LOCALLY DIVERGENT ORBITS ON HILBERT MODULAR SPACES AND MARGULIS CONJECTURES

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Abstract. We describe the closures of locally divergent orbits under the action of tori on Hilbert modular spaces of rank $r \geq 2$. In particular, we prove that if D is a maximal $\mathbb{R}$-split torus acting on a real Hilbert modular space then every locally divergent non-closed orbit is dense for $r > 2$ and its closure is a finite union of tori orbits for $r = 2$. Our results confirm an orbit rigidity conjecture of Margulis in all cases except for (i) $r = 2$ and, (ii) $r > 2$ and the Hilbert modular space corresponds to a CM-field; in the cases (i) and (ii) our results contradict the conjecture. Moreover, we show that the measure counterpart of the conjecture is not valid.

As an application, we describe the set of values at integral points of collections of non-proportional, split, binary, quadratic forms over number fields.

1. Introduction

During the last decade the problems of the descriptions of orbit closures and invariant measures for actions of maximal split tori on homogeneous spaces appear to be among the central ones in homogeneous dynamics. This interest is motivated to a large extent by number theory applications. The efficiency of the homogeneous dynamics approach in the number theory had been demonstrated in a striking way by G.A.Margulis proof of the long-standing Oppenheim conjecture dealing with density properties of values at integral points of quadratic forms in at least tree variables [M1]. In our days this approach looks quite promising regarding the still open Littlewood conjecture. Indeed, the Littlewood conjecture can be deduced from a conjecture of Margulis affirming that the maximal split tori bounded orbits on $\mathrm{SL}(3,\mathbb{R})/\mathrm{SL}(3,\mathbb{Z})$ are always compact [M2, §2]. In this direction, M.Einsiedler, A.Katok and E.Lindenstrauss classified the probability measures on $\mathrm{SL}(3,\mathbb{R})/\mathrm{SL}(3,\mathbb{Z})$ which are invariant and ergodic under the action of a maximal split torus with positive entropy. As an application, they proved that the set of exceptions to Littlewood's conjecture has Hausdorff dimension zero [E-K-L]. (See [M3], [E-L] and [L] for a collection of problems and conjectures and an account of recent achievements on this and related topics.) One of the consequences of the main results of this paper is the explicit description of the set of values at integral points of collections of non-proportional, split, binary quadratic forms over number fields (Theorem 1.10).

Let us introduce the main objects of the paper. Let K be a number field, $\mathcal{O}$ its ring of integers and K_i, $1 \leq i \leq r$, all the archimedean completions of K. Put $G = \prod_{i=1}^{r} G_i$, where $G_i = \mathrm{SL}(2, K_i)$, and let $\Gamma = \mathrm{SL}(2, \mathcal{O})$ be identified with its image in G under the diagonal embedding. *Throughout this paper we assume that* $r \geq 2$. By Margilis arithmeticity theorem [M4], [M5] (due to Selberg [S] in the case relevant to the present paper) up to conjugation and commensurability, Γ is the only irreducible non-uniform lattice in G. The quotient space G/Γ is called *the Hilbert modular space of rank* r. Denote by $\pi : G \to G/\Gamma$ the natural projection. Let D_i be the connected component of the diagonal subgroup of G_i and let $D_{i,\mathbb{R}}$ be the connected component of the subgroup of *real* matrices in D_i. (So, $D_{i,\mathbb{R}} = D_i$ if $K_i = \mathbb{R}$.) For every non-empty $I \subset \{1, \cdots, r\}$ we denote $D_I = \prod_{i \in I} D_i$ and $D_{I,\mathbb{R}} = \prod_{i \in I} D_{i,\mathbb{R}}$. When $I = \{1, \cdots, r\}$ we write D and $D_{\mathbb{R}}$ instead of D_I and $D_{I,\mathbb{R}}$, respectively. By a torus (respectively, an $\mathbb{R}$-split torus or, simply, a split torus) in G we mean a subgroup conjugated to a closed connected subgroup of D (respectively, $D_{\mathbb{R}}$). An orbit $D_I \pi(g)$ is called *locally divergent* if $D_i \pi(g)$ is divergent for all $i \in I$. (Recall that if H is a closed non-compact subgroup of G and $x \in G/\Gamma$ then the orbit Hx is divergent if the orbit map $h \mapsto hx$ is proper or, equivalently, if $\{h_n x\}$ leaves compact subsets of G/Γ whenever h_n leaves compact subsets of H.) The orbit $D_{I,\mathbb{R}}\pi(g)$ is locally divergent if and only if the orbit $D_I \pi(g)$ is locally divergent. The description of the divergent D_i-orbits (and, therefore, the divergent $D_{i,\mathbb{R}}$-orbits) follows from the general results of [T1] (see §2.2). The paper [T1] is related with [T-W]. Prior to [T-W] Margulis described the divergent orbits for the action of the full diagonal group on the space of lattices of $\mathbb{R}^n$, $n \geq 2$ [T-W, Appendix].

Let us formulate the following:

Conjectures: 1.(*Orbit rigidity*) If $\#I \geq 2$ then every orbit $D_{I,\mathbb{R}}x$, $x \in G/\Gamma$, has *homogeneous closure*, that is, $\overline{D_{I,\mathbb{R}}x} = Fx$, where F is a closed subgroup in G containing $D_{I,\mathbb{R}}$;

2.(*Measure rigidity*) If $\#I \geq 2$ then every $D_{I,\mathbb{R}}$-invariant, $D_{I,\mathbb{R}}$-ergodic, Borel measure μ on G/Γ is *algebraic*, that is, there exists a closed subgroup F in G containing $D_{I,\mathbb{R}}$ so that μ is F-invariant and $\mathrm{supp}(\mu) = Fx$ for some $x \in G/\Gamma$.

The above conjectures are special cases of the much more general [M3, Conjectures 1 and 2], respectively, about actions of split tori (in other terms, about actions of $\mathbb{R}$-diagonalizable connected subgroups) on H/Δ where H is a real Lie group and Δ its lattice. For actions of split tori on $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$, $n \geq 3$, examples of orbits with non-homogeneous closures contradicting [M3, Conjecture 1] have been constructed by F.Maucourant [Ma] and by U.Shapira [Sha]. It is our understanding that the constructions in these papers do not apply to the Hilbert modular spaces.

In the present paper we describe the closures of locally divergent D_I-orbits on the Hilbert modular spaces G/Γ. It turns out that Conjecture 1 is not valid for actions of two-dimensional tori (Theorem 1.1) and for the Hilbert modular spaces corresponding to CM-fields (Theorem 1.8) but it is valid in all remaining cases (Theorem 1.5). As a consequence from Theorem 1.1, we get counter-exemples (to the best of our knowledge the first ones) to the general measure rigidity [M3, Conjecture 2] (see Theorem 1.2 and Corollaries 1.3(b) and 1.4).

It is important to mention that both the orbit and the measure rigidities are well-known for actions of connected subgroups generated by unipotent elements on arbitrary homogeneous spaces. (Recall that a linear transformation with all its eigenvalues equal to 1 is called unipotent.) In the case of real Hilbert modular spaces of rank 2 the unipotent orbit rigidity can be proved using the methods of Dani and Margulis paper [DM1] where the orbit rigidity had been proved for generic unipotent flows on homogeneous spaces of $\mathrm{SL}(3,\mathbb{R})$. In connection with the uniform distribution of Heegner points, using the approach from [M1] and [DM1], N.Shah treated the unipotent orbit rigidity for products of several copies of SL(2) over local fields (see [Sh]). Both the unipotent orbit and measure rigidities were proved in full generality in M.Ratner substantial papers [Ra1] and [Ra2]. Note that there are deep intrinsic differences between the split tori actions and the unipotent actions. For instance, by a fundamental result of Margulis [M6] (strengthen by S.G.Dani [D]), the orbits of the unipotent groups are never divergent. The quantitative versions of this result have significant applications (see [DM2] and [KlM]).

Let us formulate the results of the paper. The cases $\#I = 2$ and $\#I > 2$ represent very different phenomena and will be considered separately.

Theorem 1.1. *Let $\#I = 2$ and $D_I\pi(g)$ be a locally divergent orbit on G/Γ. Suppose that the closure $\overline{D_I\pi(g)}$ is not an orbit of a torus. Then*

$$\overline{D_I\pi(g)} = D_I\pi(g) \cup \bigcup_{i=1}^{s} T_i\pi(h_i)$$

where $2 \leq s \leq 4$, T_i are tori containing D_I and $T_i\pi(h_i)$ are pairwise different closed non-compact orbits. In particular, if $\#I = 2$ then there are no dense locally divergent D_I-orbits.

The locally divergent orbits $D_I\pi(g)$, $\#I \geq 2$, such that $\overline{D_I\pi(g)}$ is not an orbit of a torus are explicitly described in Corollary 1.9 below.

Theorem 1.1 implies that both Conjectures 1 and 2 are not valid. More precisely we have the following.

Theorem 1.2. *Let $\#I = 2$ and $T = D_I$* or *$D_{I,\mathbb{R}}$. Suppose that $T\pi(g)$ is a locally divergent orbit such that $\overline{T\pi(g)}$ is not an orbit of a torus. Then $\overline{T\pi(g)} = \mathrm{supp}(\mu)$, where μ is a non-algebraic, T-invariant, T-ergodic, Borel measure on G/Γ. Moreover, $\overline{T\pi(g)}$ is not homogeneous.*

The maximal tori action (the so-called Weyl chamber flow) deserves special attention. The next corollary is a particular case of the Theorems 1.1 and 1.2.

Corollary 1.3. *Suppose that the Hilbert modular space G/Γ is of rank $r = 2$. Then:*

(a) *A locally divergent orbit $D\pi(g)$ is either closed or $\overline{D\pi(g)} \setminus D\pi(g) = \bigcup_{i=1}^{s} D\pi(h_i)$, where $2 \leq s \leq 4$, and $D\pi(h_i)$ are pairwise different, closed, non-compact orbits;*

(b) *There exist D-invariant, D-ergodic, non-algebraic Borel measures on G/Γ.*

Using Weil's restriction of scalars, the homogeneous space G/Γ in the formulation of Corollary 1.3 can be embedded in $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$, $n \geq 4$. In this way we obtain orbits of multidimensional tori with non-homogeneous closures which are different from the already known. We also get:

Corollary 1.4. *The measure rigidity conjecture is not valid for T-invariant, T-ergodic, Borel measures on $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$, $n \geq 4$, where T is a split torus with $\dim T = 2$ if $n = 4$ and $\dim T = n - 3$ if $n \geq 5$.*

The dynamics of the action of D_I on a Hilbert modular space G/Γ differs drastically when $\#I > 2$. In this case the so-called CM-*fields* play an important role. Recall that a number field K is called CM-field (so named for a close connection to the theory of complex multiplication) if it is a quadratic extension of a totally real number field which is totally imaginary.

