mathbf{S}$-module of highest weight λ.*

Proof. (Similar to Sections 5.12–5.15 of [Ja].) Put $\mathbf{A} = \mathbf{Q}[v, v^{-1}]$.

Let $L(\lambda)$ be the simple quotient of $\Delta(\lambda)$. Clearly $L(\lambda)$ is generated by a highest weight vector of weight λ. Throughout the following argument, we let V be either $\Delta(\lambda)$ or $L(\lambda)$. Then $V = \mathbf{S}^-x_0$ where x_0 is a maximal vector in V, so V is the $\mathbf{Q}(v)$-linear span of elements of the form $F_{i_1}\dots F_{i_r}x_0$ for various finite sequences $\underline{i} = (i_1, \dots, i_r)$ of elements in I. Put $F_{\underline{i}} = F_{i_1}\dots F_{i_r}$ for ease of notation, and write $\mathrm{wt}(\underline{i}) = \sum_j \alpha_{i_j}$. Let ${}_{\mathbf{A}}V$ be the ${}_{\mathbf{A}}\mathbf{S}$-submodule of V generated by the maximal vector x_0. Then

$$ {}_{\mathbf{A}}V = \sum_{\underline{i}} \mathbf{A}F_{\underline{i}}x_0 \quad \text{and} \quad {}_{\mathbf{A}}V_\mu = \sum_{\mathrm{wt}(\underline{i})=\lambda-\mu} \mathbf{A}F_{\underline{i}}x_0 $$

for any $\mu \in W\pi$. As $\mathbf{A}$-modules, both ${}_{\mathbf{A}}V$ and ${}_{\mathbf{A}}V_\mu$ are finitely generated and torsion free. Hence both ${}_{\mathbf{A}}V$ and ${}_{\mathbf{A}}V_{\mu,}$ are free of finite rank over $\mathbf{A}$. Clearly ${}_{\mathbf{A}}V = \sum_\mu {}_{\mathbf{A}}V_\mu$ so we get

$$ {}_{\mathbf{A}}V = \bigoplus_{\mu \in W\pi} {}_{\mathbf{A}}V_\mu. $$

Note that the natural map $\mathbf{Q}(v) \otimes_{\mathbf{A}} ({}_{\mathbf{A}}V_\mu) \to V_\mu$ is an isomorphism, for any $\mu \in W\pi$. (Here we are writing V_μ for the weight space $1_\mu V$ in V.) It follows that a basis for ${}_{\mathbf{A}}V_\mu$ over A is also a basis of V_μ over $\mathbf{Q}(v)$, and thus

$$ \mathrm{rk}_A\, {}_{\mathbf{A}}V_\mu = \dim_{\mathbf{Q}(v)} V_\mu \quad (\text{any } \mu \in W\pi). $$

We claim now that the $\mathbf{A}$-module ${}_{\mathbf{A}}V$ is stable under the action of the E_j, F_j, and 1_μ for any $j \in I$ and any $\mu \in W\pi$. This is obvious in the case of the F_j and 1_μ, since

$$ F_j(F_{i_1} \cdots F_{i_r} x_0) \in V_A \quad \text{and} \quad 1_\mu(F_{i_1} \cdots F_{i_r} x_0) = F_{i_1} \cdots F_{i_r} 1_{\mu+\mathrm{wt}(\underline{i})} x_0 $$

is zero if $\mu + \mathrm{wt}(\underline{i}) \neq \lambda$ and is $F_{i_1} \cdots F_{i_r} x_0$ if $\mu + \mathrm{wt}(\underline{i}) = \lambda$. Moreover, for the E_j we have by the defining relations 1.5 (b), (c) that

$$ E_j(F_{i_1} \cdots F_{i_r} x_0) = \sum_{1 \leqslant a \leqslant r; i_a = j} F_{i_1} \cdots F_{i_{a-1}} \sum_{\mu \in W\pi} [\langle \alpha_j^\vee,\ \mu\rangle]_j 1_\mu F_{i_{a+1}} \cdots F_{i_r} x_0 $$

and the claim follows by the preceding remarks and the observation that $[\langle \alpha_j^\vee,\ \mu\rangle]_j \in \mathbf{A}$ for any $\mu \in W\pi$, $j \in I$.

Now there is a unique homomorphism φ of $\mathbf{Q}$-algebras mapping $\mathbf{A} = \mathbf{Q}[v, v^{-1}]$ to $\mathbf{C}$ and satisfying $\varphi(v) = 1$. Regard $\mathbf{C}$ as an $\mathbf{A}$-module via φ, and put

$$ \overline{V} = \mathbf{C} \otimes_{\mathbf{A}} ({}_{\mathbf{A}}V) \quad \text{and} \quad \overline{V}_\mu = \mathbf{C} \otimes_{\mathbf{A}} ({}_{\mathbf{A}}V_\mu) $$

for any $\mu \in W\pi$. Then we have the direct sum decomposition $\overline{V} = \bigoplus_\mu \overline{V}_\mu$, where each $\overline{V}_\mu$ is a complex vector space with

$$ \dim_{\mathbf{C}} \overline{V}_\mu = \mathrm{rk}_{\mathbf{A}}({}_{\mathbf{A}}V_\mu) = \dim_{\mathbf{Q}(v)} V_\mu. $$

The actions of E_i, F_i, and 1_μ on V_A yield linear endomorphisms of $\overline{V}$ that we denote by e_i, f_i, and ι_μ. We put

$$ \overline{h}_i = \textstyle\sum_{\mu \in W\pi} \langle \alpha_i^\vee,\ \mu\rangle \iota_\mu $$

for any $i \in I$.

We claim that the endomorphisms $e_i, f_i, \overline{h}_i$ satisfy Serre's relations for the finite dimensional semisimple Lie algebra $\mathfrak{g}$ defined by the Cartan matrix $(\langle \alpha_i^\vee,\ \alpha_j\rangle)_{i,j \in I}$. Since the idempotent linear operators ι_λ

commute and are pairwise orthogonal, it follows that $\overline{h}_i$ commutes with $\overline{h}_j$; thus

$$[\overline{h}_i, \overline{h}_j] = 0, \quad \text{any } i, i \in I.$$

We have $\varphi([a]_i) = [a]_{v=1} = a$ for any integer a and any $i \in I$, so from defining relations 1.5(b) for the Schur algebra $\mathbf{S} = \mathbf{S}(\pi)$ we have

$$[e_i, f_j] = \delta_{i,j} \textstyle\sum_{\mu \in W\pi} \langle \alpha_i^\vee,\ \mu \rangle \iota_\mu = \overline{h}_i.$$

Recalling the convention that $1_\mu = 0$ for any $\mu \notin W\pi$ we put also $\iota_\mu = 0$ for any $\mu \notin W\pi$. Then we can write $\overline{h}_i = \sum_{\mu \in X} \langle \alpha_i^\vee,\ \mu \rangle \iota_\mu$ (which is still a finite sum) and by defining relation 1.6(c) we have

$$\begin{aligned}[\overline{h}_i, e_j] &= \overline{h}_i e_j - e_j \overline{h}_i = \textstyle\sum_{\mu \in X} \langle \alpha_i^\vee,\ \mu \rangle \iota_\mu e_j - \sum_{\mu \in X} \langle \alpha_i^\vee,\ \mu \rangle e_j \iota_\mu \\ &= \textstyle\sum_{\mu \in X} \langle \alpha_i^\vee,\ \mu \rangle \iota_\mu e_j - \sum_{\mu \in X} \langle \alpha_i^\vee,\ \mu \rangle \iota_{\mu+\alpha_j} e_j\end{aligned}$$

and by replacing μ by $\mu - \alpha_j$ in the second sum we obtain

$$\begin{aligned}[\overline{h}_i, e_j] &= \textstyle\sum_{\mu \in X} \langle \alpha_i^\vee,\ \mu \rangle \iota_\mu e_j - \sum_{\mu \in X} \langle \alpha_i^\vee,\ \mu - \alpha_j \rangle \iota_\mu e_j \\ &= \textstyle\sum_{\mu \in X} \langle \alpha_i^\vee,\ \alpha_j \rangle \iota_\mu e_j \\ &= \langle \alpha_i^\vee,\ \alpha_j \rangle e_j\end{aligned}$$

where we have used the second part of relation 1.6(a) to get the last line. A similar calculation proves that

$$[\overline{h}_i, f_j] = -\langle \alpha_i^\vee,\ \alpha_j \rangle f_j.$$

Finally, we have

$$\varphi\big(\left[\begin{smallmatrix} a \\ n \end{smallmatrix}\right]_i\big) = \left[\begin{smallmatrix} a \\ n \end{smallmatrix}\right]_{v=1} = \tbinom{a}{n}$$

for any integers a, n with $n \geqslant 0$. Thus relations 1.5 (d1) and (d2) imply that

$$\sum_{s=0}^{1-a_{ij}} (-1)^s \tbinom{1-a_{ij}}{s} e_i^{1-a_{ij}-s} e_j e_i^s = 0 \quad (i \neq j);$$

$$\sum_{s=0}^{1-a_{ij}} (-1)^s \tbinom{1-a_{ij}}{s} f_i^{1-a_{ij}-s} f_j f_i^s = 0 \quad (i \neq j)$$

where $a_{ij} = \langle \alpha_i^\vee,\ \alpha_j \rangle$. Thus the claim is proved.

Now let x_i, y_i, h_i ($i \in I$) be a Chevalley system of generators for the semisimple Lie algebra $\mathfrak{g}$. Then by the claim of the preceding paragraph it follows that the map $\mathfrak{g} \to \mathfrak{gl}(\overline{V})$ given by $x_i \to e_i$, $y_i \to f_i$, $h_i \to \overline{h}_i$ is a homomorphism of Lie algebras, so $\overline{V}$ is a $\mathfrak{g}$-module.

All of the preceding discussion applies equally well to $V = L(\lambda)$ or $V = \Delta(\lambda) = \mathbf{S}1_\lambda$. In either case we now see easily that $\overline{V}$ is a simple $\mathfrak{g}$-module of highest weight λ. This follows from Weyl's theorem on complete reducibility of finite dimensional representations of semisimple Lie algebras, which implies that $\overline{V}$ is completely reducible, and the observation that $\overline{V}$ is (in both cases under consideration) generated by a maximal vector, and hence has a unique simple quotient.

It follows that the weight space dimensions in $\overline{V}$ are given by Weyl's character formula, in both cases $V = L(\lambda)$ and $V = \Delta(\lambda)$. In particular, this shows that

$$\dim_{\mathbf{Q}(v)} L(\lambda) = \dim_{\mathbf{Q}(v)} \Delta(\lambda).$$

Since $L(\lambda)$ is a homomorphic image of $\Delta(\lambda)$, it follows that $L(\lambda) = \Delta(\lambda)$ and we have obtained the result. □

Note that it follows immediately from the preceding theorem that $1_\lambda \mathbf{S}^+ = \iota(\mathbf{S}^- 1_\lambda)$ is a simple right ideal in $\mathbf{S}$, for any maximal element λ in π.

3.7. We now utilize the anti-involution ι in order to define a bilinear form on $\Delta(\mu)$, for any μ in π.[1] Put $x_0 = 1_\mu + \mathbf{S}[\rhd\mu]$ in the left $\mathbf{S}$-module $M = \mathbf{S}[\unrhd\mu]/\mathbf{S}[\rhd\mu]$. By definition, $\Delta(\mu)$ is the left submodule generated by x_0. This is spanned by various elements of the form $F_A x_0$, where $F_A = F_{a_1} \cdots F_{a_r}$ for some finite sequence $A = (a_1, \dots, a_r)$ of elements of I. If $F_B x_0$, $B = (b_1, \dots, b_s)$, is another such element then

$$\text{(a)} \qquad \iota(F_B) \cdot (F_A x_0) \in 1_{\mu - \mathrm{wt}(A) + \mathrm{wt}(B)} \Delta(\mu)$$

since acting on the left by some E_j raises the left weight by α_j. Here we define $\mathrm{wt}(A) = \sum_{j=1}^{r} \alpha_{a_j}$ and $\mathrm{wt}(B) = \sum_{j=1}^{s} \alpha_{b_j}$. Hence it follows that

$$\text{(b)} \qquad 1_\mu \iota(F_B) \cdot (F_A x_0) = 1_\mu \iota(F_B) F_A 1_\mu + \mathbf{S}[\rhd\mu] \in 1_\mu \Delta(\mu)$$

is zero unless $\mathrm{wt}(A) = \mathrm{wt}(B)$. It follows that in either case there must be some scalar $c_{A,B} \in \mathbf{Q}(v)$ (necessarily zero if $\mathrm{wt}(A) \neq \mathrm{wt}(B)$) such that

$$\text{(c)} \qquad 1_\mu \iota(F_B) F_A 1_\mu = c_{A,B}\, 1_\mu \pmod{\mathbf{S}[\rhd\mu]}.$$

Now the promised bilinear form $\varphi_\mu \colon \Delta(\mu) \times \Delta(\mu) \to \mathbf{Q}(v)$ is defined by setting

$$\text{(d)} \qquad \varphi_\mu(F_A x_0, F_B x_0) = c_{A,B}$$

and extending bilinearly. We immediately record the following important properties of the bilinear form.