Theorem 1.5. *Let $\#I > 2$ and $D_I\pi(g)$ be a locally divergent orbit such that $\overline{D_I\pi(g)}$ is not an orbit of a torus. Assume that K is not a* CM*-field. Then $D_I\pi(g)$ is a dense orbit.*

In the classical case of *real* Hilbert modular spaces Theorem 1.5 implies:

Corollary 1.6. *Let K be a totally real number field of degree $r \geq 3$. Let $\#I > 2$ and $D_I\pi(g)$ be a locally divergent orbit such that $\overline{D_I\pi(g)}$ is not an orbit of a torus. Then $\overline{D_I\pi(g)} = G/\Gamma$.*

In particular, if $D_I = D$ then $D\pi(g)$ is either closed or dense.

If K *is* a CM-field then the closure of $D_I\pi(g)$ might not be homogeneous. This is related to a simple observation which we are going to explain now. Denote by $G_{i,\mathbb{R}}$, $1 \leq i \leq r$, the subgroup of real matrices in G_i and put $G_{\mathbb{R}} = \prod_{i=1}^{r} G_{i,\mathbb{R}}$. Clearly, $G_{\mathbb{R}} \supset D_{I,\mathbb{R}}$. Now let K be a CM-field which is a quadratic extension of a totally real number field F and let $\mathcal{O}_F$ be the ring of integers of F. Then $\Gamma_F = \mathrm{SL}(2, \mathcal{O}_F)$ is a lattice in $G_{\mathbb{R}}$ and the orbit $G_{\mathbb{R}}\pi(e)$ is closed and homeomorphic to $G_{\mathbb{R}}/\Gamma_F$. (It is standard to prove that this property characterize K as a CM-field, that is, if G/Γ admits a closed $G_{\mathbb{R}}$-orbit then K is a CM-field.) It follows from Corollary 1.6 that if K is a CM-field, $x \in G_{\mathbb{R}}\pi(e)$ and $D_{I,\mathbb{R}}x$ is a locally divergent orbit whose closure is not an orbit of a torus, then $\overline{D_{I,\mathbb{R}}x} = G_{\mathbb{R}}\pi(e)$. Since D_I is a compact extension of $D_{I,\mathbb{R}}$ this implies that $\overline{D_I x} = D_I G_{\mathbb{R}}\pi(e)$. So, $\overline{D_I x}$ is not homogeneous which shows that if K is a CM-field the analog of Theorem 1.5 is not valid.

Let us turn to the study of the orbits for the action of the $\mathbb{R}$-split tori $D_{I,\mathbb{R}}$ which is important from the point of view of Margulis' conjectures. Theorem 1.5 implies:

Corollary 1.7. *With the assumptions of Theorem 1.5, the orbit $D_{I,\mathbb{R}}\pi(g)$ is dense in G/Γ.*

When K is a CM-field we obtain exemples of tori orbits contradicting Conjecture 1 which are *essentially* different from those provided by Theorem 1.1.

Theorem 1.8. *Let K be a* CM*-field and $\#I > 2$. Then there exists a point $x \in G/\Gamma$ with the following properties:*

(i) $\overline{D_{I,\mathbb{R}}x} \neq G/\Gamma$;

(ii) *There exists an* $y \in \overline{D_{I,\mathbb{R}}x} \setminus D_{I,\mathbb{R}}x$ *such that* $\overline{D_{I,\mathbb{R}}x} = \overline{D_{I,\mathbb{R}}y}$ *and* Hy *is not closed for any proper subgroup* H *of* G *containing* $D_{I,\mathbb{R}}$*;*

(iii) $\overline{D_{I,\mathbb{R}}x} \setminus D_{I,\mathbb{R}}x$ *is not contain in a union of finitely many closed orbits of proper subgroups of* G*.*

In particular, $\overline{D_{I,\mathbb{R}}x}$ *is not homogeneous.*

As a by-product of the proofs of the above theorems we get the following corollary which is known for $D_I = D$ (see Theorem 2.1 below).

Corollary 1.9. *Suppose that* $D_I\pi(g)$ *is a locally divergent orbit. Then* $\overline{D_I\pi(g)}$ *(and, therefore,* $\overline{D_{I,\mathbb{R}}\pi(g)}$*) is an orbit of a torus if and only if* $g \in \mathcal{N}_G(D_I)G_K$ *where* $\mathcal{N}_G(D_I)$ *is the normalizer of* D_I *in* G*. In particular,* $D_I\pi(g)$ *is locally divergent but* $\overline{D_I\pi(g)}$*) is not an orbit of a torus if and only if*

$$g \in \big(\bigcap_{i \in I} \mathcal{N}_G(D_i)G_K\big) \setminus \mathcal{N}_G(D_I)G_K$$

In view of Theorems 1.1 and 1.8 and of [Ma] and [Sha], the following orbit rigidity conjecture is plausible:

Conjecture. Let G be a real semisimple algebraic group with no compact factors and let Γ be an irreducible lattice in G. Suppose that $\mathrm{rank}_{\mathbb{R}} G \geq 2$ and that every semisimple subgroup G_0 in G of the same $\mathbb{R}$-rank as G acts minimally on G/Γ (i.e., every G_0-orbit is dense). Then if T is a maximal $\mathbb{R}$-split torus in G and $x \in G/\Gamma$, either

(1) $\overline{Tx} = G/\Gamma$, or

(2) $\overline{Tx} \setminus Tx \subset \bigcup_{i=1}^{n} H_i x_i$ where H_i are proper reductive subgroups of G and $H_i x_i$ are closed.

We apply our method to study the values of binary quadratic forms at integral points. Denote $A = \prod_{i=1}^{r} K_i$ and $A^* = \prod_{i=1}^{r} K_i^*$. The polynomial ring $A[X,Y]$ is naturally isomorphic to $\prod_{i=1}^{r} K_i[X,Y]$. The natural embeddings of K into K_i induce embeddings of $K[X,Y]$ into $K_i[X,Y]$, $1 \leq i \leq r$, and a diagonal embedding of $K[X,Y]$ into $A[X,Y]$. *In the next theorem* $f = (f_i)_{i \in \overline{1,r}} \in A[X,Y]$*, where* $f_i \in K_i[X,Y]$ *are split, non-degenerate, quadratic forms over* K (that is, $f_i = l_{i,1} \cdot l_{i,2}$, where $l_{i,1}$ and $l_{i,2}$ are linearly independent linear forms with coefficients from K). If $(\alpha, \beta) \in \mathcal{O}^2$ then $f(\alpha, \beta)$ is an element in A with its i-th coordinate equal to $f_i(\alpha, \beta)$. It is clear that if f_i are two by two proportional (equivalently, if there exists a $g \in K[X,Y]$ such that $f_i = c_i \cdot g$, $c_i \in K$, for all i) then $f(\mathcal{O}^2)$ is a discrete subset of A. It follows from [T1, Theorem 1.8] that the opposite is also valid: the discreteness of $f(\mathcal{O}^2)$ in A implies the proportionality of f_i , $1 \leq i \leq r$. In the next theorem we describe the closure of $f(\mathcal{O}^2)$ in A when f_i, $1 \leq i \leq r$, are not proportional.

Theorem 1.10. *With the above notation and assumptions, suppose that f_i are not proportional. Then the following assertions hold:*

(a) *If $r > 2$ and K is not a* CM*-field then $f(\mathcal{O}^2)$ is dense in A;*
(b) *Let $r = 2$. Put $K'_1 = \{f_1(x, y) : (x, y) \in K_1^2 \text{ and } f_2(x, y) = 0\}$ and $K'_2 = \{f_2(x, y) : (x, y) \in K_2^2 \text{ and } f_1(x, y) = 0\}$. Then there exist K-rational quadratic forms $\phi^{(j)} \in K[X, Y]$, $1 \leq j \leq 4$, such that*

$$\overline{f(\mathcal{O}^2)} = f(\mathcal{O}^2) \bigcup \overset{4}{\underset{j=1}{\cup}} \phi^{(j)}(\mathcal{O}^2)) \bigcup K'_1 \times \{0\} \bigcup \{0\} \times K'_2.$$

So, the set $\overline{f(\mathcal{O}^2)} \cap A^*$ is countable and the set $\overline{f(\mathcal{O}^2)} \bigcap (A \setminus A^*)$ is continium. Moreover, $K'_i = \mathbb{C}$ if $K_i = \mathbb{C}$ and $K'_i = \mathbb{R}, \mathbb{R}_-$ or *$\mathbb{R}_+$ if $K_i = \mathbb{R}$.*

The main results of the paper have been announced in [T2].

2. Preliminaries

2.1. **Basic notation.** As usual $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the rational, real and complex numbers, respectively. Also, $\mathbb{R}_+$ (respectively, $\mathbb{R}_-$) is the set of non-negatives (respectively, non-positives) real numbers. Let $\mathbb{R}_{>0} = \mathbb{R}_+ \setminus \{0\}$. We denote by $| \ . \ |$ the standard norms on $\mathbb{R}$ and $\mathbb{C}$.

In this paper K is a number field and $K_1, \cdots, K_r$ are the completions of K with respect to the archimedean places of K. We denote by $| \cdot |_i$ *the normalized* valuation on K_i. So, if $x \in K$ and $K_i = \mathbb{R}$ (respectively, $K_i = \mathbb{C}$) then $|x|_i = |\sigma_i(x)|$ (respectively, $|x|_i = |\sigma_i(x)|^2$) where σ_i is the corresponding embedding of K into K_i. Note that $|\mathrm{N}_{K/\mathbb{Q}}(x)| = |x|_1 \cdots |x|_r$, where $\mathrm{N}_{K/\mathbb{Q}}(x)$ is the algebraic norm of x. The elements from K are identified with their images in K_i via the embeddings σ_i. So, if $x \in K$, with some abuse of notation, we write x instead of $\sigma_i(x)$. The exact meaning of x will be always clear from the context.

If R is a ring then R^* is its group of invertible elements.

Let $A = \prod_{i=1}^{r} K_i$ and $A^* = \prod_{i=1}^{r} K_i^*$. A (respectively, A^*) is a topological ring (respectively, topological group) endowed with the product topology. The field K (respectively, the group K^*) is diagonally embedded in A (respectively, A^*). The ring of integers $\mathcal{O}$ of K is a co-compact lattice of A and the group of units $\mathcal{O}^*$ is a discrete subgroup of A^*.

If M is a subset of a topological space X then $\overline{M}$ is the topological closure of M in X. Also, if H is a closed subgroup of a topological group L we denote by H° the connected component of H containing the identity. By $\mathcal{N}_L(H)$ we denote the normalizer of H in L.

The notation G_i, G, $G_{\mathbb{R}}$, D_I, $D_{I,\mathbb{R}}$ have been introduced in the Introduction. The group G is considered as a *real* Lie group.

The diagonal embedding of $\mathrm{SL}(2,K)$ in G will be denoted by G_K. B_K^+, B_K^- and D_K are the groups of upper triangular, lower triangular and diagonal matrices in G_K, respectively. For every $1 \leq i \leq r$ we denote by $G_{i,K}$, $B_{i,K}^+, B_{i,K}^-$ and $D_{i,K}$ the images of G_K, B_K^+, B_K^- and D_K, respectively, under the natural projection $G \to G_i$.