[1] One might prefer to define a bilinear pairing between $\Delta(\mu)$ and $\Delta^\sharp(\mu)$ instead. With that approach, which is equally natural, the use of ι is avoided.

3.8. **Lemma.** *Let $\mu \in \pi$. For any $x, y \in \Delta(\mu)$ we have:*
(a) $\varphi_\mu(x, y) = \varphi_\mu(y, x)$*;*
(b) $\varphi_\mu(u\, x, y) = \varphi_\mu(x, \iota(u)\, y)$*, for any* $u \in \mathbf{S}$*.*

Proof. (a) Applying the anti-involution ι to equation 3.7(c) proves that $c_{A,B} = c_{B,A}$ and part (a) follows.

(b) This follows from the calculation

$$1_\mu\, \iota(F_B)\, uF_A\, 1_\mu = 1_\mu\, \iota(F_B)\, \iota^2(u)\, F_A\, 1_\mu = 1_\mu\, \iota(\iota(u)F_B)\, F_A\, 1_\mu$$

which holds modulo $\mathbf{S}[\trianglerighteq\mu]$. □

The preceding lemma implies that the radical of the bilinear form

$$\operatorname{rad} \varphi_\mu = \{x \in \Delta(\mu)\colon \varphi_\mu(x, y) = 0, \text{ for all } y \in \Delta(\mu)\}$$

is an $\mathbf{S}$-submodule of $\Delta(\mu)$. Since $\varphi_\mu(x_0, x_0) = 1$, we see that this submodule is proper, hence contained in the module theoretic radical of $\Delta(\mu)$. By Theorem 3.6 we conclude that when $\lambda \in \pi$ is a maximal element, then the radical of φ_λ must be zero. In other words, the form φ_λ is nondegenerate, for any λ maximal in π. (We will soon show that in fact φ_μ is nondegenerate for any $\mu \in \pi$.)

3.9. **Theorem.** *Let $\mathbf{S} = \mathbf{S}(\pi)$. If λ is maximal in π then the natural multiplication map $\mathbf{S}1_\lambda \otimes_{\mathbf{Q}(v)} 1_\lambda\mathbf{S} \to \mathbf{S}1_\lambda\mathbf{S}$ is an isomorphism of $\mathbf{S}$-bimodules.*

Proof. The multiplication map $m : \mathbf{S}1_\lambda \otimes_{\mathbf{Q}(v)} 1_\lambda\mathbf{S} \to \mathbf{S}1_\lambda\mathbf{S}$ is clearly a surjective homomorphism of $\mathbf{S}$-bimodules, so we only need to prove injectivity. We fix bases

$$\{x1_\lambda\colon x \in \mathbf{S}^-\} \quad \text{and} \quad \{1_\lambda y\colon y \in \mathbf{S}^+\}$$

for $\mathbf{S}1_\lambda$ and $1_\lambda\mathbf{S}$, respectively. Then the set of tensors $\{x1_\lambda \otimes 1_\lambda y\}$ is a basis for $\mathbf{S}1_\lambda \otimes_{\mathbf{Q}(v)} 1_\lambda\mathbf{S}$, so it suffices to show that the corresponding set of images $\{x1_\lambda y\}$ is linearly independent over $\mathbf{Q}(v)$. Assume that

$$0 = \textstyle\sum_{x,y} c_{x,y}\, x1_\lambda y$$

where the sum is taken over the set of pairs (x, y) such that x and y independently range over the above basis elements. Let $\{x'1_\lambda\}$ be the dual basis to $\{x1_\lambda\}$ with respect to the bilinear form φ_λ, defined by the requirement $\varphi_\lambda(x'1_\lambda, x1_\lambda) = \delta x, x'$.

Fix some z such that $z1_\lambda$ is one of the above basis elements of $\mathbf{S}1_\lambda$. Then by left multiplication by $1_\lambda\iota(z')$ we obtain

$$0 = \textstyle\sum_{x,y} c_{x,y}\, 1_\lambda\iota(z')x1_\lambda y = \sum_{x,y} c_{x,y}\, \delta_{x,z} 1_\lambda y = \sum_y c_{z,y}\, 1_\lambda y$$

and by linear independence of the $\{1_\lambda y\}$ it follows that $c_{z,y} = 0$ for any y. Since z was arbitrary, this proves the desired linear independence, and thus the result. $\square$

Now we fix some maximal element $\lambda \in \pi$ and set $\pi' = \pi - \{\lambda\}$ and $\mathbf{S}' = \mathbf{S}(\pi')$. Note that π' is again saturated. It is clear from the defining relations in 1.6 that the kernel of the natural map $\mathbf{S} \to \mathbf{S}'$ is the two sided ideal generated by all idempotents 1_μ such that $\mu \in W\lambda$. In fact, this ideal is generated by a single element.

3.10. **Lemma.** *Let* $\mathbf{S} = \mathbf{S}(\pi)$. *Let* λ *be a maximal element in* π, *and set* $\pi' = \pi - \{\lambda\}$. *Then the natural quotient map* $\mathbf{S} \to \mathbf{S}' := \mathbf{S}(\pi')$ *has kernel* $\mathbf{S}1_\lambda\mathbf{S}$.