In the course of our considerations one and the same matrix with coefficients from K might be considered, according to the context, as an element from G_K or from $G_{i,K}$. For instance, if $g = (g_1, \cdots, g_r) \in G$ and $g_i \in G_{i,K}$ writing $\pi(g_i)$, where π is the map $G \to G/\Gamma$, $g \mapsto g\Gamma$, we mean that g_i is considered as an element from G and, therefore, from G_K.

Given a non-empty subset I of $\{1, \cdots, r\}$ we put $A_I^* \stackrel{def}{=} \prod_{i\in I} K_i^*$. Let $d_i : K_i^* \to G_i, x \mapsto \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$. We put $d_I \stackrel{def}{=} \prod_{i\in I} d_i$ and $d \stackrel{def}{=} d_{\{1,\cdots,r\}}$. So, $D_I = d_I((A_I^*)^\circ)$.

Let $\mathfrak{g}_i = \mathfrak{sl}(2, K_i)$, $\mathfrak{g} = \prod_{i=1}^{r} \mathfrak{g}_i$, $\mathfrak{g}_K = \mathfrak{sl}(2,K)$ and $\mathfrak{g}_{\mathcal{O}} = \mathfrak{sl}(2,\mathcal{O})$. Fixing a basis of K-rational vectors in $\mathfrak{g}_K$ we denote by $\| \cdot \|_i$ the norm max on $\mathfrak{g}_i$. Since $\mathfrak{g} = \prod_{i=1}^{r} \mathfrak{g}_i$ we can define a norm $\| \cdot \|$ on $\mathfrak{g}$ by $\| \mathbf{x} \| = \max_i \| \mathbf{x}_i \|_i$, $\mathbf{x} = (\mathbf{x}_1, \cdots, \mathbf{x}_r) \in \mathfrak{g}$.

As usual, we denote by $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ the adjoint representation of G.

2.2. **Locally divergent orbits.** The locally divergent orbits have been introduced and studied in a much more general context in [T1]. The following theorem is a very particular case of [T1, Theorem 1.4].(See also [T1, Corollary 1.7]).

Theorem 2.1. *Let $g = \{g_1, \cdots, g_r\}$ be an element in G and I be a non-empty subset of $\{1, \cdots, r\}$. The following assertions hold:*

(a) *If the orbit $D_I\pi(g)$ is closed then either I is a singleton or $I = \{1, \cdots, r\}$;*

(b) $D_i\pi(g), 1 \leq i \leq r$, *is closed (equivalently, divergent) if and only if* $g \in \mathcal{N}_G(D_i)G_K$ *(equivalently,* $g_i \in D_iG_{i,K}$*);*

(c) *The following conditions are equivalent:*

(i) $D\pi(g)$ *is closed and non-compact;*

(ii) $D\pi(g)$ *is closed and locally divergent;*

(iii) $g \in \mathcal{N}_G(D)G_K$.

We will need the following proposition:

Proposition 2.2. *If* $g \in \mathcal{N}_G(D_I)G_K$ *then* $\overline{D_I\pi(g)} = T\pi(g)$ *where* T *is a torus containing* D_I.

Proof. In view of our assumption $g = g'h$ where $h \in \mathcal{N}_G(D)G_K$ and $g' \in \prod_{i \notin I} G_i$. Let Δ be the stabilizer of $\pi(g)$ in $g'Dg'^{-1}$. It follows from Theorem 2.1(c) that $g'D\pi(h)$ is closed. Since $\overline{D_I\pi(g)} \subset g'Dg'^{-1}\pi(g)$ we get that $\overline{D_I\pi(g)} = T\pi(g)$ where T is the connected component of the closure of $D_I\Delta$. □

2.3. **Propositions about the units.** Denote $A^1 = \{(x_1, \cdots, x_r) \in A^* : |x_1|_1 \cdots |x_r|_r = 1\}$. Given a positive integer m we put $\mathcal{O}^*_m = \{\xi^m | \xi \in \mathcal{O}^*\}$.

The following lemma follows easily from the classical fact that $\mathcal{O}^*$ is a lattice in A^1.

Lemma 2.3. (*cf.*[T1, Lemma 3.2]) *Let* m *be a positive integer. There exists a real* $\kappa_m > 1$ *with the following property. Let* $x = (x_i) \in A^*$ *and for each* $1 \leq i \leq r$ *let* a_i *be a positive real number such that* $\prod_i a_i = \prod_i |x_i|_i$. *Then there exists* $\xi \in \mathcal{O}^*_m$ *such that*

$$\frac{a_i}{\kappa_m} \leq |\xi x_i|_i \leq \kappa_m a_i$$

for all i.

Proposition 2.4. *Let* $r \geq 3$, $3 \leq l \leq r$, $I = \{l, \cdots, r\}$ *and* $p_I : A^* \to A^*_I$ *be the natural projection. Denote by* H *the closure of* $p_I(\mathcal{O}^*)$ *in* A^*_I. *Then*

(a) *the projection of* H° *into each* $K^*_i, i \geq l$, *is non-trivial;*

(b) *for any real* $C > 1$ *there exists* $\xi \in \mathcal{O}^*$ *such that* $|\xi|_l > C$ *and* $|1 - |\xi|_i| < \frac{1}{C}$ *for all* $i > l$.

Proof. (a) By Dirichlet's theorem for the units there exists a positive integer m such that $\mathcal{O}^*_m$ is a free abelian group of rank $r - 1$. It is clear that H° coincides with the connected component of the closure of $p_I(\mathcal{O}^*_m)$. Since H° is open in H and $\mathcal{O}^*_m$ is diagonally embedded in H it is enough to show that $H^\circ \neq \{1\}$. Suppose on that $H^\circ = \{1\}$. Then H is a discrete subgroup of A_I^* containing a free subgroup of rank $r-1$. This is a contradiction because A_I^* is a direct product of a compact group and $\mathbb{Z}^{r-l+1}$.

(b) Consider the logarithmic representation of the group of units $\log_S : \mathcal{O}^* \to \mathbb{R}^r, \theta \mapsto (\log|\theta|_1, \cdots, \log|\theta|_r)$ (see [We]). According to the Dirichlet theorem $\log_S(\mathcal{O}^*)$ is a lattice in the hyperplane $L = \{(x_1, \cdots, x_r) \in \mathbb{R}^r : x_1 + x_2 + \cdots + x_r = 0\}$. Let $\psi : L \to \mathbb{R}^{r-1}, (x_1, \cdots, x_r) \mapsto (x_2, \cdots, x_r)$. Then $\psi(\log_S(\mathcal{O}^*))$ is a lattice in $\mathbb{R}^{r-1}$ with co-volume equal to a positive real V. For every natural n we put

$$B_n = \{(x_2, \cdots, x_r) \in \mathbb{R}^{r-1} : |x_i| \leq \frac{1}{n} \text{ if } i \neq l \text{ and } |x_l| \leq n^{r-2}V\}.$$

By Minkowski's lemma there existe a $\xi_n \in \mathcal{O}^*$ such that $\psi(\log_S(\xi_n)) \in B_n \setminus \{0\}$. If the sequence $|\xi_n|_l$ is unbounded from above then we can choose $\xi = \xi_n$ with n large enough. Let $|\xi_n|_l < C$ where C is a constant. Since $\psi(\log_S(\mathcal{O}^*))$ is discrete this implies the existence of a unit η of infinite order such that $|\eta|_l > 1$ and $|\eta|_i = 1$ if $i \neq l$ and $i > 1$. Hence we can choose $\xi = \eta^m$ with m sufficiently large. This completes the proof. □

Proposition 2.5. *Let $p_l : A^* \to K_l^*$, $1 \leq l \leq r$, be the natural projection. Assume that $K_l = \mathbb{C}$ and that the connected component of $\overline{p_l(\mathcal{O}^*)}$ coincides with $\mathbb{R}_{>0}$. Then K is a* CM*-field.*

Proof. There exists a positive integer m such that $\overline{p_l(\mathcal{O}_m^*)} = \mathbb{R}_{>0}$. Denote by F the subfield of K generated over $\mathbb{Q}$ by all $\theta \in \mathcal{O}_m^*$ and denote by $\mathcal{O}_F^*$ the group of units of F. Let s, respectively t, be the number of real, respectively complex, places of K and let s_1, respectively t_1, be the number of real, respectively complex, places of F. By Dirichlet's theorem $\mathcal{O}_m^*$ is a free group of rank $s+t-1$. Since $\mathcal{O}_m^* \subset \mathcal{O}_F^* \subset \mathcal{O}^*$ and the group of principal units of F is free of rank $s_1 + t_1 - 1$ we have

$$r - 1 = s + t - 1 = s_1 + t_1 - 1.$$

Let n be the degree of K over F. Using that $s + 2t$ is the degree of K over $\mathbb{Q}$ and $s_1 + 2t_1$ is the degree of F over $\mathbb{Q}$ we get

$$s + 2t = n(s_1 + 2t_1) \Leftrightarrow r + t = n(r + t_1) \Leftrightarrow$$
$$(n-1)r = t - t_1 n \Leftrightarrow (n-1)(t+s) = t - t_1 n.$$

Since $n > 1$ the last equality implies that $s = t_1 = 0$ and $n = 2$ which proves the proposition. □

Exemple. There are non-CM fields such that the connected component of $\overline{p_l(\mathcal{O}^*)}$ is a 1-dimensional subgroup of $\mathbb{C}^*$ different from $\mathbb{R}_{>0}$. Such fields need special treatment in the course of the proof of Proposition 5.1(a) below. An exemple of this type is provided by the field $K = \mathbb{Q}(\alpha)$ where α is a root of the equation $(x+\frac{1}{x})^2-2(x+\frac{1}{x})-1 = 0$[1]. The field K has two real and one (up to conjugation) complex completions. If $K_3 = \mathbb{C}$ then it is easy to see that $\overline{p_3(\mathcal{O}^*)}^\circ$ coincides with the unit circle.

3. Accumulations points for locally divergent orbits

Up to the end of the paper $D_I\pi(g)$ will denote a locally divergent orbit. In view of Theorem 2.1(b), *we may (and will) assume without loss of generality that $g = (g_1, \cdots, g_r)$ with $g_i \in G_{i,K}$ whenever $i \in I$.*

The following lemma is an easy consequence from the commensurability of Γ and $h\Gamma h^{-1}$ when $h \in G_K$.

Lemma 3.1. *Let $h \in G_K$. The following assertions hold:*
- (a) *There exists a positive integer m such that $d(\xi)\pi(h) = \pi(h)$ for all $\xi \in \mathcal{O}^*_m$;*
- (b) *If $\{\pi(g_i)\}$ is a converging sequence in G/Γ then there exists a converging subsequence of $\{\pi(g_i h)\}$;*
- (c) *If $\overline{D_I\pi(g)} = G/\Gamma$ then $\overline{D_I\pi(gh)} = G/\Gamma$.*

Proposition 3.2. *Let $I = \{1, 2\}$ and $(s_k, t_k) \in K_1^* \times K_2^*$ be a sequence such that $|\log|s_k|_1| + |\log|t_k|_2| \underset{k}{\to} \infty$ and $d_I(s_k, t_k)\pi(g)$ converges to an element from G/Γ. Then:*
- (a) *There exists a constant $C > 1$ such that $-C < |\log|s_k|_1| - |\log|t_k|_2| < C$;*
- (b) *Let $|s_k|_1 \to \infty$, $|t_k|_2 \to 0$ and $-C < \log|s_k|_1 + \log|t_k|_2 < C$ where C is a positive constant. Then $g_1 g_2^{-1} = b_- b_+^{-1}$, where $b_- \in B_K^-$ and $b_+ \in B_K^+$.*

Proof.(a) The remaining cases being analogous, it is enough to consider the case when $|s_k|_1 \to \infty$ and $\frac{\max\{|t_k|_2, |t_k|_2^{-1}\}}{|s_k|_1} < \infty$.

Assume on the contrary that (a) is false. Then $\frac{\max\{|t_k|_2, |t_k|_2^{-1}\}}{|s_k|_1} \underset{k}{\to} 0$. Since $\mathrm{Ad}(h)\mathfrak{g}_{\mathcal{O}}$ is commensurable with $\mathfrak{g}_{\mathcal{O}}$ for every $h \in G_K$ there exists an $\mathbf{u} \in \mathrm{Ad}(g)\mathfrak{g}_{\mathcal{O}}, \mathbf{u} \neq 0$, such that $\mathrm{pr}_1(\mathbf{u})$ is a lower triangular nilpotent matrix where pr_1 is the projection of $\mathfrak{g}$ to $\mathfrak{g}_1$. (Recall

[1]This exemple is essentially due to Yves Benoist.

that $\mathfrak{g} = \prod_{i=1}^{r} \mathfrak{g}_i$.) Let $\mathrm{Ad}(d_I(s_k,t_k))(\mathbf{u}) = (\mathbf{u}_1^{(k)}, \cdots, \mathbf{u}_r^{(k)}) \in \mathfrak{g}$. Since $\frac{\max\{|t_k|_2, |t_k|_2^{-1}\}}{|s_k|_1} \underset{k}{\to} 0$, we get that $\|\mathbf{u}_1^{(k)}\|_1 \cdots \|\mathbf{u}_r^{(k)}\|_r \underset{k}{\to} 0$. Using Lemma 2.3, we find a sequence $\xi_k \in \mathcal{O}^*$ such that $\| \mathrm{Ad}(d_I(s_k,t_k))(\xi_k\mathbf{u}) \| = \|(\xi_k\mathbf{u}_1^{(k)}, \cdots, \xi_k\mathbf{u}_r^{(k)})\| \underset{k}{\to} 0$. It follows from Mahler's compactness criterion that $d_I(s_k,t_k)\pi(g)$ tends to infinity which is a contradiction.

(b) By Bruhat decomposition

$$G_K = B_K^+ \cup B_K^+ \omega B_K^+ = \omega B_K^+ \cup B_K^- B_K^+,$$

where $\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Suppose that $g_1 g_2^{-1} \in \omega B_K^+$. Shifting g from the right by g_2^{-1} and from the left by a suitable element from $\mathcal{N}_G(D_I)$ we reduce the proof (see Lemma 3.1(b)) to the case when $g_i = e$ for all $i > 1$ and $g_1 = \omega u^+(\alpha)$, where $u^+(\alpha) = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$, $\alpha \in K$. In view of (a), there exists a constant $C > 1$ such that $\frac{1}{C} < |s_k|_1 \cdot |t_k|_2 < C$. Now using Lemma 3.1(a) and Lemma 2.3 we find a sequence $\xi_k \in \mathcal{O}^*$ and a positif constant κ such that $d(\xi_k)\pi(u^+(\alpha)g_2) = \pi(u^+(\alpha)g_2)$ and $\frac{1}{\kappa} < \frac{|s_k|_1}{|\xi_k|_1} < \kappa$, $\frac{1}{\kappa} < \frac{|t_k|_2}{|\xi_k|_2} < \kappa$ and $\frac{1}{\kappa} < |\xi_k|_i < \kappa$ for all $i > 2$. Let $(s_k, t_k, e, \cdots, e) = \xi_k a_k$ where $a_k \in A^*$. Passing to a subsequence we can suppose that a_k converges to an element from A^*. Then $d(\xi_k)\pi(g)$ converges to an element from G/Γ.

By an easy computation:

$$\begin{aligned} &d(\xi_k)\pi(g) = d(\xi_k)(\omega u^+(\alpha), e, \cdots, e)\pi(g_2) = \\ &d(\xi_k)(\omega, u^+(-\alpha), \cdots, u^+(-\alpha))\pi(u^+(\alpha)g_2) = \\ &(\omega, u^+(-\alpha\xi_k^2), \cdots, u^+(-\alpha\xi_k^2))(d_1(\xi_k^{-2}), e, \cdots, e)\pi(u^+(\alpha)g_2). \end{aligned}$$

In view of the choice of ξ_k we have that $|\xi_k|_1 \to \infty$ and $|\xi_k|_2 \to 0$. Therefore $(\omega, u^+(-\alpha\xi_k^2), \cdots, u^+(-\alpha\xi_k^2))$ converges and $d_1(\xi_k^{-2})$ diverges. Using that $u^+(-\alpha)g_2 \in G_K$, it follows from Malher's criterion that $(d_1(\xi_k^{-2}), e, \cdots, e)\pi(u^+(\alpha)g_2)$ diverges. Hence $d(\xi_k)\pi(g)$ diverges too, which is a contradiction. So, $g_1 g_2^{-1} \in B_K^- B_K^+$. □

Proposition 3.3. *Let $I = \{1, \cdots, l\}$ where $1 < l \le r$, $g_1 = \cdots = g_{l-1}$ and $g_1 g_l^{-1} = b_- b_+^{-1}$ where $b_- \in B_K^-$ and $b_+ \in B_K^+$. Denote $h = b_-^{-1} g_1 = b_+^{-1} g_l$. Then we have the following:*

(a) $(h, \cdots, h, g_{l+1}, \cdots, g_r)\pi(e) \in \overline{D_I \pi(g)}$;

(b) *Let* $s_k = (s_k^{(1)}, \cdots, s_k^{(l)}) \in A_I^*$ *be such that* $|s_k^{(i)}|_i \underset{k}{\to} \infty$ *for all* $1 \le i < l$, $|s_k^{(l)}|_l \underset{k}{\to} 0$ *and* $\frac{1}{C} < |s_k^{(1)}|_1 \cdots |s_k^{(l)}|_l < C$, *where* C *is a positive constant. Then* $d_I(s_k)\pi(g)$ *admits a converging subsequence and the limit of every such subsequence belongs to* $\overline{D_I\pi((h, \cdots, h, g_{l+1}, \cdots, g_r))}$.

Proof. Fix m such that $d(\xi)\pi(h) = \pi(h)$ for all $\xi \in \mathcal{O}_m^*$. In view of Lemma 2.3, there exists a sequence $\xi_k \in \mathcal{O}_m^*$ and a constant $C_1 > 1$ such that $\frac{1}{C_1} < |s_k^{(i)}\xi_k^{-1}|_i < C_1$ if $1 \le i \le l$ and $\frac{1}{C_1} < |\xi_k|_i < C_1$ if $i > l$. Put $a_k = (\underbrace{\xi_k, \cdots, \xi_k}_{l}, \underbrace{e, \cdots, e}_{r-l})$ and $a'_k = (\underbrace{e, \cdots, e}_{l}, \underbrace{\xi_k, \cdots, \xi_k}_{r-l})$. Passing to a subsequence we may assume that $a'_k \to a'$ where $a' \in A^*$. In view of the choice of ξ_k and the proposition hypothesis, we get

$$\lim_k d_i(\xi_k)b_- d_i(\xi_k)^{-1} = t_-, \ \forall\ 1 \le i < l,$$

and

$$\lim_k d_l(\xi_k)b_+ d_l(\xi_k)^{-1} = t_+,$$

where t_- and $t_+ \in D_K$. It is enough to prove (b) in the particular case when $s_k^{(i)} = t_-^{-1}\xi_k$, $1 \le i < l$, and $s_k^{(l)} = t_+^{-1}\xi_k$.

Using the relation $d(\xi_k)\pi(h) = \pi(h)$, we get

(1)

$$\lim_k d_I(s_k)\pi(g) = (e, \cdots, g_{l+1}h^{-1}, \cdots, g_r h^{-1})d(a'^{-1})\pi(h) \in \overline{D_I\pi(g)}.$$

Since

$$d(a_k)^{-1}\pi(h) = d(a_k)^{-1}d(\xi_k)\pi(h) = d(a'_k)\pi(h) \to d(a')\pi(h),$$

multiplying (1) by $d(a_k)^{-1}$ and passing to a limit, we obtain that

$$(h, \cdots, h, g_{l+1}, \cdots, g_r)\pi(e) \in \overline{D_I\pi(g)}$$

which proves (a). In order to prove (b) it remains to note that

$$\lim_k d_I(s_k)\pi(g) = \lim_k d(a_k)\pi((h, \cdots, h, g_{l+1}, \cdots, g_r)).$$

□

Let $h \in G_K$. A pair $(\sigma_1, \sigma_2) \in \{0, 1\}^2$ is called *admissible with respect to* h if $\omega^{\sigma_1}h\omega^{\sigma_2} \in B_K^- B_K^+$, where $\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The following lemma can be proved by a simple calculation.

Lemma 3.4. *With the above notation,* (σ_1, σ_2) *is admissible with respect to* $h = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$ *if and only if* $m_{1+\sigma_1, 1+\sigma_2} \neq 0$.

It is clear that $h \in \mathcal{N}_{G_K}(D_K)$ if and only if the number of admissible pairs is equal to 2.

Proposition 3.5. *Let $I = \{1, \cdots, l\}$, where $1 < l < r$, $g_1 = \cdots = g_{l-1}$ and $g_1 g_l^{-1} \notin \mathcal{N}_{G_K}(D_K)$. Then $\overline{D_I \pi(g)}$ contains a point*

$$(\underbrace{nh, \cdots, nh}_{l-1}, h, g_{l+1}, \cdots, g_r)\pi(e),$$

where $n \in \mathcal{N}_{G_K}(D_K), h \in G_K$ and $hg_{l+1}^{-1} \notin \mathcal{N}_{G_K}(D_K)$.