Proof. Comparing the defining presentations for $\mathbf{S}$ and $\mathbf{S}'$ observe that the kernel of the natural quotient map $\mathbf{S} \to \mathbf{S}'$ is generated by the set of idempotents of the form $1_{w(\lambda)}$ for $w \in W$. Clearly, the ideal $\mathbf{S}1_\lambda\mathbf{S}$ is contained in the kernel. On the other hand, we claim that Lemma 1.7 implies that each $1_{w(\lambda)}$ (for any $w \in W$) lies within $\mathbf{S}1_\lambda\mathbf{S}$, which gives the opposite inclusion and proves the result.

It remains to prove the claim. This is done by induction on the length of w. If $w = s_i$ is a simple reflection then by putting $\mu = s_i(\lambda)$ and setting $a = b = \langle \alpha_i^\vee,\ \lambda\rangle$ in part (iii) of Lemma 1.7 we see that

$$F_i^{(a)} E_i^{(a)} 1_{s_i(\lambda)} = 1_{s_i(\lambda)}$$

since $F_i 1_{s_i(\lambda)} = 0$ because the weight $s_i(\lambda)$ is extremal in the α_i direction. Since $E_i^{(a)} 1_{s_i(\lambda)} = E_i^{(a)} 1_{\lambda - a\alpha_i} = 1_\lambda E_i^{(a)}$ by part (i) of Lemma 1.7, we obtain the equality

$$F_i^{(a)} 1_\lambda E_i^{(a)} = 1_{s_i(\lambda)}$$

which proves the claim in case $w = s_i$ has length 1.

Now let $w \in W$ have length at least 2, and assume the claim for elements of length strictly less than the length of w. We may write w in the form $w = s_i w'$ for some $w' \in W$ such that $\ell(w') < \ell(w)$. By induction $1_{w'(\lambda)} \in \mathbf{S}1_\lambda\mathbf{S}$. Now take $\mu = w(\lambda)$ and set $a = b = \langle \alpha_i^\vee,\ w'(\lambda)\rangle$ in part (iii) of Lemma 1.7. Similar to the above, we get

$$F_i^{(a)} E_i^{(a)} 1_{w(\lambda)} = 1_{w(\lambda)}$$

since $w(\lambda)$ is extremal in the α_i direction. Then again by part (i) of Lemma 1.7 we have $E_i^{(a)} 1_{w(\lambda)} = E_i^{(a)} 1_{w'(\lambda) - a\alpha_i} = 1_{w'(\lambda)} E_i^{(a)}$, so

$$F_i^{(a)} 1_{w'(\lambda)} E_i^{(a)} = 1_{w(\lambda)}.$$

Since $1_{w'(\lambda)} \in \mathbf{S}1_\lambda\mathbf{S}$ it follows that $1_{w(\lambda)} \in \mathbf{S}1_\lambda\mathbf{S}$, as desired. This proves the claim. □

We find it convenient to use König and Xi's definition of cellularity (see 1.17) in the proof of the next result.

3.11. **Theorem.** *For any finite saturated set π the algebra $\mathbf{S} = \mathbf{S}(\pi)$ is cellular with respect to the anti-involution ι, with defining cell chain given by $\{\mathbf{S}[\trianglerighteq\mu]\colon \mu \in \pi\}$ partially ordered by set inclusion.*

Proof. It follows by Theorem 3.9 that the ideal $\mathbf{S}1_\lambda\mathbf{S}$ is a cell ideal in $\mathbf{S}$, in the sense of König and Xi, for any maximal $\lambda \in \pi$.

If $\pi = \{\lambda\}$ is a singleton set, the result now follows. Otherwise, pick some maximal element λ in π and put $\pi' = \pi - \{\lambda\}$ and $\mathbf{S}' = \mathbf{S}(\pi')$. By induction on the cardinality of the finite set π we may assume that $\mathbf{S}'$ is cellular with respect to ι and the defining cell chain given by the ideals $\mathbf{S}'[\trianglerighteq\mu]$ $(\mu \in \pi')$ partially ordered by set inclusions. Let $p\colon \mathbf{S} \to \mathbf{S}'$ be the natural quotient map. It is easily checked that p carries an ideal $\mathbf{S}[\trianglerighteq\mu]$ onto $\mathbf{S}'[\trianglerighteq\mu]$ for any $\mu \in \pi'$, and of course by Lemma 3.10 the kernel of p is $\mathbf{S}1_\lambda\mathbf{S}$. It follows that by adjoining the cell ideal $\mathbf{S}1_\lambda\mathbf{S}$ to the preimage of the defining cell chain in $\mathbf{S}'$ yields a cell chain in $\mathbf{S}$, and the result follows. □

We will now give three corollaries to the above result.

3.12. **Corollary.** *For any finite saturated π, the generalized q-Schur algebra $\mathbf{S} = \mathbf{S}(\pi)$ is quasihereditary.*

Proof. According to [KX2, Theorem 1.1] (see also [KX3, Theorem 3.3]), a cellular algebra over a field is quasihereditary if and only if the number of isomorphism classes of simple modules is the same as the length of some defining cell chain. Since this holds in our situation, the result follows. □

3.13. **Corollary.** *For any finite saturated π, the algebra $\mathbf{S} = \mathbf{S}(\pi)$ is semisimple. We have $L(\mu) = \Delta(\mu)$ for each $\mu \in \pi$; i.e., a complete set of isomorphism classes of simple left $\mathbf{S}$-modules is given by $\{\Delta(\mu)\colon \mu \in \pi\}$. The bilinear form φ_μ is nondegenerate for each $\mu \in \pi$.*

Proof. For each $\mu \in \pi$ there is some saturated subset π' of π such that μ is maximal in π'. Indeed, we can take $\pi' = \{\mu' \in \pi\colon \mu \trianglerighteq \mu'\}$. Now there is a surjective quotient map $p_{\pi,\pi'}\colon \mathbf{S}(\pi) \to \mathbf{S}(\pi')$ sending generators $E_i \to E_i$, $F_i \to F_i$, and $1_{\mu'} \to 1_{\mu'}$ for $\mu' \in W\pi'$ with $1_{\mu'} \to 0$ for all $\mu' \notin W\pi'$. By Theorem 3.6 the left ideal $\mathbf{S}(\pi')1_\mu$ is simple as an $\mathbf{S}(\pi')$-module, hence is simple when regarded as an $\mathbf{S}$-module, via the map $p_{\pi,\pi'}$. As an $\mathbf{S}$-module, we have an isomorphism between $\mathbf{S}(\pi')1_\mu$ and $\Delta(\mu)$ which is induced by the map $p_{\pi,\pi'}$, so $\Delta(\mu)$ is simple as an $\mathbf{S}$-module. The semisimplicity of $\mathbf{S}$ now follows from [GL, (3.8) Theorem]. $\square$

3.14. **Corollary.** $\dim_{\mathbf{Q}(v)} \mathbf{S}(\pi) = \sum_{\mu\in\pi} \big(\dim_{\mathbf{Q}(v)} \Delta(\mu)\big)^2$.