Proof. If the pair (σ_1, σ_2) is admissible with respect to $g_1 g_l^{-1}$ then $\omega^{\sigma_1} g_1 (\omega^{\sigma_2} g_l)^{-1} = b_- b_+^{-1}$, where $b_- \in B_K^-$ and $b_+ \in B_K^+$, and we put $h_{\sigma_1,\sigma_2} = b_-^{-1} \omega^{\sigma_1} g_1 = b_+^{-1} \omega^{\sigma_2} g_2$. Shifting $\pi(g)$ from the left by

$$(\underbrace{\omega^{\sigma_1}, \cdots, \omega^{\sigma_1}}_{l-1}, \omega^{\sigma_2}, e, \cdots, e)$$

and applying Proposition 3.3(a) we get that

$$(\underbrace{\omega^{\sigma_1} h_{\sigma_1,\sigma_2}, \cdots, \omega^{\sigma_1} h_{\sigma_1,\sigma_2}}_{l-1}, \omega^{\sigma_2} h_{\sigma_1,\sigma_2}, g_{l+1}, \cdots, g_r)\pi(e) \in \overline{D_I \pi(g)}.$$

It remains to prove that (σ_1, σ_2) can be chosen in such a way that $h_{\sigma_1,\sigma_2} g_{l+1}^{-1} \notin \mathcal{N}_{G_K}(D_K)$. Since $g_1 g_l^{-1} \notin \mathcal{N}_{G_K}(D_K)$, in view of Lemma 3.4 there are at least 3 admissible pairs with respect to $g_1 g_l^{-1}$. Shifting g from the left by an appropriate element from $\mathcal{N}_{G_K}(D_K)$, we may assume that $(0,0)$ and $(0,1)$ are admissible pairs. Then

$$h_{0,0} = b_-'^{-1} g_1 = b_+'^{-1} g_2 \text{ and } h_{1,0} = \widetilde{b}_-^{-1} \omega g_1 = \widetilde{b}_+^{-1} g_2,$$

where $b'_-, \widetilde{b}_- \in B_K^-$ and $b'_+, \widetilde{b}_+ \in B_K^+$. Suppose on the contrary that both $h_{0,0} g_{l+1}^{-1}$ and $h_{1,0} g_{l+1}^{-1} \in \mathcal{N}_{G_K}(D_K)$. In view of the above expressions for $h_{0,0}$ and $h_{1,0}$, we obtain

$$h_{0,0} h_{1,0}^{-1} \in \mathcal{N}_{G_K}(D_K) \cap B_K^+ \cap B_K^- \omega B_K^-.$$

This is a contradiction because $\mathcal{N}_{G_K}(D_K) \cap B_K^+ = D_K$ and $D_K \cap B_K^- \omega B_K^- = \emptyset$. □

4. Proofs of Theorems 1.1, 1.2 and Corollary 1.4

4.1. **Proof of Theorem 1.1.** We suppose that $I = \{1, 2\}$. It follows from Proposition 2.2 that $g_1 g_2^{-1} \notin \mathcal{N}_G(D)$. Let $(s_k, t_k) \in K_1^* \times K_2^*$ be an unbounded sequence such that $d_I(s_k, t_k)\pi(g)$ converges. In view of Proposition 3.2(a) passing to a subsequence we may assume that $|\log |s_k|_1| - |\log |t_k|_2|$ converges. There exist σ_1 and $\sigma_2 \in \{0, 1\}$ such that $\omega^{\sigma_1} d_1(s_k)\omega^{-\sigma_1} = d_1(s'_k)$, $\omega^{\sigma_2} d_2(t_k)\omega^{-\sigma_2} = d_2(t'_k)$ where $|s'_k|_1 \to \infty$ and $|t'_k|_2 \to 0$. Let $g' = (\omega^{\sigma_1} g_1, \omega^{\sigma_2} g_2, g_3, \cdots, g_r)$.

It follows from Proposition 3.2(b) that $\omega^{\sigma_1} g_1(\omega^{\sigma_2} g_2)^{-1} = b_- b_+^{-1} \in B^- B^+$, i.e., (σ_1, σ_2) is an admissible pair with respect to $g_1 g_2^{-1}$. Let

$$h_{\sigma_1,\sigma_2} = b_-^{-1}\omega^{\sigma_1} g_1 = b_+^{-1}\omega^{\sigma_2} g_2. \tag{2}$$

Using Proposition 3.3(b) we get:

$$\lim_k d_I(s'_k, t'_k)\pi(g') \in \overline{D_I \pi((h_{\sigma_1,\sigma_2}, h_{\sigma_1,\sigma_2}, g_3, \cdots, g_r))}.$$

Therefore

$$\lim_k d_I(s_k, t_k)\pi(g) \in \overline{D_I \pi((\omega^{\sigma_1} h_{\sigma_1,\sigma_2}, \omega^{\sigma_2} h_{\sigma_1,\sigma_2}, g_3, \cdots, g_r))}.$$

In view of the above and of Proposition 3.3(a) we conclude that

$$\overline{D_I\pi(g)} = D_I\pi(g) \cup \cup_{(\sigma_1,\sigma_2)\in M} \overline{D_I \pi((\omega^{\sigma_1} h_{\sigma_1,\sigma_2}, \omega^{\sigma_2} h_{\sigma_1,\sigma_2}, g_3, \cdots, g_r))} \tag{3}$$

where M is the set of all admissible pairs with respect to $g_1 g_2^{-1}$. Note that

$$\overline{D_I \pi((\omega^{\sigma_1} h_{\sigma_1,\sigma_2}, \omega^{\sigma_2} h_{\sigma_1,\sigma_2}, g_3, \cdots, g_r))} = (\omega^{\sigma_1}, \omega^{\sigma_2}, g_3 h_{\sigma_1,\sigma_2}^{-1}, \cdots, g_r h_{\sigma_1,\sigma_2}^{-1})\overline{D_I \pi(h_{\sigma_1,\sigma_2})}.$$

Since $D\pi(h_{\sigma_1,\sigma_2})$ is a closed locally divergent orbit, each of the closures in the right hand side of (3) is a non-compact orbit of a torus containing D_I. It remain to see that at least two of these orbits are different.

Since $g_1 g_2^{-1} \notin \mathcal{N}_{G_K}(D_K)$ there exists $\sigma \in \{0, 1\}$ such that $(\sigma, 0)$ and $(\sigma, 1) \in M$. Suppose on the contrary that

$$\overline{D_I \pi(\omega^\sigma h_{\sigma,0}, h_{\sigma,0}, g_3, \cdots, g_r)} = \overline{D_I \pi(\omega^\sigma h_{\sigma,1}, \omega h_{\sigma,1}, g_3, \cdots, g_r)}.$$

Then there exist tori T and T' containing D_I such that

$$T\pi((h_{\sigma,0}, h_{\sigma,0}, g_3, \cdots, g_r)) = T'\pi((h_{\sigma,1}, \omega h_{\sigma,1}, g_3, \cdots, g_r)).$$

Then

$$h_{\sigma,0} = t h_{\sigma,1}\gamma = t'\omega h_{\sigma,1}\gamma,$$

where $t, t' \in D_K$ and $\gamma \in \Gamma$, which is a contradiction. □

4.2. **Proof of Theorem 1.2.** We need the following.

Proposition 4.1. *Let H be a Lie group and Δ be a discrete subgroup of H. Let $x \in H/\Delta$ and F be a closed connected subgroup of H. Suppose that $\overline{Fx}$ is not homogeneous (i.e. $\overline{Fx}$ is not an orbit of a subgroup containing F) and Fx is an open subset in $\overline{Fx}$. Then $\overline{Fx} = \mathrm{supp}(\mu)$, where μ is a non-algebraic, F-invariant, F-ergodic, Borel measure on H/Δ.*

Proof. It follows from the Baire category theorem that Fx, endowed with the relative topology, is homeomorphic to F/Δ_F, where $\Delta_F = \{\alpha \in F : \alpha x = x\}$ (cf.[Z, Lemma 2.1.15]). Let μ_F be the F-invariant (Haar) measure on F/Δ_F. Denote by μ the measure on H/Δ supported by Fx and induced by the homeomorphism between F/Δ_F and Fx. It is clear that the measure μ satisfies the proposition. $\square$

Proof of Theorem 1.2. Let us show that both $D_I\pi(g)$ and $D_{I,\mathbb{R}}\pi(g)$ are open and proper in their closures. With the notation from the formulation of the theorem, note that if $D_I\pi(g) \cap T_i\pi(h_i) \neq \emptyset$ for some $1 \leq i \leq s$ then $\overline{D_I\pi(g)} \subset T_i\pi(h_i)$ which contradicts the fact that $s \geq 2$. Therefore, the orbit $D_I\pi(g)$ is open and proper in its closure. Suppose that there exists i such that $\overline{D_{I,\mathbb{R}}\pi(g)} \cap T_i\pi(h_i) = \emptyset$. Since $T_i \supset D_I$ this implies that $\overline{D_I\pi(g)} \cap T_i\pi(h_i) = \emptyset$ which is a contradiction. Therefore, $\overline{D_{I,\mathbb{R}}\pi(g)} \cap T_i\pi(h_i) \neq \emptyset$ for every $1 \leq i \leq s$. So, the orbit $D_{I,\mathbb{R}}\pi(g)$ is open and proper in its closure too. Now the fact that $\overline{T\pi(g)}$ is not homogeneous is an easy exercise. $\square$

4.3. **Proof of Corollary 1.4.** Let G and Γ be as in the formulation of Corollary 1.3 with K a real quadratic number field. Using Weil's restriction of scalars [Z, Ch.6], we get an injective homomorphisme $\mathrm{R}_{K/\mathbb{Q}} : G \to \mathrm{SL}(4, \mathbb{R})$ such that $\mathrm{R}_{K/\mathbb{Q}}(\Gamma) = \mathrm{R}_{K/\mathbb{Q}}(G) \cap \mathrm{SL}(4, \mathbb{Z})$. Let $\phi : G \to \mathrm{SL}(n, \mathbb{R}), g \mapsto \begin{pmatrix} \mathrm{R}_{K/\mathbb{Q}}(g) & 0 \\ 0 & \mathrm{I}_{n-4} \end{pmatrix}$, where I_{n-4} is the identity matrix of rank $n-4$. Further on we identify G and Γ with $\phi(G)$ and $\phi(\Gamma)$, respectively. Let T be the connected component of the full diagonal group in $\mathrm{SL}(n, \mathbb{R})$, H be the connected component of the centralizer of G in $\mathrm{SL}(n, \mathbb{R})$, and S be the commutator subgroup of H. Note that S is a semisimple group. Put $T' = D \times T_S$, where $T_S = S \cap T$. It is clear from the construction that there exists a T'-equivariant injective map $G/\Gamma \times S/\Gamma_S \to G/\Gamma$ where $\Gamma_S = \Gamma \cap S$. Now let $\mu = \mu_1 \times \mu_2$ where μ_1 is the D-invariant non-algebraic measure given by Corollary 1.3 and μ_2 is the Haar measure on S/Γ_S. It is clear that the measure μ convenes for the conclusion of the corollary. $\square$

5. Closures of D_I-orbits when $\#I > 2$

5.1. If K is a CM-field we denote by F the totally real subfield of K of index 2. In this case we denote by F_i the completion of F with respect to the valuation $|\ .\ |_i$ on K_i and by $\mathcal{O}_F$ the ring of integers of F. We put $A_F = \prod\limits_i F_i$.

In this section $I = \{1, \cdots, l\}$ where $3 \leq l \leq r$.