Proof. This follows easily from Corollary 3.13 and the standard theory of semisimple algebras. $\square$

The following extension of Theorem 3.9 can now be obtained.

3.15. **Theorem.** *Let $\mathbf{S} = \mathbf{S}(\pi)$ and let $\mu \in \pi$. Then the natural multiplication map $\Delta^\sharp(\mu) \otimes_{\mathbf{Q}(v)} \Delta(\mu) \to \mathbf{S}[\trianglerighteq\mu]/\mathbf{S}[\triangleright\mu]$ is an isomorphism of $\mathbf{S}$-bimodules.*

Proof. This follows from the nondegeneracy of the bilinear form φ_μ, which implies the multiplication map is injective. The surjectivity of this map is clear. $\square$

3.16. We use the preceding result to inductively build a cellular basis of $\mathbf{S} = \mathbf{S}(\pi)$, for any π. For each $\mu \in \pi$, let $\mathbf{S}' = \mathbf{S}(\pi')$ where $\pi' = \{\mu' \in X^+ \colon \mu' \trianglelefteq \mu\}$. We choose any basis

(a) $$B^-(\mu) = \{F_A 1_\mu\}_A$$

for the simple left ideal $\mathbf{S}'1_\mu = \mathbf{S}'^- 1_\mu$, where we write

(b) $F_A = F_{i_1}F_{i_2}\cdots F_{i_r}$ for any sequence $A = (i_1, i_2, \dots, i_r)$ over I.

Write $B^+(\mu) = \iota(B^-(\mu))$; this is a basis of the simple right ideal $1_\mu\mathbf{S}'$, and we have $B^+(\mu) = \{1_\mu E_A\}_A$ where

(c) $E_A = E_{i_r}E_{i_{r-1}}\cdots E_{i_1}$ for any sequence $A = (i_1, i_2, \dots, i_r)$ over I.

According to Theorem 3.9 the set of products

(d) $$B(\lambda) = \{xy \colon x \in B^-(\mu), y \in B^+(\mu)\} = \{F_A 1_\mu E_{A'}\}_{A,A'}$$

is a basis for the two sided ideal $\mathbf{S}'1_\mu\mathbf{S}'$ of $\mathbf{S}'$. In this basis, (A, A') range over all pairs of sequences indexing the basis of $\mathbf{S}'1_\mu$. By abuse of notation, we shall also denote by $B(\mu)$ the set of preimages of these elements under the quotient map $p_{\pi,\pi'}$. Note that there is a canonical choice for each of these preimages, expressed as the "same" product in $\mathbf{S}$ that was used to define them in $\mathbf{S}'$.

Having fixed a basis $B(\mu)$ as above for each $\mu \in \pi$, we put

(e) $$B_\pi = \bigsqcup_{\mu\in\pi} B(\mu) = \bigsqcup_{\mu\in\pi} \{F_A 1_\mu E_{A'}\}.$$

We note that, for any $\mu \in \pi$, it follows by induction that the set $\bigsqcup_{\mu' \trianglerighteq \mu} B(\mu') = \bigsqcup_{\mu' \trianglerighteq \mu} \{F_A 1_\mu E_{A'}\}$ is a basis for the ideal $\mathbf{S}[\trianglerighteq\mu]$, and moreover the set $\bigsqcup_{\mu' \triangleright \mu} B(\mu') = \bigsqcup_{\mu' \triangleright \mu} \{F_A 1_\mu E_{A'}\}$ is a basis for the ideal $\mathbf{S}[\triangleright\mu]$. In particular, B_π is a basis for $\mathbf{S}$.

3.17. **Theorem.** *For any π, the set B_π defined above is a cellular $\mathbf{Q}(v)$-basis of $\mathbf{S} = \mathbf{S}(\pi)$, in the sense of Graham and Lehrer* [GL].

Proof. One easily sees that $\iota(F_A 1_\mu E_{A'}) = F_{A'} 1_\mu E_A$ for all μ and all A, A'. According to the definition in [GL], we need only show that for any $x \in \mathbf{S}$, $\mu \in \pi$, and A, A' we have

$$xF_A 1_\mu E_{A'} \equiv \sum_C r_x(C, A)\, F_C 1_\mu E_{A'} \pmod{\mathbf{S}[\triangleright\mu]}$$

where $r_x(C, A) \in \mathbf{Q}(v)$ is independent of A' and where the index C ranges over the same set of sequences as A does in the definition of $B(\mu)$.

By the triangular decomposition, we may express x in the form

$$x = \textstyle\sum_{\lambda, D, D'} r_{\lambda,D,D'}\, F_D 1_\lambda E_{D'}$$

where $r_{\lambda,D,D'} \in \mathbf{Q}(v)$. Note that $F_D 1_\lambda E_{D'}$ acts as zero on $F_A 1_\mu E_{A'}$ unless the left weight of $F_A 1_\mu$ is equal to the right weight of $F_D 1_\lambda E_{D'}$. Now using the defining relations 1.5 (a), (b), (c) repeatedly we can rewrite any nonzero product $F_D 1_\lambda E_{D'} \cdot F_A 1_\mu$ as a linear combination of elements of the form $F_D 1_\lambda F_G E_{G'} 1_\mu$ and then by multiplying on the right by $E_{A'}$ and combining the F_D with the F_G we obtain the desired independence statement for the coefficients. □

3.18. **Remark.** It is easy to check at this point that the left and right $\mathbf{S}$-modules $\Delta(\mu)$ and $\Delta^\sharp(\mu)$ (for $\mu \in \pi$) are in fact isomorphic to the left and right cell modules as defined in [GL].