Proposition 5.1. *Let* $h = (e, \cdots, \underset{l}{u_l^-(\beta)u_l^+(\alpha)}, \cdots, e) \in G$ *where* $u_l^-(\beta) = \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix}$, $u_l^+(\alpha) = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$, $\alpha \in K^*$ *and* $\beta \in K_l$. *The following assertions hold:*

(a) *If* K *is not a* CM*-field then* $\overline{D_I \pi(h)} = G/\Gamma$*;*
(b) *Let* K *be a* CM*-field and* d_α *be an element in* D_l *such that* $d_\alpha^2 = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$. *Then* $\overline{D_{I,\mathbb{R}}\pi(h)} \supset d_\alpha G_\mathbb{R} d_\alpha^{-1}\pi(e)$ *and* $d_\alpha G_\mathbb{R} d_\alpha^{-1}\pi(e)$ *is closed.*

In order to prove the proposition we need the following lemma.

Lemma 5.2. *Let* K *be a* CM*-field and* $\alpha \in K^*$. *Then*

$$\overline{F_l\alpha + \mathcal{O}} = A_F\alpha + \mathcal{O}.$$

Proof. Let n be a positive integer such that $n\alpha \in \mathcal{O}$. Since $\overline{F_l + \mathcal{O}_F} = A_F$ (by the strong approximation theorem) and $A_F \cap \mathcal{O} = \mathcal{O}_F$ we have that

$$\overline{F_l + \mathcal{O}} = A_F + \mathcal{O}.$$

Therefore

$$\overline{F_l\alpha + \mathcal{O}n\alpha} = A_F\alpha + \mathcal{O}n\alpha.$$

Now the lemma follows easily from the fact that $\mathcal{O}n\alpha$ is a lattice in A. □

Proof of Proposition 5.1. Note that $U^+(A)\pi(e)$ is closed and homeomorphic to $A/\mathcal{O}$. (We denote by $U^+(A)$ the group of A-points of the upper unipotent subgroup of G.) This implies that $u_l^+(K_l)\pi(e)$ is dense in $U^+(A)\pi(e)$ and, when K is a CM-field, it follows from Lemma 5.2 that $u_l^+(F_l\alpha)\pi(e)$ is dense in the closed set $U^+(A_F\alpha)\pi(e)$.

Further the proof proceeds in several steps.

Step 1. As in the formulation of Proposition 2.4, let H° be the connected component of the closure H of the projection of $\mathcal{O}^*$ into $K_l^* \times \cdots \times K_r^*$. Let $p_j : A^* \to K_j^*$, $l \leq j \leq r$, be the natural projections. We will consider the case (a) (when K is not a CM-field) and the case (b) (when K is a CM-field) in a parallel way. Using Proposition 2.4(a), for every positive integer m we fix in H° a compact neighborhood H_m of 1 with the following properties: (i) $1 - \frac{1}{m} < |p_j(x)|_j < 1 + \frac{1}{m}$ for all $j \geq l$ and all $x \in H_m$ and, (ii) $p_l(H_m) = \{e^{(a_m + \imath b_m)t} : t \in [-\frac{1}{m}, \frac{1}{m}]\}$, where $\imath = \sqrt{-1}$ and a_m and b_m are reals such that $b_m \neq 0$ if $K_l = \mathbb{C}$ and we are in case (a), and $a_m \neq 0$ and $b_m = 0$, otherwise. In view of Proposition 2.4(b) there exists a sequence $y_n \in \mathcal{O}^*$ (respectively, $y_n \in \mathcal{O}_F^*$ in case (b)) such that $|p_l(y_n)|_l > n$ and $1 - \frac{1}{n} < |p_j(y_n)|_j < 1 + \frac{1}{n}$ for all $j > l$.

Step 2. Denote

$$L_{mn} = \{x^2 : x \in y_n H_m\}.$$

Let W_ε be the ε-neighborhood of 0 in A and $W_{\varepsilon,F}$ be the ε-neighborhood of 0 in A_F. We claim that given m for every $\varepsilon > 0$ there exists a constant $n_\circ$ such that if $n > n_\circ$ then

$$A = W_\varepsilon + p_l(L_{mn}) + \mathcal{O} \tag{4}$$

in case (a), and

$$A_F = W_{\varepsilon,F} + p_l(L_{mn}) + \mathcal{O}_F \tag{5}$$

in case (b).

Note that the projections of K_l into $A/\mathcal{O}$ and of F_l into $A_F/\mathcal{O}_F$ are dense and equidistributed. Since $|p_l(y_n)|_l > n$ this implies the claim in case (b) and in case (a) when $K_l = \mathbb{R}$.

Consider the case (a) when $K_l = \mathbb{C}$. If $\theta \in [0, 2\pi)$ we put $\mathbb{R}_\theta = e^{\imath\theta}\mathbb{R}$ and if $a < b$ we put $[a,b]_\theta = e^{\imath\phi}[a,b]$ where $\mathbb{R}$ stands for the subfield of reals in K_l. Since $\overline{K_l + \mathcal{O}} = A$ it is easy to see that for almost all $\theta \in [0, 2\pi)$ we have that $\overline{\mathbb{R}_\theta + \mathcal{O}} = A$ and, moreover, given $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that if $b - a > c_\varepsilon$ then

$$A = W_\varepsilon + z + [a,b]_\theta + \mathcal{O}, \ \forall z \in A.$$

Now let $p_l(y_n) = r_n e^{\imath\frac{\psi_n}{2}}$ and $\psi_n \underset{n}{\to} \psi$. Since the real b_m in the definition of H_m is different from 0 there exists $\frac{\theta}{2} \in (-\frac{1}{m}, \frac{1}{m})$ such that $\overline{\mathbb{R}_{\theta+\psi} + \mathcal{O}} = A$. Remark that since $r_n \to +\infty$ the curvatures at the points of the plane curve $p_l(L_{mn}) \subset \mathbb{C}$ are tending uniformly to 0 when $n \to \infty$. Therefore for every real $\beta > 0$ end every $\varepsilon > 0$ there exist a positive integer $n_\circ$ such that for every $n > n_\circ$ there exists a $z \in K_l$ such that the points of the segment $z + [0,\beta]_{\theta+\psi}$ are ε-close to $p_l(L_{nm})$. This implies the claim.

Step 3. Since $d(\xi^{-1})\pi(e) = \pi(e)$ for every $\xi \in \mathcal{O}^*$ we have that $(e, \cdots, u_l^-(\xi^{-2}\beta)u_l^+(\xi^2\alpha), d_{l+1}(\xi)^{-1}, \cdots, d_r(\xi)^{-1})\pi(e)$ belongs to $D_I\pi(h)$ (respectively, $D_{I,\mathbb{R}}\pi(h)$) if K is not (respectively, is) a CM-field. Therefore

$$X_{mn} \overset{def}{=} \{(e, \cdots, \underset{l}{u_l^-(x^{-2}\beta)u_l^+}(x^2\alpha), \cdots, d_r(x)^{-1})\pi(e) : x \in y_n H_m\} \tag{6}$$

is a subset of $\overline{D_I\pi(h)}$ in case (a) and of $\overline{D_{I,\mathbb{R}}\pi(h)}$ in case (b). Using the commensurability of $\mathcal{O}$ and $\mathcal{O}\alpha$ we deduce from (4) and (5) that for every m

$$\overline{\bigcup_n p_l(L_{mn}\alpha) + \mathcal{O}} = A \tag{7}$$

in case (a) and

$$\overline{\bigcup_n p_l(L_{mn}\alpha) + \mathcal{O}} = A_F\alpha + \mathcal{O} \tag{8}$$

in case (b). On the other hand, it follows from the definitions of H_m and y_n that for every $\delta > 0$ there exists c_δ such that if $\min\{m, n\} > c_\delta$ then $|x^{-2}\beta|_l < \delta$ and $||x|_j - 1| < \delta$ for all $x \in y_n H_m$. Now it follows from (6), (7) and (8) that $U^+(A)\pi(e) \subset \overline{D_I\pi(g)}$ in case (a) and $U^+(A_F\alpha)\pi(e) \subset \overline{D_{I,\mathbb{R}}\pi(g)}$ in case (b).

Step 4. Let B_1^+ and $B_{1,\mathbb{R}}^+$ be the upper triangular subgroup of G_1 and $G_{1,\mathbb{R}}$, respectively. In view of *Step 3* $B_1^+\pi(e) \subset \overline{D_I\pi(h)}$ in case (a) and $B_{1,\mathbb{R}}^+\pi(e) \subset \overline{D_{I,\mathbb{R}}\pi(h)})$ in case (b). Note that B_1^+ and $B_{1,\mathbb{R}}^+$ are epimorphic subgroups of G_1 and $d_\alpha G_{1,\mathbb{R}} d_\alpha^{-1}$, where d_α is as in the formulation of the proposition, respectively. It follows from [Sh-W, Theorem 1] that $\overline{B_1^+\pi(e)} = \overline{G_1\pi(e)}$ and $\overline{B_{1,\mathbb{R}}^+\pi(e)} = \overline{d_\alpha G_{1,\mathbb{R}} d_\alpha^{-1}\pi(e)}$. Since in case (b) $d_\alpha^{-1}\Gamma d_\alpha$ and Γ are commensurable subgroups of G, $d_\alpha G_{\mathbb{R}} d_\alpha^{-1}\pi(e)$ is closed. It follows from Borel's density theorem [R] that $\overline{G_1\pi(e)} = G/\Gamma$ and $\overline{d_\alpha G_{1,\mathbb{R}} d_\alpha^{-1}\pi(e)} = d_\alpha G_{\mathbb{R}} d_\alpha^{-1}\pi(e)$. Therefore $\overline{D_I\pi(h)} = G/\Gamma$ in case (a) and $\overline{D_{I,\mathbb{R}}\pi(h)} \supset d_\alpha G_{\mathbb{R}} d_\alpha^{-1}\pi(e)$ in case (b). □

5.2. **Proofs of Theorem 1.5 and Corollary 1.7.** It is enough to prove Theorem 1.5 for $I = \{1, 2, 3\}$. We may (and will) assume that $g_i \in G_{i,K},\ i \in I$. By the theorem hypothesis either $g_1 g_2^{-1} \notin \mathcal{N}_{G_K}(D_K)$ or $g_2 g_3^{-1} \notin \mathcal{N}_{G_K}(D_K)$ (see Proposition 2.2). Suppose that $g_1 g_2^{-1} \notin \mathcal{N}_{G_K}(D_K)$. In view of Proposition 3.5 there exists an element $\pi(g') \in \overline{D_I\pi(g)}, g' = \{g'_1, \cdots, g'_r\}$, such that $g'_i \in G_K$ if $1 \le i \le 3$, $g'_1 g'^{-1}_2 \in \mathcal{N}_{G_K}(D_K)$, $g'_1 g'^{-1}_3 \notin \mathcal{N}_{G_K}(D_K)$ and $g'_i = g_i$ if $i > 3$. Clearly, if $n \in D_I\pi(g)$ and $k \in G_K$ then $D_I\pi(g')$ is dense if and only if $D_I\pi(ng'k)$ is dense (see Lemma 3.1(c)). Therefore we may assume without loss of generality that $\overline{D_I\pi(g)}$ contains an element $\pi(h)$ where h is as in the formulation of Proposition 5.1. Now Theorem 1.5 follows from Proposition 5.1(a).

Let us prove Corollary 1.7. By Moore's ergodicity theorem [Z], $D_{I,\mathbb{R}}$ is ergodic on G/Γ. Therefore there exists an $y \in G/\Gamma$ such that $D_{I,\mathbb{R}}y$ is dense in G/Γ. By Theorem 1.5, $\overline{D_I\pi(g)} = G/\Gamma$. Therefore there exists a compact $M \subset D_I$ such that $M\overline{D_{I,\mathbb{R}}\pi(g)} = G/\Gamma$. Let $y = mz$, where $m \in M$ and $z \in \overline{D_{I,\mathbb{R}}\pi(g)}$. Then

$$\overline{D_{I,\mathbb{R}}\pi(g)} \supset m^{-1}\overline{D_{I,\mathbb{R}}y} = G/\Gamma$$

which completes the proof. □

5.3. **Proof of Theorem 1.8.** Recall that $I = \{1, \cdots, l\}$, $l \geq 3$. Choose $g = (e, \cdots, \underset{l}{u_l^+(\alpha)u_l^-(\beta)}, \cdots, e)$ where $\alpha \in K \setminus F$, and $\beta \in F^*$. We will prove that $x = \pi(g)$ is the point we need. First, remark that $u_l^+(\alpha)u_l^-(\beta) = tu_l^-(\beta_1)u_l^+(\alpha_1)$ where $t \in D_{l,K}$, $\beta_1 \in K$ and $\alpha_1 = \frac{\alpha}{1+\alpha\beta}$. Hence $\alpha_1 \in K \setminus F$. Let $d_{\alpha_1} \in D_l$ be such that $d_{\alpha_1}^2 = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_1^{-1} \end{pmatrix}$. It follows from Proposition 5.1(b) that

$$\overline{D_{I,\mathbb{R}}x} \supset G_{\mathbb{R}}\pi(e) \cup d_{\alpha_1}G_{\mathbb{R}}d_{\alpha_1}^{-1}\pi(e). \tag{9}$$

Note that the orbits $G_{\mathbb{R}}\pi(e)$ and $d_{\alpha_1}G_{\mathbb{R}}d_{\alpha_1}^{-1}\pi(e)$ are closed and proper.

Since $G_{\mathbb{R}}\pi(e) \supset U^-(A_F)\pi(e)$ and $d_{\alpha_1}G_{\mathbb{R}}d_{\alpha_1}^{-1}\pi(e) \supset U^+(A_F\alpha_1)\pi(e)$ we have that

$$\overline{D_{I,\mathbb{R}}x} \subset \bigcup_{0\leq\mu\leq 1}\{u_l^+(\mu\alpha)G_{\mathbb{R}}\pi(e)\}\bigcup\bigcup_{0\leq\nu\leq 1}\{tu_l^-(\nu\beta_1)d_{\alpha_1}G_{\mathbb{R}}d_{\alpha_1}^{-1}\pi(e)\}$$

where μ and $\nu \in F_l$. This implies (i).

Let us prove (ii). Using Proposition 2.4 we can choose a sequence $\xi_i \in \mathcal{O}_F^*$ such that for every $j \geq l$ the projection of ξ_i into F_j converges to some $x_j \in F_j^*$ and x_l is not an algebraic number. Put

$$y = (e, \cdots, u_l^+(x_l^2\alpha)u_l^-(x_l^{-2}\beta), d_{l+1}(x_{l+1}^{-1}), \cdots, d_r(x_r^{-1}))\pi(e).$$

Then

$$y = \lim_i d_I(\xi_i)x \in \overline{D_{I,\mathbb{R}}x}.$$

Let us show that $y \notin D_{I,\mathbb{R}}x$. Otherwise, there exist elements $d \in D_l$ and $m \in G_{l,K}$ such that $du_l^+(x_l^2\alpha)u_l^-(x_l^{-2}\beta) = u_l^+(\alpha)u_l^-(\beta)m$. Since $u_l^+(\alpha)u_l^-(\beta)m \in G_{l,K}$ the lower right coefficient of $du_l^+(x_l^2\alpha)u_l^-(x_l^{-2}\beta)$ belongs to K. This implies that $d \in D_{l,K}$ and that $x_l^2\alpha \in K$ which contradicts our choice of x_l, proving the claim.

Let H be a subgroup of G such that $H \supset D_{I,\mathbb{R}}$ and Hy be closed. It is easy to see that

$$x = \lim_i d_I(\xi_i^{-1})y.$$

In view of (9), H contains both $G_{\mathbb{R}}$ and $d_{\alpha_1}G_{\mathbb{R}}d_{\alpha_1}^{-1}$. Since $\alpha_1 \in K \setminus F$ we get that $A = A_F + A_F\alpha_1$. Therefore, $H \supset U^+(A) \cup U^-(A)$. Hence $H = G$ which proves (ii).

In order to prove (iii), suppose on the contrary that

$$\overline{D_{I,\mathbb{R}}x} \setminus D_{I,\mathbb{R}}x \subset \bigcup_i H_i x_i$$

where H_i are finitely many proper subgroups of G and $H_i x_i$ are closed orbits. Therefore there exists H_i such that $D_{I,\mathbb{R}} \subset H_i$ and $y \in H_i x_i$ which contradicts (ii). The theorem is proved. □

5.4. **Proof of Corollary 1.9.** If $g \in \mathcal{N}_G(D_I)G_K$ then it follows from Proposition 2.2 that $\overline{D_I\pi(g)}$ is an orbit of a torus. In order to prove the converse, note that $D_I\pi(g)$ is locally divergent if and only if $g \in \bigcap_{i\in I}\mathcal{N}_G(D_i)G_K$ (Theorem 2.1(b)). Since $\#I \geq 2$, it is obvious that $\bigcap_{i\in I}\mathcal{N}_G(D_i)G_K \supsetneqq \mathcal{N}_G(D_I)G_K$. If $g \in \left(\bigcap_{i\in I}\mathcal{N}_G(D_i)G_K\right) \setminus \mathcal{N}_G(D_I)G_K$ it follows from Theorem 1.1 when $\#I = 2$ and from Proposition 5.1 when $\#I > 2$ that $\overline{D_I\pi(g)}$ is not an orbit of a torus. □

6. A number theoretical application

In this section we prove Theorem 1.10. We use the notation preceding the formulation of the theorem.

We identify the elements from G/Γ with the lattices in A^2 obtained via the injective map $g\Gamma \mapsto g\mathcal{O}^2$. This map is continuous and proper with respect to the quotient topology on G/Γ and the topology of Chabauty on the space of lattices in A^2.

The group G_K is acting on $K[X,Y]$ by the law

$$(\sigma p)(X,Y) = p(\sigma^{-1}(X,Y)), \forall \sigma \in G_K,\ \forall p \in K[X,Y].$$

By the theorem hypothesis $f_i(X,Y) = l_{i,1}(X,Y)\cdot l_{i,2}(X,Y)$ where $l_{i,1}$ and $l_{i,2} \in K[X,Y]$ are linearly independent over K linear forms. There exist $g_i \in G_{i,K}$ such that $f_i(X,Y) = \alpha_i(g_i^{-1}f_0)(X,Y)$ where $\alpha_i \in K^*$ and f_0 is the form $X\cdot Y$. We may (and will) suppose that $\alpha_i = 1$ for all i. Since the forms $f_i,\ 1 \leq i \leq r$ are not proportional, $g = (g_1,\cdots,g_r)$ does not belong to $\mathcal{N}_G(D)G_K$. Therefore $D\pi(g)$ is a locally divergent non-closed orbit (Theorem 2.1(b)).

Let $r > 2$. Fix $a = (a_1,\cdots,a_r) \in A$ and choose $h \in G$ such that $he_1 = (a,1)$ where e_1 is the first vector of the canonical basis of the free A-module A^2. According to Theorem 1.5, $D\pi(g)$ is a dense orbit. Therefore there exist $d_n \in D$ and $\gamma_n \in \Gamma$ such that $\lim_n d_n g\gamma_n = h$. Put $z_n = \gamma_n e_1$. Then $z_n \in \mathcal{O}^2$ and

$$\lim_n f(z_n) = \lim_n f_0(d_n g\gamma_n e_1) = f_0(\lim_n (d_n g\gamma_n(e_1))) = f_0(a,1) = a,$$

which proves the part (a) of the theorem.

Let $r = 2$. We will prove the inclusion

$$\overline{f(\mathcal{O}^2)} \subset f(\mathcal{O}^2) \bigcup \overset{4}{\underset{j=1}{\cup}} \phi^{(j)}(\mathcal{O}^2)) \bigcup K_1' \times \{0\} \bigcup \{0\} \times K_2', \tag{10}$$

where $\phi^{(j)}$, K_1' and K_2' are as in the formulation of the theorem. Let $a = (a_1, a_2) \in \overline{f(\mathcal{O}^2)} \setminus f(\mathcal{O}^2)$. There exists a sequence $z_n = (\alpha_n, \beta_n)$ in $\mathcal{O}^2$ such that $a = \lim_n f(z_n)$ and $f(z_n) \neq 0$ for all n. Let $a_1 \neq 0$. (The case $a_1 \neq 0$ is analogous.) If $f_2(z_n) = 0$ for infinitely many n then it is easy to see that $a \in K_1' \times \{0\}$. From now on we suppose that $f_2(z_n) \neq 0$ for all n.

Let $g = (g_1, g_2) \in G$ be such that $g_i(X, Y) = (l_{i1}(X, Y), l_{i2}(X, Y))$, $i \in \{1, 2\}$. We choose sequences $s_n \in K_1^*$ and $t_n \in K_2^*$ such that

$$\left\{ \begin{array}{lcl} \lim_n s_n l_{11}(z_n) & = & a_{11} \\ \lim_n s_n^{-1} l_{12}(z_n) & = & a_{12} \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{lcl} \lim_n t_n l_{21}(z_n) & = & a_{21} \\ \lim_n t_n^{-1} l_{22}(z_n) & = & a_{22} \end{array} \right. \tag{11}$$

where $a_{11}, a_{12} \in K_1$, $a_{21}, a_{22} \in K_2$, $a_1 = a_{11} \cdot a_{12}$ and $a_2 = a_{21} \cdot a_{22}$.

If $a_2 = 0$ we choose t_n in such a way that

$$a_{21} = a_{22} = 0. \tag{12}$$

We have

$$\lim_n d(s_n, t_n) g(z_n) = (\mathbf{a_1}, \mathbf{a_2}) \tag{13}$$

where $\mathbf{a_1} = (a_{11}, a_{12}) \in K_1^2$ and $\mathbf{a_2} = (a_{21}, a_{22}) \in K_2^2$.