4. Specialization

We are now ready to study k-forms of generalized q-Schur algebras, over an arbitrary field k of characteristic zero, depending on a chosen parameter $q \in k^\times$. (We denote by $k^\times$ the multiplicative group of nonzero elements of k.) Given any such q, we regard k as an $\mathbf{A}$-algebra by means of the canonical algebra homomorphism $\mathbf{A} \to k$, given by sending v to q. We continue to fix a saturated subset π of X^+ and put $\mathbf{S} = \mathbf{S}(\pi)$ as above. The algebras $\mathbf{S}_q$ are q-deformations of the generalized Schur algebras introduced in [Do1].

4.1. In order to construct a cellular basis for the "integral" form ${}_{\mathbf{A}}\mathbf{S}$, we assume that λ is a maximal element of π and choose an arbitrary $\mathbf{A}$-basis $\mathscr{B}^-_\pi(\lambda)$ of the left ideal $({}_{\mathbf{A}}\mathbf{S}^-)1_\lambda$ which is also a $\mathbf{Q}(v)$-basis of $\mathbf{S}1_\lambda$. The existence of such a basis was established in the proof of Theorem 3.6. Similarly, we choose an arbitrary $\mathbf{A}$-basis $\mathscr{B}^+_\pi(\lambda)$ of the right ideal $1_\lambda({}_{\mathbf{A}}\mathbf{S}^+)$ which is a $\mathbf{Q}(v)$-basis of $1_\lambda\mathbf{S}^+$. (One could take $\mathscr{B}^+_\pi(\lambda) = \{\iota(x)\colon x \in \mathscr{B}^-_\pi(\lambda)\}$, for instance.) Putting

(a) $$\mathscr{B}_\pi(\lambda) = \mathscr{B}^-_\pi(\lambda)\mathscr{B}^+_\pi(\lambda) = \{xy\colon x \in \mathscr{B}^-_\pi(\lambda), y \in \mathscr{B}^+_\pi(\lambda)\}$$

gives an $\mathbf{A}$-basis of the two sided ideal $({}_{\mathbf{A}}\mathbf{S})1_\lambda({}_{\mathbf{A}}\mathbf{S})$ which is also a $\mathbf{Q}(v)$-basis of $\mathbf{S}1_\lambda\mathbf{S}$.

We may assume by induction that a cellular basis $\mathscr{B}_{\pi'}$ for the algebra ${}_{\mathbf{A}}\mathbf{S}'$ has already been constructed, where $\mathbf{S}' = \mathbf{S}(\pi - \{\lambda\})$. By adjoining the set $\mathscr{B}_\pi(\lambda)$ to that basis (regarded as a subset of ${}_{\mathbf{A}}\mathbf{S}$ by means of the canonical quotient map ${}_{\mathbf{A}}\mathbf{S} \to {}_{\mathbf{A}}\mathbf{S}'$) we obtain the desired cellular basis $\mathscr{B}_\pi$ of ${}_{\mathbf{A}}\mathbf{S}$. By construction we have $\mathscr{B}_\pi = \sqcup_{\mu\in\pi}\mathscr{B}_\pi(\mu)$.

For any $\mu \in \pi$, let ${}_{\mathbf{A}}\Delta(\mu)$ be the ${}_{\mathbf{A}}\mathbf{S}$-submodule of $\Delta(\lambda)$ with basis $\mathscr{B}^-_\pi(\mu)$. Similarly, let ${}_{\mathbf{A}}\Delta^\sharp(\mu)$ be the right ${}_{\mathbf{A}}\mathbf{S}$-submodule of the right module $\Delta^\sharp(\lambda)$ with basis $\mathscr{B}^+_\pi(\mu)$. These are the left and right cell modules for ${}_{\mathbf{A}}\mathbf{S}$.

4.2. Now we set $\mathbf{S}_q = k \otimes_{\mathbf{A}} ({}_{\mathbf{A}}\mathbf{S})$. This can also be written as $\mathbf{S}_q(\pi)$ if the indexing set π needs to be made explicit. This is the generalized q-Schur algebra *specialized* at $0 \neq q \in k^\times$. We write $\mathscr{B}_{\pi,q}$ for the set of $1 \otimes b$ as b ranges over $\mathscr{B}_\pi$, with notations $\mathscr{B}^-_{\pi,q}(\mu)$, $\mathscr{B}^+_{\pi,q}(\mu)$, and $\mathscr{B}_{\pi,q}(\mu)$ defined similarly. We shall identify a basis element b of $\mathscr{B}_\pi$ with its image $1 \otimes b$ in $\mathscr{B}_{\pi,q}$. We also write ι for the anti-involution on $\mathbf{S}_q$ induced by ι on ${}_{\mathbf{A}}\mathbf{S}$. We set

(a) $$\Delta_q(\mu) = k \otimes_{\mathbf{A}} ({}_{\mathbf{A}}\Delta(\mu)); \quad \Delta^\sharp_q(\mu) = k \otimes_{\mathbf{A}} ({}_{\mathbf{A}}\Delta^\sharp(\mu)).$$

These are left and right modules for $\mathbf{S}_q$, obtained by specializing v to q. They have k-bases $\mathscr{B}^-_{\pi,q}(\mu)$ and $\mathscr{B}^+_{\pi,q}(\mu)$, respectively.

4.3. **Proposition.** *For any field k of characteristic zero, and any specialization $v \mapsto q \in k^\times$, the algebra $\mathbf{S}_q = \mathbf{S}_q(\pi)$ is a cellular algebra over k, with anti-involution ι and cellular basis $\mathscr{B}_{\pi,q} = \sqcup_{\mu\in\pi}\mathscr{B}_{\pi,q}(\mu)$. The left and right cell modules for $\mathbf{S}_q$ are the $\Delta_q(\mu)$, $\Delta^\sharp_q(\mu)$ for $\mu \in \pi$.*

Proof. This is [GL, (1.8)], which is just the observation that cellularity is compatible with specialization. $\square$

4.4. We may now construct the simple $\mathbf{S}_q$-modules, following Graham and Lehrer. The bilinear form φ_μ on $\Delta(\mu)$ constructed in 3.7 induces a corresponding bilinear form, which we shall denote by φ_μ^q, on the cell module $\Delta_q(\mu)$. In general this form is no longer nondegenerate. However, by considering the value $\varphi_\mu^q(x_0, x_0)$ where x_0 is the basis element of $\mathscr{B}^-_{\pi,q}(\mu)$ of highest weight, it is clear that $\varphi_\mu^q \neq 0$ for each $\mu \in \pi$.

Let $\mathrm{rad}_q(\mu)$ be the radical of the form; $\mathrm{rad}_q(\mu)$ is the set of x in $\Delta_q(\mu)$ such that $\varphi_\mu^q(x,y) = 0$ for all $y \in \Delta_q(\mu)$. In [GL, (3.2)] it is proved that $\mathrm{rad}_q(\mu)$ coincides with the unique maximal submodule of $\Delta_q(\mu)$, and thus the quotient

(a) $$L_q(\mu) := \Delta_q(\mu)/\operatorname{rad}_q(\mu)$$

is a simple (in fact absolutely simple) $\mathbf{S}_q$-module of highest weight μ. By [GL, (3.4) and (3.10)] we immediately obtain our main result.

4.5. **Theorem.** *For any field k of characteristic zero, any specialization $v \mapsto q \in k^\times$, and any saturated set π put $\mathbf{S}_q = \mathbf{S}_q(\pi)$. Then:*

(a) *The algebra $\mathbf{S}_q$ is quasihereditary with respect to the ordering on simple modules induced by the ordering $\trianglelefteq$ on π. The set of standard modules is $\{\Delta_q(\mu) \colon \mu \in \pi\}$.*

(b) *The set $\{L_q(\mu) \colon \mu \in \pi\}$ is a complete set of isomorphism classes of simple $\mathbf{S}_q$-modules.*

We have the following easy application of the cellular structure of $\mathbf{S}_q$. The same result should hold in greater generality; actually it is known to hold in case q is not a root of unity.

4.6. **Corollary.** *If q is transcendental then $L_q(\mu) = \Delta_q(\mu)$ for each $\mu \in \pi$ and hence $\mathbf{S}_q$ is a semisimple algebra. In particular, any finite dimensional $\mathbf{S}_q$-module is completely reducible.*

Proof. For the bilinear form φ_μ on ${}_{\mathbf{A}}\Delta(\mu)$ and for basis elements $x, y \in \mathscr{B}^-(\mu)$, the value of $\varphi_\mu(x,y)$ is an element of $\mathbf{A} = \mathbf{Q}[v, v^{-1}]$. Thus the determinant of φ_μ is a nonzero element $f(v)$ of $\mathbf{Q}[v, v^{-1}]$, and the determinant of the corresponding form φ_μ^q on $\Delta_q(\mu)$ is the element $f(q)$ of k obtained by replacing v by q. This is a polynomial in q and q^{-1} with rational coefficients. Any such polynomial must be nonzero since q is transcendental. Hence φ_μ^q is nondegenerate and $L_q(\mu) = \Delta_q(\mu)$. Since this holds for every $\mu \in \pi$, by [GL, (3.8)] it follows that $\mathbf{S}_q$ is a semisimple algebra, as desired. $\square$

4.7. We write M^* for the linear dual space $\mathrm{Hom}_k(M, k)$ of a given k-vector space M. If M is a left $\mathbf{S}_q$-module, then M^* is naturally a right $\mathbf{S}_q$-module, with $x \in \mathbf{S}_q$ acting on $f \in M^*$ by $(f \cdot x)(m) = f(xm)$, for any $m \in M$. Similarly, the dual $(M^\sharp)^*$ of a right $\mathbf{S}_q$-module $M^\sharp$ is naturally a left $\mathbf{S}_q$-module, with $x \in \mathbf{S}_q$ acting on $f \in (M^\sharp)^*$ by $(x \cdot f)(m) = f(mx)$, for any $m \in M^\sharp$. In particular, by applying these constructions to the left and right cell modules $\Delta_q(\lambda)$, $\Delta_q^\sharp(\lambda)$ we obtain the $\mathbf{S}_q$-modules

(a) $$\nabla_q^\sharp(\lambda) := \Delta_q(\lambda)^*; \quad \nabla_q(\lambda) := \Delta_q^\sharp(\lambda)^*.$$

Note that $\nabla_q^\sharp(\lambda)$ is a right $\mathbf{S}_q$-module and $\nabla_q(\lambda)$ a left one. For each $\lambda \in \pi$, there is a well-defined homomorphism θ_λ of $\mathbf{S}_q$-modules

(b) $$\theta_\lambda \colon \Delta_q(\lambda) \to \nabla_q(\lambda).$$

It suffices to define θ_λ on the highest weight vector $x_0 \in \mathscr{B}^-_{\pi,q}(\lambda)$, since $\Delta_q(\lambda)$ is generated by x_0 as an $\mathbf{S}_q$-module. Notice that $\iota(x_0)$ is the highest weight vector of $\Delta^\sharp(\lambda)$. (In case λ is maximal in π we have $x_0 = 1_\lambda = \iota(x_0)$.) Let $f_0 \in \Delta_q^\sharp(\lambda)^*$ be the linear functional such that $f_0(\iota(x_0)) = 1$ and $f_0(\iota(x_0)e) = 0$ for every $1 \neq e \in \mathbf{S}_q^+$ such that e is the image of some element of ${}_{\mathbf{A}}\mathbf{S}_\nu^+$ where $\nu \neq 0$; see the decomposition 1.11(b). Then f_0 is a highest weight vector of $\nabla_q(\lambda)$ (of weight λ) which is killed by any E_i, and we define $\theta_\lambda(x_0) = f_0$. The image of θ_λ is the unique simple submodule of $\nabla_q(\lambda)$ and its kernel is the radical of $\Delta_q(\lambda)$.

The duality just discussed may be formalized as a covariant functor $M \mapsto M^\circ$ from left $\mathbf{S}_q$-modules to left $\mathbf{S}_q$-modules, where by definition $M^\circ = (M^\sharp)^*$ and $M^\sharp$ is the right $\mathbf{S}_q$-module which is equal to M as a vector space, but turned into a right module by twisting the given action by the anti-involution ι. Any quasihereditary algebra has a "costandard" module corresponding to each standard module; see e.g., [DR], [Xi] or [Do4, Appendix]. In the present context $\nabla_q(\lambda) = \Delta_q(\lambda)^\circ$ is the costandard module corresponding to $\Delta_q(\lambda)$.