Shifting g from the left by an element from $\mathcal{N}_{G_K}(D_K)$ if necessary, we reduce the proof to the case when $|s_n|_1 \to \infty$ and $|t_n|_2 \leq 1$. There exist μ and $\nu \in K$ such that

$$l_{22} = \mu l_{11} + \nu l_{12}.$$

We have

$$\begin{aligned} 0 < |\mathrm{N}_{K/\mathbb{Q}}(l_{22}(z_n))| &= |l_{22}(z_n)|_1 \cdot |l_{22}(z_n)|_2 = \\ &= |s_n|_1 \cdot |t_n|_2 \cdot |\mu s_n^{-1} l_{11}(z_n) + \nu s_n^{-1} l_{12}(z_n)|_1 \cdot |t_n^{-1} l_{22}(z_n)|_2. \end{aligned}$$

Since $\{\mathrm{N}_{K/\mathbb{Q}}(l_{22}(z_n))\}$ is a discrete subset of $\mathbb{R}$ which does not contain 0, in view of (11), we obtain that

$$\liminf_n |s_n|_1 \cdot |t_n|_2 > 0 \tag{14}$$

and that

$$|a_{22}|_2 = \lim_n |t_n^{-1} l_{22}(z_n)|_2 \neq 0.$$

The latter contradicts (12). Hence $a_2 \neq 0$.

Let us prove that

$$g_1 g_2^{-1} \in B_K^- B_K^+. \tag{15}$$

First we need to show that

$$\limsup_n |s_n|_1 \cdot |t_n|_2 < \infty. \tag{16}$$

There exist μ' and $\nu' \in K$ such that

$$l_{11} = \mu' l_{21} + \nu' l_{22}.$$

Then

$$0 < |\mathrm{N}_{K/\mathbb{Q}}(l_{11}(z_n))| = |l_{11}(z_n)|_1 \cdot |l_{11}(z_n)|_2 =$$
$$= |s_n|_1^{-1} \cdot |t_n|_2^{-1} \cdot |s_n l_{11}(z_n)|_1 \cdot |\mu' t_n l_{21}(z_n) + \nu' t_n l_{22}(z_n)|_2.$$

Now (16) follows from the inequality $|t_n|_2 \leq 1$ and (11).

Suppose on the contrary that $g_1 g_2^{-1} \notin B_K^- B_K^+$. Therefore $g_1 g_2^{-1} \in \omega B_K^+$. Shifting g from the left by a suitable element from D_K we reduce the proof to the case when $g_1 g_2^{-1} = \omega u$, where $u = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$. In view of (14), (16), Lemma 2.3 and Lemma 3.1 we can find a sequence $\xi_n \in \mathcal{O}^*$ and a converging to $a \in A^*$ sequence $a_n \in A^*$ such that $(s_n, t_n) = \xi_n a_n$ and $d(\xi_n) g_2 \mathcal{O}^2 = g_2 \mathcal{O}^2$. Using (13) we see that $d(\xi_n) g(z_n)$ converges to some $(\mathbf{b_1}, \mathbf{b_2}) \in A^2$ where $\mathbf{b_1} = (b_{11}, b_{12}) \in K_1^2$ and $\mathbf{b_2} = (b_{21}, b_{22}) \in K_2^2$. (Recall that A^2 is identify with $K_1^2 \times K_2^2$.) An easy computation shows that

$$d(\xi_n) g(z_n) = (h_n, e)\mathbf{w_n}$$

where $h_n = \begin{pmatrix} 0 & \xi_n^2 \\ -\xi_n^{-2} & -\alpha \end{pmatrix}$ and $\mathbf{w_n} = d(\xi_n) g_2(z_n) = (\beta_n, \gamma_n) \in g_2 \mathcal{O}^2$. So, $\big((\xi_n^2 \gamma_n, -\xi_n^{-2}\beta_n - \alpha\gamma_n), (\beta_n, \gamma_n)\big) \to (\mathbf{b_1}, \mathbf{b_2})$ which implies that $(\xi_n^2 \gamma_n, \gamma_n)$ converges to (b_{11}, b_{22}) in A. But

$$|\xi_n^2 \gamma_n|_1 \cdot |\gamma_n|_2 = |\xi_n^2|_1 \cdot |\mathrm{N}_{K/\mathbb{Q}}(\gamma_n)|.$$

Hence

$$\lim_n |\xi_n^2|_1 \cdot |\mathrm{N}_{K/\mathbb{Q}}(\gamma_n)| = |b_{11}|_1 \cdot |b_{22}|_2$$

which is a contradiction because $|\xi_n^2|_1 \to \infty$ and $\liminf_n |\mathrm{N}_{K/\mathbb{Q}}(\gamma_n)| > 0$. This complets the prove of (15).

In view of Proposition 3.3(b), there exists a subsequence of $d(s_n, t_n)\pi(g)$ converging to an element from $\bigcup_{j=1}^{s} D\pi(h_j)$, $2 \leq s \leq 4$ where $h_j \in \mathcal{N}_{G_K}(D_K)$ (see Corollary 1.3). So, there exists $d \in D$ such that $(\mathbf{a}_1, \mathbf{a}_2) \in d h_j \mathcal{O}^2$, $1 \leq j \leq s$. Hence $a \in \bigcup_{j=1}^{s} \phi^{(j)}(\mathcal{O}^2)$ where $\phi^j = h_j^{-1} f_0$. This complets the proof of (10).

The inclusion inverse to (10) is easy to prove. Let $c = \phi^{(j)}(z)$ where $z \in \mathcal{O}^2$. We have $h_j = \lim_n t_n g \sigma_n$ for some $t_n \in D$ and $\sigma_n \in \Gamma$. Therefore

$$\phi^{(j)}(z) = \lim_n f_0(t_n g \sigma_n(z)) = \lim_n f(\sigma_n(z)) \in \overline{f(\mathcal{O}^2)}.$$

It remains to prove that $\bigcup K_1' \times \{0\} \bigcup \{0\} \times K_2' \subset \overline{f(\mathcal{O}^2)}$. It is enough to prove that if $(x, y) \in K_1^2$ and $f_2(x, y) = 0$ then $(f_1(x, y), 0) \in \overline{f(\mathcal{O}^2)}$. Suppose that $l_{21}(x, y) = 0$. Since l_{11} and l_{12} are linear combinations of l_{21} and l_{22} we get that $f_1(x, y) = c \cdot l_{22}(x, y)^2$ where c is a constant. Note that the projection of the set $\{l_{22}(z) : z \in \mathcal{O}^2, l_{21}(z) = 0\}$ into K_1 is dense. Therefore $(f_1(x, y), 0) \in \overline{f(\mathcal{O}^2)}$. By similar reasons if $l_{22}(x, y) = 0$ then $f_1(x, y) = d \cdot l_{21}(x, y)^2 \in \overline{f(\mathcal{O}^2)}$, where d is a constant. Note that $K_1' = c\{\alpha^2 : \alpha \in K_1\} \cup d\{\alpha^2 : \alpha \in K_1\}$ and that, since f_1 and f_2 are not proportional, c and d can not be simultaneously equal to zero. This readily implies that $K_i' = \mathbb{C}$ if $K_i = \mathbb{C}$ and $K_i' = \mathbb{R}, \mathbb{R}_-$ or $\mathbb{R}_+$ if $K_i = \mathbb{R}$. The proof is complete. $\square$

7. Concluding remarks

1. The elements h_i in the formulation of the Theorem 1.1 can be explicitly described in terms of g. Let us give an exemple of an orbit $D_I \pi(g)$, $I = \{1, 2\}$, such that the boundary of its closure consists of *four* different closed orbits.

For simplicity we will assume that $r = 2$. (The arguments are virtually the same if $r > 2$.) Choose $g_1 = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$ and $g_2 = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$, where α and β are numbers in K such that $\alpha \cdot \beta \neq 0$, $\alpha \cdot \beta \neq 1$ and there exists a non-archimedean valuation v of K with $|\alpha|_v > 1$ and $|\beta|_v < 1$.

Since all coefficients of the matrix $g_1 g_2^{-1}$ are different from 0, all pairs $(\sigma_1, \sigma_2) \in \{0, 1\}^2$ are admissibles and, in view of (3), we need to prove that the closed orbits $D(\omega^{\sigma_1}, \omega^{\sigma_2})\pi(h_{\sigma_1, \sigma_2})$ are pairwise different. We have seen in the course of the proof of Theorem 1.1 that $D(\omega^{\sigma_1}, \omega^{\sigma_2})\pi(h_{\sigma_1, \sigma_2}) \neq D(\omega^{\sigma_1'}, \omega^{\sigma_2'})\pi(h_{\sigma_1', \sigma_2'})$ if $(\sigma_1, \sigma_2) = (0, 0)$ or $(1, 1)$ and $(\sigma_1', \sigma_2') = (0, 1)$ or $(1, 0)$. It remains to show that $D\pi(h_{0,0}) \neq D\pi(\omega h_{1,1})$ and $D(\omega, 1)\pi(h_{1,0}) \neq D(1, \omega)\pi(h_{0,1})$.

Using (2) we see that $h_{0,0} = e$ and modulo multiplication from the left by an element from D_K, $\omega h_{1,1}$ is equal to $\begin{pmatrix} \frac{1}{1-\alpha\beta} & \frac{\beta}{1-\alpha\beta} \\ \alpha & 1 \end{pmatrix}$. Since $\alpha \notin \mathcal{O}$ we conclude that $D\pi(h_{0,0}) \neq D\pi(\omega h_{1,1})$.

Modulo multiplication from the left by an element from D_K, $h_{1,0}$ (respectively, $h_{0,1}$) is equal to $\begin{pmatrix} 1 & \frac{1}{\alpha} \\ 0 & 1 \end{pmatrix}$ (respectively, $\begin{pmatrix} 1 & 0 \\ \frac{1}{\beta} & 1 \end{pmatrix}$). If $D(\omega,1)\pi(h_{1,0}) = D(1,\omega)\pi(h_{0,1})$ then

$$\frac{\xi^2\beta + \alpha}{\alpha\beta} \in \mathcal{O}$$

for some $\xi \in \mathcal{O}^*$. This leads to contradiction because in view of the choice of α and β

$$\frac{|\xi^2\beta + \alpha|_v}{|\alpha\beta|_v} = \frac{1}{|\beta|_v} > 1.$$

Therefore the boundary of $D_I\pi(g)$ consists of four pairwise different closed orbits.

2. Most of the results of this paper remain valid with small or without changes in the S-adic setting, that is, when G is a product of $\mathrm{SL}(2, K_v)$, where K_v is the completion of a number field K with respect to a place v belonging to a finite set S of places of K containing the archimedean ones. For instance, the proofs of the analogs of Theorems 1.1 and 1.10(b) remain valid in this context without any changes. The analog of Theorem 1.5 remains true with very small modifications if $K = \mathbb{Q}$ or if K is arbitrary and I contains an archimedean place. For instance, Theorem 1.5 remains true for action of maximal tori (that is, when $D = D_I$). The analog of Theorem 1.5 in the general case (for arbitrary K and I) is more delicate and will be treated later. Using the present approach, one can find tori orbits with non-homogeneous closures on spaces G/Γ where G is not a product of SL_n's. This will be treated elsewhere too.

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