This duality fixes the simple modules and interchanges projectives and injectives: $L_q(\mu)^\circ \simeq L_q(\mu)$ and $P_q(\mu)^\circ \simeq E_q(\mu)$ for any $\mu \in \pi$, where $P_q(\mu)$ is the projective cover, and $E_q(\mu)$ the injective envelope, of $L_q(\mu)$. We note that $\Delta_q(\mu)$ is the largest factor module of $P_q(\mu)$ with composition factors of the form $L_q(\mu')$ for $\mu' \trianglelefteq \mu$, and $\nabla_q(\mu)$ is the largest submodule of $E_q(\mu)$ with composition factors of the form $L_q(\mu')$ for $\mu' \trianglelefteq \mu$.

4.8. Let $\mathcal{F}(\Delta)$ be the full subcategory of the category of finite dimensional left $\mathbf{S}_q$-modules consisting of the finite dimensional left $\mathbf{S}_q$-modules M admitting a Δ-filtration. By definition, a Δ-filtration of

M is a series of submodules

$$0 = M_0 \subseteq M_1 \subseteq \cdots M_{r-1} \subseteq M_r = M$$

such that each successive subquotient M_j/M_{j-1} is isomorphic to some $\Delta_q(\mu_j)$. Dually, we have the category $\mathcal{F}(\nabla)$ defined similarly with ∇ in place of Δ. For a module $M \in \mathcal{F}(\Delta)$ we let $(M\colon \Delta_q(\mu))$ denote the number of subquotients in a Δ-filtration which are isomorphic to $\Delta_q(\mu)$, and similarly let $(M\colon \nabla_q(\mu))$ denote the number of subquotients in a ∇-filtration which are isomorphic to $\nabla_q(\mu)$, for any $M \in \mathcal{F}(\nabla)$. (These numbers are independent of the choice of filtration.) We also denote by $[M\colon L_q(\mu)]$ the multiplicity of $L_q(\mu)$ in a composition series of M.

The fact that $\mathbf{S}_q$ is quasihereditary immediately implies a number of important basic properties, some of which we list below (see [Do4, Appendix]):

(1) $\mathbf{S}_q$ has finite global dimension.
(2) For any $\mu \in \pi$, $\mathrm{End}_{\mathbf{S}_q}(\Delta_q(\mu)) \simeq k$ and $\mathrm{End}_{\mathbf{S}_q}(\nabla_q(\mu)) \simeq k$.
(3) $P_q(\mu) \in \mathcal{F}(\Delta)$, $E_q(\mu) \in \mathcal{F}(\nabla)$, $(P_q(\mu)\colon \Delta_q(\lambda)) = [\nabla_q(\lambda)\colon L_q(\mu)]$, and $(E_q(\mu)\colon \nabla_q(\lambda)) = [\Delta_q(\lambda)\colon L_q(\mu)]$, for any $\lambda, \mu \in \pi$.
(4) $E_q(\mu)/\nabla_q(\mu) \in \mathcal{F}(\nabla)$, and if $(E_q(\mu)/\nabla_q(\mu)\colon \nabla_q(\lambda)) \neq 0$ for $\lambda, \mu \in \pi$ then $\lambda \rhd \mu$.
(5) For a finite dimensional $\mathbf{S}_q$-module M and any $\mu \in \pi$, if either of the groups $\mathrm{Ext}^1_{\mathbf{S}_q}(\Delta_q(\mu), M)$ or $\mathrm{Ext}^1_{\mathbf{S}_q}(M, \nabla_q(\mu))$ is not zero, then there exists some $\mu' \in \pi$ such that $[M\colon L_q(\mu')] \neq 0$ and $\mu' \rhd \mu$.
(6) For $M \in \mathcal{F}(\Delta)$ and $N \in \mathcal{F}(\nabla)$ the k-dimension of $\mathrm{Ext}^j_{\mathbf{S}_q}(M, N)$ is $\sum_{\lambda\in\pi}(M\colon \Delta_q(\lambda))(N\colon \nabla_q(\lambda))$ if $j = 0$ and is 0 otherwise. Moreover, we have
$$(M\colon \Delta_q(\lambda)) = \dim_k \mathrm{Hom}_{\mathbf{S}_q}(M, \nabla_q(\lambda)),$$
$$(N\colon \nabla_q(\lambda)) = \dim_k \mathrm{Hom}_{\mathbf{S}_q}(\Delta_q(\lambda), N).$$
(7) A finite dimensional $\mathbf{S}_q$-module M belongs to $\mathcal{F}(\Delta)$ if and only if $\mathrm{Ext}^1_{\mathbf{S}_q}(M, \nabla_q(\mu)) = 0$ for all $\mu \in \pi$, and belongs to $\mathcal{F}(\nabla)$ if and only if $\mathrm{Ext}^1_{\mathbf{S}_q}(\Delta_q(\mu), M) = 0$ for all $\mu \in \pi$.
(8) Given a short exact sequence $0 \to M' \to M \to M'' \to 0$ of finite dimensional $\mathbf{S}_q$-modules, if $M', M \in \mathcal{F}(\nabla)$ then $M'' \in \mathcal{F}(\nabla)$, and if $M, M'' \in \mathcal{F}(\Delta)$ then $M' \in \mathcal{F}(\Delta)$.
(9) If M belongs to $\mathcal{F}(\Delta)$ then any direct summand of M also belongs to $\mathcal{F}(\Delta)$; if M belongs to $\mathcal{F}(\nabla)$ then any direct summand of M also belongs to $\mathcal{F}(\nabla)$.

(10) For $\lambda, \mu \in \pi$, $\mathrm{Ext}^i_{\mathbf{S}_q}(\Delta_q(\lambda), \nabla_q(\mu)) \simeq \begin{cases} k & \text{if } i = 0 \text{ and } \lambda = \mu \\ 0 & \text{otherwise.} \end{cases}$

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Loyola University Chicago, Chicago, Illinois 60626 U.S.A.
E-mail address: doty@math.luc.edu

Loyola University Chicago, Chicago, Illinois 60626 U.S.A.
E-mail address: tonyg@math.luc.edu